NOTES OF EXERCISE 11.2

INVERSE LAPLACE TRANSFORMATION:-

❖ ***Introduction:-***

In the previous exercise we have discussed various properties of Laplace Transformation and obtained the Laplace transform of some simple functions. However, if the Laplace transform technique is to be useful in application, we have to consider the reverse problem, i.e., we have to find the original function $f(t)$ when we know its Laplace transform F(s).

❖ ***Definition:-***

If the Laplace transform of $f(t)$ is F(s), that is,

$$\mathcal{L}\{f(t)\} = F(s)$$

Then

$$f(t) = \mathcal{L}^{-1}[F(s)]$$

EXERCISE 11.2

Compute the inverse Laplace transform of each of the following.

Question # 1:- $\frac{s-2}{s^2-2}$

Solution:-

Suppose that

$$F(s) = \frac{s-2}{s^2-2}$$

Applying $\mathcal{L}^{-1}$ on both sides, we have

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{s-2}{s^2-2}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2-2} - \frac{2}{s^2-2}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2-2}\right\} - \mathcal{L}^{-1}\left\{\frac{2}{s^2-2}\right\}$$

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$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{(s)^2-\left(\sqrt{2}\right)^2}\right\} - \mathcal{L}^{-1}\left\{\frac{\sqrt{2}.\sqrt{2}}{(s)^2-\left(\sqrt{2}\right)^2}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = cosh\sqrt{2}\,t - \sqrt{2}\,sinh\sqrt{2}\,t$$

Question # 2:- $\frac{3s+1}{s^2-6s+18}$

Solution:-

Suppose that

$$F(s) = \frac{3s+1}{s^2-6s+18}$$

Applying $\mathcal{L}^{-1}$ on both sides, we have

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{3s+1}{s^2-6s+18}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{3s+1}{(s-3)^2+9}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{3(s-3+3)+1}{(s-3)^2+3^2}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{3(s-3)}{(s-3)^2+3^2} + \frac{10}{(s-3)^2+3^2}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{3(s-3)}{(s-3)^2+3^2}\right\} + \mathcal{L}^{-1}\left\{\frac{10}{(s-3)^2+3^2}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = 3\,\mathcal{L}^{-1}\left\{\frac{(s-3)}{(s-3)^2+3^2}\right\} + \frac{10}{3}\,\mathcal{L}^{-1}\left\{\frac{3}{(s-3)^2+3^2}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = 3e^{3t}\cos 3t + \frac{10}{3}\sin 3t$$

Question # 3:- $\frac{9s-67}{s^2-16s+49}$

Solution:-

Suppose that

$$F(s) = \frac{9s-67}{s^2-16s+49}$$

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Applying $\mathcal{L}^{-1}$ on both sides, we have

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{9s-67}{s^2-16s+49}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{9s-67}{(s-8)^2-15}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{9(s-8+8)-67}{(s-8)^2-\left(\sqrt{15}\right)^2}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{9(s-8)+5}{(s-8)^2-\left(\sqrt{15}\right)^2}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{9(s-8)}{(s-8)^2-\left(\sqrt{15}\right)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{5}{(s-8)^2-\left(\sqrt{15}\right)^2}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = 9\,\mathcal{L}^{-1}\left\{\frac{(s-8)}{(s-8)^2-\left(\sqrt{15}\right)^2}\right\} + \frac{5}{\sqrt{15}}\mathcal{L}^{-1}\left\{\frac{\sqrt{15}}{(s-8)^2-\left(\sqrt{15}\right)^2}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = 9e^{8t}\cosh\sqrt{15}t + \frac{5}{\sqrt{15}}e^{8t}\sinh\sqrt{15}t$$

Question # 4:- $\boldsymbol{\frac{as+b}{s^2+2cs+d}}$

Solution:-

Suppose that

$$F(s) = \frac{as+b}{s^2+2cs+d}$$

Applying $\mathcal{L}^{-1}$ on both sides, we have

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{as+b}{s^2+2cs+d}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{as+b}{(s+c)^2+d-c^2}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{a(s+c-c)+b}{(s+c)^2+\left(\sqrt{d-c^2}\right)^2}\right\}$$

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$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{a(s+c)}{(s+c)^2+\left(\sqrt{d-c^2}\right)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{b-ac}{(s+c)^2+\left(\sqrt{d-c^2}\right)^2}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = ae^{-ct}cos\sqrt{d-c^2}t + \frac{b-ac}{\sqrt{d-c^2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{d-c^2}}{(s+c)^2+\left(\sqrt{d-c^2}\right)^2}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = ae^{-ct}cos\sqrt{d-c^2}t + \frac{b-ac}{\sqrt{d-c^2}}e^{-ct}sin\sqrt{d-c^2}t$$

Question # 5:- $\frac{\boldsymbol{s}}{(\boldsymbol{s}+\boldsymbol{a})^2+\boldsymbol{b}^2}$

Solution:-

Suppose that

$$F(s) = \frac{\boldsymbol{s}}{(\boldsymbol{s}+\boldsymbol{a})^2+\boldsymbol{b}^2}$$

Applying $\mathcal{L}^{-1}$ *on both sides, we have*

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{\boldsymbol{s}}{(\boldsymbol{s}+\boldsymbol{a})^2+\boldsymbol{b}^2}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{\boldsymbol{s}+\boldsymbol{a}-\boldsymbol{a}}{(\boldsymbol{s}+\boldsymbol{a})^2+\boldsymbol{b}^2}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{\boldsymbol{s}+\boldsymbol{a}}{(\boldsymbol{s}+\boldsymbol{a})^2+\boldsymbol{b}^2}\right\} - \mathcal{L}^{-1}\left\{\frac{\boldsymbol{a}}{(\boldsymbol{s}+\boldsymbol{a})^2+\boldsymbol{b}^2}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = e^{-at}\cos bt - \frac{a}{b}\sin bt$$

Question # 6:- $\frac{\boldsymbol{1}}{(\boldsymbol{s}^2+\boldsymbol{a}^2)(\boldsymbol{s}^2+\boldsymbol{b}^2)}$

Solution:-

Suppose that

$$F(s) = \frac{1}{(s^2+a^2)(s^2+b^2)}$$

Applying $\mathcal{L}^{-1}$ *on both sides, we have*

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)(s^2+b^2)}\right\} - - - (i)$$

Consider that

$$\frac{1}{(s^2+a^2)(s^2+b^2)} = \frac{A}{(s^2+a^2)} + \frac{B}{(s^2+b^2)}$$

$$\Longrightarrow 1 = A(s^2+b^2) + B(s^2+a^2) - - - (ii)$$

Put $s^2 = -a^2$ & $s^2 = -b^2$ *in* (ii), *we get*

$$\boxed{A = \frac{1}{b^2-a^2} \ \& \ B = \frac{1}{a^2-b^2}}$$

Therefore,

$$\frac{1}{(s^2+a^2)(s^2+b^2)} = \frac{1}{b^2-a^2}\cdot\frac{1}{(s^2+a^2)} + \frac{1}{a^2-b^2}\frac{1}{(s^2+b^2)}$$

$$\Longrightarrow \frac{1}{(s^2+a^2)(s^2+b^2)} = \frac{1}{b^2-a^2}\left\{\frac{1}{(s^2+a^2)} - \frac{1}{(s^2+b^2)}\right\}$$

Applying $\mathcal{L}^{-1}$ *on both sides, we have*

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)(s^2+b^2)}\right\} = \frac{1}{b^2-a^2}\left\{\mathcal{L}^{-1}\left[\frac{1}{(s^2+a^2)}\right] - \mathcal{L}^{-1}\left[\frac{1}{(s^2+b^2)}\right]\right\}$$

$$\Longrightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)(s^2+b^2)}\right\} = \frac{1}{b^2-a^2}\left\{\frac{1}{a}\sin at - \frac{1}{b}\sin bt\right\}$$

Thus (i) *becomes*

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{b^2-a^2}\left\{\frac{1}{a}\sin at - \frac{1}{b}\sin bt\right\}$$

Question # 7:- $\frac{1}{(s-1)(s^2+4)}$

Solution:-

Suppose that

$$F(s) = \frac{1}{(s-1)(s^2+4)}$$

Applying $\mathcal{L}^{-1}$ *on both sides, we have*

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$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s^2+4)}\right\} - - - (i)$$

