❖ *Laplace Transformation:*

Laplace transform *is essentially a mathematical tool which can be used to solve several problems in science and engineering. This transform was first introduced by Laplace in the year 1970*

MOTIVATIONS:

*The **Laplace transform** is an efficient technique for solving linear differential equations with constant co-efficient. In this chapter, we shall discuss its basic properties and will apply them to solve initial value problem.*

Laplace Transform *is an operator which transforms a function f of the variable t into a function F of the variable s*

❖ *EXERCISE 11.1*

Formulae of Laplace Transformation:-

The Laplace gave the following formulae for his transformation

(i)	$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$
(ii)	$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$
(iii)	$\mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2}$
(iv)	$\mathcal{L}\{\sinh bt\} = \frac{b}{s^2-b^2}$
(v)	$\mathcal{L}\{\cos bt\} = \frac{s}{s^2+b^2}$
(vi)	$\mathcal{L}\{\cosh bt\} = \frac{s}{s^2-b^2}$

❖ *Linearity property:-*

If c_1 and c_2 are any two constants and $F_1(s)$ and $F_2(s)$ are the Laplace Transform, respectively, of the $f_1(t)$ and $f_2(t)$, then

$$\mathcal{L}[c_1 f_1(t) + c_2 f_2(t)] = c_1\mathcal{L}\{f_1(t)\} + c_2\mathcal{L}\{f_2(t)\}$$

$$\Longrightarrow \mathcal{L}[c_1 f_1(t) + c_2 f_2(t)] = c_1 F_1(s) + c_2 F_2(s)$$

❖ *Shifting property:-*

If a function is multiplied by e^{at} *then transform of the resultant is obtained by replacing* s *by* $s-a$ *in the transform of the original function. That is, if*

(vii)	$\mathcal{L}\{e^{at}t^n\} = \frac{n!}{(s-a)^{n+1}}$
(viii)	$\mathcal{L}\{e^{at}.\sin bt\} = \frac{b}{(s-a)^2+b^2}$
(ix)	$\mathcal{L}\{e^{at}.\cos bt\} = \frac{s-a}{(s-a)^2+b^2}$
(x)	$\mathcal{L}\{e^{at}.\sinh bt\} = \frac{b}{(s-a)^2-b^2}$
(xi)	$\mathcal{L}\{e^{at}.\cosh bt\} = \frac{s-a}{(s-a)^2-b^2}$

❖ NUMERICAL PROBLEM (FROM EXERCISE+EXAMPLES)

Compute the Laplace transformation of each of the following

Question 1: $t^2+6t-17$

Solution:-

Let $f(t) = t^2+6t-17$

Taking $\mathcal{L}$ *on both sides, we have*

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^2+6t-17\}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \mathcal{L}\{t^2\} + \mathcal{L}\{6t\} - \mathcal{L}\{17\}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \mathcal{L}\{t^2\} + 6\mathcal{L}\{t\} - 17\mathcal{L}\{1\}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{2!}{s^3} + 6.\frac{1!}{s^2} - 17.\frac{0!}{s}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{2}{s^3} + \frac{6}{s^2} - \frac{17}{s}$$

Question 2: e^{3t+5}

Solution:-

Let $f(t) = e^{3t+5}$

Taking $\mathcal{L}$ on both sides, we have

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{ e^{3t+5}\}$$

$$\Longrightarrow \mathcal{L}\{f(t)\} = \mathcal{L}\{ e^{3t}.e^{5}\}$$

$$\Longrightarrow \mathcal{L}\{f(t)\} = e^{5}\mathcal{L}\{ e^{3t}\}$$

$$\Longrightarrow \mathcal{L}\{f(t)\} = e^{5}.\frac{1}{s-3} \qquad \text{since } \mathcal{L}\{ e^{at}\} = \frac{1}{s-a}$$

Question 3: $\boldsymbol{sin(7t+4)}$

Solution:-

Let $f(t) = sin(7t+4)$

$$\Longrightarrow f(t) = sin7t\cos 4 + cos7t\sin 4$$

Taking $\mathcal{L}$ on both sides, we have

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{sin7t\cos 4\} + \mathcal{L}\{cos7t\sin 4\}$$

$$\Longrightarrow \mathcal{L}\{f(t)\} = \cos 4\,\mathcal{L}\{sin7t\} + \sin 4\,\mathcal{L}\{cos7t\}$$

$$\Longrightarrow \mathcal{L}\{f(t)\} = \cos 4\,.\frac{7}{s^2+49} + \sin 4\,.\frac{s}{s^2+49}$$

Question 4: $\boldsymbol{cos(at+b)}$

Solution:-

Let $f(t) = \ cos(at+b)$

$$\Longrightarrow f(t) = \cos at\cos b + \sin at\sin b$$

Taking $\mathcal{L}$ on both sides, we have

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\cos at\cos b\} + \mathcal{L}\{\sin at\sin b\}$$

$$\mathcal{L}\{f(t)\} = \cos b\,\mathcal{L}\{\cos at\} + \sin b\,\mathcal{L}\{\sin at\}$$

$$\mathcal{L}\{f(t)\} = \cos b\,.\frac{s}{s^2+a^2} + \sin b\,.\frac{a}{s^2+a^2}$$

Question 5: $\boldsymbol{cosh(5t-3)}$

Solution:-

Let $f(t) = \ cosh(5t-3)$

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$$\Rightarrow f(t) = \cosh 5t \cosh 3 - \sinh 5t \sinh 3$$

Taking $\mathcal{L}$ on both sides, we have

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\cosh 5t \cosh 3\} - \mathcal{L}\{\sinh 5t \sinh 3\}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \cosh 3\, \mathcal{L}\{\cosh 5t\} - \sinh 3\, \mathcal{L}\{\sinh 5t\}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \cosh 3 \,.\frac{s}{s^2-25} + \sinh 3\,.\frac{5}{s^2-25}$$

Therefore,

$$\mathcal{L}\{\, cosh(5t-3)\} = \frac{s\cosh 3}{s^2-25} + \frac{5\sinh 3}{s^2-25}$$

Question 6: $(t^3-1)e^{-2t}$

Solution:-

Let $f(t) = (t^3-1)e^{-2t}$

$$\Rightarrow f(t) = t^3.e^{-2t} - e^{-2t}$$

Taking $\mathcal{L}$ on both sides, we have

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^3.e^{-2t}\} - \mathcal{L}\{e^{-2t}\}$$

Since $\mathcal{L}\{t^n.e^{at}\} = \frac{n!}{(s-a)^{n+1}}$ & $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{3!}{(s-(-2))^{3+1}} - \frac{1}{s-(-2)}$$

$$\Rightarrow \mathcal{L}\{(t^3-1)e^{-2t}\} = \frac{3!}{(s+2)^4} - \frac{1}{s+2}$$

Question: Compute the Laplace Transform of

$e^{3t}(t^3 + \sin 2t)$(EXAMPLE 9 FROM BOOK OF METHOD)

Solution:-

Let $f(t) = e^{3t}(t^3 + \sin 2t)$

$$\Rightarrow f(t) = t^3.e^{3t} + e^{3t}.\sin 2t$$

Taking $\mathcal{L}$ on both sides, we have

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^3.e^{3t}\} - \mathcal{L}\{\boldsymbol{e^{3t}.\sin 2t}\}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{3!}{(s-3)^{3+1}} - \frac{2}{(s-3)^2+4}$$

Question: Compute the Laplace Transform

$\sinh at$ and $\cosh at$. (EXAMPLE 7 FROM BOOK OF METHOD)

Solution:- *Since* $\sinh at = \frac{e^{at}-e^{-at}}{2}$

Taking $\mathcal{L}$ on both sides, we have

$$\mathcal{L}\{\sinh at\} = \mathcal{L}\left\{\frac{e^{at}-e^{-at}}{2}\right\}$$

$$\Rightarrow \mathcal{L}\{\sinh at\} = \frac{1}{2}[\mathcal{L}\{e^{at}\} - \mathcal{L}\{e^{-at}\}]$$

$$\Rightarrow \mathcal{L}\{\sinh at\} = \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$\Rightarrow \mathcal{L}\{\sinh at\} = \frac{1}{2}\left[\frac{s+a-s+a}{(s-a)(s+a)}\right]$$

$$\Rightarrow \mathcal{L}\{\sinh at\} = \frac{1}{2}\left[\frac{2a}{s^2-a^2}\right]$$

$$\Rightarrow \mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2}$$

Since $\cosh at = \dfrac{e^{at}+e^{-at}}{2}$

Taking $\mathcal{L}$ on both sides, we have

$$\mathcal{L}\{\cosh at\} = \mathcal{L}\left\{\frac{e^{at}+e^{-at}}{2}\right\}$$

$$\Rightarrow \mathcal{L}\{\cosh at\} = \frac{1}{2}[\mathcal{L}\{e^{at}\} + \mathcal{L}\{e^{-at}\}]$$

$$\Rightarrow \mathcal{L}\{\cosh at\} = \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$$

$$\Rightarrow \mathcal{L}\{\cosh at\} = \frac{1}{2}\left[\frac{s+a+s-a}{(s-a)(s+a)}\right]$$

$$\Rightarrow \mathcal{L}\{\cosh at\} = \frac{1}{2}\left[\frac{2s}{s^2 - a^2}\right]$$

$$\Rightarrow \mathcal{L}\{\cosh at\} = \frac{s}{s^2 - a^2}$$

Question 7: $e^{-t}\sin 2t$

Solution:-

Let $f(t) = e^{-t}\sin 2t$

Taking $\mathcal{L}$ on both sides, we have

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{e^{-t}\sin 2t\}$$

$$Since\ \mathcal{L}\{e^{at}.\sin bt\} = \frac{b}{(s-a)^2 + b^2}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{2}{(s-(-1))^2 + 2^2}$$

$$\Rightarrow \mathcal{L}\{e^{-t}\sin 2t\} = \frac{2}{(s+1)^2 + 4}$$

Question 8: $e^{3t}\cosh 4t$.

