Unit # 11

The Laplace Transform

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Chapter 11:- # THE LAPLACE TRANSFORM

★ **Definition:-** Let f be a real valued piecewise continuous function defined on $[0,\infty)$. Then the Laplace Transform of f denoted by $\mathcal{L}(f)$ is the function $F(S)$ denoted as:

$$\mathcal{L}\{f(t)\} = F(S) = \int_0^\infty e^{-st} f(t)\,dt$$

★ **Example 1:-**

If $f(t)=1$ on $[0,\infty)$ then $\mathcal{L}\{f(t)\}=?$

$$\mathcal{L}\{f(t)\} = \mathcal{L}(1)$$

$$= \lim_{h\to\infty} \int_0^h e^{-st}(1)\,dt$$

$$= \lim_{h\to\infty} \left.\frac{e^{-st}}{-s}\right|_0^h$$

$$= \lim_{h\to\infty} \left\{ \frac{e^{-sh}}{-s} - \frac{e^0}{-s} \right\}$$

$$= -\frac{1}{s}\left(\lim_{h\to\infty} e^{-sh}\right) + \frac{1}{s}$$

$$= -\frac{1}{s}\left(e^{-\infty}\right) + \frac{1}{s} \qquad \because e^{-S(\infty)} = e^{-\infty} = \frac{1}{e^\infty} = \frac{1}{\infty} = 0$$

$$F(S) = -\frac{1}{s}(0) + \frac{1}{s}$$

$F(S) = \frac{1}{s}$, provided $s>0$

★ **Example:-** $f(t) = t^n \quad n>0$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^n\} = \int_0^\infty e^{-st} t^n\,dt$$

$$= t^n \cdot \left.\frac{e^{-st}}{-s}\right|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} \cdot n\, t^{n-1}\,dt$$

$$= -\frac{1}{s} \left.\frac{t^n}{e^{st}}\right|_0^\infty + \frac{n}{s}\int_0^\infty e^{-st} t^{n-1}\,dt$$

$$= -\frac{1}{s}\left\{ \lim_{t\to\infty} \frac{t^n}{e^{st}} - \frac{0^n}{e^0} \right\} + \frac{n}{s}\int_0^\infty e^{-st} t^{n-1}\,dt$$

$$\because \lim_{t\to\infty} \frac{t^n}{e^{st}} = \lim_{t\to\infty} \frac{n t^{n-1}}{s\cdot e^{st}} \quad \text{l'Hospital rule}$$

$$= \lim_{t\to\infty} \frac{n!}{s^n e^{st}} \qquad \because \frac{d^n}{dx^n}(x^n) = n!$$

$$= 0$$

★ **Example:-** Compute $\mathcal{L}\{e^{at}\}$.

Let $f(t) = e^{at}$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{e^{at}\}$$

$$= \int_0^\infty e^{-st} e^{at}\,dt$$

$$= \int_0^\infty e^{-(s-a)t}\,dt$$

$$= \lim_{h\to\infty} \left.\frac{e^{-(s-a)t}}{-(s-a)}\right|_0^h$$

$$= \lim_{h\to\infty} \left\{ \frac{e^{(a-s)h}}{a-s} - \frac{e^{-(s-a)0}}{-s+a} \right\}$$

$$= \lim_{h\to\infty} \frac{e^{(a-s)h}}{a-s} - \frac{1}{a-s}$$

$$= \begin{cases} \frac{1}{s-a} & s>a \\ \infty & s=a \end{cases}$$

$$= -\frac{1}{a-s} \qquad \because \lim_{h\to\infty} \frac{e^{(a-s)h}}{a-s} = 0$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \qquad s>a$$

★ **Example 4:-**

(i) $\mathcal{L}(\cos at)$, (ii) $\mathcal{L}(\sin at)$

$$\mathcal{L}(\cos at) = \int_0^\infty e^{-st} \cos at\,dt$$

$$\mathcal{L}(\sin at) = \int_0^\infty e^{-st} \sin at\,dt$$

$$\mathcal{L}(\cos at) + i\,\mathcal{L}(\sin at) =$$

$$\int_0^\infty e^{-st}\cos at\,dt + i\int_0^\infty e^{-st} \sin at\,dt$$

$$= \int_0^\infty e^{-st}(\cos at + i \sin at)\,dt$$

$$= \int_0^\infty e^{-st} \cdot e^{iat}\,dt$$

$$= \int_0^\infty e^{(ia-s)t}\,dt$$

$$= \lim_{h\to\infty} \int_0^h e^{(ia-s)t}\,dt$$

$$= \lim_{h\to\infty} \left.\frac{e^{(ia-s)t}}{ia-s}\right|_0^h$$

$$\mathcal{L}\{f(t)\} = \frac{n}{s}\int_0^\infty e^{-st}t^{n-1}dt$$
$$= \frac{n}{s}\cdot\frac{n-1}{s}\int_0^\infty e^{-st}\cdot t^{n-2}dt$$
$$= \frac{n}{s}\cdot\frac{n-1}{s}\cdot\frac{n-2}{s}\int_0^\infty e^{-st}\cdot t^{n-3}dt$$
$$= \frac{n}{s}\cdot\frac{n-1}{s}\cdot\frac{n-2}{s}\cdots\int_0^\infty e^{-st}t^2dt$$
$$= \frac{n}{s}\cdot\frac{n-1}{s}\cdot\frac{n-2}{s}\cdots\left[t^2\cdot\frac{e^{-st}}{-s}\Big|_0^\infty + \int_0^\infty \frac{e^{-st}}{s}2t\,dt\right]$$
$$= \frac{n}{s}\cdot\frac{n-1}{s}\cdot\frac{n-2}{s}\cdots\frac{2}{s}\int_0^\infty e^{-st}t\,dt$$
$$= \frac{n}{s}\cdot\frac{n-1}{s}\cdot\frac{n-2}{s}\cdots\frac{2}{s}\left[t\cdot\frac{e^{-st}}{-s}\Big|_0^\infty + \int_0^\infty\frac{e^{-st}}{s}\cdot(1)dt\right]$$
$$= \frac{n}{s}\cdot\frac{n-1}{s}\cdot\frac{n-2}{s}\cdots\frac{2}{s}\cdot\frac{1}{s}\int_0^\infty e^{-st}\cdot 1dt$$
$$= \frac{n}{s}\cdot\frac{n-1}{s}\cdot\frac{n-2}{s}\cdots\frac{2}{s}\cdot\frac{1}{s}\left(\frac{1}{s}\right)$$
$$= \frac{n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{s\cdot s\cdot s\cdot s\cdots\cdots s\cdot s\cdot s}\left(\frac{1}{s}\right)$$
$$= \frac{n!}{s^n}\cdot\frac{1}{s} = \frac{n!}{s^{n+1}} = F(s)$$

$$= \lim_{h\to\infty}\left\{\frac{e^{(ia-s)h}}{ia-s} - \frac{1}{ia-s}\right\}$$
$$= \lim_{h\to\infty}\left\{\frac{e^{(ia-s)h}}{ia-s}\right\} - \frac{1}{ia-s}$$
$$= \frac{1}{ia-s}\left\{\lim_{h\to\infty} e^{(ia-s)h} - 1\right\}$$
$$= \begin{cases}\dfrac{1}{s-ia} & \text{for } s>0\\ \infty & \text{for } s<0\end{cases}$$
$$= \frac{s+ia}{s^2+a^2}, \quad s>0$$

Equating real and Imaginary part

$$\left.\begin{aligned}\mathcal{L}\{\cos at\} &= \frac{s}{s^2+a^2}\\ \mathcal{L}\{\sin at\} &= \frac{a}{s^2+a^2}\end{aligned}\right|\ s>0$$

★ Example 5:- $\mathcal{L}\{\frac{1}{t}\}$

$$f(t) = \frac{1}{t}$$
$$\mathcal{L}\{\tfrac{1}{t}\} = \int_0^\infty e^{-st}\frac{1}{t}dt$$

First we check the $\int_0^\infty \frac{e^{-st}}{t}dt$ for convergence

$$\int_0^\infty e^{-st}\frac{1}{t}dt = \int_0^1 e^{-st}\frac{1}{t}dt + \int_1^\infty e^{-st}\cdot\frac{1}{t}dt$$

For $0\le t\le 1$, $e^{-st}\ge e^{-s}$ if $s>0$

$$\therefore \int_0^\infty \frac{e^{-st}}{t}dt \ge \int_0^1 e^{-s}\frac{1}{t}dt + \int_1^\infty e^{-st}\cdot\frac{1}{t}dt$$

But $\int_0^1 \frac{e^{-s}}{t}dt = e^{-s}\int_0^1\frac{1}{t}dt = \frac{1}{e^s}\ln t\Big|_0^1$

$$= \frac{1}{e^s}[\ln 1 - \ln 0] = \frac{1}{e^s}[0-\infty] = -\infty$$

Hence $\int_0^1 \frac{e^{-st}}{t}dt$ also diverges to ∞.

Consequently, $\int_0^\infty \frac{e^{-st}}{t}dt$ diverges and so by definition, $\mathcal{L}\{\frac{1}{t}\}$ does not exist.

★ Example 6: $\mathcal{L}\{t^{-1/2}\}$

$$f(t) = t^{-1/2}$$

$f(t)$ does not exist at $t=0$. But we will show $\mathcal{L}\{t^{-1/2}\}$ exists.

$$\mathcal{L}(t^{-1/2}) = \int_0^\infty e^{-st}t^{-1/2}dt$$

let $st = x$

$s\,dt = dx \Rightarrow dt = \frac{dx}{s}$

when $t=0, x=0$; $st = x$

〃 $t=\infty, x=\infty$; $t = x/s$

$$t^{-1/2} = \left(\frac{x}{s}\right)^{-1/2} = \left(\frac{s}{x}\right)^{1/2} = \sqrt{\frac{s}{x}}$$
$$\therefore \mathcal{L}(t^{-1/2}) = \int_0^\infty e^{-x}\sqrt{\frac{s}{x}}\cdot\frac{dx}{s}$$
$$= \frac{1}{\sqrt{s}}\int_0^\infty e^{-x}x^{-1/2}dx$$
$$\Gamma(t) = \int_0^\infty e^{-x}x^{t-1}dx$$
$$\Gamma(\tfrac{1}{2}) = \int_0^\infty e^{-x}x^{\frac{1}{2}-1}dx$$
$$\Gamma(\tfrac{1}{2}) = \int_0^\infty e^{-x}x^{-1/2}dx$$

★ Properties of Laplace Transform:-

① $\mathcal{L}\{a g(t) + b h(t)\}$
$= a\mathcal{L}\{g(t)\} + b\mathcal{L}\{h(t)\}$

② $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$

③ If $\mathcal{L}\{f(t)\} = F(S)$

$\mathcal{L}\{f''(t)\} = s^2F(S) - sf(0) - f'(0)$

④ If $\mathcal{L}[f(t)] = F(S)$

$\mathcal{L}\{f^{(n)}(t)\} = s^nF(S) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{(n-1)}(0)$

⑤ First Translation Property:

If $\mathcal{L}\{f(t)\} = F(S)$

$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$

$\because \mathcal{L}(\cos 4t) = \frac{s}{s^2+16}$

$\mathcal{L}(e^{7t}\cos 4t) = \frac{s-7}{(s-7)^2+16}$

⑥ $\mathcal{L}\{f(t)\} = F(S)$

(i) $\mathcal{L}\{t f(t)\} = -F'(S)$

(ii) $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\{f(s)\}$

$\because \mathcal{L}(t^2 e^{mt}) = \frac{1}{s-m}$

$\mathcal{L}(t^2e^{mt}) = (-1)^2\frac{d^2}{ds^2}\left(\frac{1}{s-m}\right)$

$= 1\cdot\frac{d}{ds}\left[(-1)(s-m)^{-2}\right]$

$= (-1)(-2)(s-m)^{-3} = \frac{2}{(s-m)^3}$

$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$\Rightarrow \mathcal{L}(t^{-1/2}) = \frac{1}{\sqrt{s}}\cdot\sqrt{\pi}$

$\mathcal{L}(t^{-\frac{1}{2}}) = \sqrt{\frac{\pi}{s}}$

★ Gamma Function or Generalized ! Function:-

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$$

$\therefore x\Gamma(x) = \Gamma(x+1)$

More;

$\Gamma(1) = 1$

$\Gamma(2) = 1\cdot\Gamma(1) = 1$

$\Gamma(4) = 3\cdot\Gamma(3) = 3\cdot 2! = 3!$

In general

$\Gamma(n) = (n-1)!\,,\quad n = 1, 2, 3, \cdots$

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① $f(t) = t^2 + 6t - 17$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^2 + 6t - 17\}$

$\mathcal{L}\{f(t)\} = \mathcal{L}(t^2) + 6\mathcal{L}(t) - 17\mathcal{L}(1)$

$\because \mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$

$\therefore \mathcal{L}\{f(t)\} = \frac{2!}{s^3} + 6\cdot\frac{1!}{s^2} - 17\cdot\frac{1}{s}$

$\mathcal{L}\{f(t)\} = \frac{2}{s^3} + \frac{6}{s^2} - \frac{17}{s}$

② $f(t) = e^{3t+5}$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{e^{3t+5}\} = \mathcal{L}\{e^{3t}\cdot e^5\}$

$= e^5\mathcal{L}\{e^{3t}\}$

$= e^5\cdot\frac{1}{s-3}$ $\quad \because \mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

$\mathcal{L}\{f(t)\} = \frac{e^5}{s-3}$

③ $f(t) = \sin(7t+4)$

$f(t) = \sin 7t\cos 4 + \cos 7t \sin 4$

Taking laplace on both sides

$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin 7t\cos 4 + \cos 7t\sin 4\}$

$= \cos 4\,\mathcal{L}\{\sin 7t\} + \sin 4\,\mathcal{L}(\cos 7t)$

$= \cos 4\cdot\frac{7}{s^2+(7)^2} + \sin 4\cdot\frac{s}{s^2+(7)^2}$

$= \frac{7\cos 4}{s^2+49} + \frac{s\sin 4}{s^2+49}$

④ $f(t) = \cos(at+b)$

$f(t) = \cos at \cos b - \sin at \sin b$

$\mathcal{L}\{f(t)\} = \cos b\, \mathcal{L}\{\cos at\} - \sin b\, \mathcal{L}\{\sin at\}$

$= \cos b \left(\frac{s}{s^2+a^2}\right) - \sin b \left(\frac{a}{s^2+a^2}\right)$

$\mathcal{L}\{f(t)\} = \frac{s\cos b - a\sin b}{s^2+a^2}$

⑤ $f(t) = \cosh(5t-3)$

$f(t) = \frac{e^{5t-3} + e^{-(5t-3)}}{2}$

$\mathcal{L}\{f(t)\} = \frac{1}{2}\left[\mathcal{L}(e^{5t-3}) + \mathcal{L}(e^{-5t+3})\right]$

$= \frac{1}{2}\left[e^{-3}\mathcal{L}(e^{5t}) + e^{3}\mathcal{L}(e^{-5t})\right]$

$= \frac{1}{2}\left[e^{-3}\left(\frac{1}{s-5}\right) + e^{3}\left(\frac{1}{s+5}\right)\right]$

$= \left[\frac{e^{-3}(s+5) + e^{3}(s-5)}{2(s^2-25)}\right]$

OR

$f(t) = \cosh(5t-3)$

$= \cosh 5t \cosh 3 - \sin 5t \sinh 3$

$\mathcal{L}\{f(t)\} = \cosh 3\, \mathcal{L}\{\cosh 5t\} - \sinh 3\, \mathcal{L}\{\sin 5t\}$

$= \cosh 3 \cdot \frac{s}{s^2-(5)^2} - \sinh 3 \cdot \frac{5}{s^2-(5)^2}$

$= \frac{s\cosh 3}{s^2-25} - \frac{5\sinh 3}{s^2-25}$

⑦ $f(t) = e^{-t}\sin 2t$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{e^{-t}\sin 2t\}$

$\therefore \mathcal{L}\{e^{at}\sin bt\}$

$= \frac{s-b}{(s-a)^2+b^2}$

$\mathcal{L}\{f(t)\} = \frac{s-2}{[s-(-1)]^2+(2)^2}, s>3$

$= \frac{2}{(s+1)^2+4}, s>-1$

⑥ $f(t) = (t^3-1)e^{-2t}$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^3e^{-2t}\} - \mathcal{L}\{e^{-2t}\}$

$\because \mathcal{L}e^{at} = \frac{1}{s-a} \Rightarrow \mathcal{L}\{e^{-2t}\} = \frac{1}{s+2}$

$\because \mathcal{L}(t^n) = \frac{n!}{s^{n+1}} \Rightarrow \mathcal{L}(t^3) = \frac{3!}{s^4} = \frac{6}{s^4}$

$\because \mathcal{L}\{t^n e^{-at}\} = \frac{n!}{(s+a)^{n+1}}$

$\mathcal{L}\{t^3e^{-2t}\} = \frac{3!}{(s+2)^4} = \frac{6}{(s+2)^4}$

$\mathcal{L}[f(t)] = \frac{6}{(s+2)^4} - \frac{1}{(s+2)}, s>-2$

⑧ $f(t) = e^{3t}\cosh 4t$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{e^{3t}\cos 4t\}$

$\therefore \mathcal{L}\{e^{at}\cosh bt\} = \frac{s-a}{(s-a)^2-b^2}$

$\mathcal{L}\{f(t)\} = \frac{s-3}{(s-3)^2-16}, s>3$

⑨ $f(t) = \cos t \cos 2t$

$\because 2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$

$f(t) = \frac{1}{2}(2\cos 2t \cos t)$

$= \frac{1}{2}[\cos(2t+t) + \cos(2t-t)]$

$= \frac{1}{2}(\cos 3t + \cos t)$

$\mathcal{L}\{f(t)\} = \frac{1}{2}\mathcal{L}[\cos 3t + \cos t]$

$= \frac{1}{2}\mathcal{L}[\cos 3t] + \frac{1}{2}\mathcal{L}[\cos t]$

$= \frac{1}{2}\cdot\left(\frac{s}{s^2+9}\right) + \frac{1}{2}\left(\frac{s}{s^2+1}\right)$

$= \frac{s}{2(s^2+9)} + \frac{s}{2(s^2+1)}$

⑩ $\mathcal{L}\{\sin^3 t\}$

As $\sin 3t = 3\sin t - 4\sin^3 t$

$4\sin^3 t = 3\sin t - \sin 3t$

$\sin^3 t = \frac{3}{4}\sin t - \frac{1}{4}\sin 3t$

$\mathcal{L}\{\sin^3 t\} = \mathcal{L}\left\{\frac{3}{4}\sin t - \frac{1}{4}\sin 3t\right\}$

$$\mathcal{L}\{\sin^3 t\} = \frac{3}{4}\mathcal{L}\{\sin t\} - \frac{1}{4}\mathcal{L}\{\sin 3t\}$$
$$= \frac{3}{4}\cdot\frac{1}{s^2+(1)^2} - \frac{1}{4}\cdot\frac{3}{s^2+(3)^2}$$
$$= \frac{3}{4(s^2+1)} - \frac{3}{4(s^2+9)}$$

(11) $f(t) = te^{-3t}\sin at$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left[e^{-3t}(t\sin at)\right]$$
$$\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$$
$$\mathcal{L}\{t\sin at\} = -\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right)$$
$$= -a(-1)(s^2+a^2)^{-2}(2s)$$
$$\mathcal{L}\{t\sin at\} = \frac{2as}{(s^2+a^2)^2}$$
$$\therefore \mathcal{L}\{e^{-3t}t\sin at\} = \frac{2a(s+3)}{\left[(s+3)^2+a^2\right]^2}$$

(12) $f(t) = \sinh^2 at$

$$\cosh 2at = 1 + 2\sinh^2 at$$
$$\sinh^2 at = \frac{\cosh 2at - 1}{2}$$
$$\mathcal{L}\{f(t)\} = \frac{1}{2}\mathcal{L}\{\cosh 2at\} - \frac{1}{2}\mathcal{L}\{1\}$$
$$= \frac{1}{2}\mathcal{L}\left\{\frac{e^{2at}+e^{-2at}}{2}\right\} - \frac{1}{2}\left(\frac{1}{s}\right)$$
$$= \frac{1}{4}\mathcal{L}(e^{2at}) + \frac{1}{4}\mathcal{L}(e^{-2at}) - \frac{1}{2s}$$
$$= \frac{1}{4}\left(\frac{1}{s-2a}\right) + \frac{1}{4}\left(\frac{1}{s+2a}\right) - \frac{1}{2s}$$
$$= \frac{s(s+2a)+s(s-2a)-2(s^2-4a^2)}{4s(s^2-4a^2)}$$
$$= \frac{s^2+\cancel{2as}+s^2-\cancel{2as}-2s^2+8a^2}{4s(s^2-4a^2)}$$
$$= \frac{2a^2}{s(s^2-4a^2)}$$

