THE LAPLACE TRANSFORM

Chapter 11

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Introduction :

Piecewise Continuous :- A real valued function f defined on an interval $[a,b]$ is said to be piecewise continuous in $[a,b]$ if there exists a partition

$$P = \{a = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = b\}$$

of $[a,b]$ such that f is continuous in the interior of each subinterval $[x_i, x_{i+1}]$ and has finite one-sided limits $f(x_i+0)$ and $f(x_{i+1}-0)$ at the end points of each subinterval $(i = 0, 1, 2, \ldots, (n-1))$.

The Laplace Transform of f :-

Let f be a real-valued piecewise continuous function defined on $[0,\infty[$. The Laplace transform of f, denoted by $\mathcal{L}(f) = F$, is the function F defined by

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt \quad \text{------ (1)}$$

provided the improper integral in (1) converges. The domain of F is the set of all real numbers s for which the above integral converges.

Note that the operation transforms the given function f of the variable t into a new function F of the variable s and is written

Symbolically,

$$F(s) = \mathcal{L}\{f(t)\}.$$

Inverse Laplace transform of F :-

If $F = \mathcal{L}\{f\}$ then the original function f is called the inverse Laplace transform of F and is denoted by $f = \mathcal{L}^{-1}\{F\}$. Clearly,

$$\mathcal{L}^{-1}\{F\} = \mathcal{L}^{-1}\{\mathcal{L}(f)\} = f.$$

Thus if $\mathcal{L}\{f(t)\} = F(s)$ then $\mathcal{L}^{-1}\{F(s)\} = f(t)$.

Example :-

Let $f(t) = 1$ on $[0, \infty[$. Then

$$\mathcal{L}\{f\} = \mathcal{L}\{1\} = \int_0^\infty e^{-st}\,dt$$

$$= \lim_{h\to\infty} \int_0^h e^{-st}\,dt$$

$$= \lim_{h\to\infty} \left[-\frac{e^{-st}}{s}\right]_0^h$$

$$= \lim_{h\to\infty} \left[-\frac{e^{-hs}}{s} + \frac{1}{s}\right]$$

$$= \frac{1}{s} = F(s), \text{ provided } s > 0.$$

Example :- Let $f(t) = t^n$, n being a positive integer. Evaluate $\mathcal{L}\{f(t)\}$.

Solution :- Here $\mathcal{L}\{f(t)\} = \mathcal{L}\{t^n\} = \int_0^\infty e^{-st}\, t^n\,dt$

Integrating by parts, taking t^n as first function

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$$\mathcal{L}\{t^n\} = \left[t^n \cdot \frac{e^{-st}}{-s}\right]_0^\infty + \int_0^\infty n\, t^{n-1} \cdot \frac{e^{-st}}{s}\, dt$$

$$= -\left[\frac{t^n}{s e^{st}}\right]_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1} e^{-st}\, dt$$

$$= \frac{n}{s}\,\mathcal{L}\{t^{n-1}\},$$

Hence $\mathcal{L}\{t^n\} = \frac{n}{s}\,\mathcal{L}\{t^{n-1}\}$

$$= \frac{n}{s}\cdot\frac{n-1}{s}\cdot\mathcal{L}\{t^{n-2}\}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$= \frac{n}{s}\cdot\frac{n-1}{s}\cdots\frac{1}{s}\,\mathcal{L}\{1\}$$

$$= \frac{n!}{s^n}\cdot\frac{1}{s}, \quad \left(\text{as } \mathcal{L}\{1\} = \frac{1}{s}\right)$$

$$= \frac{n!}{s^{n+1}} = F(s).$$

Example: Compute $\mathcal{L}\{e^{at}\}$, where a is constant and $s > a$.