Solution:-

Let $f(t) = e^{3t}\cosh 4t$

Taking $\mathcal{L}$ on both sides, we have

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{e^{3t}\cosh 4t\}$$

$$Since\ \mathcal{L}\{e^{at}.\sin bt\} = \frac{b}{(s-a)^2 + b^2}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{s-3}{(s-3)^2 + 4^2}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{s-3}{(s-3)^2 + 16}$$

Question 9: $\cos t\cos 2t$.

Solution:-

Since by the formula $2cos\, \alpha\, cos\, \beta = cos\, (\alpha + \beta) + cos\, (\alpha - \beta)$*, we have*

$$cos\, t\, cos\, 2t = \frac{1}{2}[cos\, (t + 2t) + cos\, (t - 2t)]$$

$$\Longrightarrow cos\, t\, cos\, 2t = \frac{1}{2}[cos\, 3t + cos\, (-t)]$$

$$\Longrightarrow cos\, t\, cos\, 2t = \frac{1}{2}[cos\, 3t + cos\, t] \quad since\, cos\, (-\theta) = cos\, \theta$$

Taking $\mathcal{L}$ *on both sides, we have*

$$\mathcal{L}\{cos\, t\, cos\, 2t\} = \frac{1}{2}[\mathcal{L}(cos\, 3t) + \mathcal{L}(cos\, t)]$$

$$\Longrightarrow \mathcal{L}\{cos\, t\, cos\, 2t\} = \frac{1}{2}\left[\frac{s}{s^2 + 9} + \frac{s}{s^2 + 1}\right]$$

$$\Longrightarrow \mathcal{L}\{cos\, t\, cos\, 2t\} = \frac{1}{2}.\frac{s}{s^2 + 9} + \frac{1}{2}.\frac{s}{s^2 + 1}$$

Question 10: $\boldsymbol{sin^3\, t}$.

Solution:-

Since by the formula $3t = 3\, sin\, t - 4sin^3\, t$ *, we have*

$$4sin^3\, t = 3\, sin\, t - sin\, 3t$$

$$\Longrightarrow sin^3\, t = \frac{3}{4}sin\, t - \frac{1}{4}sin\, 3t$$

Taking $\mathcal{L}$ *on both sides, we have*

$$\mathcal{L}\{sin^3\, t\} = \mathcal{L}\left[\frac{3}{4}sin\, t - \frac{1}{4}sin\, 3t\right]$$

$$\mathcal{L}\{sin^3\, t\} = \frac{3}{4}.\mathcal{L}\{sin\, t\} - \frac{1}{4}.\mathcal{L}\{sin\, 3t\} \qquad By\, Linearity\, property$$

$$\Longrightarrow \mathcal{L}\{sin^3\, t\} = \frac{3}{4}.\frac{1}{s^2 + 1} - \frac{1}{4}\frac{3}{s^2 + 9}$$

$$\Longrightarrow \mathcal{L}\{sin^3\, t\} = \frac{3}{4(s^2 + 1)} - \frac{3}{4(s^2 + 9)}$$

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Question 11: $te^{-3t}\sin at$.

Solution:-

Let $f(t) = e^{-3t}\sin at$

Since we know that $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[\mathcal{L}\{f(t)\}]$. Then, we have

Here $f(t) = e^{-3t}\sin at$. Therefore,

$$\mathcal{L}\{te^{-3t}\sin at\} = (-1)^1 \frac{d}{ds}[\mathcal{L}\{e^{-3t}\sin at\}]$$

$$\Rightarrow \mathcal{L}\{te^{-3t}\sin at\} = -\frac{d}{ds}\left\{\frac{a}{(s-(-3))^2+a^2}\right\}$$

$$\Rightarrow \mathcal{L}\{te^{-3t}\sin at\} = -\frac{d}{ds}\left\{\frac{a}{(s+3)^2+a^2}\right\}$$

$$\Rightarrow \mathcal{L}\{te^{-3t}\sin at\} = (-a)\frac{d}{ds}\{((s+3)^2+a^2)^{-1}\}$$

$$\Rightarrow \mathcal{L}\{te^{-3t}\sin at\} = (-a)(-1)\left[\frac{2(s+3)}{((s+3)^2+a^2)^2}\right]$$

$$\Rightarrow \mathcal{L}\{te^{-3t}\sin at\} = \frac{2a(s+3)}{((s+3)^2+a^2)^2}$$

Question 12: $\sinh^2 at$.

Solution:-

Since by the formula$\cosh 2at = 2\sinh^2 at + 1$, we have

$$\sinh^2 at = \frac{\cosh 2at - 1}{2}$$

Taking $\mathcal{L}$ on both sides, we have

$$\mathcal{L}\{\sinh^2 at\} = \mathcal{L}\left[\frac{\cosh 2at - 1}{2}\right]$$

$$\mathcal{L}\{\sinh^2 at\} = \frac{1}{2}.\mathcal{L}\{\cosh 2at\} - \frac{1}{2}.\mathcal{L}\{1\} \qquad By\ Linearity\ property$$

$$\Rightarrow \mathcal{L}\{\sinh^2 at\} = \frac{1}{2}.\frac{s}{s^2-4a^2} - \frac{1}{2}.\frac{1}{s}$$

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$$\Rightarrow \mathcal{L}\{sinh^2\ at\} = \frac{s^2 - s^2 + 4a^2}{2s(s^2 - 4a^2)}$$

$$\Rightarrow \mathcal{L}\{sinh^2\ at\} = \frac{4a^2}{2s(s^2 - 4a^2)}$$

$$\Rightarrow \mathcal{L}\{sinh^2\ at\} = \frac{2a^2}{s(s^2 - 4a^2)}$$

Question: Compute the Laplace Transform

$cos^2\ at$. ($EXAMPLE\ 8\ FROM\ BOOK\ OF\ METHOD$)

Let $f(t) = cos^2\ at$

Since $\cos 2at = 2cos^2\ at - 1$

$$\Rightarrow f(t) = \frac{1+\cos 2at}{2}$$

$$\Rightarrow f(t) = \frac{1}{2}[1 + \cos 2at]$$

Taking $\mathcal{L}$ *on both sides, we have*

$$\mathcal{L}\{f(t)\} = \frac{1}{2}[\mathcal{L}\{1\} + \mathcal{L}\{\cos 2at\}]$$

$$\mathcal{L}\{f(t)\} = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2 + 4a^2}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{2}\left[\frac{s^2 + 4a^2 + s^2}{s(s^2 + 4a^2)}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{2}\left[\frac{2s^2 + 4a^2}{s(s^2 + 4a^2)}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{s^2 + 2a^2}{s(s^2 + 4a^2)}$$

Question 13: $cosh\ at\ sin\ at$.

