(Exercise No. 11.1)

Compute Laplace transform of each of the following (1 – 28):

Q1 $t^2 + 6t - 17$

Sol. Let $f(t) = t^2 + 6t - 17$

then $\mathcal{L}\{f(t)\} = \mathcal{L}\{t^2 + 6t - 17\}$

$$= \mathcal{L}\{t^2\} + \mathcal{L}\{6t\} - \mathcal{L}\{17\}$$

$$= \mathcal{L}\{t^2\} + 6\,\mathcal{L}\{t\} - 17\,\mathcal{L}\{1\}$$

$$= \frac{2!}{s^3} + 6 \cdot \frac{1!}{s^2} - 17 \cdot \frac{1}{s}$$

$$\mathcal{L}\{f(t)\} = \frac{2}{s^3} + \frac{6}{s^2} - \frac{17}{s} \quad \text{where } s > 0$$

Q2 e^{3t+5}

Sol. Let $f(t) = e^{3t+5}$

$$= e^{3t} \cdot e^{5}$$

$$f(t) = e^{5}\, e^{3t}$$

Then $\mathcal{L}\{f(t)\} = \mathcal{L}\{e^{5} \cdot e^{3t}\}$

$$= e^{5}\, \mathcal{L}\{e^{3t}\}$$

$$= e^{5} \cdot \frac{1}{s-3} \qquad s > 3$$

$$= \frac{e^{5}}{s-3}$$

Q3 $\sin(7t+4)$

Sol. Let $f(t) = \sin(7t+4)$

$\Rightarrow f(t) = \sin 7t \cdot \cos 4 + \cos 7t \cdot \sin 4$

then $\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin 7t \cos 4 + \cos 7t \sin 4\}$

$$= \cos 4\, \mathcal{L}\{\sin 7t\} + \sin 4\, \mathcal{L}\{\cos 7t\}$$

$$= \cos 4 \cdot \frac{7}{(s)^2+(7)^2} + \sin 4 \cdot \frac{s}{(s)^2+(7)^2} \qquad s>0$$

$$= \frac{7\cos 4}{s^2+49} + \frac{s \sin 4}{s^2+49} \qquad s>0$$

Q4 $\cos(at+b)$

Sol. Let $f(t) = \cos(at+b)$

$$f(t) = \cos at \cos b - \sin at \sin b$$

then $\mathcal{L}\{f(t)\} = \mathcal{L}\{\cos at \cos b - \sin at \sin b\}$

$$= \cos b\, \mathcal{L}\{\cos at\} - \sin b\, \mathcal{L}\{\sin at\}$$

$$= \cos b \cdot \frac{s}{s^2+a^2} - \sin b \cdot \frac{a}{s^2+a^2}$$

$$= \frac{s\cos b}{s^2+a^2} - \frac{a \sin b}{s^2+a^2}$$

Q5 $\cosh(5t-3)$

Sol. Let $f(t) = \cosh(5t-3)$

or $f(t) = \cosh 5t \cosh 3 - \sinh 5t \sinh 3$

then

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\cosh 5t \cosh 3 - \sinh 5t \sinh 3\}$$

$$= \cosh 3\, \mathcal{L}\{\cosh 5t\} - \sinh 3\, \mathcal{L}\{\sinh 5t\}$$

$$= \cosh 3 \cdot \frac{s}{s^2-(5)^2} - \sinh 3 \cdot \frac{5}{s^2-(5)^2}$$

$$= \frac{s \cosh 3}{s^2-25} - \frac{5 \sinh 3}{s^2-25}$$

<u>Q6</u> $(t^3-1)\,e^{-2t}$

<u>Sol.</u> Let $f(t) = (t^3-1)\,e^{-2t}$

or $f(t) = t^3 e^{-2t} - e^{-2t}$

then

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^3 e^{-2t} - e^{-2t}\}$$

$$= \mathcal{L}\{t^3 e^{-2t}\} - \mathcal{L}\{e^{-2t}\}$$

$$= \frac{3!}{(s-(-2))^4} - \frac{1}{s-(-2)} \qquad s > -2$$

$$= \frac{3!}{(s+2)^4} - \frac{1}{(s+2)}$$

<u>Q7</u> $e^{-t}\sin 2t$

<u>Sol.</u> Let $f(t) = e^{-t}\sin 2t$

then $\mathcal{L}\{f(t)\} = \mathcal{L}\{e^{-t}\sin 2t\}$

$$= \frac{2}{(s-(-1))^2+(2)^2} \qquad s > -1$$

$$\mathcal{L}\{f(t)\} = \frac{2}{(s+1)^2+4}$$

Q8 $e^{3t}\cosh 4t$

Sol. Let $f(t) = e^{3t}\cosh 4t$

then

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{e^{3t}\cosh 4t\}$$

$$= \frac{s-3}{(s-3)^2-(4)^2} \qquad \left(\because \mathcal{L}\{e^{at}\cosh bt\} = \frac{s-a}{(s-a)^2-b^2}\right)$$

$$= \frac{s-3}{(s-3)^2-16}$$

Q9. $\cos t\cos 2t$

Sol. Let $f(t) = \cos t\cos 2t$

or $f(t) = \frac{1}{2}(2\cos 2t\cos t)$

$$= \frac{1}{2}\left[\cos(2t+t)+\cos(2t-t)\right]$$

$$= \frac{1}{2}\left[\cos 3t+\cos t\right]$$

$$f(t) = \frac{1}{2}\cos 3t + \frac{1}{2}\cos t$$

then $\mathcal{L}\{f(t)\} = \mathcal{L}\left\{\frac{1}{2}\cos 3t+\frac{1}{2}\cos t\right\}$