(13) $f(t) = \cosh at \sin at$

$$f(t) = \left(\frac{e^{at}+e^{-at}}{2}\right)\sin at$$
$$f(t) = \frac{1}{2}\left[e^{at}\sin at + e^{-at}\sin at\right]$$
$$\mathcal{L}\{f(t)\} = \frac{1}{2}\left[\mathcal{L}\{e^{at}\sin at\} + \mathcal{L}\{e^{-at}\sin at\}\right]$$
$$= \frac{1}{2}\left[\frac{a}{(s-a)^2+a^2} + \frac{a}{(s+a)^2+a^2}\right]$$
$$= \frac{a}{2}\left[\frac{1}{(s-a)^2+a^2} + \frac{1}{(s+a)^2+a^2}\right]$$
$$= \frac{a}{2}\left[\frac{(s+a)^2+a^2+(s-a)^2+a^2}{\{(s-a)^2+a^2\}\{(s+a)^2+a^2\}}\right]$$
$$= \frac{a}{2}\left[\frac{(s+a)^2+(s-a)^2+2a^2}{(s-a)^2(s+a)^2+a^2(s-a)^2+a^2(s+a)^2+a^4}\right]$$
$$= \frac{a}{2}\left[\frac{s^2+a^2+\cancel{2as}+s^2+a^2-\cancel{2as}+2a^2}{[(s-a)(s+a)]^2+a^2\left[(s-a)^2+(s+a)^2\right]+a^4}\right]$$
$$= \frac{a}{2}\left[\frac{2s^2+4a^2}{(s^2-a^2)^2+a^2\cdot 2(a^2+s^2)+a^4}\right]$$
$$= \frac{a(s^2+2a^2)}{s^4+a^4-2\cancel{s^2}a^2+2a^4+2\cancel{a^2}s^2+a^4}$$
$$= \frac{a(s^2+2a^2)}{s^4+4a^4}$$

(14) $\sinh at \cos at = f(t)$

$$f(t) = \left(\frac{e^{at}-e^{-at}}{2}\right)\cos at$$
$$= \frac{1}{2}\left[e^{at}\cos at - e^{-at}\cos at\right]$$
$$\mathcal{L}\{f(t)\} = \frac{1}{2}\left[\mathcal{L}(e^{at}\cos at) - \mathcal{L}(e^{-at}\cos at)\right]$$
$$= \frac{1}{2}\left[\frac{s-a}{(s-a)^2+a^2} - \frac{s+a}{(s+a)^2+a^2}\right] = \frac{1}{2}\left[\frac{(s-a)\{(s+a)^2+a^2\}-(s+a)\{(s-a)^2+a^2\}}{[(s-a)^2+a^2][(s+a)^2+a^2]}\right]$$

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$$\mathcal{L}\{f(t)\} = \frac{1}{2}\left[\frac{(s-a)\{(s^2+2as+2a^2)\} - (s+a)\{(s^2-2as+2a^2)\}}{(s-a)^2(s+a)^2+a^2(s-a)^2+a^2(s+a)^2+a^4}\right]$$

$$= \frac{1}{2}\left[\frac{s^3+2as^2+2a^2s-as^2-2a^2s-2a^3-(s^3-2as^2+2a^2s+as^2-2a^2s+2a^3}{(s^2-a^2)^2+a^2\cdot 2(s^2+a^2)+a^4}\right]$$

$$\mathcal{L}\{f(t)\} = \frac{1}{2}\left[\frac{2as^2-4a^3}{s^4-2a^2s^2+a^4+2a^2s^2+2a^4+a^4}\right] = \frac{as^2-2a^3}{s^4+4a^4} = \frac{a(s^2-2a^2)}{s^4+4a^4}$$

(15) $f(t) = \cosh at \cos bt = \left(\frac{e^{at}+e^{-at}}{2}\right)\cos bt = \frac{1}{2}\left[e^{at}\cos bt + e^{-at}\cos bt\right]$

$$\mathcal{L}\{f(t)\} = \frac{1}{2}\{\mathcal{L}(e^{at}\cos bt) + \mathcal{L}(e^{-at}\cos b)\}$$

$$= \frac{1}{2}\left[\frac{s-a}{(s-a)^2+b^2} + \frac{s+a}{(s+a)^2+b^2}\right] = \frac{1}{2}\left[\frac{(s-a)\{(s+a)^2+b^2\}+(s+a)\{(s-a)^2+b^2\}}{\{(s-a)^2+b^2\}\{(s+a)^2+b^2\}}\right]$$

$$= \frac{1}{2}\left[\frac{(s-a)(s^2+2as+a^2+b^2)+(s+a)(s^2-2as+a^2+b^2)}{(s^2-2as+a^2+b^2)(s^2+2as+a^2+b^2)}\right]$$

$$= \frac{1}{2}\left[\frac{s^3+2as^2+a^2s+b^2s-as^2-2a^2s-a^3-ab^2+s^3-2as^2+a^2s+b^2s+as^2-2a^2s+a^3+ab^2}{[(s^2+a^2+b^2)-2as][(s^2+a^2+b^2)+2as]}\right]$$

$$= \frac{1}{2}\left[\frac{2s^3-2a^2s+2b^2s}{(s^2+a^2+b^2)^2-(2as)^2}\right] = \frac{1}{2}\left[\frac{2s(s^2-a^2+b^2)}{(s^2+a^2+b^2)^2-4a^2s^2}\right] = \frac{s(s^2-a^2+b^2)}{(s^2+a^2+b^2)^2-4a^2s^2}$$

(16) $f(t) = [t]$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{[t]\} = \int_0^\infty e^{-st}[t]\,dt$$

$$= \int_0^1 e^{-st}[t]\,dt + \int_1^2 e^{-st}[t]\,dt + \int_2^3 e^{-st}[t]\,dt + \int_3^4 e^{-st}\cdot[t]\,dt + \cdots$$

$$= \int_0^1 e^{-st}(0)\,dt + \int_1^2 e^{-st}(1)\,dt + \int_2^3 e^{-st}(2)\,dt + \int_3^4 e^{-st}(3)\,dt + \cdots$$

$$= \left.\frac{e^{-st}}{-s}\right|_1^2 + 2\cdot\left.\frac{e^{-st}}{-s}\right|_2^3 + 3\cdot\left.\frac{e^{-st}}{-s}\right|_3^4 + \cdots$$

$$= -\frac{1}{s}\left\{(e^{-2s}-e^{-s}) + 2(e^{-3s}-e^{-2s}) + 3(e^{-4s}-e^{-3s}) + \cdots\right\}$$

$$= -\frac{1}{s}\{-e^{-s}-e^{-2s}-e^{-3s}-e^{-4s}-\cdots\} = +\frac{1}{s}e^{-s}(1+e^{-s}+e^{-2s}+e^{-3s}+\cdots)$$

$$= \frac{1}{s}e^{-s}\{1+(e^{-s})^1+(e^{-s})^2+(e^{-s})^3+\cdots\} = \frac{e^{-s}}{s}\cdot\frac{1}{1-e^{-s}} = \frac{e^{-s}}{s(1-e^{-s})}$$

$$\because 1+x+x^2+x^3+\cdots = \frac{1}{1-r} = \frac{1}{1-x}$$

(17) $f(t) = t^\alpha$; $\alpha > -1$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^\alpha\}$$

$$= \int_0^\infty e^{-st}t^\alpha\,dt$$

(18) $f(t) = t^2 \sin at$

$$\mathcal{L}\{t^2\sin at\} = (-1)^2\frac{d^2}{ds^2}\mathcal{L}\{\sin at\}$$

$$= \frac{d^2}{ds^2}\left(\frac{a}{s^2+a^2}\right) = \frac{d}{ds}\left[\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right)\right]$$

let $st = u$. As $t=0$;
$s\,dt = du$. $u=0$
$dt = \frac{du}{s}$. and
. $t=\infty$
Also $t = \frac{u}{s}$. $u=\infty$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-u}\left(\frac{u}{s}\right)^\alpha \frac{du}{s}$$

$$= \frac{1}{s^{\alpha+1}}\int_0^\infty e^{-u}u^\alpha du \rightarrow (1)$$

$$\Gamma t = \int_0^\infty e^{-x}x^{t-1}dx$$

$$\int_0^\infty e^{-u}u^\alpha du = \int_0^\infty e^{-u}u^{(\alpha+1)-1}du$$

$$= \Gamma(\alpha+1)$$

from (1) $\mathcal{L}\{t^\alpha\} = \frac{\Gamma \alpha+1}{s^{\alpha+1}}$

Now $\mathcal{L}\{t^{5/2}\} = \frac{\Gamma \frac{5}{2}+1}{s^{\frac{5}{2}+1}}$

$$\Gamma x+1 = x\Gamma x$$

$$\Gamma \tfrac{5}{2}+1 = \tfrac{5}{2}\Gamma\tfrac{5}{2}$$

$$= \tfrac{5}{2}\left(\Gamma \tfrac{3}{2}+1\right)$$

$$= \tfrac{5}{2}\left(\tfrac{3}{2}\Gamma\tfrac{3}{2}\right)$$

$$= \tfrac{5}{2}\cdot\tfrac{3}{2}\left(\Gamma\tfrac{1}{2}+1\right)$$

$$= \tfrac{5}{2}\cdot\tfrac{3}{2}\cdot\tfrac{1}{2}\Gamma\tfrac{1}{2}$$

$$\Gamma\tfrac{5}{2}+1 = \tfrac{15}{8}\sqrt{\pi}$$

$$\therefore \mathcal{L}\{t^{5/2}\} = \frac{\frac{15}{8}\sqrt{\pi}}{s^{\frac{5}{2}+1}} = \frac{15\sqrt{\pi}}{8s^{7/2}}$$

$$\mathcal{L}\{t^2\sin at\} = \frac{d}{ds}\left[a(-1)(s^2+a^2)^{-2}(2s)\right]$$

$$= -2a\frac{d}{ds}\left\{\frac{s}{(s^2+a^2)^2}\right\}$$

$$= -2a\left[\frac{(s^2+a^2)^2.1 - 2(a^2+s^2)2s.s}{(s^2+a^2)^4}\right]$$

$$= -2a\left[\frac{(s^2+a^2)^2 - 4s^2(s^2+a^2)}{(s^2+a^2)^4}\right]$$

$$= -2a\left[\frac{(s^2+a^2)\{s^2+a^2-4s^2\}}{(s^2+a^2)^4}\right]$$

$$= -2a\left[\frac{(s^2+a^2)(a^2-3s^2)}{(s^2+a^2)^4}\right] = \frac{2a(3s^2-a^2)}{(s^2+a^2)^3}$$

(19) $f(t) = t^2\cos at$

$$\mathcal{L}\{f(t)\} = (-1)^2\frac{d^2}{ds^2}\mathcal{L}(\cos at)$$

$$= \frac{d^2}{ds^2}\left\{\frac{s}{s^2+a^2}\right\} = \frac{d}{ds}\left[\frac{d}{ds}\left\{\frac{s}{s^2+a^2}\right\}\right]$$

$$= \frac{d}{ds}\left[\frac{(s^2+a^2).1 - s(2s)}{(s^2+a^2)^2}\right] = \frac{d}{ds}\left[\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

$$= \left[\frac{(s^2+a^2)^2(-2s) - (a^2-s^2)\cdot 2(s^2+a^2)(2s)}{(s^2+a^2)^4}\right]$$

$$= \left[\frac{2s(s^2+a^2)\{-s^2-a^2-2a^2+2s^2\}}{(s^2+a^2)^4}\right]$$

$$\mathcal{L}\{f(t)\} = \frac{2s(s^2-3a^2)}{(s^2+a^2)^3}$$

(20) $f(t) = t\sin^2 at$

$$f(t) = t\cdot\left(\frac{1-\cos 2at}{2}\right)$$

$$= \tfrac{1}{2}\left[t - t\cos 2at\right]$$

$$= \tfrac{1}{2}\left[\mathcal{L}\{t\} - (-1)^1\frac{d}{ds}\mathcal{L}\{\cos 2at\}\right]$$

$$= \tfrac{1}{2}\left[\frac{1}{s^2} + \frac{d}{ds}\left(\frac{s}{s^2+4a^2}\right)\right]$$

$$= \tfrac{1}{2}\left[\frac{1}{s^2} + \frac{(s^2+4a^2).1 - s.2s}{(s^2+4a^2)^2}\right]$$

(21) $f(t) = t^2\cos^2 2t$

$f(t) = t^2.\frac{1+\cos 4t}{2}$ $\quad\because \cos^2\theta = \frac{1+\cos 2\theta}{2}$

$$f(t) = \tfrac{1}{2}\left[t^2 + t^2\cos 4t\right]$$

$$\mathcal{L}\{f(t)\} = \tfrac{1}{2}\left[\mathcal{L}\{t^2\} + \mathcal{L}\{t^2\cos 4t\}\right]$$

$$= \tfrac{1}{2}\left[\frac{2}{s^3} + (-1)^2\frac{d^2}{ds^2}\left(\frac{s}{s^2+16}\right)\right]$$

$$\mathcal{L}\{f(t)\} = \frac{1}{2}\left[\frac{1}{s^2} + \frac{s^2+4a^2-2s^2}{(s^2+4a^2)^2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s^2} + \frac{4a^2-s^2}{(s^2+4a^2)^2}\right]$$

$$= \frac{1}{2}\left[\frac{(s^2+4a^2)^2+s^2(4a^2-s^2)}{s^2(s^2+4a^2)^2}\right]$$

$$= \frac{1}{2}\left[\frac{s^4+16a^4+8a^2s^2+4a^2s^2-s^4}{s^2(s^2+4a^2)^2}\right]$$

$$= \frac{1}{2}\left[\frac{16a^4+12a^2s^2}{s^2(s^2+4a^2)^2}\right] = \frac{1}{2}\cdot\frac{4a^2(4a^2+3s^2)}{s^2(s^2+4a^2)^2}$$

$$\mathcal{L}\{f(t)\} = \frac{2a^2(4a^2+3s^2)}{s^2(s^2+4a^2)^2}$$

(22) $f(t) = \frac{\sin at}{t}$

$$\lim_{t\to 0}\frac{\sin at}{t} = \lim_{t\to 0}\frac{\sin at}{at}\cdot a$$

$$= a\lim_{t\to 0}\frac{\sin at}{at} = a(1) = a \text{ (exist)}$$

$$\mathcal{L}(\sin at) = \frac{a}{s^2+a^2}$$

$$\mathcal{L}\left(\frac{\sin at}{t}\right) = \int_s^\infty \frac{a}{u^2+a^2}\,du$$

$$= a\cdot\int_s^\infty \frac{1}{u^2+a^2}\,du = a\cdot\frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)\Big|_s^\infty$$

$$= \tan^{-1}(\infty) - \tan^{-1}\left(\frac{s}{a}\right)$$

Add and subtract $\tan^{-1}\left(\frac{a}{s}\right)$

$$= \frac{\pi}{2} - \tan^{-1}\frac{s}{a} - \tan^{-1}\frac{a}{s} + \tan^{-1}\frac{a}{s}$$

$$= \frac{\pi}{2} - \left[\tan^{-1}\left(\frac{\frac{s}{a}+\frac{a}{s}}{1-\frac{s}{a}\times\frac{a}{s}}\right) + \tan^{-1}\frac{a}{s}\right.$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{\frac{s}{a}+\frac{a}{s}}{1-1}\right) + \tan^{-1}\frac{a}{s}$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{\frac{s}{a}+\frac{a}{s}}{0}\right) + \tan^{-1}\frac{a}{s}$$

$$= \frac{\pi}{2} - \tan^{-1}\infty + \tan^{-1}\frac{a}{s}$$

$$= \cancel{\frac{\pi}{2}} - \cancel{\frac{\pi}{2}} + \tan^{-1}\frac{a}{s}$$

$$\therefore \mathcal{L}\left(\frac{\sin at}{t}\right) = \tan^{-1}\frac{a}{s}$$

$$= \frac{1}{2}\left[\frac{2}{s^3} + \frac{d}{ds}\left\{\frac{(s^2+16)\cdot 1 - s\cdot 2s}{(s^2+16)^2}\right\}\right]$$

$$= \frac{1}{2}\left[\frac{2}{s^3} + \frac{d}{ds}\left(\frac{16-s^2}{(s^2+16)^2}\right)\right.$$

$$= \frac{1}{2}\left[\frac{2}{s^3} + \left\{\frac{(s^2+16)^2(-2s) - (16-s^2)(s^2+16)2s}{(s^2+16)^4}\right\}\right]$$

$$= \frac{1}{2}\left[\frac{2}{s^3} + \frac{2s(s^2+16)\{-s^2-16-32+2s^2\}}{(s^2+16)^4}\right]$$

$$= \frac{2}{2}\left[\frac{1}{s^3} + \frac{s(s^2-48)}{(s^2+16)^3}\right] = \frac{1}{s^3} + \frac{s(s^2+16-64)}{(s^2+16)^3}$$

$$= \frac{1}{s^3} + \frac{s(s^2+16)}{(s^2+16)^3} - \frac{64s}{(s^2+16)^3}$$

$$= \frac{1}{s^3} + \frac{s}{(s^2+16)^2} - \frac{64s}{(s^2+16)^3}$$

$$= \frac{(s^2+16)^3 + s^4(s^2+16) - 64s^4}{s^3(s^2+16)^3}$$

Note :-

*1: If $\mathcal{L}\{f(t)\} = F(S)$ then

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)\,du$$

Provided $\lim_{t\to 0^+}\frac{f(t)}{t}$ exist.

*2:- $\mathcal{L}\left\{\int_0^t f(u)\,du\right\} = \frac{1}{s}\mathcal{L}\{f(t)\}$

(23) $\frac{1-\cos at}{t}$

let $f(t) = 1-\cos at$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{1-\cos at\}$$

$$= \mathcal{L}\{1\} - \mathcal{L}\{\cos at\} = \frac{1}{s} - \frac{s}{s^2+a^2}$$

By the formula $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)\,du$

$$\mathcal{L}\left\{\frac{1-\cos at}{t}\right\} = \int_s^\infty\left(\frac{1}{u} - \frac{u}{u^2+a^2}\right)du$$

$$= \int_s^\infty \frac{1}{u}\,du - \int_s^\infty \frac{u}{u^2+a^2}\,du$$

$$= \ln u\Big|_s^\infty - \frac{1}{2}\int_s^\infty \frac{2u}{u^2+a^2}\,du$$

(24) $\int_0^t \frac{\sin au}{u} du$

$f(u) = \frac{\sin au}{u}$

$\because \mathcal{L}\left\{\int_0^t f(u)du\right\} = \frac{1}{s}\mathcal{L}\{f(t)\}$

$\mathcal{L}\left\{\int_0^t \frac{\sin au}{u} du\right\} = \frac{1}{s}\mathcal{L}\left\{\frac{\sin at}{t}\right\} \quad (1)$

now $f(t) = \sin at$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$

By formula;

$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)du$

from Q-22 we have

$\mathcal{L}\left\{\frac{\sin at}{t}\right\} = \tan^{-1}\frac{a}{s}$

$\therefore$ Eq (1) becomes:

$\mathcal{L}\left\{\int_0^t \frac{\sin au}{u} du\right\} = \frac{1}{s}\tan^{-1}\left(\frac{a}{s}\right)$

(25) $\int_0^t \frac{1-\cos au}{u} du$

Let $f(u) = \frac{1-\cos au}{u}$

$\therefore$ By the formula:

$\mathcal{L}\left\{\int_0^t f(u)du\right\} = \frac{1}{s}\mathcal{L}\{f(t)\}$

$\mathcal{L}\left\{\int_0^t \frac{1-\cos au}{u} du\right\} = \frac{1}{s}\mathcal{L}\left\{\frac{1-\cos at}{t}\right\} \rightarrow (A)$

let $f(t) = 1-\cos at \rightarrow (1)$

Taking $\mathcal{L}$ on both the sides

$\Rightarrow \mathcal{L}\{f(t)\} = \mathcal{L}\{1-\cos at\}$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{1\} - \mathcal{L}\{\cos at\}$

$\mathcal{L}\{f(t)\} = \frac{1}{s} - \frac{s}{s^2+a^2} = \frac{s^2+a^2-s^2}{s(s^2+a^2)}$

$\mathcal{L}\{f(t)\} = \frac{a^2}{s(s^2+a^2)}$

By formula;

$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)\,du$

Provided $\lim_{t\to 0} \frac{f(t)}{t}$ exist.