Consider that

$$\frac{1}{(s-1)(s^2+4)} = \frac{A}{(s-1)} + \frac{Bs+C}{(s^2+4)}$$

$$\Longrightarrow 1 = A(s^2+4) + (Bs+C)(s-1) - - - (ii)$$

Put $s = 1$ *in* (ii), *we get*

$$\boxed{A = \frac{1}{5}}$$

To find B & C, *we will solve the* (ii). *So,*

$$1 = As^2 + 4A + Bs^2 - Bs + Cs - C$$

$$\Longrightarrow 1 = (A+B)s^2 + (C-B)s + 4A - C$$

Equating the co-efficient of s^2 & s, *we have*

$$A + B = 0 - - - (a) \;\&\; C - B = 0 - - - (b)$$

$$(a) \Longrightarrow \boxed{B = -\frac{1}{5}}$$

$$(b) \Longrightarrow \boxed{C = -\frac{1}{5}}$$

Therefore

$$\frac{1}{(s-1)(s^2+4)} = \frac{\frac{1}{5}}{(s-1)} + \frac{\frac{-1}{5}s - \frac{1}{5}}{(s^2+4)}$$

$$\Longrightarrow \frac{1}{(s-1)(s^2+4)} = \frac{1}{5}\frac{1}{(s-1)} - \frac{1}{5}\frac{s+1}{(s^2+4)}$$

Applying $\mathcal{L}^{-1}$ *on both sides, we have*

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s^2+4)}\right\} = \frac{1}{5}.\left\{\mathcal{L}^{-1}\left[\frac{1}{(s-1)}\right] - \mathcal{L}^{-1}\left[\frac{s+1}{(s^2+4)}\right]\right\}$$

$$\Longrightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)(s^2+b^2)}\right\} = \frac{1}{5}\left\{e^t - \mathcal{L}^{-1}\left[\frac{s}{(s^2+4)}\right] - \mathcal{L}^{-1}\left[\frac{1}{(s^2+4)}\right]\right\}$$

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$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)(s^2+b^2)}\right\} = \frac{1}{5}\left\{ e^t - \cos 2t - \frac{1}{2}\sin 2t\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)(s^2+b^2)}\right\} = \frac{1}{5}e^t - \frac{1}{5}\cos 2t - \frac{1}{10}\sin 2t$$

Thus (i) *becomes*

$$\mathcal{L}^{-1}\{F(s)\} = \frac{e^t}{5} - \frac{\cos 2t}{5} - \frac{\sin 2t}{10}$$

Question # 8:- $\frac{7s+5}{(3s-8)^2}$

Solution:-

Suppose that

$$F(s) = \frac{7s+5}{(3s-8)^2}$$

Applying $\mathcal{L}^{-1}$ *on both sides, we have*

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{7s+5}{(3s-8)^2}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = \frac{1}{9}\mathcal{L}^{-1}\left\{\frac{7s+5}{\left(s-\frac{8}{3}\right)^2}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = \frac{1}{9}\mathcal{L}^{-1}\left\{\frac{7\left(s-\frac{8}{3}+\frac{8}{3}\right)+5}{\left(s-\frac{8}{3}\right)^2}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = \frac{1}{9}\mathcal{L}^{-1}\left\{\frac{7\left(s-\frac{8}{3}\right)+\frac{56}{3}+5}{\left(s-\frac{8}{3}\right)^2}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = \frac{1}{9}\mathcal{L}^{-1}\left\{\frac{7\left(s-\frac{8}{3}\right)+\frac{71}{3}}{\left(s-\frac{8}{3}\right)^2}\right\}$$

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$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = \frac{1}{9}\mathcal{L}^{-1}\left\{\frac{7\left(s-\frac{8}{3}\right)}{\left(s-\frac{8}{3}\right)^2}\right\} + \frac{1}{9}\mathcal{L}^{-1}\left\{\frac{\frac{71}{3}}{\left(s-\frac{8}{3}\right)^2}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = \frac{7}{9}\mathcal{L}^{-1}\left\{\frac{1}{\left(s-\frac{8}{3}\right)}\right\} + \frac{71}{27}\mathcal{L}^{-1}\left\{\frac{1}{\left(s-\frac{8}{3}\right)^2}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = \frac{7}{9}e^{\frac{8t}{3}} + \frac{71}{27}te^{\frac{8t}{3}}$$

Question # 9:- $\frac{5s+3}{(s+7)^5}$

Solution:-

Suppose that

$$F(s) = \frac{5s+3}{(s+7)^5}$$

Applying $\mathcal{L}^{-1}$ on both sides, we have

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{5s+3}{(s+7)^5}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{5(s+7-7)+3}{(s+7)^5}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{5(s+7)-35+3}{(s+7)^5}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{5(s+7)-32}{(s+7)^5}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{5(s+7)}{(s+7)^5}\right\} - \mathcal{L}^{-1}\left\{\frac{32}{(s+7)^5}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = 5.\mathcal{L}^{-1}\left\{\frac{1}{(s+7)^4}\right\} - 32.\mathcal{L}^{-1}\left\{\frac{1}{(s+7)^5}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = \frac{5}{3!}.\mathcal{L}^{-1}\left\{\frac{3!}{(s+7)^4}\right\} - \frac{32}{4!}.\mathcal{L}^{-1}\left\{\frac{4!}{(s+7)^5}\right\}$$

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$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = \frac{5}{6}.t^3e^{-7t} - \frac{32}{16}.t^4e^{-7t}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = t^3e^{-7t}\left(\frac{5}{6} - \frac{4}{3}.t\right)$$

Question # 10:- $\frac{2s-3}{2s^3+3s^2-2s-3}$

Solution:-

Suppose that

$$F(s) = \frac{2s-3}{2s^3+3s^2-2s-3}$$

$$\Rightarrow F(s) = \frac{2s-3}{s^2(2s+3)-1(2s+3)}$$

$$\Rightarrow F(s) = \frac{2s-3}{(2s+3)(s^2-1)}$$

$$\Rightarrow F(s) = \frac{2s-3}{(2s+3)(s+1)(s-1)}$$

Applying $\mathcal{L}^{-1}$ on both sides, we have

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{2s-3}{(2s+3)(s+1)(s-1)}\right\} - - - (i)$$

Consider that

$$\frac{2s-3}{(2s+3)(s+1)(s-1)} = \frac{A}{2s+3} + \frac{B}{s+1} + \frac{C}{s-1}$$

$$\Rightarrow 2s-3 = A(s+1)(s-1) + B(2s+3)(s-1) + C(2s+3)(s+1) - -(ii)$$

Put $2s+3=0 \Rightarrow s = \frac{-3}{2}$ *in equation* (ii), *we have*

$$2\left(\frac{-3}{2}\right) - 3 = A\left(\frac{-3}{2}+1\right)\left(\frac{-3}{2}-1\right)$$

$$\Rightarrow -3-3 = A\left(-\frac{1}{2}\right)\left(\frac{-5}{2}\right)$$

$$\Rightarrow -6 = A\left(\frac{5}{4}\right)$$

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$\Rightarrow \boxed{A = \frac{-24}{5}}$

Put $s + 1 = 0 \Rightarrow s = -1$ *in equation* (ii), *we have*

$2(-1) - 3 = B(-2 + 3)(-1 - 1)$

$\Rightarrow -5 = B(1)(-2)$

$\Rightarrow -5 = -2B$

$\Rightarrow \boxed{B = \frac{5}{2}}$

Put $s - 1 = 0 \Rightarrow s = 1$ *in equation* (ii), *we have*

$2(1) - 3 = C(2(1) + 3)(1 + 1)$

$\Rightarrow -1 = C(5)(2)$

$\Rightarrow -1 = 10C$

$\Rightarrow \boxed{C = \frac{-1}{10}}$

Therefore,

$$\frac{2s-3}{(2s+3)(s+1)(s-1)} = \frac{\frac{-24}{5}}{2s+3} + \frac{\frac{5}{2}}{s+1} + \frac{\frac{-1}{10}}{s-1}$$

$$\Rightarrow \frac{2s-3}{(2s+3)(s+1)(s-1)} = \frac{-24}{5}.\frac{1}{2s+3} + \frac{5}{2}\frac{1}{s+1} - \frac{1}{10}\frac{1}{s-1}$$