<u>Solution:-</u>

Let $f(t) = cosh\ at\ sin\ at$

$$Since\ \cosh at = \frac{e^{at}+e^{-at}}{2}$$

$$\Rightarrow f(t) = \frac{e^{at}+e^{-at}}{2} \sin at$$

$$\Rightarrow f(t) = \frac{1}{2}[e^{at}.\sin at + e^{-at}.\sin at]$$

Taking $\mathcal{L}$ on both sides, we have

$$\mathcal{L}\{f(t)\} = \frac{1}{2}[\mathcal{L}\{e^{at}.\sin at\} + \mathcal{L}\{e^{-at}.\sin at\}]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{2}\left[\frac{a}{(s-a)^2+a^2} + \frac{a}{(s+a)^2+a^2}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{a}{2}\left[\frac{1}{(s-a)^2+a^2} + \frac{1}{(s+a)^2+a^2}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{a}{2}\left[\frac{(s+a)^2+a^2+(s-a)^2+a^2}{((s-a)^2+a^2)((s+a)^2+a^2)}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{a}{2}\left[\frac{(s+a)^2+(s-a)^2+2a^2}{(s-a)^2(s+a)^2+a^2\{(s-a)^2+(s+a)^2\}+a^4}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{a}{2}\left[\frac{2(s^2+a^2)+2a^2}{(s^2+a^2-2as)(s^2+a^2+2as)+a^2\{2(s^2+a^2)\}+a^4}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = a\left[\frac{s^2+a^2++a^2}{(s^2+a^2)^2-4a^2s^2+2a^2(s^2+a^2)+a^4}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = a\left[\frac{s^2+2a^2}{s^4+a^4+2a^2s^2-4a^2s^2+2a^2s^2+2a^4+a^4}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = a\left[\frac{s^2+a^2++a^2}{s^4+a^4+2a^4+a^4}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{a(s^2+2a^2)}{s^4+4a^4}$$

Question 14: $\sinh at \cos at$.

Solution:-

Let $f(t) = \sinh at \cos at$

$$Since\ \sinh at = \frac{e^{at}-e^{-at}}{2}$$

$$\Rightarrow f(t) = \frac{e^{at} - e^{-at}}{2}\cos at$$

$$\Rightarrow f(t) = \frac{1}{2}[e^{at}.cos\ at - e^{-at}.cos\ at]$$

Taking $\mathcal{L}$ on both sides, we have

$$\mathcal{L}\{f(t)\} = \frac{1}{2}[\mathcal{L}\{e^{at}.cos\ at\} - \mathcal{L}\{e^{-at}.cos\ at\}]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{2}\left[\frac{s-a}{(s-a)^2+a^2} - \frac{s+a}{(s+a)^2+a^2}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{2}\left[\frac{s-a\{(s+a)^2+a^2\} - (s+a)\{(s-a)^2+a^2\}}{((s-a)^2+a^2)((s+a)^2+a^2)}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\}$$

$$= \frac{1}{2}\left[\frac{(s-a)(s+a)^2 + a^2(s-a) - (s+a)(s-a)^2 - a^2(s+a)}{(s-a)^2(s+a)^2 + a^2\{(s-a)^2+(s+a)^2\} + a^4}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{2}\left[\frac{(s-a)(s+a)(s+a-s+a) + a^2(s-a-s-a)}{(s^2+a^2-2as)(s^2+a^2+2as) + a^2\{2(s^2+a^2)\} + a^4}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{2}\left[\frac{(s^2-a^2)2a + a^2(-2a)}{(s^2+a^2)^2 - 4a^2s^2 + 2a^2(s^2+a^2)+a^4}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{2}\left[\frac{2a(s^2-a^2-a^2)}{s^4+a^4+2a^2s^2-4a^2s^2+2a^2s^2+2a^4+a^4}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{2}\left[\frac{2a(s^2-2a^2)}{s^4+a^4+2a^4+a^4}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{a(s^2-2a^2)}{s^4+4a^4}$$

Question 15: cosh at cos bt.

Solution:-

Let $f(t) = cosh\ at\ cos\ bt$

$$Since\ \cosh at = \frac{e^{at}+e^{-at}}{2}$$

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$$\Rightarrow f(t) = \frac{e^{at} + e^{-at}}{2} \cos bt$$

$$\Rightarrow f(t) = \frac{1}{2}[e^{at}.\cos bt + e^{-at}.\cos bt]$$

Taking $\mathcal{L}$ on both sides, we have

$$\mathcal{L}\{f(t)\} = \frac{1}{2}[\mathcal{L}\{e^{at}.\cos bt\} + \mathcal{L}\{e^{-at}.\cos bt\}]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{2}\left[\frac{s-a}{(s-a)^2+b^2} + \frac{s+a}{(s+a)^2+b^2}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{2}\left[\frac{(s-a)\{(s+a)^2+b^2\} + (s+a)\{(s-a)^2+b^2\}}{((s-a)^2+b^2)((s+a)^2+b^2)}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\}$$

$$= \frac{1}{2}\left[\frac{(s-a)(s+a)^2 + b^2(s-a) + (s+a)(s-a)^2 + (s-a)^2 + b^2(s+a)}{(s-a)^2(s+a)^2 + b^2\{(s-a)^2 + (s+a)^2\} + b^4}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{2}\left[\frac{(s-a)(s+a)(s+a+s-a) + b^2(s-a+s+a)}{(s^2+a^2-2as)(s^2+a^2+2as) + b^2\{2(s^2+a^2)\} + b^4}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{2}\left[\frac{(s^2-a^2)2s + b^2(2s)}{(s^2+a^2)^2 - 4a^2s^2 + 2b^2(s^2+a^2)+a^4}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{2}\left[\frac{2s(s^2-a^2+b^2)}{s^4+a^4+2a^2s^2-4a^2s^2+2b^2s^2+2a^2b^2+b^4}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = 1\left[\frac{s(s^2-a^2+b^2)}{s^4+a^4+b^4+2a^2s^2+2a^2b^2+2b^2s^2-4a^2s^2}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{s(s^2-a^2+b^2)}{(s^2+a^2+b^2)^2-4a^2s^2}$$

❖ *LAPLACE BY DEFINITION:-*

*Let f be a real-valued piecewise continuous function defined on $[0,\infty[$. The **LAPLACE TRANSFORMATION** of f, denoted as $\mathcal{L}\{f\}$, is the function **F** is defined by*

$$F(s) = \int_0^\infty e^{-st} f(t)dt$$

Provided the improper integral converges.

- ***NOTE:-***

The operation transforms the given function f of the variable t into a new function F of the variable s and is written symbolically $F(s) = \mathcal{L}\{f(t)\}$ *or simply* $F = \mathcal{L}\{f\}$.

❖ ***VALUE OF BRACKET FUNCTION IN A INTEGRAL:-***

If there is any bracket function in the form $[t]$ *is in definite integral, then the value of that bracket is 1 less than the upper limit. That is,*

$$\int_a^{a+1} [t]dt = \int_a^{a+1} a.dt$$

Question: Compute the Laplace Transform of $\boldsymbol{f(t) = 1}$***. (EXAMPLE 1)***

<u>SOLUTION:-</u>

Let $f(t) = 1$

Taking $\mathcal{L}$ *on both sides, we have*

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{1\}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}.1dt \ (by\ definition)$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}dt$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \left|\frac{e^{-st}}{-s}\right|_0^\infty$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{-1}{s}(0-1)$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{s}$$

Question: Compute the Laplace Transform of $\boldsymbol{t^n}$***. (EXAMPLE 2)***

<u>Solution:-</u>

Let $f(t) = t^n$

Taking $\mathcal{L}$ *on both sides, we have*

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^n\}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}.t^n dt \ \ (by\ definition)$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \left| t^n.\frac{e^{-st}}{-s} \right|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}.nt^{n-1}dt$$

$$\Rightarrow \mathcal{L}\{f(t)\} = (0-0) + \frac{n}{s}\int_0^\infty e^{-st}.t^{n-1}dt$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{n}{s}\int_0^\infty e^{-st}.t^{n-1}dt$$

$$\Rightarrow \mathcal{L}\{t^n\} = \frac{n}{s}\mathcal{L}\{t^{n-1}\} - - - (i)$$

From (i), we have

$$\Rightarrow \mathcal{L}\{t^{n-1}\} = \frac{n-1}{s}\mathcal{L}\{t^{n-2}\}$$

$$\Rightarrow \mathcal{L}\{t^{n-1}\} = \frac{n-1}{s}.\frac{n-2}{s}\mathcal{L}\{t^{n-3}\}$$

$$\Rightarrow \mathcal{L}\{t^{n-1}\} = \frac{n-1}{s}.\frac{n-2}{s}.\frac{n-3}{s}.\mathcal{L}\{t^{n-4}\}$$

$$\Rightarrow \mathcal{L}\{t^{n-1}\} = \frac{n-1}{s}.\frac{n-2}{s}.\frac{n-3}{s} \dots\dots\dots \mathcal{L}\{t^1\}$$

$$\Rightarrow \mathcal{L}\{t^{n-1}\} = \frac{n-1}{s}.\frac{n-2}{s}.\frac{n-3}{s} \dots\dots\dots \frac{1}{s}\mathcal{L}\{t^0\}$$

$$\Rightarrow \mathcal{L}\{t^{n-1}\} = \frac{(n-1).(n-2).(n-3)\dots\dots..\mathbf{3.2.1}}{s.s.s\dots\dots\dots\dots..s}.\frac{1}{s}$$

$$\Rightarrow \mathcal{L}\{t^{n-1}\} = \frac{(n-1)!}{s^{n-1}}.\frac{1}{s}$$

$$\Rightarrow \mathcal{L}\{t^{n-1}\} = \frac{(n-1)!}{s^n} - - - (ii)$$

Using (ii) in (i), we have

$$\mathcal{L}\{t^n\} = \frac{n}{s}\frac{(n-1)!}{s^n}$$

$$\Rightarrow \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

Question: Compute the Laplace Transform of e^{at}. (EXAMPLE 3)