$$= \frac{1}{2}\mathcal{L}\{\cos 3t\} + \frac{1}{2}\mathcal{L}\{\cos t\}$$

$$= \frac{1}{2} \cdot \frac{s}{s^2+(3)^2} + \frac{1}{2} \cdot \frac{s}{s^2+(1)^2}$$

$$= \frac{1}{2} \cdot \frac{s}{s^2+9} + \frac{1}{2} \cdot \frac{s}{s^2+1}$$

$$\mathcal{L}\{f(t)\} = \frac{s}{2(s^2+9)} + \frac{s}{2(s^2+1)}$$

Q10. $\sin^3 t$

Sol. Let $f(t) = \sin^3 t$

or $f(t) = \frac{1}{4}(3\sin t - \sin 3t)$

$f(t) = \frac{3}{4}\sin t - \frac{1}{4}\sin 3t$

then $\mathcal{L}\{f(t)\} = \mathcal{L}\{\frac{3}{4}\sin t - \frac{1}{4}\sin 3t\}$

$$= \frac{3}{4}\mathcal{L}\{\sin t\} - \frac{1}{4}\mathcal{L}\{\sin 3t\}$$

$$= \frac{3}{4} \cdot \frac{1}{s^2+(1)^2} - \frac{1}{4} \cdot \frac{3}{s^2+(3)^2}$$

$$= \frac{3}{4(s^2+1)} - \frac{3}{4(s^2+9)}$$

Q11 $t e^{-3t} \sin at$

Sol. Let $f(t) = e^{-3t} \sin at$

then $\mathcal{L}\{f(t)\} = \mathcal{L}\{e^{-3t} \cdot \sin at\}$

$$= \frac{a}{(s-(-3))^2+(a)^2}$$

$$\mathcal{L}\{f(t)\} = \frac{a}{(S+3)^2+a^2} = F(S)$$

Now

$$\mathcal{L}\{t f(t)\} = \mathcal{L}\{t e^{-3t} \sin at\}$$

$$= -\frac{d}{ds}\left[\frac{a}{(S+3)^2+(a)^2}\right]$$

$$= -a \cdot \frac{-1}{\left[(S+3)^2+a^2\right]^2} \cdot 2(S+3)$$

$$= \frac{2a(S+3)}{\left[(S+3)^2+a^2\right]^2}$$

Q12 $\quad \sinh^2 at$

Sol. Let $f(t) = \sinh^2 at$

$$= \frac{\cosh 2at - 1}{2} \qquad (\because \cosh 2x = 1 + 2\sinh^2 x)$$

$$f(t) = \frac{1}{2}\cosh 2at - \frac{1}{2}$$

Then $\mathcal{L}\{f(t)\} = \mathcal{L}\left\{\frac{1}{2}\cosh 2at - \frac{1}{2}\right\}$

$$= \frac{1}{2}\mathcal{L}\{\cosh 2at\} - \frac{1}{2}\mathcal{L}\{1\}$$

$$= \frac{1}{2} \cdot \frac{S}{S^2-(2a)^2} - \frac{1}{2} \cdot \frac{1}{S}$$

$$= \frac{1}{2}\left[\frac{S}{S^2-4a^2} - \frac{1}{S}\right]$$

$$= \frac{1}{2}\left[\frac{s^2-(s^2-4a^2)}{s(s^2-4a^2)}\right]$$

$$= \frac{1}{2}\left[\frac{\cancel{s^2}-\cancel{s^2}+4a^2}{s(s^2-4a^2)}\right]$$

$$\mathcal{L}\{f(t)\} = \frac{2a^2}{s(s^2-4a^2)}$$

<u>Q13.</u> $\cosh at.\sin at$

<u>Soln</u>

Let $f(t) = \cosh at.\sin at$

$$= \left(\frac{e^{at}+e^{-at}}{2}\right).\sin at$$

$$f(t) = \frac{1}{2}\left(e^{at}\sin at + e^{-at}\sin at\right)$$

Then $\mathcal{L}\{f(t)\} = \mathcal{L}\left\{\frac{1}{2}\left(e^{at}\sin at + e^{-at}\sin at\right)\right\}$

$$= \frac{1}{2}\mathcal{L}\{e^{at}\sin at\} + \frac{1}{2}\{e^{-at}\sin at\}$$

$$= \frac{1}{2}\cdot\frac{a}{(s-a)^2+a^2} + \frac{1}{2}\frac{a}{(s+a)^2+a^2}$$

$$= \frac{a}{2}\left[\frac{1}{(s-a)^2+a^2} + \frac{1}{(s+a)^2+a^2}\right]$$

$$= \frac{a}{2}\left[\frac{(s+a)^2+a^2+(s-a)^2+a^2}{[(s-a)^2+a^2][(s+a)^2+a^2]}\right]$$

$$= \frac{a}{2}\left[\frac{s^2+2\cancel{a}s+a^2+a^2+s^2-2\cancel{a}s+a^2+a^2}{(s-a)^2.(s+a)^2+a^2(s-a)^2+a^2(s+a)^2+a^4}\right]$$

$$= \frac{a}{2}\left[\frac{2s^2+4a^2}{\left[(s-a)(s+a)\right]^2+a^2\left[(s-a)^2+(s+a)^2\right]+a^4}\right]$$

$$= \frac{a}{2}\left[\frac{2s^2+4a^2}{(s^2-a^2)^2+a^2\left(2(s^2+a^2)\right)+a^4}\right]$$

$$= a\left[\frac{s^2+2a^2}{s^4-2\cancel{s^2a^2}+a^4+2\cancel{a^2s^2}+2a^4+a^4}\right]$$