$= \frac{1}{2}\cdot 2\ln u\Big|_s^\infty - \frac{1}{2}\ln|u^2+a^2|\Big|_s^\infty$

$= \frac{1}{2}\ln u^2\Big|_s^\infty - \frac{1}{2}\ln|u^2+a^2|\Big|_s^\infty$

$= \frac{1}{2}\left|\ln\frac{u^2}{u^2+a^2}\right|_s^\infty = \frac{1}{2}\left|\ln\frac{1}{1+\frac{a^2}{u^2}}\right|_s^\infty$

$= \frac{1}{2}\left[\left\{\ln\frac{1}{1+\frac{a^2}{\infty}}\right\} - \ln\left\{\frac{1}{1+\frac{a^2}{s^2}}\right\}\right]$

$= \frac{1}{2}\left[\ln 1 - \ln\frac{s^2}{s^2+a^2}\right]$

$= \frac{1}{2}\left[0 - \ln\frac{s^2}{s^2+a^2}\right]$

$= -\frac{1}{2}\ln\frac{s^2}{s^2+a^2} = \frac{1}{2}\ln\left(\frac{s^2}{s^2+a^2}\right)^{-1}$

$= \frac{1}{2}\ln\left(\frac{s^2+a^2}{s^2}\right)$

(26) $\frac{\sinh at}{t}$

let $f(t) = \sinh at \rightarrow$ (i)

$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sinh at\}$

$\mathcal{L}\{f(t)\} = \frac{a}{s^2-a^2}$

$\because \mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)du$

$\mathcal{L}\left\{\frac{\sinh at}{t}\right\} = \int_s^\infty \frac{a}{u^2-a^2}du$

$= a\int_s^\infty \frac{1}{u^2-a^2}du$

$\because \int\frac{1}{x^2-a^2}dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$

$\mathcal{L}\left\{\frac{\sinh at}{t}\right\} = a\cdot\frac{1}{2a}\left|\ln\left(\frac{u-a}{u+a}\right)\right|_s^\infty$

$= \frac{1}{2}\left[\lim_{u\to\infty}\ln\left(\frac{u-a}{u+a}\right) - \ln\left(\frac{s-a}{s+a}\right)\right]$

$= \frac{1}{2}\left[\lim_{u\to\infty}\ln\left(\frac{1-\frac{a}{u}}{1+\frac{a}{u}}\right) - \ln\left(\frac{s-a}{s+a}\right)\right]$

$= \frac{1}{2}\left[\ln\left(\frac{1-0}{1+0}\right) - \ln\left(\frac{s-a}{s+a}\right)\right]$

$= \frac{1}{2}\left[0 - \ln\left(\frac{s-a}{s+a}\right)\right]$

$= \frac{1}{2}\ln\left(\frac{s-a}{s+a}\right)^{-1} = \frac{1}{2}\ln\left(\frac{s+a}{s-a}\right)$

$$\lim_{t\to 0}\frac{f(t)}{t} = \lim_{t\to 0}\frac{1-\cos at}{t} = \frac{2\sin^2\left(\frac{at}{2}\right)}{t} = 0$$

$$\therefore \mathcal{L}\left\{\frac{1-\cos at}{t}\right\} = \int_s^\infty \frac{a^2}{u(u^2+a^2)}du$$

$$\frac{1}{u(u^2+a^2)} = \frac{A}{u} + \frac{Bu+C}{u^2+a^2}$$

$$1 = A(u^2+a^2) + (Bu+C)u$$

$$1 = Au^2 + Aa^2 + Bu^2 + Cu$$

For A; $u=0 \Rightarrow 1 = Aa^2 \Rightarrow A = \frac{1}{a^2}$

$$\because 1 = (A+B)u^2 + Cu + Aa^2$$

$$\because A+B = 0 \Rightarrow B = -A \Rightarrow B = -\frac{1}{a^2}$$

Also $C = 0$. $\therefore$ Eq Ⓐ $\Rightarrow$

$$\mathcal{L}\left\{\int_0^t \frac{1-\cos au}{u}du\right\} = \frac{a^2}{s}\int_s^\infty\left(\frac{1}{a^2u} - \frac{u}{a^2(u^2+a^2)}\right)du$$

$$= \frac{1}{s}\left[\ln u - \frac{1}{2}\ln(u^2+a^2)\right]_s^\infty$$

$$= \frac{1}{s}\lim_{h\to\infty}\left[\ln\frac{u}{\sqrt{u^2+a^2}}\right]_s^\infty$$

$$= \frac{1}{s}\lim_{h\to\infty}\left[\ln\frac{h}{\sqrt{h^2+a^2}} - \ln\frac{s}{\sqrt{s^2+a^2}}\right]$$

$$= \frac{1}{s}\left[\ln\left\{\lim_{h\to\infty}\left(\frac{h}{\sqrt{h^2+a^2}}\right)\right\} - \ln\left(\frac{s}{\sqrt{s^2+a^2}}\right)\right]$$

$$= \frac{1}{s}\left[\ln\left\{\lim_{h\to\infty}\left(\frac{h}{h\sqrt{1+\frac{a^2}{h^2}}}\right)\right\} - \ln\left(\frac{s}{\sqrt{s^2+a^2}}\right)\right]$$

$$= \frac{1}{s}\left[(0) - \ln\left(\frac{s}{\sqrt{s^2+a^2}}\right)\right]$$

$$= -\frac{1}{s}\ln\left(\frac{s}{\sqrt{s^2+a^2}}\right)$$

$$= -\frac{1}{s}\left[\ln s - \frac{1}{2}\ln(s^2+a^2)\right]$$

$$= +\frac{1}{s}\left[\frac{1}{2}\ln(s^2+a^2) - \ln s\right]$$

$$= \frac{1}{s}\left[\frac{1}{2}\ln(s^2+a^2) - \frac{1}{2}\cdot 2\ln s\right]$$

$$= \frac{1}{s}\left[\frac{1}{2}\ln(s^2+a^2) - \frac{1}{2}\ln s^2\right]$$

$$= \frac{1}{2s}\left[\ln(s^2+a^2) + \ln s^2\right]$$

$$= \frac{1}{2s}\ln\frac{s^2+a^2}{s^2}$$

(27) $f(t) = \ln t$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\ln t\}$$

$$= \int_0^\infty e^{-st}\ln t\, dt$$

Consider $st = u$

$$s\,dt = du \Rightarrow dt = \frac{du}{s}$$

As $t\to 0$, $u \to 0$

$t\to\infty$, $u\to\infty$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-u}\ln\left(\frac{u}{s}\right)\frac{du}{s}$$

$$= \frac{1}{s}\int_0^\infty e^{-u}\ln\left(\frac{u}{s}\right)du$$

$$= \frac{1}{s}\int_0^\infty e^{-u}(\ln u - \ln s)du$$

$$= \frac{1}{s}\int_0^\infty e^{-u}\ln u\, du - \frac{\ln s}{s}\int_0^\infty e^{-u}du$$

$$= \frac{1}{s}I_1 - \frac{\ln s}{s}\cdot\frac{e^{-u}}{-1}\Big|_0^\infty$$

$$= \frac{1}{s}I_1 + \frac{\ln s}{s}\left(e^{-\infty} - e^{-0}\right)$$

$$= \frac{1}{s}I_1 + \frac{\ln s}{s}\left(\frac{1}{e^\infty} - 1\right)$$

$$= \frac{1}{s}I_1 - \frac{\ln s}{s} \longrightarrow (1)$$

$$I_1 = \int_0^\infty e^{-u}\ln u\, du$$

$$\Gamma t = \int_0^\infty e^{-u}\ln u\, du$$

$$\Gamma t = \int_0^\infty e^{-x}x^{t-1}dx$$ Gama fn

$$\Gamma t+1 = \int_0^\infty e^{-x}x^t dx$$

$$\Gamma x+1 = \int_0^\infty e^{-u}u^x du$$

$$\Gamma'(x+1) = \frac{d}{dx}\int_0^\infty e^{-u}u^x du$$

$$= \int_0^\infty e^{-u}\frac{d}{dx}u^x du$$

$$= \int_0^\infty e^{-u}\cdot u^x\ln u\, du$$

Let $x=0$; $\Gamma'(1) = \int_0^\infty e^{-u}\ln u\, du$

$\therefore$ Eq (1) $\Rightarrow$

$$\mathcal{L}\{f(t)\} = \frac{\Gamma'(1)}{s} - \frac{\ln s}{s}$$

★ **Unit Step Function:-**

Let $a \geq 0$, the function u_a defined on $(0,\infty)$ by

$$u_a(t) = \begin{cases} 0 & \text{if } t<a \\ 1 & \text{if } t>a \end{cases}$$

is called unit step function.

If $a=0$, then;

$$u_0(t) = \begin{cases} 0 & \text{if } t<0 \\ 1 & \text{if } t>0 \end{cases}$$

$$u_0(t) = 1 \quad t>0$$

★ **Theorem:-** let u_a be the step function then;

$$\mathcal{L}\{u_a(t)\} = \frac{e^{-as}}{s}$$

Proof:-

$$\mathcal{L}\{u_a(t)\} = \int_0^\infty e^{-st} u_a(t)\,dt$$

$$= \int_0^a e^{-st} u_a(t)\,dt + \int_a^\infty e^{-st} u_a(t)\,dt$$

$$= 0 + \int_a^\infty e^{-st}\cdot 1\,dt = \frac{e^{-st}}{-s}\Big|_a^\infty$$

$$= -\frac{1}{s} e^{-st}\Big|_a^\infty = -\frac{1}{s}\left(e^{-s(\infty)} - e^{-as}\right)$$

$$\mathcal{L}\{u_a(t)\} = \frac{e^{-as}}{s}$$

★ **Theorem:-** let $\mathcal{L}\{f(t)\} = F(s)$

For the function

$$u_a(t) f(t-a) = \begin{cases} 0 & 0<t<a \\ f(t-a) & t>a \end{cases}$$

$$\mathcal{L}\{u_a(t) f(t-a)] = e^{-as} F(s)$$

(28) $$f(t) = \begin{cases} 0 & \text{if } t<3 \\ (t-3)^3 & \text{if } t>3 \end{cases}$$

$$u_3(t) = \begin{cases} 0 & t<3 \\ 1 & t>3 \end{cases}$$

$$g(t) = t^3$$

$$g(t-3) = (t-3)^3$$

$$u_3(t) g(t-3) = \begin{cases} 0 & t<3 \\ (t-3)^3 & t>3 \end{cases}$$

$$u_3(t)\cdot g(t-3) = f(t)$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{u_3(t)\cdot g(t-3)\}$$

Here $a=3$

$$\mathcal{L}\{f(t)\} = e^{-3s}\mathcal{L}\{g(t)\}$$

$$= e^{-3s}\mathcal{L}(t^3)$$

$$= e^{-3s}\cdot\frac{3!}{s^{3+1}} = \frac{6e^{-3s}}{s^4}$$

(29) $\mathcal{L}[f(t)] = F(s) \quad s>a$

$$\mathcal{L}[f(ct)] = \frac{1}{c} F\left(\frac{s}{c}\right) \quad \begin{matrix} c>0 \\ s>ca \end{matrix}$$

Now.

$$\mathcal{L}[f(ct)] = \int_0^\infty e^{-st} f(ct)\,dt$$

let $ct = T$; $t\to 0,\ T\to 0$

$c\,dt = dT$; $t\to\infty,\ T\to\infty$

$dt = \frac{dT}{c}$

$$\mathcal{L}[f(ct)] = \int_0^\infty e^{-s(T/c)} f(T)\frac{dT}{c}$$

$$= \frac{1}{c}\int_0^\infty e^{-s\left(\frac{T}{c}\right)} f(T)\,dT$$

$$= \frac{1}{c}\int_0^\infty e^{\left(-\frac{s}{c}\right)T} f(T)\,dT$$

$$= \frac{1}{c}\mathcal{L}\left[f\left(\frac{T}{c}\right)\right]$$

$$= \frac{1}{c} F\left(\frac{s}{c}\right)$$

(30) $f(t) = \sin\sqrt{t}$

Powere series expansion of $\sin x$ is given by $\qquad \because \Gamma\frac{1}{2} = \sqrt{\pi}$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

replace x by $t^{1/2}$; $\sin\sqrt{t} = t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \frac{t^{7/2}}{7!} + \cdots$

$$\mathcal{L}(\sin\sqrt{t}) = \mathcal{L}(t^{1/2}) - \frac{1}{3!}\mathcal{L}(t^{3/2}) + \frac{1}{5!}\mathcal{L}(t^{5/2}) - \frac{1}{7!}\mathcal{L}(t^{7/2}) + \cdots$$

$$= \frac{\Gamma\frac{1}{2}+1}{s^{\frac{1}{2}+1}} - \frac{1}{3!}\frac{\Gamma\frac{3}{2}+1}{s^{\frac{3}{2}+1}} + \frac{1}{5!}\frac{\Gamma\frac{5}{2}+1}{s^{\frac{5}{2}+1}} - \frac{1}{7!}\frac{\Gamma\frac{7}{2}+1}{s^{\frac{7}{2}+1}} + \cdots \qquad \because t^{\alpha} = \frac{\Gamma\alpha+1}{s^{\alpha+1}}$$

$$= \frac{\frac{1}{2}\Gamma\frac{1}{2}}{s^{3/2}} - \frac{1}{3!}\frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{s^{5/2}} + \frac{1}{5!}\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{s^{7/2}} - \frac{1}{7!}\frac{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{s^{9/2}} + \cdots$$

$$= \frac{\frac{1}{2}\sqrt{\pi}}{s^{3/2}} - \frac{1}{3!}\frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{s^{5/2}} + \frac{1}{5!}\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{s^{7/2}} - \frac{1}{7!}\frac{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{s^{9/2}} + \cdots$$

$$= \frac{\frac{1}{2}\sqrt{\pi}}{s^{3/2}}\left[1 - \frac{1}{6}\,\frac{3/2}{s} + \frac{1}{120}\cdot\frac{15/4}{s^2} - \frac{1}{5040}\cdot\frac{105/8}{s^3} + \cdots\right]$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left[1 - \frac{1}{6}\times\frac{3}{2s} + \frac{1}{120}\times\frac{15}{4s^2} - \frac{1}{5040}\times\frac{105}{8s^3} + \cdots\right]$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left[1 - \frac{1}{4s} + \frac{1}{32s^2} - \frac{1}{384s^3} + \cdots\right] = \frac{\sqrt{\pi}}{2s^{3/2}}\left[1 - \frac{1}{4s} + \frac{1}{2!}\cdot\frac{1}{16s^2} - \frac{1}{3!}\cdot\frac{1}{64s^3} + \cdots\right]$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left[1 + \left(-\frac{1}{4s}\right) + \frac{1}{2!}\left(\frac{-1}{4s}\right)^2 + \frac{1}{3!}\left(-\frac{1}{4s}\right)^3 + \cdots\right]$$

$$\therefore \mathcal{L}(\sin\sqrt{t}) = \frac{\sqrt{\pi}}{2s^{3/2}}e^{-\frac{1}{4s}} \qquad \because e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Deduce $\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\}$

$f(t) = \sin\sqrt{t}$

$f'(t) = \cos\sqrt{t}\cdot\frac{1}{2\sqrt{t}} = \frac{\cos\sqrt{t}}{2\sqrt{t}}$

$\therefore \mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$

$$\mathcal{L}\left\{\frac{\cos\sqrt{t}}{2\sqrt{t}}\right\} = s\cdot\frac{\sqrt{\pi}}{2s^{3/2}}e^{-\frac{1}{4s}} - 0 \qquad \because f(0) = \sin 0 = 0$$

$$\Rightarrow \frac{1}{2}\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \frac{\pi}{2s^{1/2}}e^{-\frac{1}{4s}}$$

$$\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{s}}e^{-\frac{1}{4s}}$$

★ Inverse Laplace Transform

$$\mathcal{L}[f(t)] = F(s)$$

$$\mathcal{L}^{-1}[F(s)] = f(t)$$

★ Properties of Inverse Laplace Transform:-

If $\mathcal{L}^{-1}[F(s)] = f(t)$ then;

(i) $\mathcal{L}^{-1}[F(s-a)] = e^{at}f(t)$

(ii) $\mathcal{L}^{-1}[F(cs)] = \frac{1}{c}f\left(\frac{t}{c}\right), c>0$

(iii) $\mathcal{L}^{-1}[F^{(n)}(s)] = (-1)^n t^n f(t)$

(iv) $\mathcal{L}^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(u)\,du$

(v) $\mathcal{L}^{-1}[e^{-as}F(s)] = u_a(t)f(t-a)$

EXERCISE 11.2

① $F(s) = \frac{s-2}{s^2-2}$

$$F(s) = \frac{s}{s^2-2} - \frac{2}{s^2-2}$$

$$\mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{s}{s^2-2} - \frac{2}{s^2-2}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{s}{(s)^2-(\sqrt{2})^2}\right] - \mathcal{L}^{-1}\left[\frac{\sqrt{2}\cdot\sqrt{2}}{(s)^2-(\sqrt{2})^2}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{s}{(s)^2-(\sqrt{2})^2}\right] - \sqrt{2}\,\mathcal{L}^{-1}\left[\frac{\sqrt{2}}{(s)^2-(\sqrt{2})^2}\right]$$

$$\mathcal{L}^{-1}[F(s)] = \cosh\sqrt{2}t - \sqrt{2}\sinh\sqrt{2}t$$

② $F(s) = \frac{3s+1}{s^2-6s+18}$

$$F(s) = \frac{3s+1}{s^2-6s+9-9+18} = \frac{3s+1}{(s-3)^2+(3)^2}$$

$$= \frac{3(s-3+3)+1}{(s-3)^2+(3)^2} = \frac{3(s-3)+10}{(s-3)^2+(3)^2}$$

$$= \frac{3(s-3)}{(s-3)^2+(3)^2} + \frac{10}{(s-3)^2+(3)^2}$$

$$\mathcal{L}^{-1}F(s) = 3\mathcal{L}^{-1}\left[\frac{s-3}{(s-3)^2+(3)^2}\right] + \frac{10}{3}\mathcal{L}^{-1}\left[\frac{3}{(s-3)^2+(3)^2}\right]$$

$$= 3e^{3t}\cos 3t + \frac{10}{3}e^{3t}\sin 3t$$

③ $F(s) = \frac{9s-67}{s^2-16s+49}$

$$F(s) = \frac{9s-67}{s^2-16s+64-64+49}$$

$$= \frac{9s-67}{(s-8)^2+15} = \frac{9(s-8+8)-67}{(s-8)^2-(\sqrt{15})^2}$$

$$F(s) = \frac{9(s-8)+5}{(s-8)^2-(\sqrt{15})^2}$$

$$\mathcal{L}^{-1}[F(s)] = 9\mathcal{L}^{-1}\left[\frac{s-8}{(s-8)^2-(\sqrt{15})^2}\right] + \frac{5}{\sqrt{15}}\mathcal{L}^{-1}\left[\frac{\sqrt{15}}{(s-8)^2-(\sqrt{15})^2}\right]$$

$$= 9e^{8t}\cosh\sqrt{15}t + \sqrt{\frac{5}{3}}e^{8t}\sinh\sqrt{15}t$$

④ $F(s) = \frac{as+b}{s^2+2cs+d}, d>c^2>0$

$$\mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left\{\frac{as+b}{s^2+2cs+d}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{as+b}{s^2+2cx+c^2-c^2+d}\right\} = \mathcal{L}^{-1}\left\{\frac{as+b}{(s+c)^2+d-c^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{as+ac+b-ac}{(s+c)^2+(d-c^2)}\right\} = \mathcal{L}^{-1}\left\{\frac{a(s+c)+(b-ac)}{(s+c)^2+(\sqrt{d-c^2})^2}\right\}$$

$$\mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left\{\frac{a(s+c)}{(s+c)^2+(\sqrt{d-c^2})^2}\right\} + \mathcal{L}^{-1}\left\{\frac{b-ac}{(s+c)^2+(\sqrt{d-c^2})^2}\right\}$$

$$= a\mathcal{L}^{-1}\left\{\frac{s+c}{(s+c)^2+(\sqrt{d-c^2})^2}\right\} + \frac{b-ac}{\sqrt{d-c^2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{d-c^2}}{(s+c)^2+(\sqrt{d-c^2})^2}\right\}$$

$$\mathcal{L}^{-1}[F(s)] = a e^{-ct}\cos\sqrt{d-c^2}\, t + \frac{b-ac}{\sqrt{d-c^2}}\, e^{-ct}\sin\sqrt{d-c^2}\, t$$

