10.7 – 1

(1)

One Variable absent: Given a diff. eq. of the form $f\left(\frac{d^ny}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}, \ldots\ldots, \frac{dy}{dx}, x\right) = 0$ ——— (1)

i.e., an eq. in which the dependent variable y is missing.

In order to solve eq. (1)

we put $\boxed{\frac{dy}{dx} = p}$

then $\boxed{\frac{d^2y}{dx^2} = \frac{dp}{dx}}$

$\boxed{\frac{d^3y}{dx^3} = \frac{d^2p}{dx^2}}$

& so on.

then eq. (1) becomes

$$f\left(\frac{d^{n-1}p}{dx^{n-1}}, \frac{d^{n-2}p}{dx^{n-2}}, \ldots\ldots, p, x\right)$$

So that order of eq. (1) has been lowered by one.

Similarly Consider the eq.

$$f\left(\frac{d^ny}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}, \ldots\ldots, \frac{dy}{dx}, y\right).$$

Here x is absent.

In order to solve it, put $\boxed{\frac{dy}{dx} = p}$

then $\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\cdot\frac{dy}{dx} = \boxed{p\frac{dp}{dy}}$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(p\frac{dp}{dy}\right) = \frac{d}{dy}\left(p\frac{dp}{dy}\right)\cdot\frac{dy}{dx}$$
$$= \left[p\cdot\frac{d^2p}{dy^2} + \left(\frac{dp}{dy}\right)^2\right]p = \boxed{p^2\frac{d^2p}{dy^2} + p\left(\frac{dp}{dy}\right)^2}$$

Then eq. (2) will be transformed into an eq. in p & y of order $n-1$.

Exercise No. 10.7

Solve :

(3) $x\dfrac{d^2y}{dx^2} - \left(\dfrac{dy}{dx}\right)^3 - \dfrac{dy}{dx} = 0$

Sol. Given [redacted]

$$x\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^3 - \frac{dy}{dx} = 0 \quad \text{——— (1)}$$

Here y is absent

Put $\dfrac{dy}{dx} = p$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{dp}{dx}$$

Put values in (1)

$$x\frac{dp}{dx} - p^3 - p = 0$$ order has been lowered to '1' from '2'

or $$x\frac{dp}{dx} = p(p^2+1)$$

Separating variables

$$\int \frac{1}{p(p^2+1)}\,dp = \int \frac{dx}{x} \quad \text{——— (A)}$$

Now $$\frac{1}{p(p^2+1)} = \frac{A}{p} + \frac{Bp+C}{p^2+1}$$

$$\Rightarrow 1 = A(p^2+1) + (Bp+c)p \quad \text{(I)}$$

Put $p = 0$

So $\boxed{A=1}$

Comparing Coff. of p^2, p & Constt

$$\left.\begin{aligned} A+B &= 0 \\ C &= 0 \\ A &= 1 \end{aligned}\right\} \begin{aligned} &\Rightarrow \boxed{B=-1} \\ &\Rightarrow \boxed{C=0} \\ &\Rightarrow \boxed{A=1} \end{aligned}$$

So from (A), we have

2nd Method

$$x\frac{dp}{dx} - p^3 - p = 0$$

$$\frac{dp}{dx} - \frac{p^3}{x} - \frac{p}{x} = 0$$

$$\frac{dp}{dx} - p\cdot\frac{1}{x} = \frac{p^3}{x}$$ Bernoulli Eq.

$$\frac{1}{p^3}\frac{dp}{dx} - \frac{p}{p^3}\frac{1}{x} = \frac{1}{x}$$

$$p^{-3}\frac{dp}{dx} - p^{-2}\left(\frac{1}{x}\right) = \frac{1}{x}$$

Let $p^{-2} = z$

So $-2p^{-3}\dfrac{dp}{dx} = \dfrac{dz}{dx}$

$$p^{-3}\frac{dp}{dx} = \frac{1}{-2}\frac{dz}{dx}$$

$$\therefore \frac{-1}{2}\frac{dz}{dx} - z\left(\frac{1}{x}\right) = \frac{1}{x}$$

$$\frac{dz}{dx} + \frac{2}{x}z = -\frac{2}{x}$$ (Linear in z)

$$I.F = e^{\int\frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = \boxed{x^2}$$

$$\therefore (z)x^2 = \int\left(\frac{-2}{x}\right)x^2\,dx$$

$$= -2\int x\,dx$$

$$= -2\left(\frac{x^2}{2}\right) + c_1$$

$$z = \frac{c_1 - x^2}{x^2}$$

$$\frac{1}{p^2} = \frac{c_1 - x^2}{x^2}$$

$$p^2 = \frac{x^2}{c_1 - x^2}$$

P.T.O

10.7 – 3

③

$$\int\left(\frac{1}{p}-\frac{p}{(p^2+1)}\right)dp=\int\frac{dx}{x}$$

$$\ln p-\frac{1}{2}\ln(p^2+1)=\ln x+\ln c_1$$

$$\ln p-\ln(p^2+1)^{+1/2}=\ln c_1x$$

$$\ln\left(\frac{p}{\sqrt{p^2+1}}\right)=\ln c_1x$$

$$\Rightarrow\frac{p}{\sqrt{p^2+1}}=c_1x$$

squaring or $$\frac{p^2}{p^2+1}=c_1^2x^2$$

or $$p^2=p^2c_1^2x^2+c_1^2x^2$$

$$p^2(1-c_1^2x^2)=c_1^2x^2$$

or $$p^2=\frac{c_1^2x^2}{1-c_1^2x^2}$$

or $$p=\pm\frac{c_1x}{\sqrt{1-c_1^2x^2}}$$

$$\frac{dy}{dx}=\pm\frac{c_1x}{c_1\sqrt{\frac{1}{c_1^2}-x^2}}$$

$$dy=\pm\frac{x}{\sqrt{\left(\frac{1}{c_1}\right)^2-x^2}}dx$$

Integrating $$\int dy=\pm\int\frac{x}{\sqrt{\left(\frac{1}{c_1}\right)^2-x^2}}dx$$