Applying $\mathcal{L}^{-1}$ *on both sides, we have*

$$\mathcal{L}^{-1}\left\{\frac{2s-3}{(2s+3)(s+1)(s-1)}\right\} = \frac{-24}{5}.\mathcal{L}^{-1}\left[\frac{1}{2s+3}\right] + \frac{5}{2}\,\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] - \frac{1}{10}\,\mathcal{L}^{-1}\left[\frac{1}{s-1}\right]$$

$$\mathcal{L}^{-1}\left\{\frac{2s-3}{(2s+3)(s+1)(s-1)}\right\} = \frac{-24}{10}\,\mathcal{L}^{-1}\left[\frac{1}{s+\frac{3}{2}}\right] + \frac{5}{2}\,\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] - \frac{1}{10}\,\mathcal{L}^{-1}\left[\frac{1}{s-1}\right]$$

$$\mathcal{L}^{-1}\left\{\frac{2s-3}{(2s+3)(s+1)(s-1)}\right\} = \frac{-12}{5}\,e^{\frac{-3t}{2}} + \frac{5}{2}\,e^{-t} - \frac{1}{10}e^{t}$$

Thus (i) becomes

$$\mathcal{L}^{-1}\{F(s)\} = \frac{-12}{5}\,e^{\frac{-3t}{2}} + \frac{5}{2}\,e^{-t} - \frac{1}{10}e^{t}$$

Question # 11:- $\frac{2s^3+6s^2+21s+52}{s(s+2)(s^2+4s+13)}$

Solution:-

Suppose that

$$F(s) = \frac{2s^3+6s^2+21s+52}{s(s+2)(s^2+4s+13)}$$

Applying $\mathcal{L}^{-1}$ on both sides, we have

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{2s^3+6s^2+21s+52}{s(s+2)(s^2+4s+13)}\right\} - - - (i)$$

Consider that

$$\frac{2s^3+6s^2+21s+52}{s(s+2)(s^2+4s+13)} = \frac{A}{s} + \frac{B}{s+2} + \frac{Cs+D}{s^2+4s+13}$$

$$\Rightarrow 2s^3+6s^2+21s+52 = A(s+2)(s^2+4s+13) + Bs(s^2+4s+13) + (Cs + D)s(s+2) - -(ii)$$

Put $s = 0$ in equation (ii), we have

$$52 = A(0+2)(0+0+13)$$

$$\Rightarrow 52 = 26A$$

$$\Rightarrow A = 2$$

Put $s+2=0 \Rightarrow s=-2$ in equation (ii), we have

$$2(-2)^3 + 6(-2)^2 + 21(-2) + 52 = B(-2)(4-8+13)$$

$$\Rightarrow -16+24-42+52 = -18B$$

$$\Rightarrow 18 = -18B$$

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$\Rightarrow B = -1$

To find C & D, we have to solve equation (ii). *Therefore,*

$\Rightarrow 2s^3 + 6s^2 + 21s + 52 = (A + B + C)s^3 + (4A + 2A + 4B + 2C + D)s^2 + (13A + 8A + 13B + 2D)s + 26A$

Equating the co-efficient of s^3 & s^2, *we have*

$A + B + C = 2$

$\Rightarrow 2 - 1 + C = 2$

$\Rightarrow C = 1$

And

$4A + 2A + 4B + 2C + D = 6$

$\Rightarrow 6A + 4B + 2C + D = 6$

$\Rightarrow 6(2) + 4(-1) + 2(1) + D = 6$

$\Rightarrow 12 - 4 + 2 + D = 6$

$\Rightarrow 10 + D = 6$

$\Rightarrow D = -4$

Therefore,

$$\frac{2s^3 + 6s^2 + 21s + 52}{s(s+2)(s^2 + 4s + 13)} = \frac{2}{s} - \frac{1}{s+2} + \frac{s-4}{s^2 + 4s + 13}$$

$$\Rightarrow \frac{2s^3 + 6s^2 + 21s + 52}{s(s+2)(s^2 + 4s + 13)} = 2.\frac{1}{s} - \frac{1}{s+2} + \frac{s}{s^2 + 4s + 13} - \frac{4}{s^2 + 4s + 13}$$

$$\Rightarrow \frac{2s^3 + 6s^2 + 21s + 52}{s(s+2)(s^2 + 4s + 13)} = 2.\frac{1}{s} - \frac{1}{s+2} + \frac{s}{(s+2)^2 + 9} - \frac{4}{(s+2)^2 + 9}$$

$$\Rightarrow \frac{2s^3 + 6s^2 + 21s + 52}{s(s+2)(s^2 + 4s + 13)} = 2.\frac{1}{s} - \frac{1}{s+2} + \frac{s+2-2}{(s+2)^2 + 9} - \frac{4}{(s+2)^2 + 9}$$

$$\Rightarrow \frac{2s^3 + 6s^2 + 21s + 52}{s(s+2)(s^2 + 4s + 13)} = 2.\frac{1}{s} - \frac{1}{s+2} + \frac{s+2}{(s+2)^2 + 9} - \frac{2}{(s+2)^2 + 9} - \frac{4}{(s+2)^2 + 9}$$

$$\Rightarrow \frac{2s^3 + 6s^2 + 21s + 52}{s(s+2)(s^2 + 4s + 13)} = 2.\frac{1}{s} - \frac{1}{s+2} + \frac{s+2}{(s+2)^2 + 9} - \frac{6}{(s+2)^2 + 9}$$

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$$\Rightarrow \frac{2s^3+6s^2+21s+52}{s(s+2)(s^2+4s+13)} = 2.\frac{1}{s} - \frac{1}{s+2} + \frac{s+2}{(s+2)^2+9} - \frac{6}{3}\frac{3}{(s+2)^2+9}$$

Applying $\mathcal{L}^{-1}$ on both sides, we have

$$\mathcal{L}^{-1}\left\{\frac{2s^3+6s^2+21s+52}{s(s+2)(s^2+4s+13)}\right\} = 2.\mathcal{L}^{-1}\left[\frac{1}{s}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] + \mathcal{L}^{-1}\left[\frac{s+2}{(s+2)^2+9}\right] - 2\,\mathcal{L}^{-1}\left[\frac{3}{(s+2)^2+9}\right]$$

$$\mathcal{L}^{-1}\left\{\frac{2s^3+6s^2+21s+52}{s(s+2)(s^2+4s+13)}\right\} = 2.(1) - e^{-2t} + e^{-2t}\cos 3t - 2\,e^{-2t}\sin 3t$$

$$\mathcal{L}^{-1}\left\{\frac{2s^3+6s^2+21s+52}{s(s+2)(s^2+4s+13)}\right\} = 2 - e^{-2t} + e^{-2t}\cos 3t - 2\,e^{-2t}\sin 3t$$

Question # 12:- $\frac{\mathbf{1}}{(\boldsymbol{s^2+4})(\boldsymbol{s^2+6s-5})}$

Solution:-

Suppose that

$$F(s) = \frac{1}{(s^2+4)(s^2+6s-5)}$$

Applying $\mathcal{L}^{-1}$ on both sides, we have

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s^2+4)(s^2+6s-5)}\right\} - - - (i)$$

Consider that

$$\frac{1}{(s^2+4)(s^2+6s-5)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+6s-5}$$

$$\Rightarrow 1 = (As+B)(s^2+6s-5) + (Cs+D)(s^2+4)$$

$$\Rightarrow 1 = As^3 + 6As^2 - 5As + Bs^2 + 6Bs - 5B + Cs^3 + 4Cs + Ds^2 + 4D$$

$$\Rightarrow 1 = (A+C)s^3 + (6A+B+D)s^2 - (5A-6B-4C)s - 5B + 4D$$

Equating co-efficient of s^3 & s^2, we have

$$A + C = 0 - - - (i)$$

$$6A + B + D = 0 - - - (ii)$$

$$5A - 6B - 4C = 0 - - - (iii)$$

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$-5B + 4D = 1 - - - (iv)$

from (i) & (iii), *we have*

$$C = -A \;\&\; D = \frac{1 + 5B}{4}$$

Therefore,

$(iii) \Rightarrow 5A - 6B - 4(-A) = 0$

$\Rightarrow 5A - 6B - 4(-A) = 0$

$\Rightarrow 9A - 6B = 0 - - - (v)$

$$(ii) \Rightarrow 6A + B + \frac{1 + 5B}{4} = 0$$

$\Rightarrow 24A + 4B + 1 + 5B = 0$

$\Rightarrow 24A + 9B = -1 - - - (vi)$

from (v), *we have*

$A = \frac{2}{3}B$, *Put in* (vi), *we obtain*

$$24\left(\frac{2}{3}B\right) + 9B = -1$$

$\Rightarrow 48B + 27B = -3$

$\Rightarrow 75B = -3$

$$\Rightarrow B = \frac{-1}{25}$$

So,

$$A = \frac{2}{3}\left(\frac{-1}{25}\right)$$

$$\Rightarrow A = \frac{-2}{75}$$

Since $C = -A \;\Rightarrow C = \frac{2}{75}$

$$\& \; D = \frac{1 + 5B}{4} \;\Rightarrow D = \frac{1 + 5\left(\frac{-1}{25}\right)}{4}$$