<u>Solution:-</u>

Let $f(t) = e^{at}$

Taking $\mathcal{L}$ *on both sides, we have*

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{e^{at}\}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}.e^{at}dt \ (by\ definition)$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st+at}dt$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \int_0^\infty e^{(a-s)t}dt$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \left|\frac{e^{(a-s)t}}{a-s}\right|_0^\infty$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{a-s}\left|e^{-(s-a)t}\right|_0^\infty$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{a-s}(0-1)$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{-1}{a-s}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{s-a}$$

Question: Compute the Laplace Transform of $(i)\ \cos at\ \ (ii)\ \sin at$. (EXAMPLE 4)

<u>Solution:-</u>

$(\boldsymbol{i})$ $\quad \boldsymbol{\cos at}$

Let $f(t) = \cos at$

Taking $\mathcal{L}$ *on both sides, we have*

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\cos at\}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}.\cos at\,dt \;\; (by\ definition) - - - (a)$$

Suppose $I = \int_0^\infty e^{-st}.\cos at\,dt$

$$\Rightarrow I = \left|\cos at\,.\frac{e^{-st}}{-s}\right|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}.-a\sin at\,dt$$

$$\Rightarrow I = \left[0-\left(\frac{-1}{s}\right)\right] - \frac{a}{s}\left[\left|\sin at\,.\frac{e^{-st}}{-s}\right|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}.a\cos at\,dt\right]$$

$$\Rightarrow I = \frac{1}{s} - \frac{a}{s}\left[(0-0) + \frac{a}{s}\int_0^\infty e^{-st}.\cos at\,dt\right]$$

$$\Rightarrow I = \frac{1}{s} - \frac{a}{s}\left[\frac{a}{s}I\right]$$

$$\Rightarrow I = \frac{1}{s} - \frac{a^2}{s^2}I$$

$$\Rightarrow I + \frac{a^2}{s^2}I = \frac{1}{s}$$

$$\Rightarrow \left(\frac{s^2+a^2}{s^2}\right)I = \frac{1}{s}$$

$$\Rightarrow I = \frac{1}{s}.\left(\frac{s^2}{s^2+a^2}\right)$$

$$\Rightarrow I = \frac{s}{s^2+a^2}$$

$$\Rightarrow \int_0^\infty e^{-st}.\cos at\,dt = \frac{s}{s^2+a^2}$$

Thus equation (a) *becomes*

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{s}{s^2+a^2}$$

$(\boldsymbol{ii})$ $\mathbf{sin}\,\boldsymbol{at}$

Let $f(t) = \sin at$

Taking $\mathcal{L}$ *on both sides, we have*

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin at\}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}.\sin at\, dt \ \ (by\ definition) - - - (a)$$

Suppose $I = \int_0^\infty e^{-st}.\sin at\, dt$

$$\Rightarrow I = \left|\sin at\,.\frac{e^{-st}}{-s}\right|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}.\,a\cos at\, dt$$

$$\Rightarrow I = [0-0] + \frac{a}{s}\left[\left|\cos at\,.\frac{e^{-st}}{-s}\right|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}.-a\sin at\, dt\right]$$

$$\Rightarrow I = \frac{a}{s}\left[\left(0 - \frac{-1}{s}\right) - \frac{a}{s}\int_0^\infty e^{-st}.\cos at\, dt\right]$$

$$\Rightarrow I = \frac{a}{s}\left[\frac{1}{s} - \frac{a}{s}I\right]$$

$$\Rightarrow I = \frac{a}{s^2} - \frac{a^2}{s^2}I$$

$$\Rightarrow I + \frac{a^2}{s^2}I = \frac{a}{s^2}$$

$$\Rightarrow \left(\frac{s^2+a^2}{s^2}\right)I = \frac{a}{s^2}$$

$$\Rightarrow I = \frac{a}{s^2+a^2}$$

$$\Rightarrow \int_0^\infty e^{-st}.\sin at\, dt = \frac{a}{s^2+a^2}$$

Thus equation (a) *becomes*

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{a}{s^2+a^2}$$

Question 16: Compute the Laplace Transform of $[t]$

Solution:-

Let $f(t) = [t]$

Taking $\mathcal{L}$ *on both sides, we have*

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{[t]\}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}.[t]dt \ \ (by\ definition)$$

$$\mathcal{L}\{f(t)\} = \int_0^1 e^{-st}.[t]dt + \int_1^2 e^{-st}.[t]dt + \int_2^3 e^{-st}.[t]dt + \int_3^4 e^{-st}.[t]dt \ldots$$

$$\mathcal{L}\{f(t)\} = \int_0^1 e^{-st}.(0)dt + \int_1^2 e^{-st}.(1)dt + \int_2^3 e^{-st}.(2)dt + \int_3^4 e^{-st}.(3)dt$$

$$\Rightarrow \mathcal{L}\{f(t)\} = 0 + \int_1^2 e^{-st}dt + 2\int_2^3 e^{-st}dt + 3\int_3^4 e^{-st}dt \ldots\ldots\ldots$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \left|\frac{e^{-st}}{-s}\right|_1^2 + 2.\left|\frac{e^{-st}}{-s}\right|_2^3 + 3.\left|\frac{e^{-st}}{-s}\right|_3^4 + \cdots$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{-1}{s}|e^{-st}|_1^2 + \frac{-1}{s}.|e^{-st}|_2^3 + 3.|e^{-st}|_3^4 + \cdots$$

$$\Rightarrow \mathcal{L}\{f(t)\} = -\frac{1}{s}(e^{-2s} - e^{-s}) - \frac{1}{s}.2(e^{-3s} - e^{-2s}) - \frac{1}{s}.3(e^{-4s} - e^{-3s}) + ..$$

$$\Rightarrow \mathcal{L}\{f(t)\} = -\frac{1}{s}(e^{-2s} - e^{-s} + 2e^{-3s} - 2e^{-2s} + 3e^{-4s} - 3e^{-3s} + \cdots)$$

$$\Rightarrow \mathcal{L}\{f(t)\} = -\frac{1}{s}(-e^{-s} - e^{-2s} - e^{-3s} - e^{-4s} + \cdots)$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{s}(e^{-s} + e^{-2s} + e^{-3s} + e^{-4s} + \cdots)$$

$$\mathcal{L}\{f(t)\} = \frac{1}{s}.\frac{e^{-s}}{1 - e^{-s}}$$

$$\mathcal{L}\{f(t)\} = \frac{e^{-s}}{s(1 - e^{-s})}$$

Question 17: $t^{\alpha} \quad \alpha > -1$

<u>Solution:-</u>

Let $f(t) = \boldsymbol{t^{\alpha}}$

Taking $\mathcal{L}$ *on both sides, we have*

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\boldsymbol{t^{\alpha}}\}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}.\boldsymbol{t^\alpha} dt$$

$$Put\ st = u\ \ so\ that\ \ t = \frac{1}{s}u$$

$$dt = \frac{1}{s}.du$$

$$When\ t \to 0\ then\ u \to 0$$

$$When\ t \to \infty\ then\ u \to \infty$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \int_0^\infty e^{-u}.\left(\frac{u}{s}\right)^{\boldsymbol{\alpha}} \frac{1}{s} du$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \int_0^\infty e^{-u}.\frac{u^{\boldsymbol{\alpha}}}{s^{\boldsymbol{\alpha}}}\frac{1}{s} du$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{s^{\boldsymbol{\alpha+1}}}\int_0^\infty e^{-u}.u^{\boldsymbol{\alpha}} dt$$