$$= a\left[\frac{s^2+2a^2}{s^4+4a^4}\right]$$

$$\mathcal{L}\{f(t)\} = \frac{a(s^2+2a^2)}{s^4+4a^4}$$

Q14 Sinh at Cos at

Sol. Let $f(t) = \sinh at . \cos at$

$$= \left(\frac{e^{at}-e^{-at}}{2}\right)\cos at$$

$$f(t) = \frac{1}{2}\left(e^{at}\cos at - e^{-at}\cos at\right)$$

$$f(t) = \frac{1}{2}e^{at}\cos at - \frac{1}{2}e^{-at}\cos at$$

then $\mathcal{L}\{f(t)\} = \mathcal{L}\left\{\frac{1}{2}e^{at}\cos at - \frac{1}{2}e^{-at}\cos at\right\}$

$$= \frac{1}{2}\mathcal{L}\{e^{at}\cos at\} - \frac{1}{2}\mathcal{L}\{e^{-at}\cos at\}$$

$$\mathcal{L}\{f(t)\} = \frac{1}{2}\cdot\frac{s-a}{(s-a)^2+a^2} - \frac{1}{2}\cdot\frac{(s+a)}{(s+a)^2+a^2}$$

$$= \frac{1}{2}\left[\frac{s-a}{(s-a)^2+a^2} - \frac{s+a}{(s+a)^2+a^2}\right]$$

$$= \frac{1}{2}\left[\frac{(s-a)\left[(s+a)^2+a^2\right] - (s+a)\left[(s-a)^2+a^2\right]}{\left[(s-a)^2+a^2\right]\left[(s+a)^2+a^2\right]}\right]$$

$$= \frac{1}{2}\left[\frac{(s-a)\left[s^2+2as+2a^2\right] - (s+a)\left[s^2-2as+2a^2\right]}{(s-a)^2(s+a)^2 + a^2(s-a)^2 + a^2(s+a)^2 + a^4}\right]$$

$$= \frac{1}{2}\left[\frac{\cancel{s^3}+2as^2+\cancel{2a^2s}-as^2-\cancel{2a^2s}-2a^3-\cancel{s^3}+2as^2-\cancel{2a^2s}-as^2+\cancel{2a^2s}-2a^3}{\left[(s-a)(s+a)\right]^2 + a^2\left[(s-a)^2+(s+a)^2\right] + a^4}\right]$$

$$= \frac{1}{2}\left[\frac{2as^2-4a^3}{(s^2-a^2)^2 + a^2\left[2(s^2+a^2)\right] + a^4}\right]$$

$$= \frac{as^2-2a^3}{s^4-\cancel{2a^2s^2}+a^4+\cancel{2a^2s^2}+2a^4+a^4}$$

$$\mathcal{L}\{f(t)\} = \frac{a(s^2-2a^2)}{s^4+4a^4}$$

Q15 $\cosh at \cdot \cos bt$

Sol:- Let $f(t) = \cosh at \cdot \cos bt$

$$= \left(\frac{e^{at}+e^{-at}}{2}\right)\cos bt$$

$$f(t) = \frac{1}{2}\left(e^{at}\cos bt + e^{-at}\cos bt\right)$$

then

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{ \tfrac{1}{2}\left(e^{at}\cos bt + e^{-at}\cos bt\right)\right\}$$

$$= \tfrac{1}{2}\left(\mathcal{L}\{e^{at}\cos bt\} + \mathcal{L}\{e^{-at}\cos bt\}\right)$$

$$= \frac{1}{2}\left[\frac{s-a}{(s-a)^2+b^2} + \frac{s+a}{(s+a)^2+b^2}\right]$$

$$= \frac{1}{2}\left[\frac{(s-a)\left[(s+a)^2+b^2\right] + (s+a)\left[(s-a)^2+b^2\right]}{\left[(s-a)^2+b^2\right]\left[(s+a)^2+b^2\right]}\right]$$

$$= \frac{1}{2}\left[\frac{(s-a)(s^2+2as+a^2+b^2) + (s+a)(s^2-2as+a^2+b^2)}{(s^2-2as+a^2+b^2)(s^2+2as+a^2+b^2)}\right]$$

$$= \frac{1}{2}\left[\frac{s^3+\cancel{2as^2}+a^2s+sb^2-\cancel{as^2}-2a^2s-\cancel{a^3}-\cancel{ab^2}+s^3-\cancel{2as^2}+a^2s+b^2s+\cancel{as^2}-2a^2s+\cancel{a^3}+\cancel{ab^2}}{\left[(s^2+a^2+b^2)-2as\right]\left[(s^2+a^2+b^2)+2as\right]}\right]$$

$$= \frac{1}{2}\left[\frac{2s^3-2a^2s+2b^2s}{(s^2+a^2+b^2)^2-(2as)^2}\right]$$

$$= \frac{1}{\cancel{2}}\left[\frac{\cancel{2}(s^3-a^2s+b^2s)}{(s^2+a^2+b^2)^2-4a^2s^2}\right]$$

$$\mathcal{L}\{f(t)\} = \frac{s(s^2-a^2+b^2)}{(s^2+a^2+b^2)^2-4a^2s^2}$$

Q16 $t e^{at}\cos bt$

Sol. let $f(t) = e^{at}\cos bt$

then $\mathcal{L}\{f(t)\} = \mathcal{L}\{e^{at}\cos bt\}$

$$\mathcal{L}\{f(t)\} = \frac{s-a}{(s-a)^2+b^2} = F(s)$$

Now

$$\mathcal{L}\{t f(t)\} = \mathcal{L}\{t\, e^{at} \cos bt\}$$

$$= -\frac{d}{ds}\left(\frac{s-a}{(s-a)^2+b^2}\right)$$

$$= -\left[\frac{[(s-a)^2+b^2]\cdot 1 - (s-a)\cdot 2(s-\ )}{[(s-a)^2+b^2]^2}\right]$$