$$p=\pm\frac{x}{\sqrt{c_1-x^2}}$$

$$\frac{dy}{dx}=\pm\frac{x}{\sqrt{c_1-x^2}}$$

$$dy=\pm(c_1-x^2)^{-1/2}x\,dx$$

$$\int dy=\pm\int(c_1-x^2)^{-1/2}\frac{2x}{2}dx$$

$$y=\pm\frac{(c_1-x^2)^{1/2}}{2\cdot\frac{1}{2}}+c_2$$

$$y=\pm\sqrt{c_1-x^2}+c_2$$

Ans.

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④ 1057

$$\int dy = \pm\int\left[\left(\frac{1}{c_1}\right)^2 - x^2\right]^{-\frac{1}{2}} x\,dx$$

$$y = \mp\frac{1}{2}\int\left[\left(\frac{1}{c_1}\right)^2 - x^2\right]^{-\frac{1}{2}}(-2x)dx$$

$$= \mp\frac{1}{2}\frac{\left(\left(\frac{1}{c_1^2}\right) - x^2\right)^{\frac{1}{2}}}{1/2} + c_2$$

$$y = \mp\left(\frac{1}{c_1^2} - x^2\right)^{\frac{1}{2}} + c_2$$

$$y - c_2 = \mp\left(\frac{1}{c_1^2} - x^2\right)^{\frac{1}{2}}$$

$$(y - c_2)^2 = \frac{1}{c_1^2} - x^2$$

$$x^2 + (y - c_2)^2 = \frac{1}{c_1^2}$$

Ext ~~Q No2~~

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} = 3x^2 \quad \text{——} ①$$

y-absent.

$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 3x^2$
Cauchy Euler Eq

Put $\frac{dy}{dx} = P$ $\quad \frac{d^2y}{dx^2} = \frac{dP}{dx}$

① becomes $x\frac{dP}{dx} - P = 3x^2$

$$\frac{dP}{dx} - \frac{P}{x} = 3x \quad \text{——} ②$$

PTO

10.7-5

or $\frac{dp}{dx} - \frac{1}{x} p = 3x$ ——— (2)

It is a linear diff. eq. in p.

$$I.F. = e^{-\int \frac{1}{x} dx} = e^{-\ln x} = e^{\ln x^{-1}} = \frac{1}{x}$$

Multiply both sides of eq. (2) by I.F. $\frac{1}{x}$

$$\int d\left(p \cdot \frac{1}{x}\right) = \int 3\, dx$$

$$\frac{p}{x} = 3x + c_1$$

$$\therefore \quad p = 3x^2 + c_1 x$$

$$\frac{dy}{dx} = 3x^2 + c_1 x$$

$$\Rightarrow \quad y = \int (3x^2 + c_1 x)\, dx$$

$$\therefore \quad y = x^3 + \frac{c_1 x^2}{2} + c_2$$

$$\therefore \quad y = x^3 + c_1 x^2 + c_2$$

(8) $2\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 + 4 = 0$

Sol. Given that

$$2\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 + 4 = 0 \quad \text{——— (1)}$$

It is independent of y

Put $\frac{dy}{dx} = p$

$$\frac{d^2y}{dx^2} = \frac{dp}{dx}$$

Put in (1)

$$2\frac{dp}{dx} - p^2 + 4 = 0$$

or

$$\frac{dp}{dx} = \frac{p^2 - 4}{2}$$

11.0.7 - 6

⑥

or $\int \frac{2}{p^2-4}\,dp = \int dx$

$$2\cdot\frac{1}{2\cdot 2}\ln\left(\frac{p-2}{p+2}\right) = x + \ln c \qquad \because \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\frac{x-a}{x+a}$$

$$\frac{1}{2}\ln\left(\frac{p-2}{p+2}\right) = x+\ln c$$

$$\ln\sqrt{\frac{p-2}{p+2}} = x+\ln c$$

$$\ln\sqrt{\frac{p-2}{p+2}} = \ln e^{x}+\ln c = \ln ce^{x}$$

$$\sqrt{\frac{p-2}{p+2}} = ce^{x}$$

$$\Rightarrow \frac{p-2}{p+2} = c^2e^{2x}$$

$$\text{or}\quad \frac{p+2}{p-2} = \frac{1}{c^2e^{2x}}$$

By Comp & Dividendo

$$\frac{(p+2)+(p-2)}{(p+2)-(p-2)} = \frac{1+c^2e^{2x}}{1-c^2e^{2x}}$$

$$\frac{2p}{4} = \frac{1+c^2e^{-2x}}{1-c^2e^{2x}}$$

$$\text{or}\quad \frac{p}{2} = \frac{1+c^2e^{2x}}{1-c^2e^{2x}}$$

$$\frac{dy}{dx} = 2\left[\frac{(1-c^2e^{2x})+2c^2e^{2x}}{(1-c^2e^{2x})}\right] \qquad +,-,\,c^2e^{2x}$$

$$= 2\left[\cancel{\frac{(1-c^2e^{2x})}{(1-c^2e^{2x})}}\right]+\left[\frac{4c^2e^{2x}}{1-c^2e^{2x}}\right]$$