Solution: Here $f(t) = e^{at}$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{e^{at}\}$$

$$= \int_0^\infty e^{-st}\cdot e^{at}\, dt$$

$$= \int_0^\infty e^{(a-s)t}\, dt \qquad = \left.\frac{e^{(a-s)t}}{a-s}\right|_0^\infty$$

$$= \lim_{h\to\infty}\left[\frac{e^{(a-s)t}}{a-s}\right]_0^h \qquad = \frac{1}{a-s}\left[\frac{1}{e^{(s-a)t}}\right]_0^\infty$$

$$= \lim_{h\to\infty}\left[\frac{e^{(a-s)h}}{a-s} - \frac{1}{a-s}\right] \qquad = \frac{1}{a-s}\left[e^{-(s-a)t}\right]_0^\infty$$

$$= \frac{1}{a-s}\left(e^{-\infty} - e^{0}\right) = \frac{1}{a-s}\left(0-1\right) = \frac{-1}{a-s} = \frac{1}{s-a}$$

$$= \begin{cases} \dfrac{1}{s-a} & \text{if } s > a \\ \infty & \text{if } s \le a. \end{cases}$$

Here $e^{-(s-a)h} \to 0$ as $h \to \infty$ and $s > a$, while $e^{-(s-a)h} \to \infty$ as $h \to \infty$ and $s < a$.

$a - s \le 0$
$s - a \ge 0$
$s > a$

when $s = a$, $f(t) = e^{st}$ and

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} \cdot e^{st}\, dt = \int_0^\infty dt$$

$$= \{t\}_0^\infty = \infty.$$

Therefore, $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$, $s > a$.

Example: Find Laplace transforms of (i) $\sin at$, (ii) $\cos at$.

Solution: By definition,

$$\left.\begin{aligned} \mathcal{L}\{\cos at\} &= \int_0^\infty e^{-st} \cos at\, dt \\ \text{and } \mathcal{L}\{\sin at\} &= \int_0^\infty e^{-st} \sin at\, dt. \end{aligned}\right\}$$

Therefore, $\mathcal{L}\{\cos at\} + i\,\mathcal{L}\{\sin at\}$

$L\{e^{at}\cos bt\}$

$$= \int_0^\infty e^{-st} \cos at\, dt + i \int_0^\infty e^{-st} \sin at\, dt$$

$= \int_0^\infty e^{-st} \cos bt\, dt$

$$= \lim_{h\to\infty} \int_0^h e^{-st}\, e^{iat}\, dt = \lim_{h\to\infty} \int_0^h e^{(ia-s)t}\, dt$$

$$= \lim_{h\to\infty} \left[\frac{e^{(ia-s)t}}{ia-s} \right]_0^h = \lim_{h\to\infty} \left[\frac{e^{(ia-s)h}}{ia-s} - \frac{1}{ia-s} \right]$$

$$= \begin{cases} \dfrac{1}{s-ia} & \text{if } s > 0 \\ \text{undefined} & \text{if } s < 0 \end{cases}$$

$$= \frac{s+ia}{s^2+a^2} \quad \text{if } s > 0.$$

Here $\lim_{h\to\infty} \frac{e^{(ia-s)h}}{ia-s} = 0$ for $s>0$ and is undefined for $s<0$.

Equating real and imaginary parts, we get

(i) $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$, $s>0$

(ii) $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$, $s>0$

Example: Consider the function f defined by $f(t) = \frac{1}{t}$. For the Laplace transform of $\frac{1}{t}$, we first check the convergence of $\int_0^\infty \frac{e^{-st}}{t} dt$.

$$\int_0^\infty \frac{e^{-st}}{t} dt = \int_0^1 \frac{e^{-st}}{t} dt + \int_1^\infty \frac{e^{-st}}{t} dt$$

For $0 \le t \le 1$, we have $e^{-st} \geq e^{-s}$ if $s>0$.

