DEPARTMENT OF MATHEMATICS
University of Sargodha, Sargodha

M. Phil. Admission Test

Date: 30-08-2010; Max. Marks: 100; Time Allowed: 2 Hours

Instructions:

- **Attempt any 10 questions.**
- **You may not talk to anyone once the exam begins and keep your eyes on your own paper.**
- **Read all questions very carefully before answering them.**

Q. No.1

(a) What is $\arg Z$ of a complex number Z and what is $Arg\,Z$? Is $Arg\,Z$ a unique value? Find $Arg(-1-i)$.

(b) If $f(x)=\dfrac{z^3+2z}{(z-1)^3}$, find residue at the pole of $f(z)$. **(4+6)**

Q. No. 2

(a) In (i) and (ii) below determine whether the statements are true or false and in (iii) fill in the blank.

(i) Every non empty set can be made into a metric space by giving it the discrete metric **(T/F)**

(ii) An open map is always continuous **(T/F)**

(iii) Let $X=\{a,b,c,d\}$ and $\tau=\{\phi,X,\{a\}\}$ be a topology on X. Then the closed subsets of X are------------------

(b) Give a counter example to show that an arbitrary intersection of open sets in a metric space need not be open.

(c) Let (X,τ) and (Y,τ') be two given topological spaces. If $f:X\to Y$ is a constant map, then show that f is continuous. **(3+2+5)**

Q. No. 3

(a) Prove that every convergent sequence in a metric space is a Cauchy sequence.

(b) If T is a bounded linear operator, then we define norm on T by $\|T\|=\sup\limits_{\substack{x\in D(T)\\ x\neq 0}}\dfrac{\|Tx\|}{\|x\|}$.

Show that an alternate formula for norm on T is $\|T\|=\sup\limits_{\substack{x\in D(T)\\ x=1}}\dfrac{\|Tx\|}{\|x\|}$. **(5+5)**

Q. No. 4

(a) Solve the following differential equation by using variation of Parameters Method

$$4\frac{d^2y}{dx^2}+36y=\operatorname{cosec}3x$$

(b) Define the General Solution of the Differential Equation.

(c) What are Initial-value Problems? **(6+2+2)**

Q. No. 5

(a) Solve the system by Decomposition Method.

$$4x_1+x_2-x_3=2$$
$$x_1+3x_2+5x_3=3$$
$$x_1-x_2+x_3=3$$

(b) Write down the formula for computing Absolute errors in any number.

(c) Determine whether the following statement true or false

Taylor's series method is used to solve the higher order Ode's **(T/F)**

(6+2+2)

Q. No. 6

(a) Fill in the blanks

(i) The nilpotent groups form a class of groups lying strictly between the ----------------- groups and solvable groups.
(ii) A dihedral group D_n is nilpotent iff n is a power of --------------------.
(iii) Every finitely generated abelian group is the direct product of finite number of--------------------
(iv) If F is a field of characteristic zero, then representation over F is called --------------

(b) Give an example of non trivial group that is not of prime order and is not the direct product of its nontrivial subgroups.

(c) Is D_3 a nilpotent group? Justify your answer. Also give an example which shows that every solvable group is not nilpotent.

(4+2+4)

Q. No. 7

(a) Tick the correct choice

(i) The sum of the series $\sum_{n=1}^{\infty} \frac{n}{4^{n+1}}$ is

(a) $\frac{1}{12}$ (b) $\frac{4}{3}$ (c) $\frac{1}{9}$ (d) $\frac{1}{3}$ (e) $\frac{1}{6}$

(ii) The limit, $\lim_{x\to 0}\left[\frac{1}{x^2}\int_0^x \frac{t+t^2}{1+\sin t}dt\right]$ is equal to

(a) $\frac{1}{2\pi}$ (b) $\frac{1}{\pi}$ (c) $\frac{1}{2}$ (d) 1 (e) $\frac{\pi}{2}$

(iii) Let $[x]$ denote the greatest integer $\leq x$. If n is positive integer, then $\lim_{x\to -n-}\left(|x|-[x]\right)-\lim_{x\to n-}\left(|x|-[x]\right)$ is

(a) -2 (B) 0 (c) 2 (d) $2n-1$ (e) $2n$

(iv) In the (ε,δ) definition of the limit, $\lim_{x\to c} f(x)=L$, let $f(x)=x^3+3x^2-x+1$ and let $c=2$. Find the least upper bound on δ so that $f(x)$ is bounded with in ε on L for all sufficiently small $\varepsilon>0$.

(a) $\frac{\varepsilon}{8}$ (b) $\frac{\varepsilon^2}{16}$ (c) $\frac{\varepsilon}{23}$ (d) $\frac{\varepsilon}{19}$ (e) $\frac{\varepsilon^3}{4}$

(b) Prove that if $f\in R(\alpha)$ on $[a,b]$, then $|f|\in R(\alpha)$ on $[a,b]$ and $\left|\int_a^b f\,d\alpha\right| = \int_a^b |f|\,d\alpha$

(1+1+1+1+6)

Q. No. 8

(a) Fill in the blanks in (i) and (ii) and prove the statement in (iii).