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$$\Rightarrow D = \frac{1 + \frac{-1}{5}}{4}$$

$$\Rightarrow D = \frac{\frac{4}{5}}{4}$$

$$\Rightarrow D = \frac{1}{5}$$

Therefore,

$$\frac{1}{(s^2+4)(s^2+6s-5)} = \frac{\left(\frac{-2}{75}\right)s + \left(\frac{-1}{25}\right)}{s^2+4} + \frac{\left(\frac{2}{75}\right)s + \frac{1}{5}}{s^2+6s-5}$$

$$\Rightarrow \frac{1}{(s^2+4)(s^2+6s-5)} = \frac{\left(\frac{-2}{75}\right)s}{s^2+4} - \frac{\frac{1}{25}}{s^2+4} + \frac{\left(\frac{2}{75}\right)s}{s^2+6s-5} + + \frac{\frac{1}{5}}{s^2+6s-5}$$

$$\Rightarrow \frac{1}{(s^2+4)(s^2+6s-5)} = \frac{-2}{75}\frac{s}{s^2+4} - \frac{1}{25}\frac{1}{s^2+4} + \frac{2}{75}\frac{s}{(s+3)^2-14} + \frac{1}{5}\frac{1}{(s+3)^2-14}$$

$$\Rightarrow \frac{1}{(s^2+4)(s^2+6s-5)} = \frac{-2}{75}\frac{s}{s^2+4} - \frac{1}{25}\frac{1}{s^2+4} + \frac{2}{75}\frac{s}{(s+3)^2-14} + \frac{1}{5}\frac{1}{(s+3)^2-14}$$

$$\Rightarrow \frac{1}{(s^2+4)(s^2+6s-5)} = \frac{-2}{75}\frac{s}{s^2+4} - \frac{1}{25}\frac{1}{s^2+4} + \frac{2}{75}\frac{s+3-3}{(s+3)^2-14} - \frac{6}{75}\frac{1}{(s+3)^2-14} + \frac{1}{5}\frac{1}{(s+3)^2-14}$$

$$\Rightarrow \frac{1}{(s^2+4)(s^2+6s-5)} = \frac{-2}{75}\frac{s}{s^2+4} - \frac{1}{25}\frac{1}{s^2+4} + \frac{2}{75}\frac{s+3}{(s+3)^2-14} + \frac{9}{75}\frac{1}{(s+3)^2-14}$$

$$\Rightarrow \frac{1}{(s^2+4)(s^2+6s-5)} = \frac{-2}{75}\frac{s}{s^2+4} - \frac{1}{25}\frac{1}{s^2+4} + \frac{2}{75}\frac{s+3}{(s+3)^2-14} + \frac{3}{25}\frac{1}{(s+3)^2-14}$$

Applying $\mathcal{L}^{-1}$ on both sides, we have

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+4)(s^2+6s-5)}\right\}$$
$$= \frac{-2}{75}.\mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] - \frac{1}{25}\mathcal{L}^{-1}\left[\frac{1}{s^2+4}\right] + \frac{2}{75}\mathcal{L}^{-1}\left[\frac{s+3}{(s+3)^2-14}\right]$$
$$+ \frac{3}{25}\mathcal{L}^{-1}\left[\frac{1}{(s+3)^2-14}\right]$$

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$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s^2+4)(s^2+6s-5)}\right\}$$

$$= \frac{-2}{75}.\cos 2t - \frac{1}{50}\sin 2t + \frac{2}{75}\ \mathcal{L}^{-1}\left[\frac{s+3}{(s+3)^2-\left(\sqrt{14}\right)^2}\right]$$

$$+ \frac{3}{25\sqrt{14}}\ \mathcal{L}^{-1}\left[\frac{\sqrt{14}}{(s+3)^2-\left(\sqrt{14}\right)^2}\right]$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s^2+4)(s^2+6s-5)}\right\} = \frac{-2}{75}.\cos 2t - \frac{1}{50}\sin 2t + \frac{2}{75}e^{-3t}\cos 2t + \frac{3}{25\sqrt{14}}\ e^{-3t}\sin 2t$$

Question # 14:- $arctan\ \frac{a}{s}$

Solution:-

Suppose that

$$F(s) = arctan\ \frac{a}{s} - - - (i)$$

As we know $\mathcal{L}\{tf(t)\} = (-1)^1 \frac{d}{ds}[\mathcal{L}\{f(t)\}]$. *this implies, we have*

$$\mathcal{L}\{tf(t)\} = -\frac{d}{ds}\{F(s)\}$$

$$\Rightarrow tf(t) = -\mathcal{L}^{-1}\left[\frac{d}{ds}\{F(s)\}\right]$$

$$\Rightarrow f(t) = \frac{-1}{t}\mathcal{L}^{-1}\left[\frac{d}{ds}\{F(s)\}\right]$$

$$\Rightarrow f(t) = \frac{-1}{t}\mathcal{L}^{-1}\left[\frac{d}{ds}\left\{arctan\ \frac{a}{s}\right\}\right]$$

$$\Rightarrow f(t) = \frac{-1}{t}\mathcal{L}^{-1}\left[\frac{1}{1+\frac{a^2}{s^2}}.\frac{d}{ds}\left\{\frac{a}{s}\right\}\right]$$

$$\Rightarrow f(t) = \frac{-1}{t}\mathcal{L}^{-1}\left[\frac{s^2}{s^2+a^2}.\left\{\frac{-a}{s^2}\right\}\right]$$

$$\Rightarrow f(t) = \frac{(-1)(-1)}{t}\mathcal{L}^{-1}\left[\frac{a}{s^2+a^2}\right]$$

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$$\Rightarrow f(t) = \frac{1}{t} sin\ at$$

$$\Rightarrow f(t) = \frac{sin\ at}{t}$$

This is required.

Question # 15:- $\ln \frac{s^2+1}{(s-1)^2}$

Solution:-

Suppose that

$$F(s) = \ln \frac{s^2+1}{(s-1)^2} - - - (i)$$

As we know $\mathcal{L}\{tf(t)\} = (-1)^1 \frac{d}{ds}[\mathcal{L}\{f(t)\}]$. *this implies, we have*

$$\mathcal{L}\{tf(t)\} = -\frac{d}{ds}\{F(s)\}$$

$$\Rightarrow tf(t) = -\mathcal{L}^{-1}\left[\frac{d}{ds}\{F(s)\}\right]$$

$$\Rightarrow f(t) = \frac{-1}{t}\mathcal{L}^{-1}\left[\frac{d}{ds}\{F(s)\}\right]$$

$$\Rightarrow f(t) = \frac{-1}{t}\mathcal{L}^{-1}\left[\frac{d}{ds}\left\{\ln \frac{s^2+1}{(s-1)^2}\right\}\right]$$

$$\Rightarrow f(t) = \frac{-1}{t}\mathcal{L}^{-1}\left[\frac{d}{ds}\{\ln(s^2+1) - \ln(s-1)^2\}\right]$$

$$\Rightarrow f(t) = \frac{-1}{t}\mathcal{L}^{-1}\left[\frac{d}{ds}\{\ln(s^2+1)\} - 2\frac{d}{ds}\{\ln(s-1)\}\right]$$

$$\Rightarrow f(t) = \frac{-1}{t}\mathcal{L}^{-1}\left[\frac{2s}{s^2+1} - \frac{2}{s-1}\right]$$

$$\Rightarrow f(t) = \frac{-2}{t}\left[\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}\right]$$

$$\Rightarrow f(t) = \frac{-2}{t}[\cos t - e^t]$$

$$\Rightarrow f(t) = \frac{2e^t}{t} - \frac{2\cos t}{t}$$

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Question # 16:- $\ln\frac{s^2+a^2}{s^2+b^2}$