From calculus,

$$\int_0^\infty e^{-u}.u^{\boldsymbol{\alpha}} dt = \Gamma\alpha + 1$$

Therefore,

$$\mathcal{L}\{f(t)\} = \frac{\Gamma\alpha + 1}{s^{\alpha+1}}$$

Question 27: ln t

Solution:-

Let $f(t) = \ ln\ t$

Taking $\mathcal{L}$ *on both sides, we have*

$$\mathcal{L}\{f(t)\} = \ \mathcal{L}\{ln\ t\}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}.ln\ tdt$$

$$Put\ st = u\ \ so\ that\ \ t = \frac{1}{s}u$$

$$dt = \frac{1}{s}.du$$

$$When\ t \to 0\ then\ u \to 0$$

$$When\ t \to \infty\ then\ u \to \infty$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \int_0^\infty e^{-u}.\, ln\left(\frac{u}{s}\right)\frac{1}{s}du$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{s}\int_0^\infty e^{-u}.\{ln\,(u) - ln\,(s)\}du$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{s}\int_0^\infty \{e^{-u}.\, ln\,(u) - e^{-u}.\, ln\,(s)\}du$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{s}\int_0^\infty e^{-u}.\, ln\,(u)du - \frac{1}{s}\int_0^\infty e^{-u}.\, ln\,(s)du$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{s}\int_0^\infty e^{-u}.\, ln\,(u)du - \frac{1}{s}.\, ln\,(s)\int_0^\infty e^{-u}du$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{s}\int_0^\infty e^{-u}.\, ln\,(u)du - \frac{1}{s}.\, ln\,(s)\left|-e^{-u}\right|_0^\infty$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{s}\int_0^\infty e^{-u}.\, ln\,(u)du + \frac{1}{s}.\, ln\,(s)(0-1)$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{s}\int_0^\infty e^{-u}.\, ln\,(u)du - \frac{1}{s}.\, ln\,(s) - - - (A)$$

From calculus,

$$\int_0^\infty e^{-u}.\, u^\alpha dt = \Gamma\alpha + 1$$

Differentiating with respect to α, we have

$$\int_0^\infty e^{-u}.\, u^\alpha \ln(u)\, dt = \Gamma 1'$$

When $\alpha = 0$. *Therefore,*

$$\int_0^\infty e^{-u}.\ln(u)\, dt = \Gamma 1'$$

Thus equation (A) *becomes*

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{s}\Gamma 1' - \frac{1}{s}.\, ln\,(s)$$

❖ ***FORMULA:-***

*Let $f(t)$ be any function, then the **Laplace transformation** of $f(t)$ multiplied by t^n can be found by using the formula is given below:*

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\mathcal{L}\{f(t)\}$$

Question 18: $t^2 \sin at$

Solution:-

Let $f(t) = \sin at$

By the formula, we have

$$\mathcal{L}\{t^2 \sin at\} = (-1)^2 \frac{d^2}{ds^2}\mathcal{L}\{\sin at\}$$

$$\Rightarrow \mathcal{L}\{t^2 \sin at\} = \frac{d^2}{ds^2}\left\{\frac{a}{s^2+a^2}\right\}$$

$$\Rightarrow \mathcal{L}\{t^2 \sin at\} = \frac{d}{ds}\left[\frac{d}{ds}\left\{\frac{a}{s^2+a^2}\right\}\right]$$

$$\Rightarrow \mathcal{L}\{t^2 \sin at\} = a.\frac{d}{ds}\left[\frac{-2s}{(s^2+a^2)^2}\right]$$

$$\Rightarrow \mathcal{L}\{t^2 \sin at\} = -2a.\frac{d}{ds}\left[\frac{s}{(s^2+a^2)^2}\right]$$

$$\Rightarrow \mathcal{L}\{t^2 \sin at\} = -2a.\left[\frac{(s^2+a^2)^2.1 - 2(s^2+a^2)2s.s}{(s^2+a^2)^4}\right]$$

$$\Rightarrow \mathcal{L}\{t^2 \sin at\} = -2a.\left[\frac{(s^2+a^2)^2.1 - 4s^2(s^2+a^2)}{(s^2+a^2)^4}\right]$$

$$\Rightarrow \mathcal{L}\{t^2 \sin at\} = -2a.\left[\frac{(s^2+a^2)(s^2+a^2-4s^2)}{(s^2+a^2)^4}\right]$$

$$\Rightarrow \mathcal{L}\{t^2 \sin at\} = -2a.\left[\frac{(a^2-3s^2)}{(s^2+a^2)^3}\right]$$

$$\Rightarrow \mathcal{L}\{t^2 \sin at\} = \frac{2a(3s^2-a^2)}{(s^2+a^2)^3}$$

Question 19: $t^2 \cos at$

Solution:-

Let $f(t) = \cos at$

By the formula, we have

$$\mathcal{L}\{t^2 \cos at\} = (-1)^2 \frac{d^2}{ds^2} \mathcal{L}\{\cos at\}$$

$$\Rightarrow \mathcal{L}\{t^2 \sin at\} = \frac{d^2}{ds^2}\left\{\frac{s}{s^2+a^2}\right\}$$

$$\Rightarrow \mathcal{L}\{t^2 \sin at\} = \frac{d}{ds}\left[\frac{d}{ds}\left\{\frac{s}{s^2+a^2}\right\}\right]$$

$$\Rightarrow \mathcal{L}\{t^2 \sin at\} = \frac{d}{ds}\left[\frac{(s^2+a^2).1 - s.2s}{(s^2+a^2)^2}\right]$$

$$\Rightarrow \mathcal{L}\{t^2 \sin at\} = \frac{d}{ds}\left[\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}\right]$$

$$\Rightarrow \mathcal{L}\{t^2 \sin at\} = \frac{d}{ds}\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

$$\Rightarrow \mathcal{L}\{t^2 \sin at\} = \left[\frac{(s^2+a^2)^2.(-2s) - (a^2-s^2)2(s^2+a^2).2s}{(s^2+a^2)^4}\right]$$

$$\Rightarrow \mathcal{L}\{t^2 \sin at\} = \left[\frac{2s(s^2+a^2)(-s^2-a^2-2a^2+2s^2)}{(s^2+a^2)^4}\right]$$

$$\Rightarrow \boldsymbol{\mathcal{L}\{t^2 \sin at\}} = \frac{\boldsymbol{2s(s^2-3a^2)}}{\boldsymbol{(s^2+a^2)^3}}$$

Question 20: $t\sin^2 at$

Solution:-

Let $f(t) = t\sin^2 at$

$$f(t) = t.\frac{1-\cos 2at}{2} \qquad \text{by trigonometry} \quad \cos 2\alpha = 1 - 2\sin^2\theta$$

$$\Rightarrow f(t) = \frac{1}{2}[t - t\cos 2at]$$

Taking $\mathcal{L}$ *on both sides, we have*

$$\mathcal{L}\{f(t)\} = \frac{1}{2}[\mathcal{L}\{t\} - \mathcal{L}\{tcos\ 2at\}]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{2}\left[\frac{1}{s^2} - (-1)\frac{d}{ds}\left(\frac{s}{s^2+4a^2}\right)\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{2}\left[\frac{1}{s^2} + \frac{(s^2+4a^2).1 - s.2s}{(s^2+4a^2)^2}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{2}\left[\frac{1}{s^2} + \frac{s^2+4a^2-2s^2}{(s^2+4a^2)^2}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{2}\left[\frac{1}{s^2} + \frac{4a^2-s^2}{(s^2+4a^2)^2}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{2}\left[\frac{(s^2+4a^2)^2 + s^2(4a^2-s^2)}{s^2(s^2+4a^2)^2}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{2}\left[\frac{s^4+16a^4+8a^2s^2+4a^2s^2-s^4}{s^2(s^2+4a^2)^2}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{2}\left[\frac{16a^4+12a^2s^2}{s^2(s^2+4a^2)^2}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{2}\left[\frac{4a^2(4a^2+3s^2)}{s^2(s^2+4a^2)^2}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{2a^2(4a^2+3s^2)}{s^2(s^2+4a^2)^2}$$

Question 21: $t^2cos^2 2t$

<u>Solution:-</u>

Let $f(t) = t^2cos^2 2t$

$$f(t) = t^2.\frac{1+\cos 4t}{2} \qquad \text{by trigonometry} \quad \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\Rightarrow f(t) = \frac{1}{2}[t^2 + t^2\cos 4t]$$

Taking $\mathcal{L}$ on both sides, we have

$$\mathcal{L}\{f(t)\} = \frac{1}{2}[\mathcal{L}\{t^2\} + \mathcal{L}\{t^2\cos 4t\}]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{2}\left[\frac{2}{s^3} + (-1)^2\frac{d^2}{ds^2}\left(\frac{s}{s^2+16}\right)\right] \text{ By the formula}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{2}\left[\frac{2}{s^3} + \frac{d}{ds}\left(\frac{(s^2+16).1 - s.2s}{(s^2+16)^2}\right)\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{2}\left[\frac{2}{s^3} + \frac{d}{ds}\left(\frac{16-s^2}{(s^2+16)^2}\right)\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{2}\left[\frac{2}{s^3} + \left(\frac{(s^2+16)^2(-2s) - (16-s^2)2(s^2+16).2s}{(s^2+16)^4}\right)\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{2}\left[\frac{2}{s^3} + \frac{2s(s^2+16)(-s^2-16-32+2s^2}{(s^2+16)^4}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{2}{2}\left[\frac{1}{s^3} + \frac{s(s^2-48)}{(s^2+16)^3}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{s^3} + \frac{s(s^2+16-64)}{(s^2+16)^3}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{s^3} + \frac{s(s^2+16)}{(s^2+16)^3} - \frac{64}{(s^2+16)^3}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{s^3} + \frac{s}{(s^2+16)^2} - \frac{64}{(s^2+16)^3}$$

$$\Rightarrow \boldsymbol{\mathcal{L}\{f(t)\} = \frac{(s^2+16)^3 + s^4(s^2+16) - 64s^3}{s^3(s^2+16)^3}}$$