$$= -\left[\frac{(s-a)^2+b^2-2(s-a)^2}{[(s-a)^2+b^2]^2}\right]$$

$$= -\left[\frac{-(s-a)^2+b^2}{[(s-a)^2+b^2]^2}\right]$$

$$= \frac{(s-a)^2-b^2}{[(s-a)^2+b^2]^2}$$

Q17 t^{α}, $\alpha > -1$. Hence find $\mathcal{L}\{t^{5/2}\}$

Soln

Let $f(t) = t^{\alpha}$

then $\mathcal{L}\{f(t)\} = \mathcal{L}\{t^{\alpha}\}$

or $\mathcal{L}\{f(t)\} = \int_{-\infty}^{\infty} e^{-st}\, t^{\alpha}\, dt$

Let $st = u$

$\Rightarrow t = \frac{u}{s}$

$dt = \frac{1}{s} du$

So

So $\mathcal{L}\{t^{\alpha}\} = \int_0^{\infty} e^{-u} \cdot \left(\frac{u}{s}\right)^{\alpha} \cdot \frac{1}{s}\, du$

$= \frac{1}{s}\int_0^{\infty} e^{-u} \cdot \frac{u^{\alpha}}{s^{\alpha}} \cdot du$

$= \frac{1}{s^{\alpha+1}}\int_0^{\infty} e^{-u} \cdot u^{\alpha}\, du$

$= \frac{1}{s^{\alpha+1}}\int_0^{\infty} u^{(\alpha+1)-1} \cdot e^{-u}\, du$

$\mathcal{L}\{t^{\alpha}\} = \frac{1}{s^{\alpha+1}} \Gamma(\alpha+1)$ —— Ans. $\left(\because \Gamma(n) = \int_0^{\infty} x^{n-1} \cdot e^{-x}\, dx\right)$

Now $\mathcal{L}\{t^{5/2}\} = \frac{1}{s^{5/2+1}} \Gamma(5/2+1)$

$= \frac{1}{s^{7/2}} \Gamma(7/2)$

$= \frac{1}{s^{7/2}} \cdot \frac{5}{2} \cdot \Gamma(5/2)$ $\left(\because \Gamma(n) = (n-1)\Gamma(n-1)\right)$

$= \frac{1}{s^{7/2}} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \Gamma(3/2)$

$= \frac{1}{s^{7/2}} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma(1/2)$

$= \frac{1}{s^{7/2}} \cdot \frac{15}{8} \cdot \sqrt{\pi}$ $\left(\because \Gamma(1/2) = \sqrt{\pi}\right)$

$= \frac{15\sqrt{\pi}}{8 s^{7/2}}$

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Q18 $t^2 \sin at$

Sol:- Let $f(t) = t^2 \sin at$

then $\mathcal{L}\{f(t)\} = \mathcal{L}\{t^2 \sin at\}$

$$= (-1)^2 \frac{d^2}{ds^2} \mathcal{L}\{\sin at\} \quad \left(\because \mathcal{L}\{t^n \sin at\} = (-1)^n \frac{d^n}{ds^n} \mathcal{L}\{\sin at\}\right)$$

$$= \frac{d^2}{ds^2}\left(\frac{a}{s^2+a^2}\right) \qquad a(s^2+a^2)^{-1}$$

$$= \frac{d}{ds}\left[\frac{-a}{(s^2+a^2)^2} \cdot 2s\right] \qquad a(-1)(s^2+a^2)^{-1-1}(2s+$$

$$= \frac{d}{ds}\left[\frac{-2as}{(s^2+a^2)^2}\right] \qquad \frac{-2as}{(s^2+a^2)^2}$$

$$= -2a \frac{d}{ds}\left[\frac{s}{(s^2+a^2)^2}\right], \qquad \frac{v.u' - u.v'}{v^2}$$

$$= -2a\left[\frac{(s^2+a^2)^2 \cdot 1 - s \cdot 2(s^2+a^2)^{2-1} \cdot 2s}{(s^2+a^2)^4}\right]$$

$$= -2a\left[\frac{\cancel{(s^2+a^2)}(s^2+a^2-4s^2)}{(s^2+a^2)^{\cancel{4}}}\right]$$

$$= -2a\left[\frac{a^2-3s^2}{(s^2+a^2)^3}\right]$$

$$= \frac{2a(3s^2-a^2)}{(s^2+a^2)^3}$$

Q19 $t^2 \cos at$

Sol:- Let $f(t) = t^2 \cos at$

then $\mathcal{L}\{f(t)\} = \mathcal{L}\{t^2 \cos at\}$

$$((s^2+a^2)^{2})^2$$

$$\mathcal{L}\{f(t)\} = (-1)^2 \frac{d^2}{ds^2} \mathcal{L}\{\cos at\}$$

$$= \frac{d^2}{ds^2}\left(\frac{s}{s^2+a^2}\right)$$

$$= \frac{d}{ds}\left[\frac{(s^2+a^2).1 - s.2s}{(s^2+a^2)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

$$= \frac{(s^2+a^2)^2.(-2s) - (a^2-s^2).2(s^2+a^2).2s}{(s^2+a^2)^4}$$

$$= \frac{(s^2+a^2)^2(-2s) - 4s(a^2-s^2)(s^2+a^2)}{(s^2+a^2)^4}$$

$$= \frac{(s^2+a^2)(-2s) - 4s(a^2-s^2)}{(s^2+a^2)^3}$$

$$= \frac{-2s^3 - 2a^2s - 4a^2s + 4s^3}{(s^2+a^2)^3}$$

$$= \frac{2s^3 - 6a^2s}{(s^2+a^2)^3}$$

$$= \frac{2s(s^2-3a^2)}{(s^2+a^2)^3}$$

M **Q20** $t\sin^2 at$

Sol Let $f(t) = t\sin^2 at$

then $f'(t) = 1.\sin^2 at + t.2\sin at.\cos at.a$

$f'(t) = \sin^2 at + 2at.\sin at.\cos at$

$$f'(t) = 2a\sin at\cos at + 2\left\{1\cdot\sin at\cos at + t\cdot a\cos at\cos at + t\sin at\cdot(-a\sin at)\right\}$$