Solution:-

Suppose that

$$F(s)=\ln\frac{s^2+a^2}{s^2+b^2}---(i)$$

As we know $\mathcal{L}\{tf(t)\}=(-1)^1\frac{d}{ds}[\mathcal{L}\{f(t)\}]$. *this implies, we have*

$$\mathcal{L}\{tf(t)\}=-\frac{d}{ds}\{F(s)\}$$

$$\Rightarrow tf(t)=-\mathcal{L}^{-1}\left[\frac{d}{ds}\{F(s)\}\right]$$

$$\Rightarrow f(t)=\frac{-1}{t}\mathcal{L}^{-1}\left[\frac{d}{ds}\{F(s)\}\right]$$

$$\Rightarrow f(t)=\frac{-1}{t}\mathcal{L}^{-1}\left[\frac{d}{ds}\left\{\ln\frac{s^2+a^2}{s^2+b^2}\right\}\right]$$

$$\Rightarrow f(t)=\frac{-1}{t}\mathcal{L}^{-1}\left[\frac{d}{ds}\{\ln(s^2+a^2)-\ln s^2+b^2\}\right]$$

$$\Rightarrow f(t)=\frac{-1}{t}\mathcal{L}^{-1}\left[\frac{2s}{s^2+a^2}-\frac{2s}{s^2+b^2}\right]$$

$$\Rightarrow f(t)=\frac{-2}{t}\left[\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\}-\mathcal{L}^{-1}\left\{\frac{s}{s^2+b^2}\right\}\right]$$

$$\Rightarrow f(t)=\frac{-2}{t}[\cos at-\cos bt]$$

$$\Rightarrow f(t)=\frac{2e^t}{t}-\frac{2\cos t}{t}$$

Question # 17:- $\frac{e^{-3s}}{s^2(s^2+9)}$

Solution:-

Suppose that

$$F(s)=\frac{1}{s^2(s^2+9)}---(i)$$

Consider that

$$\frac{1}{s^2(s^2+9)} = \frac{A}{s^2} + \frac{B}{s^2+9}$$

$$\Longrightarrow 1 = A(s^2+9) + Bs^2 - - - (ii)$$

Put $s^2 = 0$ *in equation* (ii), *we have*

$$1 = A(0+9)$$

$$\Longrightarrow 9A = 1$$

$$\Longrightarrow A = \frac{1}{9}$$

Put $s^2 + 9 = 0 \Longrightarrow s^2 = -9$ *in equation* (ii), *we have*

$$1 = B(-9)$$

$$\Longrightarrow -9B = 1$$

$$\Longrightarrow B = -\frac{1}{9}$$

Therefore,

$$\frac{1}{s^2(s^2+9)} = \frac{\frac{1}{9}}{s^2} + \frac{-\frac{1}{9}}{s^2+9}$$

$$\Longrightarrow F(s) = \frac{1}{9}\left[\frac{1}{s^2} - \frac{1}{s^2+9}\right]$$

Taking $\mathcal{L}^{-1}$ *on both sides, we have*

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{9}\left[\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}\right]$$

$$\Longrightarrow f(t) = \frac{1}{9}\left[t - \frac{1}{3}\sin 3t\right]$$

$$\Longrightarrow f(t) = \frac{1}{9}t - \frac{1}{3}\sin 3t$$

As we know that

$$\mathcal{L}\{u_a(t)f(t-a)\} = e^{-as}F(s)$$

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$$\Rightarrow \mathcal{L}^{-1}\{e^{-as}F(s)\} = u_a(t)f(t-a) - - - (iii)$$

For $a = 3$, we have

$$\mathcal{L}^{-1}\left\{e^{-3s}\frac{1}{s^2(s^2+9)}\right\} = u_3(t)f(t-3)$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^2(s^2+9)}\right\} = u_3(t)\left[\frac{1}{9}(t-3) - \frac{1}{27}sin\,3(t-3)\right]$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^2(s^2+9)}\right\} = \frac{1}{9}u_3(t)(t-3) - \frac{1}{27}u_3(t)sin\,3(t-3)$$

Question # 18:- $e^{-\pi s}\frac{s}{s^2-4s+5}$

Solution:-

Suppose that

$$F(s) = \frac{s}{s^2-4s+5}$$

$$\Rightarrow F(s) = \frac{s}{(s-2)^2+1}$$

$$\Rightarrow F(s) = \frac{s-2+2}{(s-2)^2+1}$$

$$\Rightarrow F(s) = \frac{s-2}{(s-2)^2+1} + \frac{2}{(s-2)^2+1}$$

Taking $\mathcal{L}^{-1}$ on both sides, we have

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{(s-2)^2+1}\right\}$$

$$\Rightarrow f(t) = e^{2t}\cos t + 2e^{2t}\sin t$$

$$\Rightarrow f(t-\pi) = e^{2(t-\pi)}\cos(t-\pi) + 2e^{2(t-\pi)}\sin(t-\pi)$$

As we know that

$$\mathcal{L}\{u_a(t)f(t-a)\} = e^{-as}F(s)$$

$$\Rightarrow \mathcal{L}^{-1}\{e^{-as}F(s)\} = u_a(t)f(t-a) - - - (iii)$$

For $a = \pi$, we have

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$$\mathcal{L}^{-1}\left\{e^{-\pi s}\frac{s}{s^2-4s+5}\right\} = u_\pi(t)f(t-\pi)$$

$$\Rightarrow \mathcal{L}^{-1}\left\{e^{-\pi s}\frac{s}{s^2-4s+5}\right\} = u_\pi(t)\left[e^{2(t-\pi)}\cos(t-\pi) + 2e^{2(t-\pi)}\sin(t-\pi)\right]$$

$$\Rightarrow \mathcal{L}^{-1}\left\{e^{-\pi s}\frac{s}{s^2-4s+5}\right\} = u_\pi(t)e^{2(t-\pi)}[\cos(t-\pi) + 2\sin(t-\pi)]$$

$$\Rightarrow \mathcal{L}^{-1}\left\{e^{-\pi s}\frac{s}{s^2-4s+5}\right\} = u_\pi(t)e^{2(t-\pi)}[\cos(\pi-t) - 2\sin(\pi-t)]$$

$$\Rightarrow \mathcal{L}^{-1}\left\{e^{-\pi s}\frac{s}{s^2-4s+5}\right\} = u_\pi(t)e^{2(t-\pi)}[-\cos t - 2\sin t]$$

$$\Rightarrow \mathcal{L}^{-1}\left\{e^{-\pi s}\frac{s}{s^2-4s+5}\right\} = -u_\pi(t)e^{2(t-\pi)}[\cos t + 2\sin t]$$

Question # 19:- $e^{-2s}\frac{s+6}{s^3-5s^2+6s}$

Solution:-

Suppose that

$$F(s) = \frac{s+6}{s^3-5s^2+6s}$$

$$\Rightarrow F(s) = \frac{s+6}{s(s^2-5s+6)}$$

$$\Rightarrow F(s) = \frac{s+6}{s(s-2)(s-3)}$$

Consider that

$$\frac{s+6}{s(s-2)(s-3)} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s-3}$$

$$\Rightarrow s+6 = A(s-2)(s-3) + Bs(s-3) + Cs(s-2) \;---\; (i)$$

Put $s = 0$ in equation (i), we have

$$6 = A(0-2)(0-3)$$

$$\Rightarrow 6A = 6$$

$$\Rightarrow A = 1$$

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Put $s-2=0 \Longrightarrow s=2 in\ equation\ (i)$, *we have*

$$8=B(-2)$$

$$\Longrightarrow -2B=8$$

$$\Longrightarrow B=-4$$

Put $s-3=0 \Longrightarrow s=2 in\ equation\ (i)$, *we have*

$$9=C(3)$$

$$\Longrightarrow 3C=9$$

$$\Longrightarrow C=3$$

Therefore,

$$\frac{s+6}{s(s-2)(s-3)}=\frac{1}{s}-\frac{4}{s-2}+\frac{3}{s-3}$$

$$\Longrightarrow F(s)=\frac{1}{s}-\frac{4}{s-2}+\frac{3}{s-3}$$

Taking $\mathcal{L}^{-1}$ *on both sides, we have*

$$\mathcal{L}^{-1}\{F(s)\}=\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}-4\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}+3\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$$

$$\Longrightarrow f(t)=1-4e^{2t}+3e^{3t}$$

$$\Longrightarrow f(t-2)=1-4e^{2(t-2)}+3e^{3(t-2)}$$

As we know that

$$\mathcal{L}\{u_a(t)f(t-a)\}=e^{-as}F(s)$$

$$\Longrightarrow \mathcal{L}^{-1}\{e^{-as}F(s)\}=u_a(t)f(t-a) --- (iii)$$

For $a=2$, *we have*

$$\mathcal{L}^{-1}\left\{e^{-2s}\frac{s+6}{s(s-2)(s-3)}\right\}=u_2(t)f(t-2)$$

$$\Longrightarrow \mathcal{L}^{-1}\left\{e^{-2s}\frac{s+6}{s(s-2)(s-3)}\right\}=u_2(t)\left[1-4e^{2(t-2)}+3e^{3(t-2)}\right]$$

$$\Longrightarrow \mathcal{L}^{-1}\left\{e^{-\pi s}\frac{s}{s^2-4s+5}\right\}=u_2(t)[1+3e^{3t-6}-4e^{2t-4}]$$