⑤ $F(s) = \dfrac{s}{(s+a)^2+b^2}$

$$= \frac{(s+a)-a}{(s+a)^2+b^2} = \frac{s+a}{(s+a)^2+b^2} - \frac{a}{(s+a)^2+b^2}$$

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{s+a}{(s+a)^2+b^2}\right] - \frac{a}{b}\mathcal{L}^{-1}\left[\frac{b}{(s+a)^2+b^2}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{s-(-a)}{(s-(-a))^2+b^2}\right] - \frac{a}{b}\mathcal{L}^{-1}\left[\frac{b}{(s+a)^2+b^2}\right] = e^{-at}\cos bt - \frac{a}{b}e^{-at}\sin bt$$

⑥ $F(s) = \dfrac{1}{(s^2+a^2)(s^2+b^2)} = \dfrac{1}{a^2-b^2}\left(\dfrac{a^2-b^2}{(s^2+a^2)(s^2+b^2)}\right)$

$$= \frac{1}{a^2-b^2}\left[\frac{s^2+a^2-s^2-b^2}{(s^2+a^2)(s^2+b^2)}\right] = \frac{1}{a^2-b^2}\left[\frac{(s^2+a^2)-(s^2+b^2)}{(s^2+a^2)(s^2+b^2)}\right]$$

$$= \frac{1}{a^2-b^2}\left(\frac{1}{s^2+b^2}\right) - \frac{1}{a^2-b^2}\left(\frac{1}{s^2+b^2}\right)$$

$$\mathcal{L}^{-1}[F(s)] = \frac{1}{a^2-b^2}\mathcal{L}^{-1}\left(\frac{1}{s^2+b^2}\right) - \frac{1}{a^2-b^2}\mathcal{L}^{-1}\left(\frac{1}{s^2+a^2}\right)$$

$$= \frac{1}{b(a^2-b^2)}\mathcal{L}^{-1}\left(\frac{b}{s^2+b^2}\right) - \frac{1}{a(a^2-b^2)}\mathcal{L}^{-1}\left(\frac{a}{s^2+a^2}\right)$$

$$= \frac{1}{b(a^2-b^2)}\cdot \sin bt - \frac{1}{a(a^2-b^2)}\sin at = \frac{1}{a^2-b^2}\left(\frac{1}{b}\sin bt - \frac{1}{a}\sin at\right)$$

$$\mathcal{L}^{-1}[F(s)] = \frac{1}{b^2-a^2}\left(\frac{1}{a}\sin at - \frac{1}{b}\sin bt\right)$$ (Can also be solve using partial fractions)

⑦ $F(s) = \dfrac{1}{(s-1)(s^2+4)}$; By Partial fraction; Consider $\dfrac{1}{(s-1)(s^2+4)} = \dfrac{1}{5(s-1)} - \dfrac{s+1}{5(s^2+4)}$

$$\therefore F(s) = \frac{1}{5(s-1)} - \frac{s+1}{5(s^2+4)}$$

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+4}\right\}$$

$$= \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4} + \frac{1}{s^2+4}\right\}$$

$$= \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - \frac{1}{10}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$= \frac{1}{5}e^{t} - \frac{1}{5}\cos 2t - \frac{1}{10}\sin 2t$$

(8) $F(s) = \dfrac{7s+5}{(3s-8)^2} = \dfrac{7}{3(3s-8)} + \dfrac{71}{3(3s-8)^2}$ By partial fraction

$\mathcal{L}^{-1}\{F(s)\} = \frac{7}{3}\mathcal{L}^{-1}\left\{\frac{1}{3s-8}\right\} + \frac{71}{3}\mathcal{L}^{-1}\left\{\frac{1}{(3s-8)^2}\right\} = \frac{7}{3}\cdot\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-\frac{8}{3}}\right\} + \frac{71}{3}\cdot\frac{1}{9}\mathcal{L}^{-1}\left\{\frac{1}{(s-\frac{8}{3})^2}\right\}$

$= \frac{7}{9}e^{\frac{8t}{3}}\mathcal{L}^{-1}\left(\frac{1}{s}\right) + \frac{71}{27}e^{\frac{8}{3}t}\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = \frac{7}{9}e^{\frac{8}{3}t}(1) + \frac{71}{27}e^{\frac{8}{3}t}(t)$

$= \frac{7}{9}e^{\frac{8}{3}t} + \frac{71}{27}te^{\frac{8}{3}t}$.

(9) $F(s) = \dfrac{5s+3}{(s+7)^5} = \dfrac{5s+35+3-35}{(s+7)^5} = \dfrac{5(s+7)-32}{(s+7)^5}$

$F(s) = \dfrac{5(s+7)}{(s+7)^5} - \dfrac{32}{(s+7)^5} = \dfrac{5}{(s+7)^4} - \dfrac{32}{(s+7)^5}$

$\mathcal{L}^{-1}\{F(s)\} = 5\mathcal{L}^{-1}\left\{\frac{1}{(s+7)^4}\right\} - 32\mathcal{L}^{-1}\left\{\frac{1}{(s+7)^5}\right\}$

$= \frac{5}{3!}\mathcal{L}^{-1}\left\{\frac{3!}{(s-(-7))^4}\right\} - \frac{32}{4!}\mathcal{L}^{-1}\left\{\frac{4!}{(s-(-7))^5}\right\}$

$= \frac{5}{6}e^{-7t}\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\} - \frac{32}{4!}e^{-7t}\left\{\frac{4!}{s^5}\right\} = \frac{5}{6}t^3e^{-7t} - \frac{4}{3}t^4e^{-7t}$

$\therefore \mathcal{L}^{-1}\{F(s)\} = t^3e^{-7t}\left(\frac{5}{6} - \frac{4}{3}t\right)$

(10) $F(s) = \dfrac{2s-3}{2s^3+3s^2-2s-3} = \dfrac{2s-3}{s^2(2s+3)-1(2s+3)}$

$= \dfrac{2s-3}{(2s+3)(s^2-1)} = \dfrac{2s-3}{(2s+3)(s-1)(s+1)}$

$\because \dfrac{2s-3}{(2s+3)(s-1)(s+1)} = \dfrac{A}{2s+3} + \dfrac{B}{s-1} + \dfrac{C}{s+1} = \dfrac{-24/5}{2s+3} + \dfrac{-1/10}{s-1} + \dfrac{5/2}{s+1}$

$\mathcal{L}^{-1}\{F(s)\} = -\frac{24}{5}\mathcal{L}^{-1}\left(\frac{1}{2s+3}\right) - \frac{1}{10}\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) + \frac{5}{2}\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)$

$= \frac{-24}{5}\cdot\frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{s+\frac{3}{2}}\right) - \frac{1}{10}e^{t} + \frac{5}{2}e^{-t} = -\frac{12}{5}e^{-\frac{3}{2}t} - \frac{1}{10}e^{t} + \frac{5}{2}e^{-t}$

(11) $F(s) = \dfrac{2s^3+6s^2+21s+52}{s(s+2)(s^2+4s+13)} = \dfrac{2}{s} - \dfrac{1}{s+2} + \dfrac{(1)s-4}{s^2+4s+13}$

$\mathcal{L}^{-1}\{F(s)\} = 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \mathcal{L}^{-1}\left\{\frac{s-4}{s^2+4s+13}\right\}$

$= 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s-(-2)}\right\} + \mathcal{L}^{-1}\left\{\frac{(s+2)-6}{s^2+4s+4+9}\right\}$

$= 2(1) - e^{-2t} + \mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+(3)^2}\right\} - 2\mathcal{L}^{-1}\left\{\frac{3}{(s+2)^2+(3)^2}\right\}$

$\mathcal{L}^{-1}\{F(s)\} = 2 - e^{-2t} + e^{-2t}\cos 3t - 2e^{-2t}\sin 3t$

(12) $F(s) = \frac{1}{(s^2+4)(s^2+6s-5)}$ using partial fractions

$$F(s) = \frac{-\frac{2}{75}s - \frac{1}{25}}{s^2+4} + \frac{\frac{2}{75}s + \frac{1}{5}}{s^2+6s-5} = \frac{-2s-3}{75(s^2+4)} + \frac{2s+15}{75(s^2+6s-5)}$$

$$= \frac{-2s}{75(s^2+4)} - \frac{3}{75(s^2+4)} + \frac{2s}{75(s^2+6s-5)} + \frac{15}{75(s^2+6s-5)}$$

$$= \frac{-2}{75}\cdot\frac{s}{s^2+(2)^2} - \frac{1}{25}\cdot\frac{1}{s^2+(2)^2} + \frac{2}{75}\cdot\frac{(s+3)-3}{s^2+6s+9-14} + \frac{1}{5}\cdot\frac{1}{s^2+6s+9-14}$$

$$= \frac{-2}{75}\cdot\frac{s}{s^2+(2)^2} - \frac{1}{25}\cdot\frac{1}{s^2+(2)^2} + \frac{2}{75}\cdot\frac{(s+3)}{(s+3)^2-(\sqrt{14})^2} - \frac{6}{75}\cdot\frac{1}{(s+3)^2-(\sqrt{14})^2} + \frac{1}{5}\frac{1}{(s+3)^2-(\sqrt{14})^2}$$

$$= \frac{-2}{75}\cdot\frac{s}{s^2+(2)^2} - \frac{1}{25}\cdot\frac{1}{s^2+(2)^2} + \frac{2}{75}\cdot\frac{s+3}{(s+3)^2-(\sqrt{14})^2} + \frac{3}{25}\cdot\frac{1}{(s+3)^2-(\sqrt{14})^2}$$

Applying Laplace Inverse

$$\mathcal{L}^{-1}\{F(s)\} = \frac{-2}{75}\mathcal{L}^{-1}\left\{\frac{s}{s^2+(2)^2}\right\} - \frac{1}{50}\mathcal{L}^{-1}\left\{\frac{2}{s^2+(2)^2}\right\} + \frac{2}{75}\mathcal{L}^{-1}\left\{\frac{s+3}{(s+3)^2-(\sqrt{14})^2}\right\} + \frac{3}{25\sqrt{14}}\mathcal{L}^{-1}\left\{\frac{\sqrt{14}}{(s+3)^2-(\sqrt{14})^2}\right\}$$

$$= -\frac{2}{75}\cos 2t - \frac{1}{50}\sin 2t + \frac{2}{75}e^{-3t}\cosh(\sqrt{14}\,t) + \frac{3}{25\sqrt{14}}e^{-3t}\sinh(\sqrt{14}\,t)$$

(13) $F(s) = \frac{s^3+3s^2-s-3}{(s^2+2s+5)^2} = \frac{s+1}{s^2+2s+5} + \frac{-8s-8}{(s^2+2s+5)^2}$

$$= \frac{s+1}{s^2+2s+5} - \frac{8(s+1)}{(s^2+2s+5)^2}$$

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{s+1}{s^2+2s+5}\right\} - 8\mathcal{L}^{-1}\left\{\frac{s+1}{(s^2+2s+5)^2}\right\} \longrightarrow (A)$$

Now

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+2s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{s^2+2s+1+4}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+(2)^2}\right\}$$

$$= e^{-t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\} = e^{-t}\cos 2t$$

Consider;

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+1+4}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+(2)^2}\right\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+(2)^2}\right\} = \frac{1}{2}e^{-t}\sin 2t.$$

$\because \mathcal{L}\{f(t)\} = F(s)$

$$\mathcal{L}\{t\,f(t)\} = -\frac{d}{ds}\{F(s)\} \Rightarrow t\mathcal{L}\{f(t)\} = -\frac{d}{ds}\{F(s)\}$$

$$F(s) = -\frac{1}{t}\frac{d}{ds}\{F(s)\}$$

$$\mathcal{L}^{-1}\{F(s)\} = -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}F(s)\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+5}\right\} = -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2+2s+5}\right)\right\}$$

$$\frac{1}{2}e^{-t}\sin 2t = -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{-1}{(s^2+2s+5)^2}(2s+2)\right\}$$

$$\frac{1}{2}te^{-t}\sin 2t = \mathcal{L}^{-1}\left\{\frac{2s+2}{(s^2+2s+5)^2}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{s+1}{(s^2+2s+5)^2}\right\} = \frac{1}{4}te^{-t}\sin 2t$$

$\therefore$ Eq. (A) becomes:

$$\mathcal{L}^{-1}\{F(s)\} = e^{-t}\cos 2t - 8\cdot\frac{1}{4}te^{-t}\sin 2t$$

$$= e^{-t}\cos 2t - 2te^{-t}\sin 2t.$$

(14) $F(s) = \tan^{-1}\left(\frac{a}{s}\right)$

$\mathcal{L}^{-1}[F^{(n)}(s)] = (-1)^n t^n [f(t)]$ where $\mathcal{L}[f(t)] = F(s)$

let $n=1$;

$$\mathcal{L}^{-1}[F'(s)] = -t\,f(t)$$

$$f(t) = -\frac{1}{t}\mathcal{L}^{-1}[F'(s)]$$

$$\mathcal{L}^{-1}[F(s)] = -\frac{1}{t}\mathcal{L}^{-1}[F'(s)]$$

$$= -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}F(s)\right\}$$

$$\therefore \mathcal{L}^{-1}\left[\tan^{-1}\left(\frac{a}{s}\right)\right] = -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}\left(\tan^{-1}\frac{a}{s}\right)\right\}$$

$$= -\frac{1}{t}\mathcal{L}^{-1}\left[\frac{1}{1+\frac{a^2}{s^2}}\left(-\frac{a}{s^2}\right)\right]$$

$$= \frac{1}{t}\mathcal{L}^{-1}\left(\frac{a}{s^2+a^2}\right) = \frac{1}{t}\cdot\sin at$$

(15) $F(s) = \ln\frac{s^2+1}{(s-1)^2}$

$$F(s) = \ln(s^2+1) - \ln(s-1)^2$$

$$= \ln(s^2+1) - 2\ln(s-1)$$

$$F'(s) = \frac{1}{s^2+1}(2s) - 2\cdot\frac{1}{s-1}(1)$$

$$= \frac{2s}{s^2+1} - \frac{2}{s-1}$$

we know:

$$\mathcal{L}^{-1}[F(s)] = -\frac{1}{t}\mathcal{L}^{-1}[F'(s)]$$

$$= \frac{-2}{t}\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) + \frac{2}{t}\mathcal{L}^{-1}\left(\frac{1}{s-1}\right)$$

$$= -\frac{2}{t}\cos t + \frac{2}{t}e^t$$

$$\mathcal{L}^{-1}[F(s)] = \frac{-2}{t}\cos t + \frac{2}{t}e^t$$

(16) $F(s) = \ln\frac{s^2+a^2}{s^2+b^2}$

By formula;

$$\mathcal{L}^{-1}\{F(s)\} = -\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\}$$

$$\Rightarrow f(t) = -\frac{1}{t}\mathcal{L}^{-1}\left[\frac{d}{ds}\{F(s)\}\right]$$

$$= -\frac{1}{t}\mathcal{L}^{-1}\left[\frac{d}{ds}\left\{\ln\frac{s^2+a^2}{s^2+b^2}\right\}\right]$$

$$= -\frac{1}{t}\mathcal{L}^{-1}\left[\frac{d}{ds}\{\ln(s^2+a^2) - \ln(s^2+b^2)\}\right]$$

$$= -\frac{1}{t}\mathcal{L}^{-1}\left[\frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}\right]$$

$$= -\frac{2}{t}\left[\mathcal{L}^{-1}\left(\frac{s}{s^2+a^2}\right) - \mathcal{L}^{-1}\left(\frac{s}{s^2+b^2}\right)\right]$$

$$\Rightarrow f(t) = -\frac{2}{t}(\cos at - \cos bt)$$

(17) $F(s) = \frac{e^{-3s}}{s^2(s^2+9)}$

$$= e^{-3s}\left\{\frac{1}{s^2(s^2+9)}\right\} = e^{-3s}\left(\frac{1/9}{s^2} + \frac{(-1/9)}{s^2+9}\right)$$

$$= \frac{1}{9}\left(\frac{e^{-3s}}{s^2}\right) - \frac{1}{9}\left(\frac{e^{-3s}}{s^2+9}\right)$$

we know:

$$\mathcal{L}^{-1}[e^{-as}F(s)] = u_a(t)f(t-a)$$

where $f(t) = \mathcal{L}^{-1}[F(s)]$

(18) $F(s) = e^{-\pi s}\frac{s}{s^2-4s+5}$

$$F(s) = e^{-\pi s}\left[\frac{(s-2)+2}{(s^2-4s+4+1}\right]$$

$$= e^{-\pi s}\left[\frac{s-2}{(s-2)^2+(1)^2} + \frac{2}{(s-2)^2+(1)^2}\right]$$

$$= e^{-\pi s}\left(\frac{s-2}{(s-2)^2+1}\right) + 2e^{-\pi s}\left(\frac{1}{(s-2)^2+1}\right)$$

$$\because \mathcal{L}^{-1}\left(\frac{e^{-3s}}{s^2}\right) = u_3(t)\cdot\mathcal{L}^{-1}\left(\frac{1}{s^2}\right)\Big|_{t=t-3}$$

$$= u_3(t)\cdot t\Big|_{t-3} = u_3(t)(t-3)$$

$$\mathcal{L}^{-1}\left(\frac{e^{-3s}}{s^2+9}\right) = \frac{1}{3}u_3(t)\cdot\mathcal{L}^{-1}\left(\frac{3}{s^2+9}\right)$$

$$= \frac{1}{3}u_3(t)\sin 3t\Big|_{t=t-3}$$

$$= \frac{1}{3}u_3(t)\sin 3(t-3)$$

$$= \frac{1}{3}u_3(t)\sin(3t-9)$$

$$\therefore \mathcal{L}^{-1}[F(s)] = \frac{1}{9}u_3(t)(t-3) + \frac{1}{27}u_3(t)\sin(3t-9)$$

$$= \frac{1}{9}u_3(t)\left[(t-3) + \frac{1}{3}\sin(3t-9)\right]$$

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{e^{-\pi s}\frac{s-2}{(s-2)^2+(1)^2}\right\} + 2\mathcal{L}^{-1}\left\{e^{-\pi s}\cdot\frac{1}{(s-2)^2+(1)^2}\right\}$$

We know

$$\mathcal{L}\{u_a(t)f(t-a)\} = e^{-as}F(s)$$

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = u_a(t)f(t-a)$$

$$\therefore \mathcal{L}^{-1}\{F(s)\} = u_\pi(t)e^{2(t-\pi)}\cos(t-\pi) + 2u_\pi(t)e^{2(t-\pi)}\sin(t-\pi)$$

$$= u_\pi(t)e^{2(t-\pi)}\left[\cos(\pi-t) - 2\sin(\pi-t)\right]$$

$$= u_\pi(t)e^{2(t-\pi)}\left[-\cos t - 2\sin t\right]$$

$$= -u_\pi(t)e^{2(t-\pi)}\left[\cos t + 2\sin t\right]$$

(19) $F(s) = e^{-2s}\dfrac{s+6}{s^3-5s^2+6s}$

$$= \frac{e^{-2s}(s+6)}{s(s^2-5s+6)} = \frac{e^{-2s}(s+6)}{s(s^2-3s-2s+6)} = \frac{e^{-2s}(s+6)}{s(s(s-3)-2(s-3))} = \frac{e^{-2s}(s+6)}{s(s-2)(s-3)}$$

$$= e^{-2s}\left[\frac{1}{s} - \frac{4}{s-2} + \frac{3}{s-3}\right] = e^{-2s}\cdot\frac{1}{s} - 4e^{-2s}\cdot\frac{1}{s-2} + 3e^{-2s}\cdot\frac{1}{s-3}$$