10.7-7

⑦

~~$\int dy = \int 2 + \frac{4c^2 e^{2x}}{(1-c^2e^{2x})} dx$~~

$$\int dy = \int 2dx - 2\int \frac{-2c^2e^{2x}}{1-c^2e^{2x}} dx$$

or $y = 2x - 2\ln(1-c^2e^{2x}) + C_2$

$y = 2x - 2\ln(1-C_1e^{2x}) + C_2$

④ $x\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} = 12x^3$; $y(1) = 0$, $y'(1) = 1$, $y''(1) = 0$

Sol. Given

$$x\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} = 12x^3 \quad \text{—— ①}$$

Here y is absent

$\left(\because \frac{d^2y}{dx^2} \text{ is lowest order in eq ①}\right)$ ∴ Put $\frac{d^2y}{dx^2} = p$

then $\frac{d^3y}{dx^3} = \frac{dp}{dx}$

Put in ①

$$x\frac{dp}{dx} - 2p = 12x^3$$

or

$$\frac{dp}{dx} - \frac{2}{x}p = 12x^2 \quad \text{—— ②}$$

It is a linear diff eq in p

I.F. $= e^{-\int 2/x\, dx} = e^{-2\ln x} = e^{\ln x^{-2}} = \frac{1}{x^2}$

Multiply both sides of ② by I.F. $\frac{1}{x^2}$

$$\int d\left(p.\frac{1}{x^2}\right) = \int 12\, dx$$

$$\frac{p}{x^2} = 12x + C$$

or $p = 12x^3 + Cx^2$

$$\frac{d^2y}{dx^2} = 12x^3 + Cx^2$$

~~But $y''(1) = 0$~~

$\Rightarrow \quad 0 = 12 + C \quad \Rightarrow \quad \boxed{C = -12}$

So

$$\frac{d^2y}{dx^2} = 12x^3 - 12x^2$$

$$\Rightarrow \int \frac{d^2y}{dx^2}\,dx = \int (12x^3 - 12x^2)\,dx$$

or $$\frac{dy}{dx} = 3x^4 - 4x^3 + C_2$$

But $y'(1) = 1$

So $1 = 3 - 4 + C_2 \quad \Rightarrow \quad \boxed{C_2 = 2}$

So

$$\frac{dy}{dx} = 3x^4 - 4x^3 + 2$$

$$\Rightarrow \int \frac{dy}{dx}\,dx = \int (3x^4 - 4x^3 + 2)\,dx$$

$$y = \frac{3}{5}x^5 - x^4 + 2x + C_3$$

But $y(1) = 0$

So $0 = \frac{3}{5} - 1 + 2 + C_3$

$0 = \frac{8}{5} + C_3$

$\Rightarrow \boxed{C_3 = -8/5}$

So req. Soln. is

$$y = \frac{3}{5}x^5 - x^4 + 2x - \frac{8}{5}$$

$$\therefore y = \frac{1}{5}(3x^5 - 5x^4 + 10x - 8)$$

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10.7

10.7-9

1062

Ext 5 $\dfrac{dy}{dx}\dfrac{d^3y}{dx^3}+\left(\dfrac{dy}{dx}\right)^2=2\left(\dfrac{d^2y}{dx^2}\right)^2$

Sol. Given

$$\frac{dy}{dx}\cdot\frac{d^3y}{dx^3}+\left(\frac{dy}{dx}\right)^2=2\left(\frac{d^2y}{dx^2}\right)^2 \quad \text{—— (1)}$$

Put $\dfrac{dy}{dx}=p$

then

$$\frac{d^2y}{dx^2}=\frac{dp}{dx}=\frac{dp}{dy}\cdot\frac{dy}{dx}=p\frac{dp}{dy}$$

&

$$\frac{d^3y}{dx^3}=\frac{d}{dx}\left(p\frac{dp}{dy}\right)=\frac{d}{dy}\left(p\frac{dp}{dy}\right)\cdot\frac{dy}{dx}$$

$$=\left[p\frac{d^2p}{dy^2}+\left(\frac{dp}{dy}\right)^2\right]p$$

Put in (1)

$$p^2\left[\left(\frac{dp}{dy}\right)^2+p\frac{d^2p}{dy^2}\right]+p^2=2p^2\left(\frac{dp}{dy}\right)^2$$

or

$$\left(\frac{dp}{dy}\right)^2+p\frac{d^2p}{dy^2}+1=2\left(\frac{dp}{dy}\right)^2$$

or $$p\frac{d^2p}{dy^2}=\left(\frac{dp}{dy}\right)^2-1 \quad \text{—— (2)}$$

Again

put $\dfrac{dp}{dy}=q$

$$\Rightarrow \frac{d^2p}{dy^2}=\frac{dq}{dy}=\frac{dq}{dp}\cdot\frac{dp}{dy}=q\frac{dq}{dp}$$