$$= a(2\sin at\cos at) + a(2\sin at\cos at) + 2a^2t\cos^2 at - 2a^2t\sin^2 at$$

$$= a\sin 2at + a\sin 2at + 2a^2t(\cos^2 at - \sin^2 at)$$

$$= 2a\sin 2at + 2a^2t(1 - 2\sin^2 at)$$

$$f''(t) = 2a\sin 2at + 2a^2t - 4a^2t\sin^2 at$$

using formula

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

$$\mathcal{L}\{2a\sin 2at + 2a^2t - 4a^2t\sin^2 at\} = s^2\mathcal{L}\{f(t)\} - s(0) - 0$$

$$2a\mathcal{L}\{\sin 2at\} + 2a^2\mathcal{L}\{t\} - 4a^2\mathcal{L}\{f(t)\} = s^2\mathcal{L}\{f(t)\}$$

$$2a\cdot\frac{2a}{s^2+(2a)^2} + 2a^2\cdot\frac{1}{s^2} = (4a^2+s^2)\mathcal{L}\{f(t)\}$$

$$\frac{4a^2}{s^2+4a^2} + \frac{2a^2}{s^2} = (4a^2+s^2)\mathcal{L}\{f(t)\}$$

$$(4a^2+s^2)\cdot\mathcal{L}\{f(t)\} = \frac{4a^2}{s^2+4a^2} + \frac{2a^2}{s^2}$$

$$= \frac{4a^2s^2 + 2a^2(s^2+4a^2)}{s^2(s^2+4a^2)}$$

$$(s^2+4a^2)\mathcal{L}\{f(t)\} = \frac{4a^2s^2 + 2a^2s^2 + 8a^4}{s^2(s^2+4a^2)}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{6a^2s^2 + 8a^4}{s^2(s^2+4a^2)^2}$$

$$= \frac{2a^2(3s^2+4a^2)}{s^2(s^2+4a^2)^2}$$

Q21 $t^2\cos^2 2t$

Sol: Let $f(t) = t^2\cos^2 2t$

$$= t^2\left[\frac{1+\cos 4t}{2}\right]$$

$$f(t) = \frac{1}{2}t^2 + \frac{1}{2}t^2\cos 4t$$

$$\Rightarrow f'(t) = \frac{1}{2}(2t) + \frac{1}{2}\left[t^2 \cdot -4\sin 4t + 2t\cos 4t\right]$$

$$f'(t) = t - 2t^2\sin 4t + t\cos 4t$$

$$f''(t) = 1 - 2\left(t^2 \cdot 4\cos 4t + 2t\sin 4t\right) + t\left(-4\sin 4t\right) + \cos 4t \cdot 1$$

$$= 1 - 8t^2\cos 4t - 4t\sin 4t - 4t\sin 4t + \cos 4t$$

$$= 1 + \cos 4t - 8t\sin 4t - 8t^2\left(2\cos^2 2t - 1\right)$$

$$= 1 + \cos 4t - 8t\sin 4t - 16t^2\cos^2 2t + 8t^2$$

$(-1)^n \frac{d}{ds}\mathcal{L}\{\sin 4t\}$

$$f''(t) = 1 + 8t^2 + \cos 4t - 8t\sin 4t - 16t^2\cos^2 2t$$

$$\mathcal{L}\{f''(t)\} = \mathcal{L}\{1\} + 8\mathcal{L}\{t^2\} + \mathcal{L}\{\cos 4t\} - 8(-1)\frac{d}{ds}\mathcal{L}\{\sin 4t\} - 16\mathcal{L}\{f(t)\}$$

$$= \frac{1}{s} + 8\cdot\frac{2}{s^3} + \frac{s}{s^2+(4)^2} + 8\frac{d}{ds}\left(\frac{4}{s^2+16}\right) - 16\mathcal{L}\{f(t)\}$$

$$= \frac{1}{s} + \frac{16}{s^3} + \frac{s}{s^2+16} + 8\cdot\frac{-4\cdot 2s}{(s^2+16)^2} - 16\mathcal{L}\{f(t)\}$$

$$= \frac{s^2+16}{s^3} + \frac{s}{s^2+16} - \frac{64s}{(s^2+16)^2} - 16\mathcal{L}\{f(t)\}$$

$$\mathcal{L}\{f''(t)\} = \frac{(s^2+16)^3 + s^4(s^2+16) - 64s^4}{s^3(s^2+16)^2} - 16\mathcal{L}\{f(t)\}$$