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{e^{-2s}\cdot\frac{1}{s}\right\} - 4\mathcal{L}^{-1}\left\{e^{-2s}\cdot\frac{1}{s-2}\right\} + 3\mathcal{L}^{-1}\left\{e^{-2s}\cdot\frac{1}{s-3}\right\}$$

$$= u_2(t)\mathcal{L}\left\{\frac{1}{s}\right\} - 4u_2(t)\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + 3u_2(t)\mathcal{L}^{-1}\left(\frac{1}{s-3}\right)$$

$$= u_2(t)(1) - 4u_2(t)e^{2t}\Big|_{t=t-2} + 3u_2(t)e^{3t}\Big|_{t=t-2}$$

$$= u_2(t) - 4u_2(t)e^{2(t-2)} + 3u_2(t)e^{3(t-2)} = u_2(t)\left(1 - 4e^{2t-4} - 3e^{3t-6}\right)$$

(20) $F(s) = \dfrac{e^{-3s}(3s-7)}{s^2-10s+26} = \dfrac{e^{-3s}(3s-7)}{s^2-10s+25+1} = \dfrac{e^{-3s}\{3(s-5+5)-7\}}{(s-5)^2+(1)^2}$

$$= \frac{3e^{-3s}(s-5)}{(s-5)^2+(1)^2} + 8\left[\frac{e^{-3s}}{(s-5)^2+(1)^2}\right]$$

$$\mathcal{L}^{-1}\{F(s)\} = 3\mathcal{L}^{-1}\left\{e^{-3s}\cdot\frac{(s-5)}{(s-5)^2+1^2}\right\} + 8\mathcal{L}^{-1}\left\{e^{-3s}\cdot\frac{1}{(s-5)^2+1^2}\right\}$$

$$= 3u_3(t)\mathcal{L}^{-1}\left\{\frac{s-5}{(s-5)^2+(1)^2}\right\}_{t=t-3} + 8u_3(t)\mathcal{L}^{-1}\left\{\frac{1}{(s-5)^2+(1)^2}\right\}_{t=t-3}$$

$$= 3u_3(t)e^{5t}\cos t\Big|_{t=t-3} + 8u_3(t)e^{5t}\sin t\Big|_{t=t-3}$$

$$= 3u_3(t)e^{5(t-3)}\cos(t-3) + 8u_3(t)e^{5(t-3)}\sin(t-3)$$

★ **Definition:-** Let $f(t)$ and $g(t)$ be piecewise continuous function on $[0,\infty)$. The Convolution of f and g written as $f*g$

$$(f*g)(t)=\int_0^t f(t-u)\,g(u)\,du$$

★ **Theorem:-**

(i) $\mathcal{L}^{-1}[(f*g)t]=F(s)\cdot G(s)$

where $F(s)=\mathcal{L}[f(t)]$

and $G(s)=\mathcal{L}[g(t)]$

(ii) $\mathcal{L}^{-1}[F(s)\cdot G(s)](t)=(f*g)(t)$

$$=\int_0^t f(t-u)\,g(u)\,du$$

(21) $\dfrac{1}{s^2(s+5)}$

$F(s)=\dfrac{1}{s^2}$, $G(s)=\dfrac{1}{s+5}$

$f(t)=\mathcal{L}^{-1}[F(s)]=\mathcal{L}^{-1}\left(\frac{1}{s^2}\right)=t$

$g(t)=\mathcal{L}^{-1}[G(s)]=\mathcal{L}^{-1}\left(\frac{1}{s+5}\right)=e^{-5t}$

$\Rightarrow f(t-u)=t-u$

By Convolution Theorem;

$$\mathcal{L}^{-1}\{F(s)\cdot G(s)\}=(f*g)(t)=\int_0^t f(t-u)g(u)\,du$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s^2}\cdot\frac{1}{s+5}\right\}=\int_0^t (t-u)e^{-5u}\,du$$

$$=\int_0^t t e^{-5u}\,du-\int_0^t u e^{-5u}\,du$$

$$=t\cdot\frac{e^{-5u}}{-5}\Big|_0^t-\left[u\cdot\frac{e^{-5u}}{-5}\Big|_0^t-\int_0^t\frac{e^{-5u}}{-5}\cdot 1\,du\right]$$

$$=\frac{-1}{5}t\left(e^{-5t}-e^{0}\right)-\left[\left(-\frac{te^{-5t}}{5}+0\right)+\frac{1}{5}\cdot\frac{e^{-5u}}{-5}\Big|_0^t\right]$$

$$=\frac{-te^{-5t}}{5}+\frac{1}{5}t-\left[-\frac{te^{-5t}}{5}-\frac{1}{25}\left(e^{-5t}-1\right)\right]$$

(22) $\dfrac{s}{(s+1)(s^2+4)}$

Suppose $F(s)=\dfrac{1}{s+1}$ & $G(s)=\dfrac{s}{s^2+4}$

$\mathcal{L}^{-1}\{F(s)\}=\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}\Rightarrow f(t)=e^{-t}$

$\mathcal{L}^{-1}\{G(s)\}=\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}\Rightarrow g(t)=\cos 2t$

$\Rightarrow f(t-u)=e^{-(t-u)}$ & $g(u)=\cos 2u$

By Convolution Theorem:

$$\mathcal{L}^{-1}\{F(s)\cdot G(s)\}=(f*g)(t)=\int_0^t f(t-u)g(u)\,du$$

$$\Rightarrow\mathcal{L}^{-1}\left\{\frac{1}{s+1}\cdot\frac{s}{s^2+4}\right\}=\int_0^t e^{-(t-u)}\cos 2u\,du$$

$$=\int_0^t e^{-t}\cdot e^{u}\cos 2u\,du$$

$$=e^{-t}\int_0^t e^{u}\cos 2u\,du=e^{-t}I\to(1)$$

Now

$I=\int e^{u}\cos 2u\,du$

$=\cos 2u\cdot e^{u}-\int e^{u}(-2\sin 2u)\,du$

$=e^{u}\cos 2u+2\int e^{u}\sin 2u\,du$

$=e^{u}\cos 2u+2\left[\sin 2u\cdot e^{u}-\int e^{u}\cdot 2\cos 2u\,du\right]$

$=e^{u}\cos 2u+2e^{u}\sin 2u-4\int e^{u}\cos 2u\,du$

$=e^{u}\cos 2u+2e^{u}\sin 2u-4I$

$\Rightarrow 5I=e^{u}\cos 2u+2e^{u}\sin 2u$

$I=\dfrac{e^{u}\cos 2u}{5}+\dfrac{2e^{u}\sin 2u}{5}$

$\Rightarrow\int e^{u}\cos 2u\,du=\dfrac{e^{u}\cos 2u}{5}+\dfrac{2}{5}e^{u}\sin 2u$

$$\int_0^t e^{u}\cos 2u\,du=\frac{1}{5}\left|e^{u}\cos 2u\right|_0^t+\frac{2}{5}\left|e^{u}\sin 2u\right|_0^t$$

$=\frac{1}{5}\left[e^{t}\cos 2t-e^{0}(1)\right]+\frac{2}{5}\left[e^{t}\sin 2t-e^{0}(0)\right]$

$=\frac{1}{5}\left(e^{t}\cos 2t-1\right)+2e^{t}\sin 2t$

$\therefore$ Eq(1) $\Rightarrow$

$=e^{-t}\cdot\frac{1}{5}\left(e^{t}\cos 2t-1+2e^{t}\sin 2t\right)$

$=\frac{1}{5}\left(\cos 2t-e^{-t}+2\sin 2t\right)$

$$= -\cancel{\frac{te^{-5t}}{5}} + \frac{t}{5} + \cancel{\frac{te^{-5t}}{5}} + \frac{1}{25}\left(e^{-5t}-1\right)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+5)}\right\} = \frac{t}{5} + \frac{1}{25}\left(e^{-5t}-1\right)$$

$$= \frac{1}{25}\left(e^{-5t}+5t-1\right)$$

(23) $\dfrac{1}{(s^2+1)(s^2+4s+5)}$

Let $F(s) = \dfrac{1}{s^2+1}$, $G(s) = \dfrac{1}{s^2+4s+5}$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$f(t) = \sin t$$

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+4+1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+1}\right\} = e^{-2t}\sin t$$

$\Rightarrow f(u) = \sin u$; $g(t-u) = e^{-2(t-u)}\sin(t-u)$

By Convolution Theorem;

$$\mathcal{L}^{-1}\{F(s)\cdot G(s)\} = (f*g)(t) = \int_0^t f(u)\,g(t-u)\,du$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4s+5)}\right\} = \int_0^t \sin u\cdot e^{-2(t-u)}\sin(t-u)\,du$$

$$= \int_0^t \sin u\cdot e^{-2t+2u}\cdot\sin(t-u)\,du = \int_0^t e^{-2t}\cdot e^{2u}\left(\sin u\cdot\sin(t-u)\right)du$$

$$= -\frac{e^{-2t}}{2}\int_0^t e^{2u}\left(-2\sin u\sin(t-u)\right)du = -\frac{e^{-2t}}{2}\int_0^t e^{2u}\{\cos(u+t-u)-\cos(u-t+u)\}\,du$$

$$= -\frac{e^{-2t}}{2}\int_0^t e^{2u}\left(\cos t-\cos(2u-t)\right)du = -\frac{e^{-2t}}{2}\int_0^t e^{2u}\cos t\,du + \frac{e^{-2t}}{2}\int_0^t e^{2u}\cos(2u-t)\,du$$

$$= \frac{-e^{-2t}\cos t}{2}\int_0^t e^{2u}\,du + \frac{e^{-2t}}{2}\int_0^t e^{2u}\cos(2u-t)\,du = -\frac{e^{-2t}\cos t}{2}I_1 + \frac{e^{-2t}}{2}I_2 \rightarrow (1)$$

$$I_1 = \int_0^t e^{2u}\,du = \frac{e^{2u}}{2}\Big|_0^t = \frac{1}{2}\left(e^{2t}-e^0\right) = \frac{1}{2}\left(e^{2t}-1\right)$$

$$\because \int e^{ax}\cos(bx+c)\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos(bx+c)+b\sin(bx+c)\right]$$

$$\therefore I_2 = \int_0^t e^{2u}\cos(2u-t)\,du = \left[\frac{e^{2u}}{2^2+2^2}\{2\cos(2u-t)+2\sin(2u-t)\}\right]_0^t$$

$$= \frac{e^{2t}}{8}\{2\cos(2t-t)+2\sin(2t-t)\} - \left\{\frac{e^0}{8}\{2\cos(0-t)+2\sin(0-t)\}\right\}$$

$$= \frac{e^{2t}}{4}(\cos t+\sin t) - \frac{1}{4}(\cos t-\sin t)$$

$$\therefore \text{Eq }(1) \Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4s+5)}\right\} = \frac{-e^{-2t}\cos t}{2}\left(\frac{1}{2}(e^{2t}-1)\right) + \frac{e^{-2t}}{2}\left\{\frac{e^{2t}}{4}(\cos t+\sin t) - \frac{1}{4}(\cos t-\sin t)\right\}$$

$$= \frac{-\cos t}{4} + \frac{e^{-2t}\cos t}{4} + \frac{\cos t}{8} + \frac{\sin t}{8} - \frac{e^{-2t}\cos t}{8} + \frac{e^{-2t}\sin t}{8}$$

$$= \left(-\frac{1}{4}+\frac{1}{8}\right)\cos t + \frac{\sin t}{8} + e^{-2t}\cos t\left(\frac{1}{4}-\frac{1}{8}\right) + e^{-2t}\frac{\sin t}{8}$$

$$= -\frac{1}{8}\cos t + \frac{1}{8}\sin t + \frac{1}{8}e^{-2t}\cos t + \frac{1}{8}e^{-2t}\sin t$$

$$= \frac{1}{8}(\sin t-\cos t) + \frac{e^{-2t}}{8}(\sin t+\cos t)$$

(24) Show $\mathcal{L}^{-1}\left\{\frac{s^3}{s^4+4a^4}\right\} = \cosh at \cos at$

Consider

$\mathcal{L}\{\cosh at \cos at\} = \mathcal{L}\left\{\left(\frac{e^{at}+e^{-at}}{2}\right)\cos at\right\}$

$\cancel{= \mathcal{L}\left\{\frac{1}{2}(e^{at}\cos at)\right\} + \frac{1}{2}(e^{-at}\cos at)\}}$

$= \frac{1}{2}\mathcal{L}\{e^{at}\cos at\} + \frac{1}{2}\mathcal{L}\{e^{-at}\cos at\}$

$= \frac{1}{2}\left[\frac{s-a}{(s-a)^2+a^2}\right] + \frac{1}{2}\left[\frac{s+a}{(s+a)^2+a^2}\right] = \frac{1}{2}\left[\frac{(s-a)\{(s+a)^2+a^2\}+(s+a)\{(s-a)^2+a^2\}}{((s-a)^2+a^2)((s+a)^2+a^2)}\right]$

$= \frac{1}{2}\left[\frac{(s-a)(s+a)^2+a^2(s-a)+(s+a)(s-a)^2+a^2(s+a)}{(s-a)^2(s+a)^2+a^2(s-a)^2+a^2(s+a)^2+a^2\cdot a^2}\right]$

$= \frac{1}{2}\left[\frac{(s-a)(s+a)^2+(s+a)(s-a)^2+a^2(s-a)+a^2(s+a)}{(s-a)^2(s+a)^2+a^2\{(s-a)^2+(s+a)^2\}+a^4}\right]$

$= \frac{1}{2}\left[\frac{(s-a)(s+a)(s+a+s-a)+a^2(s-a+s+a)}{(s^2+a^2-2as)(s^2+a^2+2as)+a^2\{2(s^2+a^2)\}+a^4}\right]$

$= \frac{1}{2}\left[\frac{(s^2-a^2)2s+a^2(2s)}{s^4+2a^2s^2+a^4-4a^2s^2+2a^2s^2+2a^4+a^4}\right] = \frac{1}{2}\left[\frac{2s(s^2-a^2+a^2)}{s^4+4a^4}\right] = \frac{s^3}{s^4+4a^4}$

$\mathcal{L}^{-1}\left\{\frac{s^3}{s^4+4a^4}\right\} = \cosh at \cos at$

(25) Show that $\mathcal{L}^{-1}\left\{\frac{s}{s^4+4a^4}\right\} = \frac{1}{2a^2}\sinh at \sin at$

$\mathcal{L}\{\sinh at \sin at\} = \mathcal{L}\left\{\left(\frac{e^{at}-e^{-at}}{2}\right)\sin at\right\} = \mathcal{L}\left\{\frac{1}{2}e^{at}\sin at - \frac{1}{2}e^{-at}\sin at\right\}$

$= \frac{1}{2}\mathcal{L}\{e^{at}\sin at\} - \frac{1}{2}\mathcal{L}\{e^{-at}\sin at\} = \frac{1}{2}\left[\frac{a}{(s-a)^2+a^2}\right] - \frac{1}{2}\left[\frac{a}{(s+a)^2+a^2}\right]$

$= \frac{1}{2}\left[\frac{a[(s+a)^2+a^2]-a[(s-a)^2+a^2]}{[(s-a)^2+a^2][(s+a)^2+a^2]}\right] = \frac{1}{2}a\left[\frac{(s^2+a^2+2as+a^2)-(s^2+a^2-2as+a^2)}{(s^2+a^2-2as)(s^2+a^2+2as)}\right]$

$= \frac{1}{2}a\left[\frac{4as}{s^4+2a^2s^2+a^4-4a^2s^2+2a^2s^2+2a^4+a^4}\right] = \frac{2a^2s}{s^4+4a^4}$

$\sinh at \sin at = \mathcal{L}^{-1}\left\{\frac{2a^2s}{s^4+4a^4}\right\}$

$\mathcal{L}^{-1}\left\{\frac{s}{s^4+4a^4}\right\} = \frac{1}{2a^2}\sinh at \sin at$

EXERCISE 11·3

① $\frac{dy}{dt} - ky = ce^{kt}$, $y(0)=0$

$\mathcal{L}\left(\frac{dy}{dt}\right) - k\mathcal{L}(y) = \mathcal{L}(ce^{kt}) \rightarrow ①$

$\mathcal{L}(y) = Y(s)$

$\mathcal{L}[f'(t)] = sF(s) - f(0)$

So from eq ①;

$sY(s) - y(0) - kY(s) = c\mathcal{L}(e^{kt})$

$sY(s) - 0 - kY(s) = c\left(\frac{1}{s-k}\right)$

$(s-k)Y(s) = \frac{c}{s-k}$

$Y(s) = \frac{c}{(s-k)^2}$

$\mathcal{L}^{-1}[Y(s)] = c\,\mathcal{L}^{-1}\left[\frac{1}{(s-k)^2}\right]$

$= c\left[e^{kt}\mathcal{L}^{-1}\left(\frac{1}{s^2}\right)\right.$

$y(t) = ce^{kt}t$

② $\frac{dy}{dt} + 4y = 2e^{t} - 4e^{-t}$, $y(0)=0$

$\mathcal{L}\left(\frac{dy}{dt}\right) + 4\mathcal{L}(y) = 2\mathcal{L}(e^{t}) - 4\mathcal{L}(e^{-t})$

$\{sY(s) - y(0)\} + 4Y(s) = \frac{2}{s-1} - 4\cdot\frac{1}{s+1}$

$sY(s) - 0 + 4Y(s) = \frac{2s+2-4s+4}{(s-1)(s+1)}$

$(s+4)Y(s) = \frac{6-2s}{(s-1)(s+1)}$

$Y(s) = \frac{6-2s}{(s+4)(s-1)(s+1)}$

$= \frac{2}{5(s-1)} + \frac{14}{15(s+4)} - \frac{4}{3(s+1)}$

$\mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left(\frac{2}{5(s-1)}\right) + \frac{14}{15}\mathcal{L}^{-1}\left(\frac{1}{s+4}\right) - \frac{4}{3}\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)$

$y(t) = \frac{2}{5}e^{t} + \frac{14}{15}e^{-4t} - \frac{4}{3}e^{-t}$

③ $\frac{dy}{dt} + y = f(t)$ where

$f(t) = \begin{cases} 0 & \text{if } 0 \le t < 1 \\ 5 & \text{if } t \ge 1 \end{cases}$, $y(0)=0$

$f(t) = \begin{cases} 0 & \text{if } 0 \le t < 1 \\ 5 & \text{if } t \ge 1 \end{cases}$

$= 5\begin{cases} 0 & \text{if } 0 \le t < 1 \\ 1 & \text{if } t \ge 1 \end{cases}$

$= 5u_1(t)$

$\therefore \frac{dy}{dt} + y = 5u_1(t)$

$\mathcal{L}\left(\frac{dy}{dt}\right) + \mathcal{L}(y) = 5\mathcal{L}[u_1(t)]$

$\because \mathcal{L}[u_a(t)] = \frac{e^{-as}}{s}$

$sY(s) - y(0) + Y(s) = \frac{5e^{-s}}{s}$

$(s+1)Y(s) = \frac{5e^{-s}}{s}$

$Y(s) = \frac{5e^{-s}}{s(s+1)}$

$= \frac{5e^{-s}}{s} - \frac{5e^{-s}}{s+1}$

$\mathcal{L}^{-1}[Y(s)] = 5\mathcal{L}^{-1}\left(\frac{e^{-s}}{s}\right) - 5\mathcal{L}^{-1}\left(\frac{e^{-s}}{s+1}\right)$

④ $\frac{dy}{dt} + 2y = f(t)$ where $f(t) = \begin{cases} t & 0 \le t < 1 \\ 0 & t \ge 1 \end{cases}$,

$y(0)=0$

$u_1(t) = \begin{cases} 0 & t < 1 \\ 1 & t \ge 1 \end{cases}$

$tu_1(t) = \begin{cases} 0 & 0 \le t < 1 \\ t & t \ge 1 \end{cases}$

$-tu_1(t) = \begin{cases} 0 & t < 1 \\ -t & t \ge 1 \end{cases}$

$t - tu_1(t) = \begin{cases} t & 0 \le t < 1 \\ 0 & t \ge 1 \end{cases}$

$t - tu_1(t) = f(t)$

$f(t) = t - tu_1(t)$

$= t - (t-1+1)u_1(t)$

$= t - (t-1)u_1(t) - u_1(t)$

$\therefore \frac{dy}{dt} + 2y = t - (t-1)u_1(t) - u_1(t)$

$sY(s) - y(0) + 2Y(s) = \frac{1}{s^2} -$

$\mathcal{L}[u_1(t)(t-1)] - \mathcal{L}[u_1(t)]$

$(s+2)Y(s) = \frac{1}{s^2} - \mathcal{L}[u_1(t)(t-1)] - \mathcal{L}[u_1(t)]$

$$y(t) = 5u_1(t) - 5u_1(t)\,\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)\Bigg|_{t=t-1}$$