Put in (2)

$$pq\frac{dq}{dp}=q^2-1$$

$$\int\frac{q}{q^2-1}\,dq=\int\frac{dp}{p}$$

10.7-10

$$\frac{1}{2}\ln(q^2-1) = \ln p + \ln c_1$$

$$\ln\sqrt{q^2-1} = \ln pc_1$$

$$\Rightarrow \sqrt{q^2-1} = pc_1$$

or $q^2-1 = p^2c_1^2$

$$q^2 = 1+p^2c_1^2$$

$$q = \sqrt{1+p^2c_1^2}$$

$$\frac{dp}{dy} = \sqrt{1+p^2c_1^2}$$

$$\int\frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\left(\frac{x}{a}\right)$$

$$\frac{dp}{\sqrt{1+p^2c_1^2}} = dy$$

$$\frac{1}{c_1}\int\frac{dp}{\sqrt{\left(\frac{1}{c_1}\right)^2+p^2}} = \int dy$$

$$\frac{1}{c_1}\sinh^{-1}(c_1p) = y + c_2$$

or

$$\sinh^{-1}(c_1p) = c_1y + c_2c_1$$

$$c_1p = \sinh(c_1y + c_1c_2)$$

$$p = \frac{1}{c_1}\sinh(c_1y + c_1c_2)$$

$$\frac{dy}{dx} = \frac{1}{c_1}\sinh(c_1y + c_1c_2)$$

$$\int\frac{dy}{\sinh(c_1y + c_1c_2)} = \frac{1}{c_1}\int dx$$

$$\int \operatorname{cosech}(c_1y + c_1c_2)\,dy = \frac{1}{c_1}\int dx$$

$$\ln\left[\cosh(c_1y + c_1c_2) + \tanh(c_1y + c_1c_2)\right] = \frac{1}{c_1}x + c_3$$

10-7-11

Type 2 ⑨

(5) $2y\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = 1$

Sol. Given

$$2y\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = 1 \quad \text{—— (1)}$$

Here x is absent

Put $\frac{dy}{dx} = p$

$$\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\cdot\frac{dy}{dx} = p\frac{dp}{dy}$$

Put in (1)

$$2yp\frac{dp}{dy} - p^2 = 1$$

$$2yp\frac{dp}{dy} = p^2+1$$

Separating variables

$$\int \frac{2p}{p^2+1}\,dp = \int \frac{dy}{y}$$

$$\ln(p^2+1) = \ln y + \ln c_1$$

$$\ln(p^2+1) = \ln c_1 y$$

$$\Rightarrow \quad p^2+1 = c_1 y$$

or $\quad p^2 = c_1 y - 1$

or

$$p = \pm\sqrt{c_1 y - 1}$$

$$\frac{dy}{dx} = \pm\sqrt{c_1 y - 1}$$

$$\int \frac{dy}{\sqrt{c_1 y - 1}} = \pm\int dx$$

$$\frac{1}{c_1}\int (c_1 y - 1)^{-1/2}(c_1)\,dy = \pm x + c_2$$

$$\frac{1}{c_1}\frac{(c_1y-1)^{1/2}}{1/2} = \pm x + c_2$$

$$\frac{1}{c_1}\, 2\sqrt{c_1y-1} = \pm x + c_2$$

$$2\sqrt{c_1y-1} = \pm c_1 x + c_1c_2$$

$$2\sqrt{c_1y-1} = \pm c_1 x + c_2$$

(b) $y\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = 4y^2 \ln y$ $\qquad y(1) = e,\ y'(1) = 2e$

Sol. Given

$$y\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = 4y^2 \ln y \quad \text{——— (1)}$$

Here x is absent.

Put $\frac{dy}{dx} = p$

$$\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\cdot\frac{dy}{dx} = p\frac{dp}{dy}$$

Put in (1)

$$yp\frac{dp}{dy} - p^2 = 4y^2 \ln y$$

$$yp\,dp - (p^2 + 4y^2 \ln y)\,dy = 0 \quad \text{——— (2)}$$

It is non exact diff. eq.

$M = yp$ | $N = -p^2 - 4y^2 \ln y$

$M_y = p$ | $N_p = -2p$

Now

$$\frac{N_p - M_y}{M} = \frac{-2p - p}{yp} = -\frac{3}{y} \quad (\text{a fn. of } y)$$

So

$$\text{I.F.} = e^{\int -\frac{3}{y}dy} = e^{-3\ln y} = e^{\ln y^{-3}} = \frac{1}{y^3}$$

Multiplying eq. (2) by I.F. $= \frac{1}{y^3}$

10.7-13

(11)

$$\cancel{\frac{p}{y^2}dp - \left(\frac{p^2+4y^2\ln y}{y^3}\right)dy = 0}$$

It is an exact diff. eq.

So $\int M\,dp + \int(\text{terms of } N \text{ free from } p)\,dy = c_1$

$$\int \frac{p}{y^2}\,dp + \int -4\frac{\ln y}{y}\,dy = c_1$$

$$\frac{p^2}{2y^2} - 4\left[\frac{(\ln y)^2}{2}\right] = c_1$$

or $\frac{p^2}{2y^2} - 2(\ln y)^2 = c_1$

$$p^2 - 4y^2(\ln y)^2 = 2c_1y^2$$

But $y'(1) = 2e$ & $y(1) = e$

So when $x = 1$, $y = e$, $y' = p = 2e$

So $\cancel{4}e^2 - \cancel{4}e^2(\ln e)^2 = 2c_1e^2$

$\Rightarrow \boxed{c_1 = 0}$

Hence $p^2 - 4y^2(\ln y)^2 = 0$

$$p^2 = 4y^2(\ln y)^2$$

$$p = 2y\ln y$$

$$\frac{dy}{dx} = 2y\ln y$$

$$\int \frac{1}{y\ln y}\,dy = 2\int dx$$

$$\ln\ln y = 2x + c_2$$

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But $y(1) = e^{\cdots}$

So $\ln \ln e = 2 + C_2$

$0 = 2 + C_2$

$\Rightarrow \boxed{C_2 = -2}$

Hence

$\ln \ln y = 2x - 2$

Ext 1 $\dfrac{d^2y}{dx^2} = \left(\dfrac{dy}{dx}\right)^2 + 1$ 'y' absent

Soln. Given

$$\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2 + 1 \quad \text{——— (1)}$$