❖ ***FORMULA***

If $\mathcal{L}\{f(t)\} = F(s)$ *then*

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)du \text{ provided } \lim_{t\to 0^+}\frac{f(t)}{t} \text{ exists}$$

Question # 22:- $\boldsymbol{\frac{\sin at}{t}}$

SOLUTION:-

Let $f(t) = \sin at --- (i)$

Taking $\mathcal{L}$ *on both sides, we have*

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin at\}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{a}{s^2 + a^2}$$

By the formula $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)du$, *we have*

$$\mathcal{L}\left\{\frac{\sin at}{t}\right\} = \int_s^\infty \frac{a}{u^2 + a^2} du$$

$$\Rightarrow \mathcal{L}\left\{\frac{\sin at}{t}\right\} = a.\int_s^\infty \frac{1}{u^2 + a^2} du$$

$$\Rightarrow \mathcal{L}\left\{\frac{\sin at}{t}\right\} = a.\int_s^\infty \frac{1}{u^2 + a^2} du$$

$$\Rightarrow \mathcal{L}\left\{\frac{\sin at}{t}\right\} = a.\frac{1}{a}\left|tan^{-1}\left(\frac{u}{a}\right)\right|_s^\infty$$

$$\Rightarrow \mathcal{L}\left\{\frac{\sin at}{t}\right\} = \left[tan^{-1}(\infty) - tan^{-1}\left(\frac{s}{a}\right)\right]$$

$$\Rightarrow \mathcal{L}\left\{\frac{\sin at}{t}\right\} = \frac{\pi}{2} - tan^{-1}\left(\frac{s}{a}\right)$$

$$\Rightarrow \mathcal{L}\left\{\frac{\sin at}{t}\right\} = tan^{-1}\left(\frac{a}{s}\right)$$

Question # 23:- $\frac{1 - \cos at}{t}$

SOLUTION:-

Let $f(t) = 1 - \cos at - - - (i)$

Taking $\mathcal{L}$ *on both sides, we have*

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{1 - \cos at\}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \mathcal{L}\{1\} - \mathcal{L}\{\cos at\}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{s} - \frac{s}{s^2 + a^2}$$

By the formula $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)du$, *we have*

$$\mathcal{L}\left\{\frac{1 - \cos at}{t}\right\} = \int_s^\infty \left(\frac{1}{u} - \frac{u}{u^2 + a^2}\right) du$$

$$\Rightarrow \mathcal{L}\left\{\frac{1 - \cos at}{t}\right\} = \int_s^\infty \frac{1}{u} du - \int_s^\infty \frac{u}{u^2 + a^2} du$$

$$\Rightarrow \mathcal{L}\left\{\frac{1-\cos at}{t}\right\} = \frac{1}{2}.\left|\ln u^2\right|_s^\infty - \frac{1}{2}\left|\ln(u^2+a^2)\right|_s^\infty$$

$$\Rightarrow \mathcal{L}\left\{\frac{1-\cos at}{t}\right\} = \frac{1}{2}.\left|\ln\frac{u^2}{u^2+a^2}\right|_s^\infty$$

$$\Rightarrow \mathcal{L}\left\{\frac{1-\cos at}{t}\right\} = \frac{1}{2}.\left|\ln\frac{1}{1+\frac{a^2}{u^2}}\right|_s^\infty$$

$$\Rightarrow \mathcal{L}\left\{\frac{1-\cos at}{t}\right\} = \frac{1}{2}.\left[\ln 1 - \ln\frac{1}{1+\frac{a^2}{s^2}}\right]$$

$$\Rightarrow \mathcal{L}\left\{\frac{1-\cos at}{t}\right\} = \frac{1}{2}.\left[0 - \ln\frac{s^2}{s^2+a^2}\right]$$

$$\Rightarrow \mathcal{L}\left\{\frac{1-\cos at}{t}\right\} = -\frac{1}{2}\ln\frac{s^2}{s^2+a^2}$$

$$\Rightarrow \mathcal{L}\left\{\frac{1-\cos at}{t}\right\} = \frac{1}{2}\ln\frac{s^2+a^2}{s^2}$$

Question # 26:- $\frac{\sinh at}{t}$

SOLUTION:-

Let $f(t) = \sinh at - - - (i)$

Taking $\mathcal{L}$ *on both sides, we have*

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sinh at\}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{a}{s^2-a^2}$$

By the formula $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)du$, *we have*

$$\mathcal{L}\left\{\frac{\sinh at}{t}\right\} = \int_s^\infty \frac{a}{u^2-a^2}du$$

$$\Rightarrow \mathcal{L}\left\{\frac{\sinh at}{t}\right\} = a.\int_s^\infty \frac{1}{u^2-a^2}du$$

$$\Rightarrow \mathcal{L}\left\{\frac{\sinh at}{t}\right\} = a.\frac{1}{2a}\left|ln\left(\frac{u-a}{u+a}\right)\right|_s^\infty$$

$$\Rightarrow \mathcal{L}\left\{\frac{\sinh at}{t}\right\} = \frac{1}{2}\left|ln\left(\frac{1-\frac{a}{u}}{1+\frac{a}{u}}\right)\right|_s^\infty$$

$$\Rightarrow \mathcal{L}\left\{\frac{\sinh at}{t}\right\} = \frac{1}{2}\left[\ln 1 - ln\left(\frac{1-\frac{a}{s}}{1+\frac{a}{s}}\right)\right]$$

$$\Rightarrow \mathcal{L}\left\{\frac{\sinh at}{t}\right\} = \frac{1}{2}\left[0 - ln\left(\frac{s-a}{s+a}\right)\right]$$

$$\Rightarrow \mathcal{L}\left\{\frac{\sinh at}{t}\right\} = -\frac{1}{2}ln\left(\frac{s-a}{s+a}\right)$$

$$\Rightarrow \mathcal{L}\left\{\frac{\sinh at}{t}\right\} = \frac{1}{2}ln\left(\frac{s+a}{s-a}\right)$$

Question : Compute the laplace transform of $\frac{\sin t}{t}$ (Example 12)

<u>SOLUTION:-</u>

Let $f(t) = \sin t - - - (i)$

Taking $\mathcal{L}$ *on both sides, we have*

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin t\}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{s^2+1}$$

By the formula $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)du$, *we have*

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{u^2+1}du$$

$$\Rightarrow \mathcal{L}\left\{\frac{\sin t}{t}\right\} = \frac{1}{1}.\left|tan^{-1}\left(\frac{u}{1}\right)\right|_s^\infty$$

$$\Rightarrow \mathcal{L}\left\{\frac{\sin t}{t}\right\} = [tan^{-1}(\infty) - tan^{-1}(s)]$$

$$\Rightarrow \mathcal{L}\left\{\frac{\sin at}{t}\right\} = \frac{\pi}{2} - tan^{-1}(s)$$

❖ ***FORMULA***

If g is piecewise continuous and is of exponential order a, then

$$\mathcal{L}\left\{\int_0^t g(u)\,du\right\} = \frac{1}{s}\mathcal{L}\{g(t)\}$$

Question # 24:- $\int_0^t \frac{\sin au}{u}\,du$

SOLUTION:-

Let $g(u) = \frac{\sin au}{u}$

Therefore, by the formula $\mathcal{L}\left\{\int_0^t g(u)\,du\right\} = \frac{1}{s}\mathcal{L}\{g(t)\}$, *we have*

$$\mathcal{L}\left\{\int_0^t \frac{\sin au}{u}\,du\right\} = \frac{1}{s}\mathcal{L}\left\{\frac{\sin at}{t}\right\} - - - (A)$$

Let $f(t) = \sin at - - - (i)$

Taking $\mathcal{L}$ *on both sides, we have*

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin at\}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{a}{s^2 + a^2}$$