$\because \mathcal{L}^{-1}[e^{-as}F(s)] = u_a(t)\,f(t-a)$

where $f(t) = \mathcal{L}^{-1}F(s)$

$$\mathcal{L}^{-1}\left[e^{-s}\left(\frac{1}{s+1}\right)\right] = ?$$

$a = 1$, $F(s) = \frac{1}{s+1}$

$f(t) = \mathcal{L}^{-1}[F(s)] = e^{-t}$

$f(t) = e^{-t}$

$$\mathcal{L}^{-1}\left(\frac{e^{-s}}{s+1}\right) = u_1(t)\,f(t-1)$$
$$= u_1(t)\,e^{-(t-1)}$$

$$\therefore y(t) = 5u_1(t) - 5u_1(t)\,e^{-(t-1)}$$

⑤ $\frac{dy}{dt} = \cos t + \int_0^t y(u)\cos(t-u)\,du$, $y(0)=1$

If $f(t) = y(t)$

and $g(t) = \cos t$

$f*g = \int_0^t y(u)\cos(t-u)\,du$

$\mathcal{L}(f*g) = \mathcal{L}\{\int_0^t y(u)\cos(t-u)\,du$

$= Y(s)\cdot G(s)$

where

$Y(s) = \mathcal{L}[y(t)]$; $G(s) = \mathcal{L}[g(t)]$

$\mathcal{L}\left(\frac{dy}{dt}\right) = \mathcal{L}(\cos t) + \mathcal{L}(f*g)$

$sY(s) - y(0) = \frac{s}{s^2+1} + Y(s)\cdot\frac{s}{s^2+1}$

$sY(s) - 1 = \frac{s}{s^2+1} + Y(s)\cdot\frac{s}{s^2+1}$

$sY(s) - Y(s)\frac{\cdot s}{s^2+1} = \frac{s}{s^2+1} + 1$

$\left(\frac{s^3+s-s}{s^2+1}\right)Y(s) = \frac{s+s^2+1}{s^2+1}$

$s^3Y(s) = s^2+s+1$

$Y(s) = \frac{s^2+s+1}{s^3} = \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3}$

$\mathcal{L}^{-1}\{Y(s)\} = 1 + t + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2!}{s^{2+1}}\right\}$

$y(t) = 1 + t + \frac{1}{2}t^2$

$\because \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$ $\therefore \mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$

$\because \mathcal{L}[u_a(t)] = \frac{e^{-as}}{s}$ $\therefore \mathcal{L}[u_1(t)] = \frac{e^{-s}}{s}$

$\because \mathcal{L}[u_a(t)f(t-a)] = e^{-as}\mathcal{L}[f(t)]$

$\therefore \mathcal{L}[u_1(t)(t-1)] = e^{-s}\mathcal{L}(t) = e^{-s}\left(\frac{1}{s^2}\right)$

$\therefore (s+2)Y(s) = \frac{1}{s^2} - e^{-s}\cdot\frac{1}{s^2} - e^{-s}\cdot\frac{1}{s}$

$$Y(s) = \frac{1}{s^2(s+2)} - \frac{e^{-s}}{s^2(s+2)} - \frac{e^{-s}}{s(s+2)}$$

$$Y(s) = \frac{1}{s^2(s+2)} - e^{-s}\left(\frac{s+1}{s^2(s+2)}\right)$$

by Partial fraction;

$$= \left[-\frac{1}{4s} + \frac{1}{2s^2} + \frac{1}{4(s+2)}\right] - e^{-s}\left[\frac{1}{4s} + \frac{1}{2s^2} - \frac{1}{4(s+2)}\right]$$

$$= \frac{-1}{4s} + \frac{1}{2s^2} + \frac{1}{4(s+2)} - \frac{e^{-s}}{4s} - \frac{e^{-s}}{2s^2} + \frac{e^{-s}}{4(s+2)}$$

$$\mathcal{L}^{-1}\{Y(s)\} = -\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} - \frac{1}{4}\mathcal{L}^{-1}\left\{e^{-s}\cdot\frac{1}{s}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{e^{-s}\cdot\frac{1}{s^2}\right\} + \frac{1}{4}\mathcal{L}^{-1}\left\{e^{-s}\cdot\frac{1}{s+2}\right\}$$

$$\Rightarrow y(t) = -\frac{1}{4}(1) + \frac{1}{2}t + \frac{1}{4}e^{-2t} - \frac{1}{4}u_1(t) - \frac{1}{2}u_1(t)\,\mathcal{L}^{-1}\left(\frac{1}{s^2}\right)\Bigg|_{t=t-1} + \frac{1}{4}u_1(t)\,\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}\Bigg|_{t=t-1}$$

$$= -\frac{1}{4} + \frac{1}{2}t + \frac{1}{4}e^{-2t} - \frac{1}{4}u_1(t) - \frac{1}{2}u_1(t)(t-1) + \frac{1}{4}u_1(t)\,e^{-2(t-1)}$$

$$y(t) = -\frac{1}{4} + \frac{1}{2}t + \frac{1}{4}e^{-2t} + \frac{1}{4}u_1(t)\left[1 - 2t + e^{-2(t-1)}\right]$$

⑥ $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = e^t$, $y(0)=1$, $y'(0)=0$

$\mathcal{L}[y''(t)] + 2\mathcal{L}[y'(t)] - 3\mathcal{L}[y(t)] = \mathcal{L}(e^t)$

$\mathcal{L}[y''(t)] = s^2Y(s) - sy(0) - y'(0)$

$= s^2Y(s) - s(1) - 0$

$= s^2Y(s) - s$

$\mathcal{L}[y'(t)] = sY(s) - y(0) = sY(s) - 1$

$\therefore \{s^2Y(s) - s\} + 2\{sY(s) - 1\} - 3Y(s) = \frac{1}{s-1}$

$(s^2+2s-3)Y(s) - s - 2 = \frac{1}{s-1}$

$\therefore Y(s) = \frac{1}{(s-1)(s^2+2s-3)} + \frac{s+2}{s^2+2s-3}$

⑦ $\frac{d^2y}{dt^2}+y=\cos t,\ y(0)=0,\ y'(0)=-1$

$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\}+\mathcal{L}\{y(t)\}=\mathcal{L}\{\cos t\}$

$s^2Y(s)-sy(0)-y'(0)+Y(s)=\frac{s}{s^2+1}$

$s^2Y(s)-0-(-1)+Y(s)=\frac{s}{s^2+1}$

$(s^2+1)Y(s)=\frac{s}{s^2+1}-1$

$Y(s)=\frac{s}{(s^2+1)^2}-\frac{1}{s^2+1}$

$Y(s)=-\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s^2+1}\right)-\frac{1}{s^2+1}$

$\mathcal{L}^{-1}[Y(s)]=\frac{1}{2}\mathcal{L}^{-1}\left\{(-1)^1\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right\}-\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right)$

$\therefore y(t)=\frac{1}{2}t\sin t-\sin t$

$\because \mathcal{L}[t^n f(t)]=(-1)^n\frac{d^n}{ds^n}[F(s)]$

$Y(s)=\frac{1}{(s-1)^2(s+3)}+\frac{s+2}{(s-1)(s+3)}$

$=\left[\frac{-1}{16(s-1)}+\frac{1}{4(s-1)^2}+\frac{1}{16(s+3)}\right]+\left[\frac{1}{4(s+3)}+\frac{1}{4(s-1)}\right]$

$\mathcal{L}^{-1}[Y(s)]=\frac{-1}{16}e^t+\frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{(s-1)^2}\right]+\frac{1}{16}e^{-3t}+\frac{1}{4}e^{-3t}+\frac{3}{4}e^t$

$\because \mathcal{L}^{-1}\left[\frac{1}{(s-1)^2}\right]=e^t\mathcal{L}\left(\frac{1}{s^2}\right)=e^t.t$

$\therefore y(t)=\frac{11}{16}e^t+\frac{1}{4}te^t+\frac{5}{16}e^{-3t}$

⑧ $\frac{d^2y}{dt^2}+y=4t\sin t,\ y(0)=0=y'(0)$

$\because \mathcal{L}[t^n f(t)]=(-1)^n\frac{d^n}{ds^n}[F(s)]$ where $F(s)=\mathcal{L}[f(t)]$

$\therefore s^2Y(s)+sy(0)-y'(0)+Y(s)=4\mathcal{L}(t\sin t)=4(-1)\frac{d}{ds}\left(\frac{1}{s^2+1}\right)$

$s^2Y(s)+Y(s)=-4(-1)(s^2+1)^{-2}(2s)$

$(s^2+1)Y(s)=\frac{8s}{(s^2+1)^2}\Longrightarrow Y(s)=\frac{8s}{(s^2+1)^3}$

$y(t)=\mathcal{L}^{-1}[Y(s)]=\mathcal{L}^{-1}\left[\frac{8s}{(s^2+1)^2}\cdot\frac{1}{s^2+1}\right]=f(t)*g(t)\rightarrow$ ① where

$\mathcal{L}^{-1}\left[\frac{8s}{(s^2+1)^2}\right]=f(t),\ \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right)=g(t)$

$f(t)=\mathcal{L}^{-1}\left\{\frac{8s}{(s^2+1)^2}\right\}=-4\mathcal{L}^{-1}\left\{\frac{-2s}{(s^2+1)^2}\right\}=-4\mathcal{L}^{-1}\left[\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right]$

$f(t)=-4(-1)^1\cdot t^1\sin t=4t\sin t\quad \because \mathcal{L}^{-1}[F^{(n)}(s)]=(-1)^n t^n f(t)$

$g(t)=\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right)=\sin t$

$\because y(t)=f(t)*g(t)$ where $f(t)=4t\sin t$ and $g(t)=\sin t$

$=\int_0^t f(u)\cdot g(t-u)\,du=\int_0^t 4u\sin u\cdot\sin(t-u)\,du$

$=\frac{4}{-2}\int_0^t(-2\sin(t-u)\sin u\,du\quad \because -2\sin\left(\frac{\alpha+\beta}{2}\right)\cdot\sin\left(\frac{\alpha-\beta}{2}\right)=\cos\alpha-\cos\beta$

$=-2\int_0^t u\{\cos t-\cos(t-2u)\}du=2\int_0^t u\{\cos(t-2u)-\cos t\}du$

$=2\int_0^t u\cos(2u-t)-2\cos t\int_0^t u\,du$

$=2\left\{u\cdot\frac{\sin(2u-t)}{2}\Big|_0^t-\int_0^t\frac{\sin(2u-t)}{2}\cdot 1\,du\right\}-2\cos t\,\frac{u^2}{2}\Big|_0^t$

$$y(t) = t\sin t - \int_0^t \sin(2u-t)\,du - 2\cos t \cdot \frac{t^2}{2}$$

$$= t\sin t + \frac{\cos(2u-t)}{2}\Big|_0^t - t^2\cos t = t\sin t + \frac{1}{2}(\cos t - \cos t) - t^2\cos t$$

$$y(t) = t\sin t - t^2\cos t$$

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(9) $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} = 20e^{-t}\cos t$, $y(0) = 0 = y'(0)$

$$\mathcal{L}\{y''(t)\} - 2\mathcal{L}\{y'(t)\} = 20\,\mathcal{L}\{e^{-t}\cos t\}$$

$$s^2Y(s) - sy(0) - y'(0) - 2\{sY(s) - y(0)\} = 20\frac{s+1}{(s+1)^2+1}$$

$$s^2Y(s) - s(0) - 0 - 2sY(s) + 0 = 20\frac{s+1}{(s+1)^2+1} \Rightarrow (s^2-2s)Y(s) = 20\frac{s+1}{s^2+2s+2}$$

$$Y(s) = 20\frac{s+1}{(s^2+2s)(s^2+2s+2)} = \frac{20s+20}{s(s-2)(s^2+2s+2)} = \frac{-5}{s} + \frac{3}{s-2} + \frac{2s-2}{s^2+2s+2}$$

$$\mathcal{L}^{-1}\{Y(s)\} = -5\mathcal{L}^{-1}\left(\frac{1}{s}\right) + 3\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) + \mathcal{L}^{-1}\left\{\frac{2(s-1)}{s^2+2s+2}\right\}$$

$$y(t) = -5(1) + 3e^{2t} + \mathcal{L}^{-1}\left\{\frac{2[(s+1)-1]-2}{(s+1)^2+1}\right\} = -5 + 3e^{2t} + \mathcal{L}^{-1}\left\{\frac{2(s+1)-4}{(s+1)^2+1}\right\}$$

$$= -5 + 3e^{2t} + 2\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} - 4\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

$$= -5 + 3e^{2t} + 2e^{-t}\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) - 4e^{-t}\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right)$$

$$y(t) = -5 + 3e^{2t} + 2e^{-t}\cos t - 4e^{-t}\sin t.$$

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(10) $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} - 4y = 12e^{-3t}\sin 2t$, $y(0) = 1$, $y'(0) = 0$

$$\mathcal{L}\{y''(t)\} - 3\mathcal{L}\{y'(t)\} - 4\mathcal{L}\{y(t)\} = 12\,\mathcal{L}\{e^{-3t}\sin 2t\}$$

$$s^2Y(s) - sy(0) - y'(0) - 3\{sY(s) - y(0)\} - 4Y(s) = 12\cdot\frac{2}{(s+3)^2+(2)^2}$$

$$s^2Y(s) - s - 0 - 3\{sY(s) - 1\} - 4Y(s) = \frac{24}{(s^2+6s+13)}$$

$$(s^2-3s-4)Y(s) - s + 3 = \frac{24}{s^2+6s+13} \Rightarrow (s^2-3s-4)Y(s) = s - 3 + \frac{24}{s^2+6s+13}$$

$$Y(s) = \frac{s-3}{s^2-3s-4} + \frac{24}{(s^2-3s-4)(s^2+6s+13)}$$

$$\therefore Y(s) = \left[\frac{1}{5(s-4)} + \frac{4}{5(s+1)}\right] + \left[\frac{24}{265(s-4)} - \frac{3}{5(s+1)} + \frac{27s+11}{53(s^2+6s+13)}\right]$$

$$= \frac{77}{265(s-4)} + \frac{1}{5(s+1)} + \frac{1}{53}\cdot\frac{27(s+3)+111-81}{(s+3)^2+(2)^2}$$

$$= \frac{77}{265}\cdot\frac{1}{s-4} + \frac{1}{5}\cdot\frac{1}{s+1} + \frac{1}{53}\cdot\frac{27(s+3)+30}{(s+3)^2+(2)^2}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \frac{77}{265}\mathcal{L}^{-1}\left(\frac{1}{s-4}\right) + \frac{1}{5}\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) + \frac{27}{53}\mathcal{L}^{-1}\left\{\frac{s+3}{(s+3)^2+(2)^2}\right\} + \frac{15}{53}\mathcal{L}^{-1}\left\{\frac{2}{(s+3)^2+4}\right\}$$

$$y(t) = \frac{77}{265}e^{4t} + \frac{1}{5}e^{-t} + \frac{27}{53}e^{-3t}\cos 2t + \frac{15}{53}e^{-3t}\sin 2t.$$

(11) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = u_3(t)$, $y(0)=0$, $y'(0)=1$

$$\mathcal{L}\{y''(t)\} - 4\mathcal{L}\{y'(t)\} + 4\mathcal{L}\{y(t)\} = \mathcal{L}\{u_3(t)\}$$

$$s^2Y(s) - sy(0) - y'(0) - 4\{sY(s) - y(0)\} + 4Y(s) = \frac{e^{-3s}}{s}$$

$$s^2Y(s) - 0 - 1 - 4sY(s) - 0 + 4Y(s) = e^{-3s}/3$$

$$(s^2-4s+4)Y(s) = 1 + \frac{e^{-3s}}{s} \Rightarrow (s-2)^2Y(s) = 1 + \frac{e^{-3s}}{s}$$

$$Y(s) = \frac{1}{(s-2)^2} + \frac{e^{-3s}}{s(s-2)^2} = -\frac{d}{ds}\left(\frac{1}{s-2}\right) + e^{-3s}\left[\frac{1}{s(s-2)^2}\right] -$$

$$\therefore Y(s) = -\frac{d}{ds}\left(\frac{1}{s-2}\right) + \frac{1}{4}\cdot\frac{e^{-3s}}{s} - \frac{1}{4}e^{-3s}\frac{1}{s-2} + \frac{1}{2}e^{-3s}\cdot\frac{1}{(s-2)^2}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{(-1)'\frac{d}{ds}\left(\frac{1}{s-2}\right)\right\} + \frac{1}{4}\mathcal{L}^{-1}\left(\frac{e^{-3s}}{s}\right) - \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s-2}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{(s-2)^2}\right\}$$

$$y(t) = te^{2t} + \frac{1}{4}u_3(t)\mathcal{L}^{-1}\left(\frac{1}{s}\right) - \frac{1}{4}u_3(t)\mathcal{L}^{-1}\left(\frac{1}{s-2}\right)\Big|_{t=t-3} + \frac{1}{2}u_3(t)\mathcal{L}^{-1}\frac{1}{(s-2)^2}\Big|_{t-3}$$

$$y(t) = te^{2t} + \frac{1}{4}u_3(t)(1) - \frac{1}{4}u_3(t)e^{2(t-3)} + \frac{1}{2}u_3(t)(t-3)e^{2(t-3)}$$

$$= te^{2t} + \frac{1}{4}u_3(t)\left[1 - e^{2(t-3)} + 2(t-3)e^{2(t-3)}\right]$$ is required Solution.