Put $\dfrac{dy}{dx} = p$

$\dfrac{d^2y}{dx^2} = \dfrac{dp}{dx}$

Put in (1)

$$\frac{dp}{dx} = p^2 + 1$$

separating

$$\int \frac{1}{p^2+1}\, dp = \int dx$$

$$\tan^{-1} p = x + C_1$$

$$p = \tan(x + C_1)$$

$$\int \frac{dy}{dx} dx = \int \tan(x + C_1)\, dx$$

$$y = \ln \sec(x + C_1) + \ln C_2$$

$$y = \ln C_2 \sec(x + C_1)$$

$$\Rightarrow e^y = \frac{C_2}{\cos(x + C_1)}$$

10.7-15

1966 (B)

(7) $(1+y^2)\dfrac{d^2y}{dx^2}+\left(\dfrac{dy}{dx}\right)^3+\dfrac{dy}{dx}=0$

Sol. Given

$$(1+y^2)\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3+\frac{dy}{dx}=0 \quad \text{--- (1)}$$

Here x is absent.

Put $\dfrac{dy}{dx}=p$

$$\Rightarrow \frac{d^2y}{dx^2}=\frac{dp}{dx}=\frac{dp}{dy}\,\frac{dy}{dx}=p\frac{dp}{dy}$$

Put in (1)

$$(1+y^2)\,p\frac{dp}{dy}+p^3+p=0$$

$$p\left\{(1+y^2)\frac{dp}{dy}+(p^2+1)\right\}=0$$

$$\text{or } (1+y^2)\frac{dp}{dy}+(p^2+1)=0$$

$$(1+y^2)\frac{dp}{dy}=-(p^2+1)$$

Separating variables $\displaystyle\int\frac{dp}{p^2+1}=-\int\frac{dy}{y^2+1}$

$$\tan^{-1}p=-\tan^{-1}y+c_1$$

$$\tan^{-1}p+\tan^{-1}y=c_1$$

$$\tan^{-1}\left(\frac{p+y}{1-py}\right)=c_1$$

$$\frac{p+y}{1-py}=\tan c_1=c$$

$$p+y=c-cpy$$

$$p(1+cy)=c-y$$

$$p=\frac{c-y}{1+cy}$$

$$\frac{dy}{dx}=\frac{c-y}{cy+1}$$

$$\frac{dy}{dx}=-\frac{y-c}{cy+1}$$

10.7-16

$$\int \frac{cy+1}{y-c}\,dy = -\int dx$$

$$y-c \,\big)\, cy+1 \,\big(\, c$$
$$\underline{-cy \mp c^2}$$
$$1+c^2$$

$$\int c + \frac{1+c^2}{y-c}\,dy = -\int dx$$

$$cy + (1+c^2)\ln(y-c) = -x + c_2$$

$$\therefore\; x = c_2 - cy - (1+c^2)\ln|y-c|$$

$$x = c_1 + c_1 y - (1+c_1^2)\ln|y+c_1| \quad \text{where } c_1 = -c$$

(8) $y\dfrac{d^2y}{dx^2} + 4y^2 - \dfrac{1}{2}\left(\dfrac{dy}{dx}\right)^2 = 0$; $y(0) = 1$, $y'(0) = \sqrt{8}$

Sol. Given that

$$y\frac{d^2y}{dx^2} + 4y^2 - \frac{1}{2}\left(\frac{dy}{dx}\right)^2 = 0 \qquad (1)$$

Here x is absent

Put $\dfrac{dy}{dx} = p$

$$\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\cdot\frac{dy}{dx} = p\frac{dp}{dy}$$

Put in (1)

$$yp\frac{dp}{dy} + 4y^2 - \frac{1}{2}p^2 = 0$$

$$yp\,dp + \left(4y^2 - \frac{1}{2}p^2\right)dy = 0 \qquad (2)$$

Here

$M = yp$	$N = 4y^2 - \frac{1}{2}p^2$
$M_y = p$	$N_p = -p$

So eq. (2) is non exact

Now $\dfrac{N_p - M_y}{M} = \dfrac{-p-p}{yp} = -\dfrac{2}{y}$ (fn. of y).

10.7-17

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So I.F. $= e^{-2\int \frac{1}{y} dy} = e^{-2\ln y} = e^{\ln y^{-2}} = \frac{1}{y^2}$

Multiply both sides of ② by I.F. $\frac{1}{y^2}$

$$\left(\frac{p}{y}\right) dp + \left(4 - \frac{p^2}{2y^2}\right) dy = 0$$

It is an exact diff. eq.