By the formula $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)du$, *we have*

$$\mathcal{L}\left\{\frac{\sin at}{t}\right\} = \int_s^\infty \frac{a}{u^2 + a^2}\,du$$

$$\Rightarrow \mathcal{L}\left\{\frac{\sin at}{t}\right\} = a.\int_s^\infty \frac{1}{u^2 + a^2}\,du$$

$$\Rightarrow \mathcal{L}\left\{\frac{\sin at}{t}\right\} = a.\int_s^\infty \frac{1}{u^2 + a^2}\,du$$

$$\Rightarrow \mathcal{L}\left\{\frac{\sin at}{t}\right\} = a.\frac{1}{a}\left|tan^{-1}\left(\frac{u}{a}\right)\right|_s^\infty$$

$$\Rightarrow \mathcal{L}\left\{\frac{\sin at}{t}\right\} = \left[tan^{-1}(\infty) - tan^{-1}\left(\frac{s}{a}\right)\right]$$

$$\Rightarrow \mathcal{L}\left\{\frac{\sin at}{t}\right\} = \frac{\pi}{2} - tan^{-1}\left(\frac{s}{a}\right)$$

$$\Rightarrow \mathcal{L}\left\{\frac{\sin at}{t}\right\} = tan^{-1}\left(\frac{a}{s}\right)$$

Therefore equation (A) becomes

$$\mathcal{L}\left\{\int_0^t \frac{sin\ au}{u}du\right\} = \frac{1}{s}tan^{-1}\left(\frac{a}{s}\right)$$

Question # 25:- $\int_0^t \frac{1-cos\ au}{u}du$

Let $g(u) = \frac{1-cos\ au}{u}$

Therefore, by the formula $\mathcal{L}\left\{\int_0^t g(u)\,du\right\} = \frac{1}{s}\mathcal{L}\{g(t)\}$*, we have*

$$\mathcal{L}\left\{\int_0^t \frac{1-cos\ au}{u}du\right\} = \frac{1}{s}\mathcal{L}\left\{\frac{1-cos\ at}{t}\right\} - - - (A)$$

Let $f(t) = 1 - \cos at - - - (i)$

Taking $\mathcal{L}$ *on both sides, we have*

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{1 - \cos at\}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \mathcal{L}\{1\} - \mathcal{L}\{\cos at\}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{s} - \frac{s}{s^2+a^2}$$

By the formula $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)du$*, we have*

$$\mathcal{L}\left\{\frac{1-cos\ at}{t}\right\} = \int_s^\infty \left(\frac{1}{u} - \frac{u}{u^2+a^2}\right)du$$

$$\Rightarrow \mathcal{L}\left\{\frac{1-cos\ at}{t}\right\} = \int_s^\infty \frac{1}{u}du - \int_s^\infty \frac{u}{u^2+a^2}du$$

$$\Rightarrow \mathcal{L}\left\{\frac{1-cos\ at}{t}\right\} = \frac{1}{2}.\left|\ln u^2\right|_s^\infty - \frac{1}{2}\left|\ln(u^2+a^2)\right|_s^\infty$$

$$\Rightarrow \mathcal{L}\left\{\frac{1-cos\ at}{t}\right\} = \frac{1}{2}.\left|\ln\frac{u^2}{u^2+a^2}\right|_s^\infty$$

$$\Rightarrow \mathcal{L}\left\{\frac{1-\cos at}{t}\right\} = \frac{1}{2}.\left|\ln\frac{1}{1+\frac{a^2}{u^2}}\right|_s^\infty$$

$$\Rightarrow \mathcal{L}\left\{\frac{1-\cos at}{t}\right\} = \frac{1}{2}.\left[\ln 1 - \ln\frac{1}{1+\frac{a^2}{s^2}}\right]$$

$$\Rightarrow \mathcal{L}\left\{\frac{1-\cos at}{t}\right\} = \frac{1}{2}.\left[0 - \ln\frac{s^2}{s^2+a^2}\right]$$

$$\Rightarrow \mathcal{L}\left\{\frac{1-\cos at}{t}\right\} = -\frac{1}{2}\ln\frac{s^2}{s^2+a^2}$$

$$\Rightarrow \mathcal{L}\left\{\frac{1-\cos at}{t}\right\} = \frac{1}{2}\ln\frac{s^2+a^2}{s^2}$$

Therefore equation (A) becomes

$$\mathcal{L}\left\{\int_0^t \frac{1-\cos au}{u}du\right\} = \frac{1}{2s}\ln\frac{s^2+a^2}{s^2}$$

Question: - Compute the Laplace Transform of $\int_0^t \frac{1-\cosh au}{u}du$. (Example 13)

Solution:

Let $g(u) = \frac{1-\cosh au}{u}$

Therefore, by the formula $\mathcal{L}\left\{\int_0^t g(u)\,du\right\} = \frac{1}{s}\mathcal{L}\{g(t)\}$*, we have*

$$\mathcal{L}\left\{\int_0^t \frac{1-\cosh au}{u}du\right\} = \frac{1}{s}\mathcal{L}\left\{\frac{1-\cosh at}{t}\right\} - - - (A)$$

Let $f(t) = 1-\cosh at - - - (i)$

Taking $\mathcal{L}$ *on both sides, we have*

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{1-\cosh at\}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \mathcal{L}\{1\} - \mathcal{L}\{\cosh at\}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{s} - \frac{s}{s^2-a^2}$$

By the formula $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)du$*, we have*

$$\mathcal{L}\left\{\frac{1-cosh\,at}{t}\right\} = \int_s^\infty \left(\frac{1}{u} - \frac{u}{u^2-a^2}\right) du$$

$$\Rightarrow \mathcal{L}\left\{\frac{1-cosh\,at}{t}\right\} = \int_s^\infty \frac{1}{u}du - \int_s^\infty \frac{u}{u^2-a^2}du$$

$$\Rightarrow \mathcal{L}\left\{\frac{1-cosh\,at}{t}\right\} = \frac{1}{2}.\left|\ln u^2\right|_s^\infty - \frac{1}{2}\left|\ln(u^2-a^2)\right|_s^\infty$$

$$\Rightarrow \mathcal{L}\left\{\frac{1-cosh\,at}{t}\right\} = \frac{1}{2}.\left|\ln\frac{u^2}{u^2-a^2}\right|_s^\infty$$

$$\Rightarrow \mathcal{L}\left\{\frac{1-cosh\,at}{t}\right\} = \frac{1}{2}.\left|\ln\frac{1}{1-\frac{a^2}{u^2}}\right|_s^\infty$$

$$\Rightarrow \mathcal{L}\left\{\frac{1-cosh\,at}{t}\right\} = \frac{1}{2}.\left[\ln 1 - \ln\frac{1}{1-\frac{a^2}{s^2}}\right]$$

$$\Rightarrow \mathcal{L}\left\{\frac{1-cosh\,at}{t}\right\} = \frac{1}{2}.\left[0 - \ln\frac{s^2}{s^2-a^2}\right]$$

$$\Rightarrow \mathcal{L}\left\{\frac{1-cosh\,at}{t}\right\} = -\frac{1}{2}\ln\frac{s^2}{s^2-a^2}$$

$$\Rightarrow \mathcal{L}\left\{\frac{1-cosh\,at}{t}\right\} = \frac{1}{2}\ln\frac{s^2-a^2}{s^2}$$

Therefore equation (A) becomes

$$\mathcal{L}\left\{\int_0^t \frac{1-cosh\,au}{u}du\right\} = \frac{1}{2s}\ln\frac{s^2-a^2}{s^2}$$

❖ *UNIT STEP FUNCTION*

Definition:-

Let $a \geq 0$. The function u_a defined on $]0,\infty[$ by

$$u_a(t) = \begin{cases} 0 & if\ \ t < a \\ 1 & if\ \ t > a \end{cases}$$

*is called the **unit step function.***

Theorem:- *Let u_a be the unit step function. Then,*

$$\mathcal{L}\{u_a(t)\} = \frac{e^{-as}}{s}$$

Theorem: - *Let f be a function of exponential order a and* $\mathcal{L}\{f(t)\} = F(s)$. *For the function*

$$u_a(t)f(t-a) = \begin{cases} 0 & if\ \ t < a \\ f(t-a) & if\ \ t > a \end{cases}$$

We have,

$$\mathcal{L}\{u_a(t)f(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}.$$

Question # 28:- Compute the Laplace transform of

$$f(t) = \begin{cases} 0 & if\ \ t < 3 \\ (t-3)^3 & if\ \ t > 3 \end{cases}$$

Solution:-

Given function is

$$f(t) = \begin{cases} 0 & if\ \ t < 3 \\ (t-3)^3 & if\ \ t > 3 \end{cases}$$

Then we have,

$$f(t) = u_3(t)f(t-3)$$

Taking $\mathcal{L}$ *on both sides, we have*

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{u_3(t)f(t-3)\}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = e^{-3s}\mathcal{L}\{t^3\}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = e^{-3s}.\frac{3!}{s^4}$$

$$\Rightarrow \boldsymbol{\mathcal{L}\{f(t)\} = \frac{6e^{-3s}}{s^4}}$$

Question:- Compute the Laplace transform of (Example 14)

$$f(t) = \begin{cases} 0 & if\ \ t < \frac{\pi}{2} \\ \cos t & if\ \ t > \frac{\pi}{2} \end{cases}$$