Therefore, $\int_0^\infty \frac{e^{-st}}{t} dt \geq \int_0^1 \frac{e^{-s}}{t} dt + \int_1^\infty \frac{e^{-st}}{t} dt$

But $\int_0^1 \frac{e^{-s}}{t} dt = e^{-s} \lim_{h\to 0} [\ln t]_h^1$

$$= \frac{1}{e^s} \lim_{h\to 0} (\ln 1 - \ln h)$$

$$= \infty$$

Hence $\int_0^1 \frac{e^{-st}}{t} dt$, also diverges to ∞. Consequently, $\int_0^\infty \frac{e^{-st}}{t} dt$ diverges and so by definition, $\mathcal{L}\left\{\frac{1}{t}\right\}$ does not exist.

Exponential Order a:

A function f defined on $[0,\infty[$ is said to be of exponential order a as $t\to\infty$. If there exists real constants a, $M>0$ and $T>0$ such that

$$|f(t)| \le Me^{at} \quad \text{for} \quad t \ge T.$$

Theorem: Let f be a piecewise continuous function defined on $[0,\infty[$. If f is of exponential order a as $t\to\infty$ then $\mathcal{L}\{f(t)\}$ exists for all $s>a$.

Proof: Since f is of exponential order a, there exists positive real numbers M and T such that

$$|f(t)| \le Me^{at} \quad , \quad t \ge T \qquad \text{——— (1)}$$

The theorem will be proved if we show that $\int_0^\infty e^{-st} f(t)\,dt$ converges.

Now
$$\int_0^\infty e^{-st} f(t)\,dt = \int_0^T e^{-st} f(t)\,dt + \int_T^\infty e^{-st} f(t)\,dt \qquad (2)$$

Since f is piecewise continuous, the first integral on the right of (2) exists. Thus the convergence of $\int_0^\infty e^{-st} f(t)\,dt$ depends on the convergence of $\int_T^\infty e^{-st} f(t)\,dt$.. But

$$\left|\int_T^\infty e^{-st} f(t)\,dt\right| \le \int_T^\infty |e^{-st} f(t)|\,dt$$

$$\le \int_T^\infty e^{-st} Me^{at}\,dt, \qquad \text{by (1)}$$

$$= M \int_T^\infty e^{(a-s)t}\,dt$$

$$= M \lim_{h\to\infty}\left[\frac{e^{(a-s)t}}{a-s}\right]_T^h$$

$$= \lim_{h\to\infty} M\left[\frac{e^{(a-s)h}}{a-s} - \frac{e^{(a-s)T}}{a-s}\right]$$

$$= \begin{cases} M\left(0 - \dfrac{e^{(a-s)T}}{a-s}\right) & \text{if } a-s<0 \\ \infty & \text{if } a-s\geq 0. \end{cases}$$

Here $\lim_{h\to\infty} \dfrac{e^{(a-s)h}}{a-s} = 0$ if $a-s<0$, and

$\lim_{h\to\infty} \dfrac{e^{(a-s)h}}{a-s} = \infty$ if $a-s\geq 0$.

Thus $\mathcal{L}\{f(t)\}$ exists for all $s>a$.

Note: The conditions stated above are sufficient but not neccessary. There exists functions which do not satisfy the hypothesis of above theorem.

Example: Consider the function f, defined by

$$f(t) = t^{-1/2}$$

Clearly f is not defined at $t=0$, but it will be shown that $\mathcal{L}\{t^{-1/2}\}$ exists. By definition, we have

$$\mathcal{L}\{t^{-1/2}\} = \int_0^\infty e^{-st}\, t^{-1/2}\,dt \quad \text{——— (i)}$$

Let $st = x$. Then $s\,dt = dx$.

or $dt = \frac{1}{s} dx$. So

$$t^{-1/2} = \left(\frac{x}{s}\right)^{-1/2} = \sqrt{\frac{s}{x}}.$$

Substituting these values of 't' and dt into (1), we have

$$\mathcal{L}\{t^{-1/2}\} = \frac{1}{\sqrt{s}} \int_0^\infty e^{-x} x^{-1/2} dx$$

$$= \frac{1}{\sqrt{s}} \Gamma\left(\frac{1}{2}\right) = \sqrt{\frac{\pi}{s}}$$

Thus $\mathcal{L}\{t^{-1/2}\}$ exists.