Using formula

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

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Q23 $\frac{1-\cos at}{t}$

Sol: Let $f(t) = 1-\cos at$

then $\mathcal{L}\{f(t)\} = \mathcal{L}\{1-\cos at\}$

$= \mathcal{L}\{1\} - \mathcal{L}\{\cos at\}$

So $\mathcal{L}\{f(t)\} = \frac{1}{s} - \frac{s}{s^2+a^2} = F(s)$

Now $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \mathcal{L}\left\{\frac{1-\cos at}{t}\right\}$

$$= \int_s^\infty F(u)\,du$$

$$= \int_s^\infty \left(\frac{1}{u} - \frac{u}{u^2+a^2}\right) du$$

$$= \int_s^\infty \left(\frac{1}{u} - \frac{2u}{2(u^2+a^2)}\right) du$$

$$= \left| \ln u - \frac{1}{2}\ln(u^2+a^2) \right|_s^\infty$$

$$= \left| \ln u - \ln\sqrt{u^2+a^2} \right|_s^\infty$$

$$= \left| \frac{1}{2}\ln u^2 - \frac{1}{2}\ln(u^2+a^2) \right|_s^\infty$$

$$= \frac{1}{2}\left| \ln u^2 - \ln(u^2+a^2) \right|_s^\infty$$

$$= \frac{1}{2}\left| \ln\left(\frac{u^2}{u^2+a^2}\right) \right|_s^\infty$$

$$= \lim_{u\to\infty}\left[\frac{1}{2}\ln\left(\frac{u^2}{u^2+a^2}\right) - \frac{1}{2}\ln\left(\frac{s^2}{s^2+a^2}\right)\right]$$

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$$\frac{(s^2+16)^3+s^4(s^2+16)-64s^4}{s^3(s^2+16)^2}-16\,\mathcal{L}\{f(t)\}=s^2\mathcal{L}\{f(t)\}-s(0)-0$$

$$\frac{(s^2+16)^3+s^4(s^2+16)-64s^4}{s^3(s^2+16)^2}=(16+s^2)\,\mathcal{L}\{f(t)\}$$

$$\Rightarrow \mathcal{L}\{f(t)\}=\frac{(s^2+16)^3+s^4(s^2+16)-64s^4}{s^3(s^2+16)^3}$$

Q22. $\dfrac{\sin at}{t}$

Sol:- Let $f(t)=\sin at$

then $\mathcal{L}\{f(t)\}=\dfrac{a}{s^2+a^2}=F(s)$

then $\mathcal{L}\left\{\dfrac{f(t)}{t}\right\}=\mathcal{L}\left\{\dfrac{\sin at}{t}\right\}$

$$=\int_s^\infty F(u)\,du$$

$$=\int_s^\infty \frac{a}{u^2+a^2}\,du$$

$$=a\int_s^\infty \frac{1}{a^2+u^2}\,du$$

$$=a\left|\frac{1}{a}\tan^{-1}\frac{u}{a}\right|_s^\infty$$

$$=\tan^{-1}\infty-\tan^{-1}\left(\frac{s}{a}\right)$$

$$=\frac{\pi}{2}-\tan^{-1}\left(\frac{s}{a}\right)$$

$$=\tan^{-1}\left(\frac{a}{s}\right)$$

$$= \lim_{u\to\infty}\left[\frac{1}{2}\ln\left(\frac{1}{1+a^2/u^2}\right) - \frac{1}{2}\ln\left(\frac{s^2}{s^2+a^2}\right)\right]$$

$$= \frac{1}{2}\ln\left(\frac{1}{1+0}\right) - \frac{1}{2}\ln\left(\frac{s^2}{s^2+a^2}\right)$$

$$= 0 - \frac{1}{2}\ln\left(\frac{s^2+a^2}{s^2}\right)^{-1}$$

$$= (-1)\left(-\frac{1}{2}\right)\ln\left(\frac{s^2+a^2}{s^2}\right)$$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \frac{1}{2}\ln\left(\frac{s^2+a^2}{s^2}\right)$$

<u>Q24</u> $\int_0^t \frac{\sin au}{u}\,du$

<u>Sol</u> Consider $\int_0^t \frac{\sin au}{u}\,du$

First we find $\mathcal{L}\left\{\frac{\sin at}{t}\right\}$

let $f(t) = \sin at$

then $\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin at\}$

$$= \frac{a}{s^2+a^2} = F(s)$$

then $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \mathcal{L}\left\{\frac{\sin at}{t}\right\}$

$$= \int_s^\infty F(u)\,du$$

$$= \int_s^\infty \frac{a}{u^2+a^2}\,du$$

$$= \left|\tan^{-1}\frac{u}{a}\right|_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1}(s/a)$$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right)$$

$$\mathcal{L}\left\{\frac{\sin at}{t}\right\} = \tan^{-1}\left(\frac{a}{s}\right) \quad \text{——— (1)}$$

We know that

$$\mathcal{L}\left\{\int_0^t f(u)\,du\right\} = \frac{1}{s}\mathcal{L}\{f(t)\}$$

So

$$\mathcal{L}\left\{\int_0^t \frac{\sin au}{u}\,du\right\} = \frac{1}{s}\mathcal{L}\left\{\frac{\sin at}{t}\right\}$$

$$= \frac{1}{s}\tan^{-1}\left(\frac{a}{s}\right) \quad \text{using (1)}$$

<u>Q25</u> $\int_0^t \frac{1-\cos au}{u}\,du$

<u>Sol.</u>
Consider $\int_0^t \frac{1-\cos au}{u}\,du$

First we will find $\mathcal{L}\left\{\frac{1-\cos at}{t}\right\}$

Let $f(t) = 1-\cos at$

then
$$\mathcal{L}\{f(t)\} = \mathcal{L}\{1-\cos at\}$$
$$= \mathcal{L}\{1\} - \mathcal{L}\{\cos at\}$$
$$= \frac{1}{s} - \frac{s}{s^2+a^2} = F(s)$$