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Question # 20:- $e^{-3s}\frac{3s-7}{s^2-10s+26}$

Solution:-

Suppose that

$$F(s)=\frac{3s-7}{s^2-10s+26}$$

$$\Longrightarrow F(s)=\frac{3s-7}{(s-5)^2+26-25}$$

$$\Longrightarrow F(s)=\frac{3s-7}{(s-5)^2+1}$$

$$\Longrightarrow F(s)=\frac{3(s-5+5)-7}{(s-5)^2+1}$$

$$\Longrightarrow F(s)=\frac{3(s-5)+15-7}{(s-5)^2+1}$$

$$\Longrightarrow F(s)=\frac{3(s-5)+8}{(s-5)^2+1}$$

$$\Longrightarrow F(s)=\frac{3(s-5)}{(s-5)^2+1}+\frac{8}{(s-5)^2+1}$$

Taking $\mathcal{L}^{-1}$ on both sides, we have

$$\mathcal{L}^{-1}\{F(s)\}=3\mathcal{L}^{-1}\left\{\frac{(s-5)}{(s-5)^2+1}\right\}+8\mathcal{L}^{-1}\left\{\frac{1}{(s-5)^2+1}\right\}$$

$$\Longrightarrow f(t)=3e^{5t}\cos t+8e^{5t}\sin t$$

$$\Longrightarrow f(t-3)=3e^{5(t-3)}\cos(t-3)+8e^{5(t-3)}\sin(t-3)$$

As we know that

$$\mathcal{L}\{u_a(t)f(t-a)\}=e^{-as}F(s)$$

$$\Longrightarrow \mathcal{L}^{-1}\{e^{-as}F(s)\}=u_a(t)f(t-a)---(iii)$$

For $a=3$, we have

$$\mathcal{L}^{-1}\left\{e^{-3s}\frac{3s-7}{s^2-10s+26}\right\}=u_3(t)f(t-3)$$

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$$\Rightarrow \mathcal{L}^{-1}\left\{e^{-3s}\frac{3s-7}{s^2-10s+26}\right\} = u_3(t)\left[3e^{5(t-3)}\cos(t-3) + 8e^{5(t-3)}\sin(t-3)\right]$$

$$\Rightarrow \mathcal{L}^{-1}\left\{e^{-3s}\frac{3s-7}{s^2-10s+26}\right\} = u_3(t)e^{5(t-3)}[3\cos(t-3) + 8\sin(t-3)]$$

❖ CONVOLUTION THEOREM:-

STATEMENT:-

Let $F(s)$ & $G(s)$ denote the Laplace transforms of $f(t)$ & $g(t)$, respectively. Then the product $F(s).G(s)$ is the Laplace transform of the convolution of $f(t)$& $g(t)$, and is denoted by $(f * g)\,(t)$ and has the integral representation as follow:

$$(f * g)\,(t) = \int_0^t f(t-u)g(u)\,du$$

Question # 21:- Use Convolution theorem to evaluate the inverse Laplace transform of $\frac{1}{s^2(s+5)}$

Solution:-

Suppose that

$$F(s) = \frac{1}{s^2} \;\&\; G(s) = \frac{1}{s+5}$$

Taking $\mathcal{L}^{-1}$ of $F(s)$ & $G(s)$, we have

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} \;\&\mathcal{L}^{-1}\{\,G(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\}$$

$$\Rightarrow f(t) = t \;\&\; g(t) = e^{-5t}$$

$$\Rightarrow f(t-u) = t-u \;\&\; g(u) = e^{-5u}$$

<u>CONVOLUTION THEOREM:-</u>

$$\mathcal{L}^{-1}\{F(s).G(s)\} = (f * g)\,(t) = \int_0^t f(t-u)g(u)\,du$$

Substituting the required values, we have

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$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s^2}.\frac{1}{s+5}\right\} = \int_0^t (t-u)e^{-5u}\,du$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s^2(s+5)}\right\} = \int_0^t te^{-5u}\,du - \int_0^t ue^{-5u}\,du$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s^2(s+5)}\right\} = t.\left|\frac{e^{-5u}}{-5}\right|_0^t - \left[\left|u.\frac{e^{-5u}}{-5}\right|_0^t - \int_0^t \frac{e^{-5u}}{-5}\,du\right]$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s^2(s+5)}\right\} = -\frac{1}{5}t(e^{-5t}-1) - \left[-\frac{te^{-5t}}{5} + \frac{1}{5}\int_0^t e^{-5u}\,du\right]$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s^2(s+5)}\right\} = -\frac{te^{-5t}}{5} + \frac{t}{5} - \left[-\frac{te^{-5t}}{5} - \frac{1}{25}(e^{-5t}-1)\right]$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s^2(s+5)}\right\} = -\frac{te^{-5t}}{5} + \frac{t}{5} + \frac{te^{-5t}}{5} + \frac{1}{25}(e^{-5t}-1)$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s^2(s+5)}\right\} = \frac{t}{5} + \frac{1}{25}(e^{-5t}-1)$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s^2(s+5)}\right\} = \frac{1}{25}(e^{-5t}+5t-1)$$

Question # 22:- Use Convolution theorem to evaluate the inverse Laplace transform of $\frac{s}{(s+1)(s^2+4)}$

Solution:-

Suppose that

$$F(s) = \frac{1}{s+1} \;\&\; G(s) = \frac{s}{s^2+4}$$

Taking $\mathcal{L}^{-1}$ of $F(s)$ & $G(s)$, we have

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} \;\&\mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}$$

$$\Rightarrow f(t) = e^{-t} \;\&\; g(t) = \cos 2t$$

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$$\Rightarrow f(t-u) = e^{-(t-u)} \;\&\; g(u) = \cos 2u$$

CONVOLUTION THEOREM:-

$$\mathcal{L}^{-1}\{F(s).G(s)\} = (f * g)\,(t) = \int_0^t f(t-u)g(u)\,du$$

Substituting the required values, we have

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s+1}.\frac{s}{s^2+4}\right\} = \int_0^t e^{-(t-u)}.\cos 2u\,du$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{\boldsymbol{s}}{(\boldsymbol{s}+\mathbf{1})(\boldsymbol{s}^{\mathbf{2}}+\mathbf{4})}\right\} = \int_0^t e^{-t}.e^{u}\cos 2u\,du$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{\boldsymbol{s}}{(\boldsymbol{s}+\mathbf{1})(\boldsymbol{s}^{\mathbf{2}}+\mathbf{4})}\right\} = e^{-t}.\int_0^t e^{u}\cos 2u\,du - - - (i)$$

Consider

$$I = \int e^{u}\cos 2u\,du$$

$$\Rightarrow I = \cos 2u\,.\,e^{u} - \int e^{u}(-2)\sin 2u\,du \quad \textit{integration by parts}$$

$$\Rightarrow I = e^{u}\cos 2u + 2\int e^{u}\sin 2u\,du$$

$$\Rightarrow I = e^{u}\cos 2u + 2\sin 2u\,.\,e^{u} - 2\int e^{u}\,2\cos 2u\,du$$

$$\Rightarrow I = e^{u}\cos 2u + 2e^{u}\sin 2u - 4\int e^{u}\cos 2u\,du$$

$$\Rightarrow I = e^{u}\cos 2u + 2e^{u}\sin 2u - 4I$$

$$\Rightarrow 5I = e^{u}\cos 2u + 2e^{u}\sin 2u$$

$$\Rightarrow I = \frac{e^{u}\cos 2u}{5} + \frac{2}{5}e^{u}\sin 2u$$

$$\Rightarrow \int e^{u}\cos 2u\,du = \frac{e^{u}\cos 2u}{5} + \frac{2}{5}e^{u}\sin 2u$$

Umer Asghar (umermth2016@gmail.com) For Online Skype Tuition (Skype ID): sp15mmth06678