Properties Of The Laplace Transform:

Theorem: (The linearity property).

Let $f(t) = a\,g(t) + b\,h(t)$, where a, b are constants and $\mathcal{L}\{g(t)\}$ and $\mathcal{L}\{h(t)\}$ exists. Then $\mathcal{L}\{f(t)\}$ exists and

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{a g(t) + b h(t)\}$$

$$= a\mathcal{L}\{g(t)\} + b\mathcal{L}\{h(t)\}$$

Proof: By definition,

$$\mathcal{L}\{a\,g(t) + b\,h(t)\} = \int_0^\infty e^{-st}\{a g(t) + b h(t)\} dt$$

$$= a\int_0^\infty e^{-st} g(t)\, dt + b\int_0^\infty e^{-st} h(t)\, dt$$

$$= a\,\mathcal{L}\{g(t)\} + b\,\mathcal{L}\{h(t)\}$$

Theorem: (The Differentiation Formula). Let f be continuous on $[0,\infty[$ and of exponential order a. Let f' be piecewise continuous on every finite closed interval $0 \le t \le b$. Then $\mathcal{L}\{f'(t)\}$ exists for $s > a$ and

$$\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0).$$

Proof: By previous theorem, $\mathcal{L}\{f'(t)\}$ exists.

Let $F(s) = \mathcal{L}\{f(t)\}$. Then by definition,

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= \left[e^{-st} f(t)\right]_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= -f(0) + s\,\mathcal{L}\{f(t)\}$$

$$= s\,\mathcal{L}\{f(t)\} - f(0)$$

$$= sF(s) - f(0).$$

Corrollary: If $F(s) = \mathcal{L}\{f(t)\}$, then

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

Proof: $\mathcal{L}\{f''(t)\} = s\,\mathcal{L}\{f'(t)\} - f'(0),$

$$= s\left[s\,\mathcal{L}\{f(t)\} - f(0)\right] - f'(0)$$

$$= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

$$= s^2F(s) - sf(0) - f'(0).$$

<u>Corollary</u>: $\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{(n-1)}(0)$.

<u>Proof</u>: $\mathcal{L}\{f^{(n)}(t)\} = s\,\mathcal{L}\{f^{(n-1)}(t)\} - f^{(n-1)}(0)$

$$= s\left[s\{f^{(n-2)}(t)\} - f^{(n-2)}(0)\right] - f^{(n-1)}(0)$$

$$= s^2\mathcal{L}\{f^{(n-2)}(t)\} - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

$$= s^2\left[s\,\mathcal{L}\{f^{(n-3)}(t)\} - f^{(n-3)}(0)\right] - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

$$= s^3\mathcal{L}\{f^{(n-3)}(t)\} - s^2f^{(n-3)}(0) - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

Continuing in this way we get the required result.

<u>Theorem</u>: (First Shifting property)(or Translation property).

Let $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > b$. For any constant a,

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a) \text{ for } s > a+b.$$

<u>Proof</u>: By definition,

$$F(s) = \int_0^\infty e^{-st}f(t)\,dt$$

$$\mathcal{L}\{e^{at}f(t)\} = \int_0^\infty e^{-st}e^{at}f(t)\,dt$$

$$= \int_0^\infty e^{-(s-a)t}f(t)\,dt$$

$$= F(s-a) \text{ if } s-a > b$$

or if $s > a+b$.

Example: Compute $\mathcal{L}\{\sinh at\}$ and $\mathcal{L}(\cosh at)$.