Now
$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \mathcal{L}\left\{\frac{1-\cos at}{t}\right\}$$
$$= \int_s^\infty F(u)\,du$$
$$= \int_s^\infty \left(\frac{1}{u} - \frac{u}{u^2+a^2}\right)du$$

$$= \int_s^\infty \left(\frac{1}{u} - \frac{2u}{2(u^2+a^2)} \right) du$$

$$= \left| \ln u - \frac{1}{2} \ln(u^2+a^2) \right|_s^\infty$$

$$= \left| \frac{1}{2} \ln u^2 - \frac{1}{2} \ln(u^2+a^2) \right|_s^\infty$$

$$= \frac{1}{2} \left| \ln\left(\frac{u^2}{u^2+a^2} \right) \right|_s^\infty$$

$$= \lim_{u\to\infty} \left[\frac{1}{2} \ln\left(\frac{u^2}{u^2+a^2} \right) - \frac{1}{2} \ln\left(\frac{s^2}{s^2+a^2} \right) \right]$$

$$= \lim_{u\to\infty} \left[\frac{1}{2} \ln\left(\frac{1}{1+\frac{a^2}{u^2}} \right) - \frac{1}{2} \ln\left(\frac{s^2}{s^2+a^2} \right) \right]$$

$$= \frac{1}{2} \ln\left(\frac{1}{1+0} \right) - \frac{1}{2} \ln\left(\frac{s^2}{s^2+a^2} \right)$$

$$= \frac{1}{2}(0) - \frac{1}{2} \ln\left(\frac{s^2}{s^2+a^2} \right)$$

$$= -\frac{1}{2} \ln\left(\frac{s^2}{s^2+a^2} \right)$$

$$= \frac{1}{2} \ln\left(\frac{s^2}{s^2+a^2} \right)^{-1}$$

$$\mathcal{L}\left\{ \frac{1-\cos at}{t} \right\} = \frac{1}{2} \ln\left(\frac{s^2+a^2}{s^2} \right)$$

We know that

$$\mathcal{L}\left\{ \int_0^t f(u)\,du \right\} = \frac{1}{s} \mathcal{L}\{f(t)\}$$

So. $$\mathcal{L}\left\{ \int_0^t \frac{1-\cos au}{u}\,du \right\} = \frac{1}{s} \cdot \frac{1}{2} \ln\left(\frac{s^2+a^2}{s^2} \right)$$

$$= \frac{1}{2s} \ln\left(\frac{s^2+a^2}{s^2} \right)$$

Q26 : $\frac{\sinh at}{t}$

Sol: Let $f(t) = \sinh at$

then $\mathcal{L}\{f(t)\} = \mathcal{L}\{\sinh at\}$

$$= \frac{a}{s^2-a^2} = F(s)$$

Now $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \mathcal{L}\left\{\frac{\sinh at}{t}\right\}$

$$= \int_s^\infty F(u)\,du$$

$$= \int_s^\infty \frac{a}{u^2-a^2}\,du$$

$$= a\int_s^\infty \frac{1}{u^2-a^2}\,du$$

$$= a\left|\frac{1}{2a}\ln\left(\frac{u-a}{u+a}\right)\right|_s^\infty$$

$$= \frac{1}{2}\left|\ln\left(\frac{u-a}{u+a}\right)\right|_s^\infty$$

$$= \frac{1}{2}\lim_{u\to\infty}\left[\ln\left(\frac{u-a}{u+a}\right) - \ln\left(\frac{s-a}{s+a}\right)\right]$$

$$= \frac{1}{2}\lim_{u\to\infty}\left[\ln\left(\frac{1-a/u}{1+a/u}\right) - \ln\left(\frac{s-a}{s+a}\right)\right]$$

$$= \frac{1}{2}\left[\ln\left(\frac{1-0}{1+0}\right) - \ln\left(\frac{s-a}{s+a}\right)\right]$$

$$= \frac{1}{2}\left[0 - \ln\left(\frac{s-a}{s+a}\right)\right]$$

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$$= -\frac{1}{2}\ln\left(\frac{s+a}{s-a}\right)^{-1}$$

$$= \frac{1}{2}\ln\left(\frac{s+a}{s-a}\right)$$

Q. ? $\ln t$

Sol: let $f(t) = \ln t$

then $\mathcal{L}\{f(t)\} = \mathcal{L}\{\ln t\}$

$$= \int_0^\infty e^{-st}\cdot \ln t \, dt$$

$$= \int_0^\infty e^{-u}\cdot \ln\left(\frac{u}{s}\right)\cdot \frac{1}{s}\,du$$

Put $st = u$

$t = \frac{u}{s}$

$dt = \frac{1}{s}du$

$dt = \frac{1}{s}du$

$$= \frac{1}{s}\int_0^\infty e^{-u}(\ln u - \ln s)\,du$$

$$= \frac{1}{s}\int_0^\infty e^{-u}\cdot \ln u\,du - \frac{1}{s}\int_0^\infty e^{-u}\cdot \ln s\,du$$

$$= \frac{1}{s}\int_0^\infty e^{-u}\cdot \ln u\,du - \frac{\ln s}{s}\int_0^\infty e^{-u}\,du$$

$$= \frac{1}{s}\int_0^\infty e^{-u}\cdot \ln u\,du - \frac{\ln s}{s}\Big| -e^{-u}\Big|_0^\infty$$

$$= \frac{1}{s}\int_0^\infty e^{-u}\cdot \ln u\,du - \frac{\ln s}{s}\left(-e^{-\infty} + e^{0}\right)$$

$$= \frac{1}{s}\int_0^\infty e^{-u}\cdot \ln u\,du - \frac{\ln s}{s}(0+1)$$

$$\mathcal{L}\{f(t)\} = \frac{1}{s}\int_0^\infty e^{-u}\cdot \ln u\,du - \frac{\ln s}{s} \quad \text{———} \; (1)$$

we know that

$$\Gamma(x+1) = \int_0^\infty u^x \cdot e^{-u}\, du$$

Diff. w.r.t. x

$$\Gamma'(x+1) = \frac{d}{dx}\int_0^\infty u^x \cdot e^{-u}\, du$$

$$= \int_0^\infty \frac{d}{dx}(u^x \cdot e^{-u})\, du$$

$$\Gamma'(x+1) = \int_0^\infty (u^x \cdot \ln u) \cdot e^{-u}\, du$$

Put $x = 0$

$$\Gamma'(1) = \int_0^\infty (u^0 \cdot \ln u)\, e^{-u}\, du$$

or $$\Gamma'(1) = \int_0^\infty e^{-u} \cdot \ln u\, du$$

Put in ①

$$\mathcal{L}\{f(t)\} = \frac{1}{s}\Gamma'(1) - \frac{\ln s}{s}$$

Q28 $f(t) = \begin{cases} 0 & \text{if } t < 3 \\ (t-3)^3 & \text{if } t > 3 \end{cases}$