(i) The order of the product of two tensors is the --------------of the orders of the given tensors.

(ii) The transformation law for the components of the first order (contravariant) tensor is mathematically given by ------------------

(iii) Prove that $\varepsilon_{ijk}\varepsilon_{ijk}=6$

(b) If $\vec{A}=(x-3y)\hat{i}+(y-2x)\hat{j}$, evaluate $\oint_C \vec{A}.\vec{dr}$, where C is an ellipse

$\frac{x^2}{9}+\frac{y^2}{4}=1$ in xy-plane traversed in positive direction **(1+1+2+6)**

Q. No. 9

(a) In (i) and (ii) tick the correct choice, in (iii) fill in the blank and in (iv) answer the short question.

(i) If V is the velocity, l is length, and ν the kinematical viscosity of the fluid then, which of the following combinations is dimensionless.

(a) $Vl\nu$ (b) Vl/ν (c) $V^2\nu$ (d) V^2l/ν

(ii) Equation of continuity for a steady, compressible flow is

(a) $\frac{\partial \rho}{\partial t}=0$ (b) $\nabla\cdot V=0$ (c) $\nabla\cdot\rho V=0$ (d) $\nabla\cdot\rho V+\frac{\partial\rho}{\partial t}=0$

(iii) In a fluid, angular deformation occurs due to .... stresses.

(iv) A stream function is given by the expression $\psi=2x^2-y^2$. What are the components of velocity at a point (3, 1).

(b) The following velocity field describes the motion of an incompressible, steady state two-dimensional flow

$$V=\left(x^2y-y^2\right)i-xy^2j$$

Find out

(i) Pressure gradient in the x- and y-direction (neglecting viscous effects)

(ii) Values of pressure gradient at a point (2, 1) if the fluid is water.

(1+1+1+2+5)

Q. No. 10

(a) Tick the correct choice

(i)The number of solutions (equivalence classes) of the congruence

$3x+11\equiv 20(\mathrm{mod}\,12)$ is

(a) No solution (b) 1 (c) 3 (d) 4

(ii) Find the reminder on dividing 3^{20} by 7

(a) 1 (b) 2 (c) 3 (d) 4

(iii) For any positive integer n, n^7-n is divisible by

(a) 4 (b) 6 (c) 7 (d) 18

(b) There is only one integer, x between 100 and 200 such that $144x \equiv 22 \pmod{71}$.
What is x?

(1+1+1+7)

Q. No. 11

(a) Tick the correct choice.

(i) The number of generator of cyclic group of order 8 is

(a) 6 (b) 4 (c) 3 (d) 2

(ii) Up to isomorphism, how many abelian groups are there of order 36?

(a) 1 (b) 4 (c) 9 (d) 12

(iii) Which of the following sets, together with the given binary operation $*$, does not form a group?

(a) $G = \left\{ a + b\sqrt{2} \in \mathbb{R} \setminus \{0\} \middle| a, b \in \mathbb{Q} \right\}$, $*$: usual multiplication of real number.

(b) $G = \left\{ a + bi\sqrt{2} \in \mathbb{C} \setminus \{0\} \middle| a, b \in \mathbb{Q} \right\}$, $*$: usual multiplication of complex number.

(c) $G = \left\{ \sqrt[3]{a} \in \mathbb{R} \middle| a \in \mathbb{Z} \right\}$, $*$: for $a, b \in G$, $\sqrt[3]{a} * \sqrt[3]{b} = \sqrt[3]{a+b}$.

(d) $G = \mathbb{R} \setminus \{0\}$, $*$: for $a, b \in G$, $a * b = |a|b$.

(iv) A subgroup H in group G is invariant if $gH = Hg$ for every g in G. If H and K are both invariant subgroups of G, which of the following is also an invariant subgroup?

(a) $H \cap K$ (b) HK (c) $H \cup K$ (d) Both (a) and (b)

(b) Let $\phi : G \to G$ be a function such that $\phi(g) = g^{-1}$ for every g in G. Show that G is abelian if and only if ϕ is an endomorphism.

Q. No. 12

(a) Fill in the blanks in (i) and (ii) and answer the short question in (iii)

(i) The Idea of 4-dim space-time was given by ------------------------

(ii) The event is said to be time-like if ds^2 -------------------------------

(iii) Calculate the speed with which a bus must move in order that its length may be shortened to one-fourth of its proper length.

(b) Prove that if two vectors are perpendicular in a frame **S**, then these may not be necessarily perpendicular in the frame **S'**, which is moving with a constant velocity **v** relative to **S**. Also, find the conditions under which the perpendicularity condition may sustain.

(1+1+2+6)