Solution: Here $\sinh at = \dfrac{e^{at} - e^{-at}}{2}$

we have

$$\mathcal{L}\{\sinh at\} = \mathcal{L}\left\{\frac{e^{at} - e^{-at}}{2}\right\}$$

$$= \frac{1}{2}\mathcal{L}\{e^{at}\} - \frac{1}{2}\mathcal{L}\{e^{-at}\},$$

$$= \frac{1}{2}\cdot\frac{1}{s-a} - \frac{1}{2}\cdot\frac{1}{s+a}$$

$$= \frac{a}{s^2-a^2}$$

Similarly, $\mathcal{L}\{\cosh at\} = \dfrac{s}{s^2-a^2}$

Example: Compute $\mathcal{L}\{\cos^2 at\}$

Solution: Let $f(t) = \cos^2 at$

$$f'(t) = -2a\cos at \sin at$$

$$= -a\sin 2at$$

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

or $$s\mathcal{L}\{f(t)\} = \mathcal{L}\{f'(t)\} + f(0)$$

i.e., $$s\mathcal{L}\{\cos^2 at\} = -a\mathcal{L}\{\sin 2at\} + f(0)$$

$$= -a\cdot\frac{2a}{s^2+4a^2} + 1$$

$$= \frac{s^2+2a^2}{s^2+4a^2}$$

Therefore, $\mathcal{L}\{f(t)\} = \mathcal{L}\{\cos^2 at\}$

$$= \frac{s^2+2a^2}{s(s^2+4a^2)}$$

Example: Evaluate $\mathcal{L}\{e^{3t}(t^3+\sin 2t)\}$.

Solution:

$$\mathcal{L}\{e^{3t}(t^3+\sin 2t)\} = \mathcal{L}\{e^{3t}t^3 + e^{3t}\sin 2t\}$$
$$= \mathcal{L}\{e^{3t}t^3\} + \mathcal{L}\{e^{3t}\sin 2t\}$$

Now $\mathcal{L}\{t^3\} = \dfrac{3!}{s^4}$

$$\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4}$$

Now $\mathcal{L}\{e^{3t}t^3\} = \dfrac{3!}{(s-3)^4}$.

and $\mathcal{L}\{e^{3t}\sin 2t\} = \dfrac{2}{(s-3)^2+4}$

Therefore, $\mathcal{L}\{e^{3t}(t^3+\sin 2t)\} = \dfrac{3!}{(s-3)^4} + \dfrac{2}{(s-3)^2+4}$

Example: Compute $\mathcal{L}\{te^{at}\cos bt\}$.

Solution: Consider $te^{at}e^{ibt} = te^{(a+ib)t}$

Let $f(t) = t$, then $\mathcal{L}\{f(t)\} = \mathcal{L}\{t\} = \dfrac{1}{s^2}$.

Now,

$$\mathcal{L}\{te^{(a+ib)t}\} = \frac{1}{[s-(a+ib)]^2} \quad (\because \mathcal{L}(e^{at}f(t)) = F(s-a))$$

$$= \frac{1}{[(s-a)-ib]^2}$$

$$= \frac{[(s-a)+ib]^2}{[\{(s-a)-ib\}\{(s-a)+ib\}]^2}$$

$$= \frac{(s-a)^2-b^2+2ib(s-a)}{[(s-a)^2+b^2]^2}$$

Equating real parts, we have

$$\mathcal{L}\{t\,e^{at}\cos bt\} = \frac{(s-a)^2 - b^2}{[(s-a)^2 + b^2]^2}$$

Theorem: Suppose $\mathcal{L}\{f(t)\} = F(s)$ exists for $s > a$. Then $\mathcal{L}\{t f(t)\} = -F'(s)$.