$$\int M\, dP + \int (\text{terms of N free from p})\, dy = c$$

$$\int \frac{p}{y}\, dp + \int 4\, dy = c$$

$$\frac{p^2}{2y} + 4y = c$$

But $y(0) = 1$ & $y'(0) = \sqrt{8}$

So $\frac{8}{2} + 4 = c$

$\Rightarrow \boxed{c = 8}$

Hence

$$\frac{p^2}{2y} + 4y = 8$$

$$p^2 + 8y^2 = 16y$$

$$p^2 = 16y - 8y^2$$

$$p^2 = 8(2y - y^2)$$

$$p = \sqrt{8}\,\sqrt{2y - y^2}$$

$$\frac{dy}{dx} = \sqrt{8}\,\sqrt{2y - y^2}$$

$$\int \frac{dy}{\sqrt{-(y^2 - 2y)}} = \sqrt{8}\int dx$$

$$\int \frac{dy}{\sqrt{-(y^2 - 2y + 1) + 1}} = \sqrt{8}\int dx$$

$$\int \frac{dy}{\sqrt{1 - (y-1)^2}} = \sqrt{8}\int dx$$

~~$\sin^{-1}(y-1) = \sqrt{8}x + C$~~

But $y(0) = 1$

So $\sin^{-1}(1-1) = C$

$\Rightarrow \boxed{C=0}$

So $\sin^{-1}(y-1) = \sqrt{8}\,x$

$\Rightarrow \sin^{-1}(y-1) = \sqrt{8}\,x$

$y-1 = \sin(\sqrt{8}\,x)$

or $y = 1 + \sin(\sqrt{8}\,x)$

Example 22

$$(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} + ax = 0$$

Sol. Given that

$$(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} + ax = 0 \quad (1)$$

Here y is absent

Put $\frac{dy}{dx} = p$

$\frac{d^2y}{dx^2} = \frac{dp}{dx}$

Put in (1)

$$(1+x^2)\frac{dp}{dx} + xp + ax = 0$$

$$(1+x^2)\frac{dp}{dx} = -x(a+p)$$

$$\int\frac{dp}{a+p} = -\int\frac{x}{1+x^2}\,dx$$

$$\ln(a+p) = -\frac{1}{2}\ln(1+x^2) + \ln C_1$$

$$\ln(a+p) = \ln C_1(1+x^2)^{-1/2}$$

$$\Rightarrow a+p = \frac{C_1}{\sqrt{1+x^2}}$$

10.7-19

~~or $a+\frac{dy}{dx} = \frac{c_1}{\sqrt{1+x^2}}$~~

$$\frac{dy}{dx} = \frac{c_1}{\sqrt{1+x^2}} - a$$

$$\int dy = \int\left(\frac{c_1}{\sqrt{1+x^2}} - a\right)dx$$

$\therefore$ $y = c_1 \sinh^{-1}x - ax + c_2$ is the general soln.

Example 23 $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = \frac{dy}{dx}$

Sol:. Given that

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = \frac{dy}{dx} \quad \text{——— (1)}$$

Here x is absent

Put $\frac{dy}{dx} = p$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\cdot\frac{dy}{dx} = p\frac{dp}{dy}$$

Put in (1)

$$yp\frac{dp}{dy} + p^2 = p$$

or

$$y\frac{dp}{dy} + p = 1$$

or

$$\frac{dp}{dy} + \frac{1}{y}p = \frac{1}{y} \quad \text{——— (2)}$$

It is a linear diff. eq.

I.F. $= e^{\int \frac{1}{y}dy} = e^{\ln y} = y$

Multiply both sides of (2) by I.F. y

$$\int d(p.y) = \int 1\,dy$$

~~$py = y + c_1$~~

or $y\dfrac{dy}{dx} = y + c_1$

$$\int \frac{y}{y+c_1}\,dy = \int dx$$

$$\int \frac{(y+c_1)-c_1}{y+c_1}\,dy = \int dx$$

$$\int 1 - \frac{c_1}{y+c_1}\,dy = \int dx$$

$$y - c_1 \ln(y+c_1) = x + c_2$$

or $y = x + c_1 \ln(y+c_1) + c_2$

Exact linear equations: If we differentiate the linear eq.

$$R_0 y^{(n-1)} + R_1 y^{(n-2)} + \cdots + R_{n-2} y' + R_{n-1} y = S(x) + c \quad \text{—(1)}$$

where

$R_0, R_1, R_2, \ldots, R_{n-2}, R_{n-1}$ are functions of x, we obtain another linear eq.

$$P_0 y^{n} + P_1 y^{n-1} + \cdots + P_{n-2} y'' + P_{n-1} y' + P_n y = Q(x) \quad \text{—(2)}$$

where

$P_0, P_1, P_2, \ldots, P_{n-2}, P_{n-1}, P_n$ are functions of x s.t.

$$\left.\begin{aligned}
Q(x) &= S'(x) \\
P_0 &= R_0 \\
P_1 &= R_0' + R_1 \\
P_2 &= R_1' + R_2 \\
P_3 &= R_2' + R_3 \\
&\ \,\vdots \\
P_{n-1} &= R_{n-2}' + R_{n-1} \\
P_n &= R_{n-1}'
\end{aligned}\right\} \longrightarrow (3)$$

From (3), we obtain

$$S(x) = \int Q(x)\,dx + c$$

$$\boxed{R_0} = P_0$$

$$\boxed{R_1} = P_1 - R_0' = \boxed{P_1 - P_0'}$$

$$\boxed{R_2} = P_2 - R_1' = \boxed{P_2 - P_1' + P_0''}$$

$$\boxed{R_3} = P_3 - R_2' = \boxed{P_3 - P_2' + P_1'' - P_0'''}$$

$$R_{n-2} = P_{n-2} - R'_{n-3}$$

$$= P_{n-2} - P'_{n-3} + P''_{n-4} - \cdots + (-1)^{n-2} P_0^{(n-2)}$$

$$R_{n-1} = P_{n-1} - R'_{n-2}$$

$$= P_{n-1} - P'_{n-2} + P''_{n-3} - \cdots + (-1)^{n-1} P_0^{(n-1)}$$

Under this conditions, the linear eq. (2) of order n is said to be exact & (1) is called its first integral.