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By applying the limits, we have

$$\int_0^t e^u \cos 2u\, du = \frac{1}{5}|e^u \cos 2u|_0^t + \frac{2}{5}|e^u \sin 2u|_0^t$$

$$\Rightarrow \int_0^t e^u \cos 2u\, du = \frac{1}{5}(e^t \cos 2t - 1) + \frac{2}{5}(e^t \sin 2t)$$

$$\Rightarrow \int_0^t e^u \cos 2u\, du = \frac{1}{5}(e^t \cos 2t - 1 + 2e^t \sin 2t)$$

Thus equation (i) will become

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{\boldsymbol{s}}{(\boldsymbol{s}+\mathbf{1})(\boldsymbol{s}^2+\mathbf{4})}\right\} = e^{-t}.\frac{1}{5}(e^t \cos 2t - 1 + 2e^t \sin 2t)$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{\boldsymbol{s}}{(\boldsymbol{s}+\mathbf{1})(\boldsymbol{s}^2+\mathbf{4})}\right\} = \frac{1}{5}(\cos 2t - e^{-t} + 2\sin 2t)$$

Question # 23:- Use Convolution theorem to evaluate the inverse Laplace transform of $\frac{\mathbf{1}}{(\boldsymbol{s}^2+\mathbf{1})(\boldsymbol{s}^2+\mathbf{4}\boldsymbol{s}+\mathbf{5})}$

Solution:-

Suppose that

$$F(s) = \frac{1}{s^2+1} \;\&\; G(s) = \frac{1}{s^2+4s+5}$$

Taking $\mathcal{L}^{-1}$ of $F(s)$ & $G(s)$, we have

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \;\&\mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+5}\right\}$$

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \;\&\mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+1}\right\}$$

$$\Rightarrow f(t) = \sin t \;\&\; g(t) = e^{-2t}\sin t$$

$$\Rightarrow f(t-u) = \sin(t-u) \;\&\; g(u) = e^{-2u}\sin u$$

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CONVOLUTION THEOREM:-

$$\mathcal{L}^{-1}\{F(s).G(s)\} = (f * g)\ (t) = \int_0^t f(t-u)g(u)\,du$$

Substituting the required values, we have

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\cdot\frac{1}{s^2+4s+5}\right\} = \int_0^t \sin(t-u)\,.\,e^{-2u}\sin u\,du$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4s+5)}\right\} = \int_0^t (\sin t\cos u - \cos t\sin u).\,e^{-2u}\sin u\,du$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4s+5)}\right\} = \int_0^t \sin t\cos u\,.\,e^{-2u}\sin u\,du - \int_0^t \cos t\sin u\,.\,e^{-2u}\sin u\,du$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4s+5)}\right\} = \sin t\int_0^t e^{-2u}\sin u\cos u\,du - \cos t\int_0^t e^{-2u}\sin^2 u\,du$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4s+5)}\right\} = \sin t\int_0^t e^{-2u}\frac{\sin 2u}{2}\,du - \cos t\int_0^t e^{-2u}\left(\frac{1-\cos 2u}{2}\right)du$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4s+5)}\right\} = \frac{\sin t}{2}\int_0^t e^{-2u}\sin 2u\,du - \frac{\cos t}{2}\int_0^t e^{-2u}(1-\cos 2u)\,du - - - (i)$$

Consider

$$I_1 = \int e^{-2u}\sin 2u\,du$$

$$\Rightarrow I_1 = \sin 2u\,.\frac{e^{-2u}}{-2} - \int \frac{e^{-2u}}{-2}(2)\cos 2u\,du \quad integration\ by\ parts$$

$$\Rightarrow I_1 = -\frac{e^{-2u}\sin 2u}{2} + \int e^{-2u}\cos 2u\,du$$

$$\Rightarrow I_1 = -\frac{e^{-2u}\sin 2u}{2} + \cos 2u\,.\frac{e^{-2u}}{-2} - \int \frac{e^{-2u}}{-2}(-2)\sin 2u\,du$$

$$\Rightarrow I_1 = -\frac{e^{-2u}\sin 2u}{2} - \frac{e^{-2u}\cos 2u}{2} - \int e^{-2u}\sin 2u\,du$$

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$$\Rightarrow I_1 + I_1 = -\frac{e^{-2u}\sin 2u}{2} - \frac{e^{-2u}\cos 2u}{2}$$

$$\Rightarrow 2I_1 = -\frac{e^{-2u}\sin 2u}{2} - \frac{e^{-2u}\cos 2u}{2}$$

$$\Rightarrow I_1 = -\frac{e^{-2u}\sin 2u}{4} - \frac{e^{-2u}\cos 2u}{4}$$

$$\Rightarrow \int e^{-2u}\sin 2u\, du = -\frac{e^{-2u}\sin 2u}{4} - \frac{e^{-2u}\cos 2u}{4}$$

Applying the limit on both sides, we have

$$\int_0^t e^{-2u}\sin 2u\, du = -\left|\frac{e^{-2u}\sin 2u}{4}\right|_0^t - \left|\frac{e^{-2u}\cos 2u}{4}\right|_0^t$$

$$\Rightarrow \int_0^t e^{-2u}\sin 2u\, du = -\frac{1}{4}(e^{-2t}\sin 2t - 0) - \frac{1}{4}(e^{-2t}\cos 2t - 1)$$

$$\Rightarrow \int_0^t e^{-2u}\sin 2u\, du = -\frac{1}{4}(e^{-2t}\sin 2t + e^{-2t}\cos 2t - 1) - -(ii)$$

$$I_2 = \int_0^t e^{-2u}(1 - \cos 2u)\, du$$

$$\Rightarrow I_2 = \int_0^t e^{-2u}\, du - \int_0^t e^{-2u}\cos 2u\, du$$

$$\Rightarrow I_2 = \int_0^t e^{-2u}\, du - I_3$$

$$\Rightarrow I_2 = \left|\frac{e^{-2u}}{-2}\right|_0^t - I_3$$

$$\Rightarrow I_2 = -\frac{1}{2}(e^{-2t} - 1) - I_3 - - - (iii)$$

Now consider

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$$I_3 = \int_0^t e^{-2u} \cos 2u \, du$$

$$\Rightarrow I_3 = \left|\cos 2u \, . \frac{e^{-2u}}{-2}\right|_0^t - \int_0^t \frac{e^{-2u}}{-2}(-2)\sin 2u \, du$$

$$\Rightarrow I_3 = -\frac{1}{2}(e^{-2t}\cos 2t - 1) - \int_0^t e^{-2u}\sin 2u \, du$$

$$\Rightarrow I_3 = -\frac{1}{2}(e^{-2t}\cos 2t - 1) - \left[\left|\sin 2u \, . \frac{e^{-2u}}{-2}\right|_0^t - \int_0^t \frac{e^{-2u}}{-2} 2\cos 2u \, du\right]$$

$$\Rightarrow I_3 = -\frac{1}{2}(e^{-2t}\cos 2t - 1) - \left[-\frac{1}{2}(e^{-2t}\sin 2t - 0) + \int_0^t e^{-2u}\cos 2u \, du\right]$$

$$\Rightarrow I_3 = -\frac{1}{2}(e^{-2t}\cos 2t - 1) + \frac{1}{2}e^{-2t}\sin 2t - \int_0^t e^{-2u}\cos 2u \, du$$

$$\Rightarrow I_3 = -\frac{1}{2}(e^{-2t}\cos 2t - 1) + \frac{1}{2}e^{-2t}\sin 2t - I_3$$

$$\Rightarrow 2I_3 = -\frac{1}{2}(e^{-2t}\cos 2t - 1) + \frac{1}{2}e^{-2t}\sin 2t$$

$$\Rightarrow I_3 = -\frac{1}{4}(e^{-2t}\cos 2t - 1) + \frac{1}{4}e^{-2t}\sin 2t$$

Therefore,

$$I_2 = -\frac{1}{2}(e^{-2t} - 1) + \frac{1}{4}(e^{-2t}\cos 2t - 1) - \frac{1}{4}e^{-2t}\sin 2t$$

$$\Rightarrow \int_0^t e^{-2u}(1 - \cos 2u)\, du = -\frac{1}{2}(e^{-2t} - 1) + \frac{1}{4}(e^{-2t}\cos 2t - 1) - \frac{1}{4}e^{-2t}\sin 2t - -(iv)$$