Proof: Consider

$$\frac{d}{ds}\mathcal{L}\{f(t)\} = \frac{d}{ds}F(s)$$

$$= \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\infty \frac{d}{ds} e^{-st} f(t)\,dt$$

$$= \int_0^\infty -t e^{-st} f(t)\,dt$$

$$= -\mathcal{L}\{t f(t)\}$$

$$F'(s) = -\mathcal{L}\{t f(t)\}$$

or $\mathcal{L}\{t f(t)\} = -F'(s)$ as desired.

Using the above result repeatedly, we have

$$\mathcal{L}\{t^{(n)} f(t)\} = -\frac{d}{ds}\mathcal{L}\{t^{(n-1)} f(t)\}$$

$$= (-1)^2 \frac{d^2}{ds^2}\mathcal{L}\{t^{n-2} f(t)\}$$

$$\vdots$$

$$= (-1)^n \frac{d^n}{ds^n}\mathcal{L}\{f(t)\}.$$

Example: Compute $\mathcal{L}\{t^3 e^{-t}\}$.

Solution: Let $f(t) = e^{-t}$. Then

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{e^{-t}\} = \frac{1}{s+1}$$

Now, $\mathcal{L}\{t^3 e^{-t}\} = (-1)^3 \cdot \dfrac{d^3}{ds^3}\, \mathcal{L}\{e^{-t}\}$

$$= -\frac{d^3}{ds^3}\left(\frac{1}{s+1}\right)$$

$$= -\frac{(-1)^3\, 3!}{(s+1)^4}$$

$$= \frac{6}{(s+1)^4}$$

<u>Theorem</u>: If $\mathcal{L}\{f(t)\} = F(s)$ then

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)\,du \quad \text{provided } \lim_{t\to\infty} \frac{f(t)}{t} \text{ exists}$$

exists.

<u>Proof</u>: Let $\dfrac{f(t)}{t} = g(t)$. Then $f(t) = t\,g(t)$.

Now $F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{t\,g(t)\}$

$$= -\frac{d}{ds}\mathcal{L}\{g(t)\},$$

Integrating, we have

$$\mathcal{L}\{g(t)\} = -\int_\infty^s F(u)\,du$$

$$= \int_s^\infty F(u)\,du.$$

<u>Theorem</u>: If f is piecewise continuous and is of exponential order a, then

$$\mathcal{L}\left\{\int_0^t f(u)\,du\right\} = \frac{1}{s}\,\mathcal{L}\{f(t)\}.$$

<u>Proof</u>: The integral $g(t) = \displaystyle\int_0^t f(u)\,du$

is a continuous function of t. Since $f(t)$ is of exponential order a, $|f(t)| \le Me^{at}$. Therefore,

$$|g(t)| = \left|\int_0^t f(u)\,du\right| \le M\int_0^t e^{au}\,du$$

$$\le \frac{M}{a}\left\{e^{at}-1\right\}.$$

By the fundamental theorem of integral calculus, $g'(t) = f(t)$ except at points where f is discontinuous. Hence $g'(t)$ is piecewise continuous.

$\therefore$ We have $\mathcal{L}\{f(t)\} = \mathcal{L}\{g'(t)\}$

$$= s\mathcal{L}\{g(t)\} - g(0), \quad s > a$$

$$= s\mathcal{L}\{g(t)\}, \text{ since } g(0) = 0.$$

Thus $\mathcal{L}\{g(t)\} = \frac{1}{s}\mathcal{L}\{f(t)\}$.

Example: Compute $\mathcal{L}\left\{\frac{\sin t}{t}\right\}$.

Solution: Let $f(t) = \sin t$ so that

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin t\} = \frac{1}{s^2+1} = F(s).$$

Set $g(t) = \frac{\sin t}{t}$, so we have

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^\infty F(u)\,du$$

$$= \int_s^\infty \frac{1}{1+u^2}\,du = \left[\arctan u\right]_s^\infty$$

$$= \frac{\pi}{2} - \text{arc}\tan s.$$

Example: Evaluate $\mathcal{L}\left\{\int_0^t \frac{1-\cosh au}{u}\,du\right\}$.