Sol: Given that

$$f(t) = \begin{cases} 0 & \text{if } t < 3 \\ (t-3)^3 & \text{if } t > 3 \end{cases}$$

then $f(t) = u_3(t)(t-3)^3$

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$$\text{then } \mathcal{L}\{f(t)\} = \mathcal{L}\left\{u_3(t)(t-3)^3\right\}$$

$$= e^{-3s}\mathcal{L}\{t^3\} \qquad \because \mathcal{L}\{u_a(t)f(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}$$

$$= e^{-3s}\cdot\frac{3!}{s^4}$$

$$= \frac{6e^{-3s}}{s^4}$$

V.Imp

Q29 If $\mathcal{L}\{f(t)\} = F(s)$ for $s > a$ then show that

$$\mathcal{L}\{f(ct)\} = \frac{1}{c}F\left(\frac{s}{c}\right), \quad c > 0 \text{ \& } s > ca$$

Soln. Given that $\mathcal{L}\{f(t)\} = F(s)$

Now $\mathcal{L}\{f(ct)\} = \int_0^\infty e^{-st} f(ct)\,dt$

$$= \int_0^\infty e^{-s\left(\frac{T}{c}\right)} f(T)\cdot\frac{1}{c}\,dT \qquad \text{Put } ct = T \Rightarrow t = \frac{T}{c} \text{ \& } dt = \frac{1}{c}dT$$

$$= \frac{1}{c}\int_0^\infty e^{\left(-\frac{s}{c}\right)T}\cdot f(T)\,dT$$

$$= \frac{1}{c}\cdot F\left(\frac{s}{c}\right) \quad \text{where } \frac{s}{c} > a \text{ or } s > ca$$

Q30 Compute $\mathcal{L}\{\sin\sqrt{t}\}$. Deduce $\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\}$

Soln. Let $f(t) = \sin\sqrt{t}$

$$= \sin t^{1/2}$$

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$$= t^{1/2} - \frac{(t^{1/2})^3}{3!} + \frac{(t^{1/2})^5}{5!} - \cdots\cdots$$

$$f(t) = t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \cdots\cdots$$

Then

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{ t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \cdots\cdots \right\}$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^{1/2}\} - \frac{1}{3!}\mathcal{L}\{t^{3/2}\} + \frac{1}{5!}\mathcal{L}\{t^{5/2}\} - \cdots\cdots$$

We know $\mathcal{L}\{t^{\alpha}\} = \dfrac{\Gamma(\alpha+1)}{s^{\alpha+1}}$

So
$$\mathcal{L}\{f(t)\} = \frac{\Gamma(3/2)}{s^{3/2}} - \frac{1}{3!}\frac{\Gamma(5/2)}{s^{5/2}} + \frac{1}{5!}\frac{\Gamma(7/2)}{s^{7/2}} - \cdots\cdots$$

$$= \frac{\frac{1}{2}\Gamma(1/2)}{s^{3/2}} - \frac{1}{3!}\frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma(1/2)}{s^{5/2}} + \frac{1}{5!}\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma(1/2)}{s^{7/2}} - \cdots\cdots$$

$$= \frac{\Gamma(1/2)}{2s^{3/2}} - \frac{3\Gamma(1/2)}{24s^{5/2}} + \frac{15\Gamma(1/2)}{(120)(8)s^{7/2}} - \cdots\cdots$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}} - \frac{\sqrt{\pi}}{8s^{5/2}} + \frac{\sqrt{\pi}}{64s^{7/2}} - \cdots\cdots$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left[1 - \frac{1}{4s} + \frac{1}{32s^2} - \cdots\cdots\right]$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left[1 - \left(\frac{1}{4s}\right) + \frac{\left(-\frac{1}{4s}\right)^2}{2!} - \cdots\cdots\right]$$

$$\mathcal{L}\{f(t)\} = \frac{\sqrt{\pi}}{2s^{3/2}}\, e^{-\frac{1}{4s}}$$

or
$$\mathcal{L}\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}\, e^{-\frac{1}{4s}}}{2s^{3/2}}$$ ——— Ans.

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Deduction

Here $f(t) = \sin\sqrt{t}$

$$f'(t) = \cos\sqrt{t}\cdot\frac{1}{2\sqrt{t}} = \frac{\cos\sqrt{t}}{2\sqrt{t}}$$

Using [illegible]

$$\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)$$

$$\mathcal{L}\left\{\frac{\cos\sqrt{t}}{2\sqrt{t}}\right\} = s\,\mathcal{L}\{\sin\sqrt{t}\} - 0$$

$$\mathcal{L}\left\{\frac{\cos\sqrt{t}}{2\sqrt{t}}\right\} = s\cdot\frac{\sqrt{\pi}\,e^{-\frac{1}{4s}}}{2\,s^{3/2}}$$

$$\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \frac{2\sqrt{\pi}}{2\sqrt{s}}\,e^{-\frac{1}{4s}}$$

$$= \sqrt{\frac{\pi}{s}}\,e^{-\frac{1}{4s}}$$

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