Using equations (ii) & (iv) *in* (i), *we have*

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$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4s+5)}\right\}$$
$$= \frac{\sin t}{2}\left[-\frac{1}{4}(e^{-2t}\sin 2t + e^{-2t}\cos 2t - 1)\right]$$
$$-\frac{\cos t}{2}\left[-\frac{1}{2}(\mathrm{e}^{-2t}-1) + \frac{1}{4}(\mathrm{e}^{-2t}\cos 2\mathrm{t} - 1) - \frac{1}{4}\mathrm{e}^{-2t}\sin 2\mathrm{t}\right]$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4s+5)}\right\}$$
$$= \frac{\sin t}{2}\left[-\frac{e^{-2t}\sin 2t}{4} - \frac{e^{-2t}\cos 2t}{4} + \frac{1}{4}\right]$$
$$-\frac{\cos t}{2}\left[-\frac{\mathrm{e}^{-2t}}{2} + \frac{1}{2} + \frac{\mathrm{e}^{-2t}\cos 2\mathrm{t}}{4} - \frac{1}{4} - \frac{\mathrm{e}^{-2t}\sin 2\mathrm{t}}{4}\right]$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4s+5)}\right\}$$
$$= -\frac{e^{-2t}\sin 2t\sin t}{8} - \frac{e^{-2t}\cos 2t\sin t}{8} + \frac{\sin t}{8} + \frac{\mathrm{e}^{-2t}\cos t}{4} - \frac{\cos t}{4}$$
$$-\frac{\mathrm{e}^{-2t}\cos 2\mathrm{t}\cos t}{8} + \frac{\cos t}{8} + \frac{\mathrm{e}^{-2t}\sin 2\mathrm{t}\cos t}{8}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4s+5)}\right\}$$
$$= -\frac{e^{-2t}}{8}\{\cos 2\mathrm{t}\cos t + \sin 2t\sin t\} + \frac{e^{-2t}}{8}\{\sin 2\mathrm{t}\cos t - \cos 2t\sin t\}$$
$$+\frac{\mathrm{e}^{-2t}\cos t}{4} + \frac{\sin t}{8} - \frac{\cos t}{8}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4s+5)}\right\}$$
$$= -\frac{e^{-2t}}{8}\ cos\ (2t-t) + \frac{e^{-2t}}{8} sin\ (2t-t) + \frac{\mathrm{e}^{-2t}\cos t}{4} + \frac{\sin t}{8} - \frac{\cos t}{8}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4s+5)}\right\} = -\frac{e^{-2t}}{8}\ cos\ t + \frac{e^{-2t}}{8} sin\ t + \frac{\mathrm{e}^{-2t}\cos t}{4} + \frac{\sin t}{8} - \frac{\cos t}{8}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4s+5)}\right\} = \frac{e^{-2t}}{8} sin\ t + \frac{\mathrm{e}^{-2t}\cos t}{8} + \frac{\sin t}{8} - \frac{\cos t}{8}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4s+5)}\right\} = \frac{e^{-2t}}{8}(sin\ t + cos\ t) + \frac{1}{8}(sin\ t - cos\ t)$$

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Question # 24:- Show that $\mathcal{L}^{-1}\left\{\frac{s^3}{s^4+4a^4}\right\} = \cosh at \cos at$

Solution:-

Let $f(t) = cosh\ at\ cos\ at$

$$Since \quad \cosh at = \frac{e^{at}+e^{-at}}{2}$$

$$\Rightarrow f(t) = \frac{e^{at}+e^{-at}}{2}\cos at$$

$$\Rightarrow f(t) = \frac{1}{2}[e^{at}.cos\ at + e^{-at}.cos\ at]$$

Taking $\mathcal{L}$ on both sides, we have

$$\mathcal{L}\{f(t)\} = \frac{1}{2}[\mathcal{L}\{e^{at}.cos\ at\} + \mathcal{L}\{e^{-at}.cos\ at\}]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{2}\left[\frac{s-a}{(s-a)^2+a^2} + \frac{s+a}{(s+a)^2+a^2}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{2}\left[\frac{s-a\{(s+a)^2+a^2\}+(s+a)\{(s-a)^2+a^2\}}{((s-a)^2+a^2)((s+a)^2+a^2)}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{2}\left[\frac{(s-a)(s+a)^2+a^2(s-a)+(s+a)(s-a)^2+a^2(s+a)}{(s-a)^2(s+a)^2+a^2\{(s-a)^2+(s+a)^2\}+a^4}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{2}\left[\frac{(s-a)(s+a)(s+a+s-a)+a^2(s-a+s+a)}{(s^2+a^2-2as)(s^2+a^2+2as)+a^2\{2(s^2+a^2)\}+a^4}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{2}\left[\frac{(s^2-a^2)2s+a^2(2s)}{(s^2+a^2)^2-4a^2s^2+2a^2(s^2+a^2)+a^4}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{2}\left[\frac{2s(s^2-a^2+a^2)}{s^4+a^4+2a^2s^2-4a^2s^2+2a^2s^2+2a^4+a^4}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{2}\left[\frac{2s^3}{s^4+a^4+2a^4+a^4}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{s^3}{s^4+4a^4}$$

$$\Rightarrow \mathcal{L}\{cosh\ at\ cos\ at\} = \frac{s^3}{s^4+4a^4}$$

$$\Rightarrow \mathcal{L}\left\{\frac{s^3}{s^4+4a^4}\right\} = cosh\ at\ cos\ at$$

This completes the proof.

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Question # 25:- Show that $\mathcal{L}^{-1}\left\{\frac{s}{s^4+4a^4}\right\} = \sinh at \sin at$

SOLUTION:

Let $f(t) = \sinh at \sin at$

Since $\sinh at = \frac{e^{at}-e^{-at}}{2}$

$$\Rightarrow f(t) = \frac{e^{at}-e^{-at}}{2} \sin at$$

$$\Rightarrow f(t) = \frac{1}{2}[e^{at}.\sin at - e^{-at}.\sin at]$$

Taking $\mathcal{L}$ *on both sides, we have*

$$\mathcal{L}\{f(t)\} = \frac{1}{2}[\mathcal{L}\{e^{at}.\sin at\} - \mathcal{L}\{e^{-at}.\sin at\}]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{2}\left[\frac{a}{(s-a)^2+a^2} - \frac{a}{(s+a)^2+a^2}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{a}{2}\left[\frac{1}{(s-a)^2+a^2} - \frac{1}{(s+a)^2+a^2}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{a}{2}\left[\frac{(s+a)^2+a^2-(s-a)^2-a^2}{((s-a)^2+a^2)((s+a)^2+a^2)}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{a}{2}\left[\frac{(s+a)^2-(s-a)^2}{(s-a)^2(s+a)^2+a^2\{(s-a)^2+(s+a)^2\}+a^4}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{a}{2}\left[\frac{4as}{(s^2+a^2-2as)(s^2+a^2+2as)+a^2\{2(s^2+a^2)\}+a^4}\right]$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{2a^2s}{(s^2+a^2)^2-4a^2s^2+2a^2(s^2+a^2)+a^4}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{2a^2s}{s^4+a^4+2a^2s^2-4a^2s^2+2a^2s^2+2a^4+a^4}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{2a^2s}{s^4+a^4+2a^4+a^4}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{2a^2s}{s^4+4a^4}$$

$$\Rightarrow \mathcal{L}\{\sinh at \sin at\} = \frac{2a^2s}{s^4+4a^4}$$

Multiplying both sides by $\frac{1}{2a^2}$*, we have*

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$$\frac{1}{2a^2}\mathcal{L}\{\sinh at \sin at\} = \frac{1}{2a^2}\frac{2a^2 s}{s^4 + 4a^4}$$

$$\Longrightarrow \mathcal{L}\left\{\frac{1}{2a^2}\sinh at \sin at\right\} = \frac{s}{s^4 + 4a^4}$$

$$\Longrightarrow \mathcal{L}^{-1}\left\{\frac{s}{s^4 + 4a^4}\right\} = \frac{1}{2a^2}\sinh at \sin at$$

This completes the proof.

NOTES OF INVERSE LAPLACE TRANSFORMATION
BY
MUHAMMAD UMER ASGHAR (0307-4896454)
B.Sc. (DOUBLE MATH+ PHYSICS)
M.Sc. Mathematics: (In Progress)
&
MUHAMMAD ALI MALIK
(GOVT. DEGREE COLLEGE NAUSHERA)
PUBLISHERS: www,mathcity.org

publisher: www.mathcity.org