From above, we obtain

$$\boxed{P_n - R'_{n-1} = P_n - P'_{n-1} + P''_{n-2} - P'''_{n-3} + \cdots + (-1)^n P_0^{(n)} = 0}$$

which is a necessary & sufficient condition for eq (2) to be exact.

Linear D.Eq. of Ist order

$$\frac{dy}{dx} + P(x)\, y = Q(x)$$

$$I.F = e^{\int P\,dx}$$

10.7.23

1076 (21)

(4) ~~$(2x^2+3x)\frac{d^2y}{dx^2}+(6x+3)\frac{dy}{dx}+2y=(x+1)e^x$~~

Sol. Given that

$$(2x^2+3x)\frac{d^2y}{dx^2}+(6x+3)\frac{dy}{dx}+2y=(x+1)e^x \quad \text{---(1)}$$

Here

$\boxed{n=2}$, $P_0 = 2x^2+3x$, $P_1 = 6x+3$, $P_2 = 2$, $Q(x) = (x+1)e^x$

Also $\boxed{P_2 - P_1' + P_0''} = 2-6+4 = \boxed{0}$

Hence (1) is exact & its first Integral is $\boxed{R_0\frac{dy}{dx} + R_1 y = S x}$ ---(2)

Now $\boxed{R_0} = P_0 = \boxed{2x^2+3x}$

$\boxed{R_1} = P_1 - P_0' = (6x+3)-(4x+3) = \boxed{2x}$

$$R_0\frac{dy}{dx} + R_1 y = S(x)$$

$S(x) = \int Q(x)dx$

$S(x) = \int \underset{I}{(x+1)}\underset{II}{e^x}\,dx$ IBP

from (2) $(2x^2+3x)\frac{dy}{dx} + 2xy = (x+1)e^x - \int e^x.1\,dx$

$= (x+1)e^x - e^x + c_1$

$\therefore (2x^2+3x)\frac{dy}{dx} + 2xy = xe^x + c_1$

or

$$\frac{dy}{dx} + \frac{2xy}{2x^2+3x} = \frac{xe^x}{2x^2+3x} + \frac{c_1}{2x^2+3x}$$

or $\frac{dy}{dx} + \frac{2\cancel{x}}{\cancel{x}(2x+3)}y = \frac{\cancel{x}e^x}{\cancel{x}(2x+3)} + \frac{c_1}{x(2x+3)}$ ---(3) L.D.Eq

L.D.Eq

$\therefore$ I.F. $= e^{\int\frac{2}{2x+3}dx} = e^{\ln(2x+3)} = 2x+3$

Multiplying both sides of (3) by I.F. $2x+3$

$\therefore \int d(y(2x+3)) = \int e^x\,dx + \int\frac{c_1}{x}dx$

$y(2x+3) = e^x + c_1\ln x + c_2$ Ans.

(19) $\sin x \frac{d^2y}{dx^2} - \cos x \frac{dy}{dx} + 2y \sin x = 0$

Sol. Given that

$$\sin x \frac{d^2y}{dx^2} - \cos x \frac{dy}{dx} + 2y \sin x = 0 \quad \text{--- (1)}$$

Here $\boxed{n=2}$, $P_0 = \sin x$, $P_1 = -\cos x$, $P_2 = 2\sin x$, $Q(x) = 0$

Also

$\boxed{P_2 - P_1' + P_0''} = 2\sin x - \sin x - \sin x = \boxed{0}$

$\therefore$ eq (1) is exact & its first integral is

$$\boxed{R_0 \frac{dy}{dx} + R_1 y = S(x)} \quad \text{--- (2)} \qquad S(x) = \int Q(x) = \int 0\, dx = c_1$$

where

$\boxed{R_0} = P_0 = \boxed{\sin x}$

$\boxed{R_1} = P_1 - P_0' = -\cos x - \cos x = \boxed{-2\cos x}$

from (2) $\sin x \frac{dy}{dx} - 2\cos x . y = c_1$

or

$$\frac{dy}{dx} - 2\cot x . y = \frac{c_1}{\sin x} \quad \text{--- (3)} \quad \text{L.D.Eq}$$

$\therefore$ I.F. $= e^{-2\int \cot x} = e^{-2\ln \sin x} = e^{\ln \sin^{-2} x} = \frac{1}{\sin^2 x}$

Multiplying both sides of eq (3) by I.F. $\frac{1}{\sin^2 x}$

$$\int d\left(\frac{y}{\sin^2 x}\right) = c_1 \int \csc^3 x\, dx$$

Now $\int \csc^3 x\, dx = \int \csc x . \csc^2 x\, dx = -\csc x \cot x - \int (-\cot x)(-\csc x \cot x) dx$

$= -\csc x \cot x - \int \csc x (\csc^2 x - 1)\, dx$

$\int \csc^3 x\, dx = -\csc x \cot x - \int \csc^3 x\, dx + \int \csc x\, dx$

or $\int \csc^3 x\, dx = \frac{1}{2}\left[-\csc x \cot x + \ln(\csc x - \cot x)\right]$

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1078 (23)

~~So $\frac{y}{\sin^2 x} = -\frac{\operatorname{Cosec} x \operatorname{Cot} x}{2} + \frac{1}{2}\ln(\operatorname{Cosec} x - \operatorname{Cot} x) + C_2$~~

or $y = -\frac{\operatorname{Cot} x \sin x}{2} + \frac{1}{2}\sin^2 x \ln(\operatorname{Cosec} x - \operatorname{Cot} x) + C_2 \sin x$

(11) $(x+\sin x)\frac{d^3y}{dx^3} + 3(1+\cos x)\frac{d^2y}{dx^2} - 3\sin x\frac{dy}{dx} - y\cos x = -\sin x$