(12) $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = f(t)$ where $f(t) = \begin{cases} 0 & \text{if } 0<t<2 \\ 3 & \text{if } 2<t<5 \\ 0 & \text{if } t>5 \end{cases}$, $y(0) = 0 = y'(0)$

$$f(t) = \begin{cases} 0-0 & 0<t<2 \\ 3-0 & 2<t<5 \\ 3-3 & t>5 \end{cases}$$

$$= \begin{cases} 0 & 0<t<2 \\ 3 & t>2 \end{cases} - \begin{cases} 0 & 0<t<5 \\ 3 & t>5 \end{cases}$$

$$= 3\begin{cases} 0 & 0<t<2 \\ 1 & t>2 \end{cases} - 3\begin{cases} 0 & 0<t<5 \\ 1 & t>5 \end{cases}$$

$$f(t) = 3u_2(t) - 3u_5(t)$$

$$\therefore \{s^2Y(s) - s\,y(0) - y'(0)\} - 3\{sY(s) - y(0)\} + 2Y(s) = 3\mathcal{L}\{u_2(t)\} - 3\mathcal{L}\{u_5(t)\}$$

$$(s^2-3s+2)Y(s) = \frac{e^{-2s}}{s} - \frac{e^{-5s}}{s}$$

$$Y(s) = 3(e^{-2s} - e^{-5s})\left\{\frac{1}{2s} - \frac{1}{s-1} + \frac{1}{2(s-2)}\right\}$$

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \frac{3}{2}\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s}\right) - 3\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s-1}\right) + \frac{3}{2}\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s-2}\right) -$$

$$\frac{3}{2}\mathcal{L}^{-1}\left(\frac{e^{-5s}}{s}\right) + 3\mathcal{L}^{-1}\left(\frac{e^{-5s}}{s-1}\right) - \frac{3}{2}\mathcal{L}^{-1}\left(\frac{e^{-5s}}{s-2}\right)$$

$$y(t) = \frac{3}{2}u_2(t) - 3u_2(t)\left[\mathcal{L}^{-1}\left(\frac{1}{s-1}\right)\right]_{t=t-2} + \frac{3}{2}u_2(t)\left[\mathcal{L}^{-1}\left(\frac{1}{s-2}\right)\right]_{t=t-2} -$$

$$\frac{3}{2}u_5(t)\mathcal{L}^{-1}\left(\frac{1}{s}\right) + 3u_5(t)\left[\mathcal{L}^{-1}\left(\frac{1}{s-1}\right)\right]_{t=t-5} - \frac{3}{2}u_5(t)\left[\mathcal{L}^{-1}\left(\frac{1}{s-2}\right)\right]_{t=t-5}$$

$$y(t) = \frac{3}{2}u_2(t) - 3u_2(t)e^{t}\Big|_{t-2} + \frac{3}{2}u_2(t)e^{2t}\Big|_{t-2} - \frac{3}{2}u_5(t)(1) + 3u_5(t)e^{t}\Big|_{t-5} - \frac{3}{2}u_5(t)e^{2t}\Big|_{t-5}$$

$$= \frac{3}{2}u_2(t) - 3u_2(t)e^{(t-2)} + \frac{3}{2}u_2(t)e^{2(t-2)} - \frac{3}{2}u_5(t) + 3u_5(t)e^{t-5} - \frac{3}{2}u_5(t)e^{2(t-5)}$$

$$= \frac{3}{2}u_2(t)\left[1 - 2e^{t-2} + e^{2(t-2)}\right] - \frac{3}{2}u_5(t)\left[1 - 2e^{t-5} + e^{2(t-5)}\right]$$

(13) $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 2(t-3)u_3(t)$, $y(0) = 2$, $y'(0) = 1$

$$s^2Y(s) - sy(0) - y'(0) + 2[sY(s) - y(0)] + Y(s) = 2 \cdot \frac{e^{-3s}}{s^2}$$

$$s^2Y(s) - s(2) - 1 + 2sY(s) - 2(2) + Y(s) = 2e^{-3s}/s^2$$

$$(s^2+2s+1)Y(s) = 2s - 5 = \frac{2e^{-3s}}{s^2} \Rightarrow (s+1)^2Y(s) = 2s+5 + \frac{2e^{-3s}}{s^2}$$

$$Y(s) = \frac{2s+5}{(s+1)^2} + e^{-3s}\left[\frac{2}{s^2(s+1)^2}\right]$$ using partial fraction, we get

$$= \frac{2}{s+1} + \frac{3}{(s+1)^2} + e^{-3s}\left[-\frac{4}{s} + \frac{2}{s^2} + \frac{4}{s+1} + \frac{2}{(s+1)^2}\right]$$

$$\mathcal{L}^{-1}\{Y(s)\} = 2\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) + 3\mathcal{L}^{-1}\left(\frac{1}{(s+1)^2}\right) - 4\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s}\right\} + 2\mathcal{L}^{-1}\left(\frac{e^{-3s}}{s^2}\right) + 4\mathcal{L}^{-1}\left(\frac{e^{-3s}}{s+1}\right) + 2\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{(s+1)^2}\right\}$$

$$= 2e^{-t} + 3e^{-t}\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) - 4u_3(t)\mathcal{L}^{-1}(1/s) + 2u_3(t)\mathcal{L}^{-1}\left(\frac{1}{s^2}\right)\Big|_{t=t-3} + 4u_3(t)\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) + 2u_3(t)\mathcal{L}^{-1}\left[\frac{1}{(s+1)^2}\right]_{t=t-3}$$

$$= 2e^{-t} + 3e^{-t}(t) - 4u_3(t)(1) + 2u_3(t)\,t\big|_{t-3} + 4u_3(t)(1) + 2u_3(t)e^{-t}t\big|_{t-3}$$

$$= 2e^{-t} + 3te^{-t} + u_3(t)\left[-4 + 2(t-3) + 4e^{-(t-3)} + 2e^{-(t-3)}(t-3)\right]$$

(14) $\frac{d^2y}{dt^2} + y = \begin{cases} \cos t & \text{if } 0 \le t < \frac{\pi}{2} \\ 0 & \text{if } \frac{\pi}{2} < t < \infty \end{cases}$, $y(0) = 3$, $y'(0) = -1$

$$f(t) = \begin{cases} \cos t & 0 \le t < \frac{\pi}{2} \\ 0 & t > \frac{\pi}{2} \end{cases}$$

$$= \begin{cases} \cos t + 0 & 0 \le t < \frac{\pi}{2} \\ \cos t - \cos t & t > \frac{\pi}{2} \end{cases}$$

$$= \cos t + \begin{cases} 0 & 0 \le t < \frac{\pi}{2} \\ -\cos t & t > \frac{\pi}{2} \end{cases}$$

$$= \cos t + \begin{cases} 0 & 0 \le t < \frac{\pi}{2} \\ \sin(t - \frac{\pi}{2}) & t > \frac{\pi}{2} \end{cases}$$

$$= \cos t + \sin(t - \tfrac{\pi}{2})\begin{cases} 0 & 0 \le t < \pi/2 \\ 1 & t > \pi/2 \end{cases}$$

$$f(t) = \cos t - u_{\frac{\pi}{2}}(t)\sin(t - \tfrac{\pi}{2})$$

$$\therefore \frac{d^2y}{dt^2} + y = \cos t + u_{\frac{\pi}{2}}(t) \cdot \sin(t - \tfrac{\pi}{2})$$

$$\Rightarrow s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{s}{s^2+1} + \frac{e^{-\frac{\pi}{2}s}}{s^2+1}$$

$$s^2Y(s) - s(3) + 1 + Y(s) = \quad ''$$

$$(s^2+1)Y(s) = \frac{s}{s^2+1} + \frac{e^{-\frac{\pi}{2}s}}{s^2+1} + 3s - 1$$

$$Y(s) = \frac{3s}{s^2+1} - \frac{1}{s^2+1} + \frac{s}{(s^2+1)^2} + \frac{e^{-\frac{\pi}{2}s}}{(s^2+1)^2}$$

$$y(t) = \mathcal{L}^{-1}[Y(s)] = 3\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) - \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) + \mathcal{L}^{-1}\left[\frac{s}{(s^2+1)^2}\right] + \mathcal{L}^{-1}\left[\frac{e^{-\frac{\pi}{2}s}}{(s^2+1)^2}\right]$$

$$y(t) = 3\cos t - \sin t - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{-2s}{(s^2+1)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{e^{-\frac{\pi}{2}s}}{(s^2+1)^2}\right\} \to (1)$$

$$\mathcal{L}^{-1}\left\{\frac{-2s}{(s^2+1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{d}{ds}(s^2+1)^{-1}\right\} = (-1)^1 t^1 \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = -t\sin t$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-\frac{\pi}{2}s}}{(s^2+1)^2}\right\} = u_{\frac{\pi}{2}}(t)\,\mathcal{L}^{-1}\left[\frac{1}{(s^2+1)^2}\right]\Bigg|_{t=t-\frac{\pi}{2}} = \frac{1}{2}.u_{\frac{\pi}{2}}(t)\,\mathcal{L}^{-1}\left\{\frac{2(1)^3}{(s^2+1)^2}\right\}\Bigg|_{t=t-\frac{\pi}{2}}$$

$$= \frac{1}{2}u_{\frac{\pi}{2}}(t)\left(\sin t - t\cos t\right)\Big|_{t=t-\frac{\pi}{2}} = \frac{1}{2}u_{\frac{\pi}{2}}(t)\left[\sin\left(t-\frac{\pi}{2}\right) - \left(t-\frac{\pi}{2}\right)\cos\left(t-\frac{\pi}{2}\right)\right]$$

from (1)

$$y(t) = 3\cos t - \sin t - \frac{1}{2}(-t\sin t) + \frac{1}{2}u_{\frac{\pi}{2}}(t)\left[\sin\left(t-\frac{\pi}{2}\right) - \left(t-\frac{\pi}{2}\right)\cos\left(t-\frac{\pi}{2}\right)\right]$$

$$= 3\cos t - \sin t + \frac{1}{2}t\sin t - \frac{1}{2}u_{\frac{\pi}{2}}(t)\left[\sin\left(\frac{\pi}{2}-t\right) + \left(t-\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}-t\right)\right]$$

$$= 3\cos t - \sin t + \frac{1}{2}t\sin t - \frac{1}{2}u_{\frac{\pi}{2}}(t)\left[\cos t + \left(t-\frac{\pi}{2}\right)\sin t\right]$$

(15) $\frac{d^2y}{dt^2} + 4y = \sin t - u_{2\pi}(t)\sin(t-2\pi)$, $\quad y(0) = 0 = y'(0)$

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = \frac{1}{s^2+1} - e^{-2\pi s}\mathcal{L}(\sin t)$$

$$(s^2+4)Y(s) - 0 - 0 = \frac{1}{s^2+1} - \frac{e^{-2\pi s}}{s^2+1} \Rightarrow Y(s) = \frac{1}{(s^2+1)(s^2+4)} - \frac{e^{-2\pi s}}{(s^2+4)(s^2+1)}$$

$$Y(s) = \frac{1}{3(s^2+1)} - \frac{1}{3(s^2+4)} - \frac{e^{-2\pi s}}{3(s^2+1)} + \frac{e^{-2\pi s}}{3(s^2+4)}$$

$$\mathcal{L}^{-1}[Y(s)] = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \frac{1}{6}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{s^2+1}\right\} + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{s^2+4}\right\}$$

$$y(t) = \frac{1}{3}\sin t - \frac{1}{6}\sin 2t - \frac{1}{3}u_{2\pi}(t)\,\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right]\Bigg|_{t=t-2\pi} + \frac{1}{6}u_{2\pi}(t)\,\mathcal{L}^{-1}\frac{2}{s^2+4}\Bigg|_{t=t-2\pi}$$

$$= \frac{1}{3}\sin t - \frac{1}{6}\sin 2t - \frac{1}{3}u_{2\pi}(t)\sin t\Big|_{t=t-2\pi} + \frac{1}{6}u_{2\pi}(t)\sin 2t\Big|_{t=t-2\pi}$$

$$= \frac{1}{3}\sin t - \frac{1}{6}\sin 2t - \frac{1}{3}u_{2\pi}(t)\sin(t-2\pi) + \frac{1}{6}u_{2\pi}(t)\sin 2(t-2\pi)$$

$$= \frac{1}{3}\sin t - \frac{1}{6}\sin 2t - \frac{1}{3}u_{2\pi}(t)\sin t + \frac{1}{6}u_{2\pi}(t)\sin 2t$$

$$= \frac{1}{6}(2\sin t - \sin 2t) - \frac{1}{6}u_{2\pi}(t)(2\sin t - \sin 2t)$$

$$= \frac{1}{6}(2\sin t - \sin 2t)\left(1 - u_{2\pi}(t)\right)$$

(16) $\frac{d^3y}{dt^3} - 4\frac{d^2y}{dt^2} + \frac{dy}{dt} + 6y = te^t$, $\quad y(0) = 0,\ y'(0) = 0,\ y''(0) = 1$

$$\left[s^3Y(s) - s^2y(0) - sy'(0) - y''(0)\right] - 4\left[s^2Y(s) - sy(0) - y'(0)\right] + \left[sY(s) - y(0)\right] + 6Y(s) = (-1)^1\frac{d}{ds}\left(\frac{1}{s-1}\right)$$

$$\left[s^3Y(s) - 0 - 0 - 1\right] - 4\left[s^2Y(s) - 0 - 0\right] + sY(s) + 0 + 6Y(s) = -\frac{-1}{(s-1)^2}$$

$$(s^3 - 4s^2 + s + 6)Y(s) - 1 = \frac{1}{(s-1)^2} \Rightarrow (s^3 - 4s^2 + s + 6)Y(s) = 1 + \frac{1}{(s-1)^2} \to (1)$$

Consider s^3-4s^2+s+6.
$s=-1$ is one of its root so $s+1$ is its one factor.
By Synthetic division;

-1	1	-4	1	6
	$\downarrow$	-1	5	-6
	1	-5	6	0

$\because s^2-5s+6=0$

$\therefore (s+1)(s^2-5s+6)=0$

$$\therefore (s+1)(s^2-5s+6)Y(s)=\frac{(s-1)^2+1}{(s-1)^2}$$

$$Y(s)=\frac{s^2-2s+2}{(s-1)^2(s+1)(s^2-5s+6)}$$

By partial fraction;

$$Y(s)=\frac{1}{4(s-1)}+\frac{1}{4(s-1)^2}+\frac{5}{48(s+1)}-\frac{2}{3(s-2)}+\frac{5}{16(s-3)}$$

$$y(t)=\tfrac{1}{4}e^t+\tfrac{1}{4}te^t+\tfrac{5}{48}e^{-t}-\tfrac{2}{3}e^{2t}+\tfrac{5}{16}e^{3t}$$

(17) $\dfrac{d^3y}{dt^3}-5\dfrac{d^2y}{dt^2}+7\dfrac{dy}{dt}-3y=20\sin t;\quad y(0)=0=y'(0),\ y''(0)=-2$

$$[s^3Y(s)-s^2y(0)-sy'(0)-y''(0)]-5[s^2Y(s)-sy(0)-y'(0)]+7[sY(s)-y(0)]-3Y(s)=20\cdot\frac{1}{s^2+1}$$

$$[s^3Y(s)-0-0+2]-5[s^2Y(s)-0-0]+7[sY(s)-0]-3Y(s)=\frac{20}{s^2+1}$$

$$(s^3-5s^2+7s-3)Y(s)=-2+\frac{20}{s^2+1};$$

Solving by Synthetic div.

1	1	-5	7	-3
	$\downarrow$	1	-4	3
1	1	-4	3	0
	$\downarrow$	1	-3	
	1	-3	0	

$$\therefore (s-1)(s-1)(s-3)Y(s)=\frac{-2(s^2+1)+20}{s^2+1}$$

$$Y(s)=\frac{-2(s^2+1-10)}{(s-1)^2(s-3)(s^2+1)}=\frac{-2(s^2-9)}{''}$$

$$=\frac{-2(s-3)(s+3)}{(s-1)^2(s-3)(s^2+1)}=\frac{-2s-6}{(s-1)^2(s^2+1)}\rightarrow(1)$$

$$=\frac{3}{s-1}-\frac{4}{(s-1)^2}-\frac{3s+1}{s^2+1}=3\mathcal{L}^{-1}\left(\frac{1}{s-1}\right)-4\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}+\mathcal{L}^{-1}\left\{\frac{-3s+1}{s^2+1}\right\}$$

$$y(t)=\mathcal{L}^{-1}[Y(s)]=3\mathcal{L}^{-1}\left(\frac{1}{s-1}\right)-4\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}-3\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$y(t)=3e^t-4te^t-3\cos t+\sin t.$$

(18) $\left(\dfrac{d^2}{dt^2}+6\dfrac{d}{dt}+7\right)^2y=0,\quad y(0)=0=y'(0)=y''(0),\ y'''(0)=4\sqrt{2}$

$$\frac{d^4y}{dt^4}+36\frac{d^2y}{dt^2}+49y+12\frac{d^3y}{dt^3}+84\frac{dy}{dt}+14\frac{d^2y}{dt^2}=0$$

$$\frac{d^4y}{dt^4}+12\frac{d^3y}{dt^3}+50\frac{d^2y}{dt^2}+84\frac{dy}{dt}+49y=0$$

$$[s^4Y(s)-s^3y(0)-s^2y'(0)-sy''(0)-y'''(0)]+12[s^3Y(s)-s^2y(0)-sy'(0)-y''(0)]+50[s^2Y(s)-sy(0)-y'(0)]+84[sY(s)-y(0)]+49Y(s)=0$$

$s^4 Y(s) - 4\sqrt{2} + 12s^3 Y(s) + 50s^2 Y(s) + 84s Y(s) + 49 Y(s) = 0$

$[s^4 + 12s^3 + 50s^2 + 84s + 49] Y(s) = 4\sqrt{2}$

$[(s^2)^2 + 36s^2 + 49 + 2(s^2)(7) + 2(s^2)(6s) + 2(6s)(7)] Y(s) = 4\sqrt{2}$

$(s^2+6s+7)^2 Y(s) = 4\sqrt{2} \Rightarrow Y(s) = \dfrac{4\sqrt{2}}{(s^2+6s+7)^2}$

$\mathcal{L}^{-1}\{Y(s)\} = 4\sqrt{2}\,\mathcal{L}^{-1}\left\{\dfrac{1}{(s^2+6s+7)^2}\right\} = 4\sqrt{2}\,\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+6s+7} \cdot \dfrac{1}{s^2+6s+7}\right\}$

$y(t) = 4\sqrt{2}\,[f(t) * g(t)] \rightarrow (1)$ where

$f(t) = \mathcal{L}^{-1}\left(\dfrac{1}{s^2+6s+7}\right) = \mathcal{L}^{-1}\left(\dfrac{1}{s^2+6s+9-9+7}\right) = \mathcal{L}^{-1}\left(\dfrac{1}{(s+3)^2-(\sqrt{2})^2}\right)$

$= e^{-3t}\mathcal{L}^{-1}\left(\dfrac{1}{s^2-(\sqrt{2})^2}\right) = \dfrac{1}{\sqrt{2}} e^{-3t}\mathcal{L}^{-1}\left\{\dfrac{\sqrt{2}}{s^2-(\sqrt{2})^2}\right\} = \dfrac{1}{\sqrt{2}} e^{-3t}\sinh\sqrt{2}t$

and $g(t) = \dfrac{1}{\sqrt{2}} e^{-3t}\sinh\sqrt{2}t$

from (1) $y(t) = 4\sqrt{2}\,(f*g)$

$= 4\sqrt{2}\displaystyle\int_0^t \frac{1}{\sqrt{2}} e^{-3u}\sinh\sqrt{2}u \cdot \frac{1}{\sqrt{2}} e^{-3(t-u)}\sinh[\sqrt{2}(t-u)]\,du$

$= 2\sqrt{2}\, e^{-3t}\displaystyle\int_0^t e^{-3u+3u}\sinh\sqrt{2}u\,\sinh\sqrt{2}(t-u)\,du$

$= 2\sqrt{2}\, e^{-3t}\displaystyle\int_0^t \sinh\sqrt{2}(t-u)\cdot\sinh\sqrt{2}u\,du$

$= \dfrac{2\sqrt{2}}{-2} e^{-3t}\displaystyle\int_0^t -2\sinh[\sqrt{2}(t-u)]\sinh(\sqrt{2}u)\,du$

$= -\sqrt{2}e^{-3t}\displaystyle\int_0^t [\cosh(\sqrt{2}t-\sqrt{2}u+\sqrt{2}u) - \cosh(\sqrt{2}t-\sqrt{2}u-\sqrt{2}u)]\,du$

$= -\sqrt{2}e^{-3t}\displaystyle\int_0^t [\cosh(\sqrt{2}t) - \cosh(\sqrt{2}t-2\sqrt{2}u)]\,du$

$= -\sqrt{2}e^{-3t}\displaystyle\int_0^t [-\cosh(\sqrt{2}t) + \cosh(2\sqrt{2}u-\sqrt{2}t)]\,du$

$= +\sqrt{2}e^{-3t}\cosh(\sqrt{2}t)\displaystyle\int_0^t du - \sqrt{2}e^{-3t}\int_0^t \cosh(2\sqrt{2}u-\sqrt{2}t)\,du$

$= \sqrt{2}e^{-3t}\cosh\sqrt{2}t\; u\Big|_0^t - \sqrt{2}e^{-3t}\dfrac{\sinh(2\sqrt{2}u-\sqrt{2}t)}{2\sqrt{2}}\Big|_0^t$

$= \sqrt{2}e^{-3t}t\cosh\sqrt{2}t - \dfrac{1}{2}e^{-3t}\{\sinh(2\sqrt{2}t-\sqrt{2}t) - \sinh(-\sqrt{2}t)\}$

$= \sqrt{2}e^{-3t}t\cosh\sqrt{2}t - \dfrac{1}{2}e^{-3t}(\sinh\sqrt{2}t + \sinh\sqrt{2}t)$

$= e^{-3t}\sqrt{2}t\cosh\sqrt{2}t - e^{-3t}\sinh\sqrt{2}t = e^{-3t}\{\sqrt{2}t\cosh\sqrt{2}t - \sinh\sqrt{2}t\}$

(19) $\dfrac{d^4y}{dt^4} + 5\dfrac{d^2y}{dt^2} + 4y = 1 - u_\pi(t)$, $y(0) = 0 = y'(0) = y''(0) = y'''(0)$

$\mathcal{L}[y^{(4)}(t)] + 5\mathcal{L}\{y''(t)] + 4\mathcal{L}\{y(t)] = \mathcal{L}\{1 - u_\pi(t)\}$

$\Rightarrow [s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)] + 5\{s^2Y(s) - sy(0) - y'(0)\} +$

$+4Y(s) = \frac{1}{s} - \frac{e^{-\pi s}}{s} \Rightarrow (s^4+5s^2+4)Y(s) = \frac{1}{s} - \frac{e^{-\pi s}}{s}$