Solution: Let $f(t) = 1 - \cosh at$ so that

$$\mathcal{L}\{f(t)\} = \frac{1}{s} - \frac{s}{s^2-a^2} = F(s)$$

Set $g(t) = \frac{1-\cosh at}{t}$. Then, we get

$$\mathcal{L}\left\{\frac{1-\cosh at}{t}\right\} = \int_s^\infty \left(\frac{1}{u} - \frac{u}{u^2-a^2}\right) du$$

$$= \left[\ln u - \frac{1}{2}\ln(u^2-a^2)\right]_s^\infty$$

$$= \lim_{u\to\infty}\left[\frac{1}{2}\ln u^2 - \frac{1}{2}\ln(u^2-a^2)\right] + \frac{1}{2}\ln(s^2-a^2) - \ln s$$

$$= \lim_{u\to\infty} \ln\left(\frac{u^2}{u^2-a^2}\right) + \frac{1}{2}\ln\left(\frac{s^2-a^2}{s^2}\right)$$

$$= \ln \lim_{u\to\infty}\left(\frac{u^2}{u^2-a^2}\right) + \frac{1}{2}\ln\left(\frac{s^2-a^2}{s^2}\right)$$

$$= \ln 1 + \frac{1}{2}\ln\left(\frac{s^2-a^2}{s^2}\right) = \frac{1}{2}\ln\left(\frac{s^2-a^2}{s^2}\right)$$

So $\mathcal{L}\left\{\int_0^t \frac{1-\cosh au}{u}\,du\right\} = \frac{1}{s}\mathcal{L}\left\{\frac{1-\cosh at}{t}\right\}$

$$= \frac{1}{2s}\ln\left(\frac{s^2-a^2}{s^2}\right).$$

Unit Step Function: Let $a \geq 0$. The function u_a defined on $]0,\infty[$ by

$$u_a(t) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases}$$

is called the unit step function. If $a = 0$, then

$$u_0(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t > 0. \end{cases}$$

Theorem: Let u_a be the unit step function. Then $\mathcal{L}\{u_a(t)\} = \dfrac{e^{-as}}{s}$.

Proof: By definition,

$$\mathcal{L}\{u_a(t)\} = \int_0^\infty e^{-ts}\, u_a(t)\, dt$$

$$= \int_0^a 0\, e^{-st}\, dt + \int_a^\infty e^{-ts}\, dt = \int_a^\infty e^{-st}\, dt$$

$$= \lim_{h\to\infty} \left[\frac{e^{-ts}}{-s} \right]_a^h$$

$$= \lim_{h\to\infty} -\frac{e^{-hs}}{s} + \frac{e^{-as}}{s}$$

$$= \frac{e^{-as}}{s}, \quad s > 0, \text{ because } \lim_{h\to\infty} \frac{e^{-hs}}{s} = 0.$$

Theorem: Let f be a function of exponential order 'a' and $\mathcal{L}\{f(t)\} = F(s)$. For the function

$$u_a(t) f(t-a) = \begin{cases} 0 & \text{if } 0 < t < a \\ f(t-a) & \text{if } t > a \end{cases}$$

$$\mathcal{L}\{u_a(t) f(t-a)\} = e^{-as} F(s).$$

Proof: $\mathcal{L}\{u_a(t) f(t-a)\} = \int_0^\infty e^{-st} u_a(t) f(t-a)\, dt$

$$= \int_0^a e^{-st} \cdot 0\, dt + \int_a^\infty e^{-st} f(t-a)\, dt$$

$$= \int_a^\infty e^{-st} f(t-a)\, dt \quad \text{———} \quad (1)$$

putting $t-a = z$ in (1), we get

$\Rightarrow dt = dz$

$$\mathcal{L}\{u_a(t) f(t-a)\} = e^{-as} \int_0^\infty e^{-sz} f(z)\, dz$$

$$= e^{-as} F(s).$$

This is known as the second translation property.