Solution:-

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Given function is

$$f(t)=\begin{cases}0 & if\ \ t<\frac{\pi}{2}\\ \cos t & if\ \ t>\frac{\pi}{2}\end{cases}$$

Firstly, we have to express $\cos t$ *in terms of* $t-\frac{\pi}{2}$*, so as to apply the formula.*

As, $\cos t=-\sin(t-\frac{\pi}{2})$*, let*

$$g(t)=\begin{cases}0 & if\ \ t<\frac{\pi}{2}\\ \sin(t-\frac{\pi}{2}) & if\ \ t>\frac{\pi}{2}\end{cases}$$

Then $f(t)=-u_{\frac{\pi}{2}}(t)g(t)$

Taking $\mathcal{L}$ *on both sides, we have*

$$\mathcal{L}\{f(t)\}=-\mathcal{L}\left\{u_{\frac{\pi}{2}}(t).\sin(t-\frac{\pi}{2})\right\}$$

$$\Rightarrow \mathcal{L}\{f(t)\}=-e^{-\frac{\pi}{2}s}\mathcal{L}\{\sin t\}$$

$$\Rightarrow \boldsymbol{\mathcal{L}\{f(t)\}=-e^{-\frac{\pi}{2}s}\frac{1}{s^2+1}}$$

Question # 29: If $\boldsymbol{\mathcal{L}\{f(t)\}=F(s)\ for\ s>a}$***, show that***

$$\boldsymbol{\mathcal{L}\{f(ct)\}=\frac{1}{c}F\left(\frac{s}{c}\right), c>0\ and\ s>ca}$$

<u>SOLUTION: -</u>

Given that $\mathcal{L}\{f(t)\}=F(s)$

Then by definition

$$\mathcal{L}\{f(ct)\}=\int_0^\infty e^{-st}.f(ct)dt$$

$$Put\ ct=T\ \ so\ that\ \ t=\frac{1}{c}T$$

$$dt=\frac{1}{c}.dT$$

$$When\ t\to 0\ then\ T\to 0$$

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$$When\ t \to \infty\ then\ T \to \infty$$

$$\mathcal{L}\{f(ct)\} = \int_0^\infty e^{-s\frac{T}{c}}.f(T)\frac{1}{c}.dT$$

$$\mathcal{L}\{f(ct)\} = \frac{1}{c}.\int_0^\infty e^{-s\frac{T}{c}}.f(T)dT$$

$$\mathcal{L}\{f(ct)\} = \frac{1}{c}.\mathcal{L}\left\{f(\frac{T}{c})\right\}$$

$$\mathcal{L}\{f(ct)\} = \frac{1}{c}.F\left(\frac{s}{c}\right)$$

This completes the proof.

❖ ***F0RMULA:***

MOTIVATION:-

If there is a function such that it is a derivative of any other function.to finds the Laplace transformation of such kind of function we use the following formula which is stated below.

STATEMENT:-

Let $f(t)$ is any function, then the Laplace transformation of $f'(t)$ can be found by the following formula.

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

The application of this formula is stated in the following question.

Question # 32:- Compute $\mathcal{L}\{\sin\sqrt{t}\}$. Deduce $\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\}$.

SOLUTION:-

$$Let\ f(t) = sin\sqrt{t}$$

The power series expansion of $\sin x$ *is given by*

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} - - - - - - -$$

Replacing x by $t^{\frac{1}{2}}$, we have

$$sin\sqrt{t} = t^{\frac{1}{2}} - \frac{t^{\frac{3}{2}}}{3!} + \frac{t^{\frac{5}{2}}}{5!} - \frac{t^{\frac{7}{2}}}{7!} - - - - - - -$$

Taking $\mathcal{L}$ on both sides, we have

$$\mathcal{L}\{sin\sqrt{t}\} = \mathcal{L}\left[t^{\frac{1}{2}} - \frac{t^{\frac{3}{2}}}{3!} + \frac{t^{\frac{5}{2}}}{5!} - \frac{t^{\frac{7}{2}}}{7!} - - - - - - -\right]$$

$$\Rightarrow \mathcal{L}\{sin\sqrt{t}\} = \mathcal{L}\left\{t^{\frac{1}{2}}\right\} - \frac{1}{3!}\mathcal{L}\left\{t^{\frac{3}{2}}\right\} + \frac{1}{5!}\mathcal{L}\left\{t^{\frac{5}{2}}\right\} - \frac{1}{7!}\mathcal{L}\left\{t^{\frac{7}{2}}\right\} - - - -$$

$$\Rightarrow \mathcal{L}\{sin\sqrt{t}\} = \frac{\Gamma\frac{1}{2}+1}{s^{\frac{1}{2}+1}} - \frac{1}{3!}\frac{\Gamma\frac{3}{2}+1}{s^{\frac{3}{2}+1}} + \frac{1}{5!}\frac{\Gamma\frac{5}{2}+1}{s^{\frac{5}{2}+1}} - \frac{1}{7!}\frac{\Gamma\frac{7}{2}+1}{s^{\frac{7}{2}+1}} - - - -$$

Here the sign "Γ" denote the Gamma function.

$$\Rightarrow \mathcal{L}\{sin\sqrt{t}\} = \frac{\frac{1}{2}\Gamma\frac{1}{2}}{s^{\frac{3}{2}}} - \frac{1}{3!}\frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{s^{\frac{5}{2}}} + \frac{1}{5!}\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{s^{\frac{7}{2}}} - \frac{1}{7!}\frac{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{s^{\frac{9}{2}}} - - - -$$

Since $\Gamma\frac{1}{2} = \sqrt{\pi}$. Therefore,

$$\Rightarrow \mathcal{L}\{sin\sqrt{t}\} = \frac{\frac{1}{2}\sqrt{\pi}}{s^{\frac{3}{2}}} - \frac{1}{3!}\frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{s^{\frac{5}{2}}} + \frac{1}{5!}\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{s^{\frac{7}{2}}} - \frac{1}{7!}\frac{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{s^{\frac{9}{2}}} - - - -$$

$$\Rightarrow \mathcal{L}\{sin\sqrt{t}\} = \frac{\frac{1}{2}\sqrt{\pi}}{s^{\frac{3}{2}}}\left[1 - \frac{1}{6}\frac{\frac{3}{2}}{s} + \frac{1}{120}\frac{\frac{15}{4}}{s^2} - \frac{1}{5040}\frac{\frac{105}{8}}{s^3} - - - -\right]$$

$$\Rightarrow \mathcal{L}\{sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}\left[1 - \frac{1}{2.2}\frac{1}{s} + \frac{1}{8.4}\frac{1}{s^2} - \frac{1}{48.8}\frac{1}{s^3} - - - -\right]$$

$$\Rightarrow \mathcal{L}\{sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}\left[1 - \frac{1}{4s} + \frac{1}{32s^2} - \frac{1}{384s^3} - - - -\right]$$

$$\Rightarrow \mathcal{L}\{sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}\left[1 - \frac{1}{4s} + \frac{1}{2!}\frac{1}{16s^2} - \frac{1}{3!}\frac{1}{64s^3} - - - -\right]$$

$$\Rightarrow \mathcal{L}\{sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}\left[1 + \left(-\frac{1}{4s}\right) + \frac{1}{2!}\left(-\frac{1}{4s}\right)^2 + \frac{1}{3!}\left(-\frac{1}{4s}\right)^3 - - - -\right]$$

$$\Rightarrow \mathcal{L}\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}.e^{-\frac{1}{4s}}$$

Now, we have to deduce $\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\}$.

Since $f(t)=\sin\sqrt{t}$. *this implies that* $f(0)=0$ *and*

$$f'(t) = \frac{\sin\sqrt{t}.}{2\sqrt{t}}$$

Using the formula $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$, *we have*

$$\mathcal{L}\left\{\frac{\cos\sqrt{t}.}{2\sqrt{t}}\right\} = s\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}.e^{-\frac{1}{4s}} - 0$$

$$\Rightarrow \frac{1}{2}\mathcal{L}\left\{\frac{\cos\sqrt{t}.}{\sqrt{t}}\right\} = \frac{\sqrt{\pi}}{2s^{\frac{1}{2}}}.e^{-\frac{1}{4s}}$$

$$\Rightarrow \mathcal{L}\left\{\frac{\cos\sqrt{t}.}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{s}}.e^{-\frac{1}{4s}}$$