$Y(s) = (1-e^{-\pi s})\left[\frac{1}{s(s^4+5s^2+4)}\right] = (1-e^{-\pi s})\left[\frac{1}{4s} - \frac{s}{3(s^2+1)} + \frac{s}{12(s^2+4)}\right]$

$= \left(\frac{1}{4s} - \frac{s}{3(s^2+1)} + \frac{s}{12(s^2+4)}\right) - \frac{e^{-\pi s}}{4s} + e^{-\pi s}\cdot\frac{s}{3(s^2+1)} - \frac{1}{12}\cdot\frac{e^{-\pi s}\, s}{s^2+4}$

$y(t) = \frac{1}{4}(1) - \frac{1}{3}\cos t + \frac{1}{12}\cos 2t - \frac{1}{4}u_\pi(t)\,\mathcal{L}^{-1}\left(\frac{1}{s}\right) -$

$\frac{1}{3}u_\pi(t)\left[\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right)\right]_{t=t-\pi} - \frac{1}{12}u_\pi(t)\left[\mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right)\right]_{t=t-\pi}$

$= \frac{1}{4} - \frac{1}{3}\cos t + \frac{1}{12}\cos 2t - \frac{1}{4}u_\pi(t)(1) - \frac{1}{3}u_\pi(t)\cos(t-\pi) - \frac{1}{12}u_\pi(t)\cos 2(t-\pi)$

$= \frac{1}{4} - \frac{1}{3}\cos t + \frac{1}{12}\cos 2t - \frac{1}{4}u_\pi(t) - \frac{1}{3}u_\pi(t)\cos(\pi-t) - \frac{1}{12}u_\pi(t)\cos 2(\pi-t)$

$= \frac{1}{4} - \frac{1}{3}\cos t + \frac{1}{12}\cos 2t - \frac{1}{4}u_\pi(t) + \frac{1}{3}u_\pi(t)\cos t - \frac{1}{12}u_\pi(t)\cos 2t$

$= \frac{1}{4}\{1-u_\pi(t)\} - \frac{1}{3}\{1-u_\pi(t)\}\cos t + \frac{1}{12}\{1-u_\pi(t)\}\cos 2t$

$= \left(\frac{1}{4} - \frac{1}{3}\cos t + \frac{1}{12}\cos 2t\right)\{1-u_\pi(t)\}$

(20) $t\frac{d^2y}{dt^2} + (t-1)\frac{dy}{dt} - y = 0$, $y(0) = 5$, $y(\infty) = 0$

$\mathcal{L}\{t\,y''(t)\} + \mathcal{L}\{t\,y'(t)\} - \mathcal{L}\{y'(t)\} - \mathcal{L}\{y(t)\} = 0$

we know $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\{F(s)\}$ where $F(s) = \mathcal{L}\{f(t)\}$

$(-1)^1\frac{d}{ds}\{\mathcal{L}y''(t)\} + (-1)^1\frac{d}{ds}\mathcal{L}\{y'(t)\} - \{sY(s) - y(0)\} - Y(s) = 0$

$-\frac{d}{ds}[s^2Y(s) - sy(0) - y'(0)] - \frac{d}{ds}[sY(s) - y(0)\} - sY(s) + y(0) - Y(s) = 0$

$-\frac{d}{ds}[s^2Y(s) - 5s\} - \frac{d}{ds}\{sY(s) - 5\} - sY(s) + 5 - Y(s) = 0$

$-2sY(s) - s^2Y'(s) + 5 - Y(s) - sY'(s) - sY(s) + 5 - Y(s) = 0$

$-(s^2+s)Y'(s) - (3s+2)Y(s) + 10 = 0$

$(s^2+s)Y'(s) + (3s+2)Y(s) = 10$

$Y'(s) + \frac{3s+2}{s^2+s}Y(s) = \frac{10}{s^2+s} \Rightarrow Y'(s) + \left[\frac{3s+2}{s(s+1)}\right]Y(s) = \frac{10}{s(s+1)} \rightarrow (1)$

It is linear D.E in $Y'(s)$

Integ. factor $= I.F = \exp\int\frac{3s+2}{s(s+1)}ds = e^{\int\left(\frac{2}{s}+\frac{1}{s+1}\right)ds} = e^{[2\ln s + \ln s+1]}$

$= e^{\ln s^2 + \ln s+1} = e^{\ln[s^2(s+1)]} = s^2(s+1)$

Eq (1) $\Rightarrow s^2(s+1)Y'(s) + s(3s+2)Y(s) = 10s$

$(s^3+s^2)Y'(s) + (3s^2+2s)Y(s) = 10s$

$\frac{d}{ds}\left[(s^3+s^2)Y(s)\right] = 10s$

$$\int \frac{d}{ds}\left[(s^3+s^2)Y(s)\right]ds = 10\int s\,ds \Rightarrow (s^3+s^2)Y(s) = \frac{10s^2}{2}+c$$

$$(s^3+s^2)Y(s) = 5s^2+c \Rightarrow Y(s) = \frac{5}{s+1} + \frac{c}{s^2(s+1)}$$

$$Y(s) = \frac{5}{s+1} + \left(-\frac{c}{s}+\frac{c}{s^2}+\frac{c}{s+1}\right) = \frac{5+c}{s+1} - c\left(\frac{1}{s}-\frac{1}{s^2}\right)$$

$$y(t) = (c+5)e^{-t} - c(1-t)$$

Now $y(\infty)=0 \Rightarrow y=0$ as $t\to\infty$

$$0 = (c+5)e^{-\infty} - c(1-t) \Rightarrow 0 = 0 - c(1-t) \Rightarrow c=0$$

$\therefore y(t) = (0+5)e^{-t} - 0 \Rightarrow y(t) = 5e^{-t}$ is required sol.

(21) $\frac{dx}{dt} - x - 3y = 0$, $x(0)=2$

$\frac{dy}{dt} - 5x - 3y = 0$, $y(0)=1$

Taking laplace on bothsides

$$\mathcal{L}\left\{\frac{dx}{dt}\right\} - \mathcal{L}\{x(t)\} - 3\mathcal{L}\{y(t)\} = 0$$

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} - 5\mathcal{L}\{x(t)\} - 3\mathcal{L}\{y(t)\} = 0$$

Using initial conds, we get

$$sX(s) - x(0) - X(s) - 3Y(s) = 0$$

$$sY(s) - y(0) - 5X(s) - 3Y(s) = 0$$

$$(s-1)X(s) - 3Y(s) - 2 = 0 \rightarrow (1)$$

$$-5X(s) + (s-3)Y(s) - 1 = 0 \rightarrow (2)$$

Solving the equations;

$$\frac{X(s)}{(-3)(-1)-(-2)(s-3)} = \frac{-Y(s)}{(-1)(s-1)-(-2)(-5)} = \frac{1}{(s-1)(s-3)-15}$$

$$\frac{X(s)}{2s-3} = \frac{Y(s)}{s+9} = \frac{1}{s^2-4s-12}$$

$$\Rightarrow X(s) = \frac{2s-3}{s^2-4s-12} = \frac{2s-3}{(s+2)(s-6)} \rightarrow (A)$$

$$Y(s) = \frac{s+9}{s^2-4s-12} = \frac{s+9}{(s+2)(s-6)} \rightarrow (B)$$

$$X(s) = \frac{7}{8(s+2)} + \frac{9}{8(s-6)}$$

$$Y(s) = \frac{-7}{8(s+2)} + \frac{15}{8(s-6)}$$

Taking laplace inverse Transform

(22) $\frac{dx}{dt} - 4x - 5y = e^{-4t}$, $x(0)=0$

$\frac{dy}{dt} + 4x + 4y = e^{4t}$, $y(0)=0$

$$sX(s) - x(0) - 4X(s) - 5Y(s) = \frac{1}{s+4}$$

$$sY(s) - y(0) + 4X(s) + 4Y(s) = \frac{1}{s-4}$$

$$(s-4)X(s) - 5Y(s) = \frac{1}{s+4}$$

$$4X(s) + (s+4)Y(s) = 1/s-4$$

$$(s-4)X(s) - 5Y(s) - \frac{1}{s+4} = 0 \rightarrow (1)$$

$$4X(s) + (s+4)Y(s) - \frac{1}{s-4} = 0 \rightarrow (2)$$

Solve by Cramer's Rule

$$\frac{X(s)}{\begin{vmatrix} -5 & \frac{-1}{s+4} \\ s+4 & \frac{-1}{s-4} \end{vmatrix}} = \frac{-Y(s)}{\begin{vmatrix} s-4 & \frac{-1}{s+4} \\ 4 & \frac{-1}{s-4} \end{vmatrix}} = \frac{1}{\begin{vmatrix} s-4 & -5 \\ 4 & s+4 \end{vmatrix}}$$

$$\frac{X(s)}{\frac{s+1}{s-4}} = \frac{Y(s)}{\frac{s}{s+4}} = \frac{1}{s^2+4}$$

$$\Rightarrow X(s) = \frac{s+1}{(s-4)(s^2+4)} \rightarrow (A)$$

$$Y(s) = \frac{3}{(s+4)(s^2+4)} \rightarrow (B)$$

$$X(s) = \frac{1}{4(s-4)} - \frac{s}{4(s^2+4)}$$

$$Y(s) = \frac{-1}{5(s+4)} + \frac{s+1}{5(s^2+4)}$$

Taking Inverse Laplace Transform

$$\mathcal{L}^{-1}\{X(s)\} = \frac{7}{8}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \frac{9}{8}\mathcal{L}^{-1}\left\{\frac{1}{s-6}\right\}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \frac{-7}{8}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \frac{15}{8}\mathcal{L}^{-1}\left\{\frac{1}{s-6}\right\}$$

$$\left.\begin{aligned} x(t) &= \frac{7}{8}e^{-2t} + \frac{9}{8}e^{6t} \\ y(t) &= \frac{-7}{8}e^{-2t} + \frac{15}{8}e^{6t} \end{aligned}\right\}$$ is req. sol.

$$\mathcal{L}^{-1}\{X(s)\} = \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} - \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}$$

$$\mathcal{L}^{-1}\{Y(s)\} = -\frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} + \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{1}{10}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$x(t) = \frac{1}{4}e^{4t} - \frac{1}{4}\cos 2t$$

$$y(t) = -\frac{1}{5}e^{-4t} + \frac{1}{5}\cos 2t + \frac{1}{10}\sin 2t$$

(23) $2\frac{dx}{dt} + \frac{dy}{dt} - x - y = 1$, $x(0)=0$

$\frac{dx}{dt} + \frac{dy}{dt} + 2x - y = t$, $y(0)=0$,

$$2\mathcal{L}\left\{\frac{dx}{dt}\right\} + \mathcal{L}\left\{\frac{dy}{dt}\right\} - \mathcal{L}\{x(t)\} - \mathcal{L}\{y(t)\} = \mathcal{L}\{1\}$$

$$\mathcal{L}\left\{\frac{dx}{dt}\right\} + \mathcal{L}\left\{\frac{dy}{dt}\right\} + 2\mathcal{L}\{x(t)\} - \mathcal{L}\{y(t)\} = \mathcal{L}\{t\}$$

$$2\{sX(s) - x(0)\} + sY(s) - y(0) - X(s) - Y(s) = 1/s$$

$$sX(s) - x(0) + sY(s) - y(0) + 2X(s) - Y(s) = 1/s^2$$

$$2sX(s) + sY(s) - X(s) - Y(s) = 1/s$$
$$sX(s) + sY(s) + 2X(s) - Y(s) = \frac{1}{s^2}$$
$$\Rightarrow \quad (2s-1)X(s) + (s-1)Y(s) = \frac{1}{s} \rightarrow (1)$$
$$(s+2)X(s) + (s-1)Y(s) = \frac{1}{s^2} \rightarrow (2)$$

$$(1) - (2) \Rightarrow (2s-1-s-2)X(s) = \frac{1}{s} - \frac{1}{s^2}$$

$$(s-3)X(s) = \frac{s-1}{s^2} \Rightarrow X(s) = \frac{s-1}{s^2(s-3)} \rightarrow (A)$$ put in (2)

$$(s+2)\cdot\frac{(s-1)}{s^2(s-3)} + (s-1)Y(s) = \frac{1}{s^2} \Rightarrow \frac{s+2}{s^2(s-3)} + Y(s) = \frac{1}{s^2(s-1)}$$

$$Y(s) = \frac{1}{s^2(s-1)} - \frac{s+2}{s^2(s-3)} = \frac{1}{s^2}\left[\frac{s-3-(s-1)(s+2)}{(s-1)(s-3)}\right]$$

$$Y(s) = \frac{1}{s^2}\left[\frac{s-3-s^2-s+2}{(s-1)(s-3)}\right] = \frac{-s^2-1}{s^2(s-1)(s-3)} \rightarrow (B)$$

$$X(s) = \frac{-2}{9s} + \frac{1}{3s^2} + \frac{2}{9(s-3)}$$

$$Y(s) = \frac{-4}{9s} - \frac{1}{3s^2} + \frac{1}{s-1} - \frac{5}{9(s-3)}$$

$$\left.\begin{aligned} x(t) &= -\frac{2}{9} + \frac{1}{3}t + \frac{2}{9}e^{3t} \\ y(t) &= \frac{-4}{9} - \frac{1}{3}t + e^{t} - \frac{5}{9}e^{3t} \end{aligned}\right\}$$ is required Solution.

(24) $\frac{dx}{dt} + \frac{dy}{dt} = t$, $x(0)=3$, $x'(0)=-2$

$\frac{d^2x}{dt^2} - y = e^{-t}$, $y(0)=0$

(25) $\frac{dx}{dt} + 2\frac{d^2y}{dt^2} = e^{-t}$, $x(0)=0$

$\frac{dx}{dt} + 2x - y = 1$, $y(0)=0=y'(0)$

$$\mathcal{L}\left\{\frac{dx}{dt}\right\}+\mathcal{L}\left\{\frac{dy}{dt}\right\}=\mathcal{L}\{t\}$$

$$\mathcal{L}\left\{\frac{d^2x}{dt^2}\right\}-\mathcal{L}\{y(t)\}=\mathcal{L}\{e^{-t}\}$$

$$sX(s)-x(0)+sY(s)-y(0)=\frac{1}{s^2}$$

$$s^2X(s)-sx(0)-x'(0)-Y(s)=\frac{1}{s+1}$$

$$sX(s)-3+sY(s)-0=\frac{1}{s^2}$$

$$s^2X(s)-3s+2-Y(s)=\frac{1}{s+1}$$

$$sX(s)+sY(s)=\frac{1}{s^2}+3$$

$$s^2X(s)-Y(s)=\frac{1}{s+1}+3s-2$$

$$sX(s)+sY(s)=\frac{1+3s^2}{s^2}\rightarrow ①$$

$$s^2X(s)-Y(s)=\frac{1+(3s-2)(s+1)}{s+1}$$

$$=\frac{3s^2+s-1}{s+1}\rightarrow ②$$

multiply eq ① by s^2 and ② by s

$$s^3X(s)+s^3Y(s)=1+3s^2\rightarrow Ⓐ$$

$$s^3X(s)-sY(s)=\frac{3s^3+s^2-s}{s+1}\rightarrow Ⓑ$$

Now Ⓐ − Ⓑ

$$s^3Y(s)+sY(s)=3s^2+1-\frac{3s^3+s^2-s}{s+1}$$

$$(s^3+s)Y(s)=\frac{(3s^2+1)(s+1)-(3s^3+s^2-s)}{s+1}$$

$$s(s^2+1)Y(s)=\frac{2s^2+2s+1}{s+1}$$

$$Y(s)=\frac{2s^2+2s+1}{s(s+1)(s^2+1)}$$

By partial fraction;

$$Y(s)=\frac{1}{s}-\frac{1}{2(s+1)}-\frac{s-3}{2(s^2+1)}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}-\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}-$$

$$\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}+\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$y(t)=1-\frac{1}{2}e^{-t}-\frac{1}{2}\cos t+\frac{3}{2}\sin t$$

diff. w.r.t 't'

$$\frac{dy}{dt}=\frac{1}{2}e^{-t}+\frac{1}{2}\sin t+\frac{3}{2}\cos t$$

put in given eq.

$$\frac{dx}{dt}+\frac{1}{2}e^{-t}+\frac{1}{2}\sin t+\frac{3}{2}\cos t=t$$

$$sX(s)-x(0)+2[s^2Y(s)-sy(0)-y'(0)]=\frac{1}{s+1}$$

$$sX(s)-x(0)+2X(s)-Y(s)=\frac{1}{s}$$

$$sX(s)-0+2s^2Y(s)=\frac{1}{s+1}$$

$$sX(s)-0+2X(s)-Y(s)=\frac{1}{s}$$

$$sX(s)+2s^2Y(s)=\frac{1}{s+1}\rightarrow ①$$

$$(s+2)X(s)-Y(s)=\frac{1}{s}\rightarrow ②$$

Multiplying eq ② by $2s^2$

$$sX(s)+2s^2Y(s)=\frac{1}{s+1}\rightarrow Ⓐ$$

$$2s^2(s+2)X(s)-2s^2Y(s)=2s\rightarrow Ⓑ$$

Add Ⓐ and Ⓑ

$$[s+2s^2(s+2)]X(s)=2s+\frac{1}{s+1}$$

$$s[1+2s(s+2)]X(s)=\frac{2s^2+2s+1}{s+1}$$

$$s(1+2s^2+4s)X(s)=\text{"}$$

$$X(s)=\frac{2s^2+2s+1}{s(s+1)(2s^2+4s+1)}$$

By Partial Fraction;

$$X(s)=\frac{1}{s}+\frac{1}{s+1}-\frac{4(s+1)}{2s^2+4s+1}$$

$$=\frac{1}{s}+\frac{1}{s+1}-\frac{2(s+1)}{s^2+2s+\frac{1}{2}}$$

$$=\frac{1}{s}+\frac{1}{s+1}-\frac{2(s+1)}{s^2+2s+1-\frac{1}{2}-1}$$

$$=\frac{1}{s}+\frac{1}{s+1}-\frac{2(s+1)}{(s+1)^2-\left(\frac{1}{\sqrt{2}}\right)^2}$$

$$x(t)=1+e^{-t}-2e^{-t}\cosh\left(\frac{1}{\sqrt{2}}t\right)\rightarrow(i)$$

diff. w.r.t 't'

$$\frac{dx}{dt}=-e^{-t}-\left(2e^{-t}(-1)\cosh\frac{t}{\sqrt{2}}\right)-$$

$$\left(2e^{-t}\sinh\frac{t}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\right)$$

$$=-e^{-t}+2e^{-t}\cosh\frac{t}{\sqrt{2}}-\sqrt{2}e^{-t}\sinh\frac{t}{\sqrt{2}}$$

put in given eq.

$$\frac{dx}{dt}+2x-y=1$$

$$\Rightarrow y=\frac{dx}{dt}+2x-1$$

$$\frac{dx}{dt} = t - \frac{1}{2}e^{-t} - \frac{1}{2}\sin t - \frac{3}{2}\cos t$$

Integrating w.r.t 't'

$$x(t) = \frac{t^2}{2} - \frac{1}{2}\frac{e^{-t}}{-1} - \frac{1}{2}(-\cos t) - \frac{3}{2}\sin t + c$$

$$x(t) = \frac{t^2}{2} + \frac{1}{2}e^{-t} + \frac{1}{2}\cos t - \frac{3}{2}\sin t + c$$

As $x(0) = 3$

$$3 = 0 + \frac{1}{2}e^{0} + \frac{1}{2}(1) - 0 + c \Rightarrow c = 2$$

So Required solution is;

$$x(t) = \frac{t^2}{2} + \frac{1}{2}e^{-t} + \frac{1}{2}\cos t - \frac{3}{2}\sin t + 2$$

$$y(t) = 1 - \frac{1}{2}e^{-t} - \frac{1}{2}\cos t + \frac{3}{2}\sin t$$

$$\Rightarrow y(t) = -e^{-t} + 2e^{-t}\cosh\frac{t}{\sqrt{2}} - \sqrt{2}e^{-t}\sinh\frac{t}{\sqrt{2}} + 2\left(1 - e^{-t} - 2e^{-t}\cosh\frac{t}{\sqrt{2}}\right)^{-1}$$

$$y(t) = 1 + e^{t} - 2e^{-t}\cosh\frac{t}{\sqrt{2}} - \sqrt{2}\,e^{-t}\sinh\frac{t}{\sqrt{2}} \rightarrow \text{(ii)}$$

$\therefore$ (i) and (ii) constitute the sol.