Example: Find the Laplace transform of

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$$f(t) = \begin{cases} 0 & \text{if } 0 < t < \frac{\pi}{2} \\ \cos t & \text{if } t > \frac{\pi}{2} \end{cases}$$

Solution: By definition,

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^{\pi/2} e^{-st} f(t)\, dt + \int_{\pi/2}^\infty e^{-st} f(t)\, dt$$

$$= 0 + \int_{\pi/2}^{\infty} e^{-st} \cos t \, dt$$

$$= \left[\frac{e^{-st}}{-s} \cos t \right]_{\pi/2}^{\infty} + \frac{1}{s} \int_{\pi/2}^{\infty} e^{-st} (-\sin t) \, dt$$

$$= -\frac{1}{s} \int_{\pi/2}^{\infty} e^{-st} . \sin t \, dt$$

$$= \frac{1}{s^2} \left[e^{-st} \sin t \right]_{\pi/2}^{\infty} - \frac{1}{s^2} \int_{\pi/2}^{\infty} e^{-st} \cos t \, dt$$

$$= \frac{e^{-\pi/2 \, s}}{s^2} - \frac{1}{s^2} F(s)$$

Therefore, $\left(1 + \frac{1}{s^2}\right) F(s) = -\frac{e^{-\pi/2 \, s}}{s^2}$

or $F(s) = -\frac{e^{-\frac{\pi}{2} s}}{s^2 + 1}$

Table Of Some Laplace Transforms:

	$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
1	1	$\frac{1}{s}$, $s > 0$
2	t	$\frac{1}{s^2}$
3	t^n	$\frac{n!}{s^{n+1}}$, $s > 0$
4	t^{α}, $\alpha > -1$	$\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$
5	$\frac{1}{\sqrt{\pi t}}$	$\frac{1}{\sqrt{s}}$

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$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
(4) e^{at}	$\frac{1}{s-a}$, $s > a$
(7) te^{at}	$\frac{1}{(s-a)^2}$
(8) $\sin at$	$\frac{a}{s^2+a^2}$, $s > 0$
(9) $\cos at$	$\frac{s}{s^2+a^2}$, $s > 0$
(10) $\sinh at$	$\frac{a}{s^2-a^2}$, $s > \lvert a \rvert$
(11) $\cosh at$	$\frac{s}{s^2-a^2}$, $s > \lvert a \rvert$
(12) $t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$, $s > a$
(13) $e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$, $s > a$
(14) $e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$, $s > a$
(15) $t\sin at$	$\frac{2as}{(s^2+a^2)^2}$, $s > 0$
(16) $t\cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$, $s > 0$
(17) $f(ct)$	$\frac{1}{c} F\left(\frac{s}{c}\right)$, $c > 0$
(18) $\int_0^t f(u)\,du$	$\frac{1}{s} F(s)$

19	$t^n f(t)$	$(-1)^n \dfrac{d^n}{ds^n} F(s)$
20	$\dfrac{f(t)}{t}$ If limit exists, If Laplace exists	$\int_s^\infty F(u)\,du$
21	$u_a(t)$	$\dfrac{e^{-as}}{s}$
22	$u_a(t)\, f(t-a)$	$e^{-as} F(s)$
23	$\sin at - at\cos at$	$\dfrac{2a^3}{(s^2+a^2)^2}$
24	$f'(t)$	$sF(s) - f(0)$ or $s\mathcal{L}\{f(t)\} - f(0)$
25	$1-\cos at$	$\dfrac{a^2}{s(s^2+a^2)}$
26	$at - \sin at$	$\dfrac{a^3}{s^2(s^2+a^2)}$
27	$\sinh at - \sin at$	$\dfrac{2a^3}{s^4-a^4}$
28	$\cosh at - \cos at$	$\dfrac{2a^2 s}{s^4-a^4}$