Sol. Given that

$$(x+\sin x)\frac{d^3y}{dx^3} + 3(1+\cos x)\frac{d^2y}{dx^2} - 3\sin x\frac{dy}{dx} - y\cos x = -\sin x \quad \text{——(1)}$$

Here $\boxed{n=3}$

$P_0 = x+\sin x$, $P_1 = 3(1+\cos x)$, $P_2 = -3\sin x$, $P_3 = -\cos x$ and $Q(x) = -\sin x$

Also

$\boxed{P_3 - P_2' + P_1'' - P_0'''} = -\cos x + 3\cos x - 3\cos x + \cos x = 0$

Hence given eq. (1) is exact & its first integral is

$$\boxed{R_0\frac{d^2y}{dx^2} + R_1\frac{dy}{dx} + R_2 y = S(x)} \quad \text{——(2)}$$

$S(x) = \int Q(x) = \int -\sin x\,dx$

where

$\boxed{R_0} = P_0 = \boxed{x+\sin x}$

$\boxed{R_1} = P_1 - P_0' = 3+3\cos x - 1 - \cos x = \boxed{2(1+\cos x)}$

$\boxed{R_2} = P_2 - P_1' + P_0'' = -\cancel{3}\sin x + \cancel{3}\sin x - \sin x = \boxed{-\sin x}$

from (2) $(x+\sin x)\frac{d^2y}{dx^2} + 2(1+\cos x)\frac{dy}{dx} - \sin x \cdot y = \cos x + C_1$ ——(3)

Here in (3)

$\boxed{n=2}$, $P_0 = x+\sin x$, $P_1 = 2(1+\cos x)$, $P_2 = -\sin x$ & $Q(x) = \cos x + C_1$

Also $\boxed{P_2 - P_1' + P_0''} = -\sin x + 2\sin x - \sin x = \boxed{0}$ Hence exact

and integral is

$$\boxed{R_0\frac{dy}{dx} + R_1 y = S(x)} \quad \text{——(4)}$$

$S(x) = \int Q(x)dx = \int(\cos x + C_1)dx$

~~where~~ $\boxed{R_0} = P_0 = \boxed{x+\sin x}$

$\boxed{R_1} = P_1 - P_0 = 2(1+\cos x) - (1+\cos x) = 1+\cos x$

from (4) $(x+\sin x)\dfrac{dy}{dx} + (1+\cos x)\,y = \sin x + c_1 x + c_2$

$\div$ by $(x+\sin x)$

$$\frac{dy}{dx} + \frac{(1+\cos x)}{(x+\sin x)}\,y = \frac{\sin x}{x+\sin x} + \frac{c_1 x}{x+\sin x} + \frac{c_2}{x+\sin x} \quad (5)$$

L.D.Eq

(5) is L.D.E

$$\therefore \text{I.F.} = e^{\int \frac{1+\cos x}{x+\sin x}dx} = e^{\ln(x+\sin x)} = (x+\sin x)$$

Multiply both sides of above eq. by I.F. $(x+\sin x)$

$$\int d\big(y(x+\sin x)\big) = \int \sin x + c_1 x + c_2 \; dx$$

$$y(x+\sin x) = -\cos x + c_1\frac{x^2}{2} + c_2 x + c_3$$

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(23)

Ext $~~\sin x\frac{d^3y}{dx^3}+(2\cos x+1)\frac{d^2y}{dx^2}-\sin x\frac{dy}{dx}=\cos x~~$

Sol:- Given that

$$\sin x\frac{d^3y}{dx^3}+(2\cos x+1)\frac{d^2y}{dx^2}-\sin x\frac{dy}{dx}=\cos x \quad \text{---} (1)$$

Here $\boxed{n=3}$

$P_0=\sin x$, $P_1=2\cos x+1$, $P_2=-\sin x$, $P_3=0$, $Q(x)=\cos x$

Also $\boxed{P_3-P_2'+P_1''-P_0'''}=0+\cos x-2\cos x+\cos x=\boxed{0}$

Hence given eq (1) is exact & its integral is

$$\boxed{R_0\frac{d^2y}{dx^2}+R_1\frac{dy}{dx}+R_2y}=S(x) \quad \text{---} (2)$$

$S(x)=\int Q(x)\,dx$
$=\int \cos x\,dx$

where

$\boxed{R_0}=P_0=\sin x$

$\boxed{R_1}=P_1-P_0'=2\cos x+1-\cos x=1+\cos x$

$\boxed{R_2}=P_2-P_1'+P_0''=-\sin x+2\sin x-\sin x=0$

(2) becomes

$$\sin x\frac{d^2y}{dx^2}+(1+\cos x)\frac{dy}{dx}=\sin x+C_1 \quad \text{---} (3)$$

Here

$n=2$, $P_0=\sin x$, $P_1=1+\cos x$, $P_2=0$ $Q(x)=\sin x+C_1$

Also $\boxed{P_2-P_1'+P_0''}=0+\sin x-\sin x=\boxed{0}$ Hence exact & its integral

Hence eq (3) is exact & its integral is

is $\boxed{R_0\frac{dy}{dx}+R_1y=S(x)} \quad \text{---} (4)$

$S(x)=\int Q(x)\,dx=\int(\sin x+C_1)\,dx$
$=-\cos x+C_1x+C_2$

where

$R_0=P_0=\sin x$ & $R_1=P_1-P_0'=1+\cancel{\cos x}-\cancel{\cos x}=1$

from (4) $\sin x\frac{dy}{dx}+y=-\cos x+C_1x+C_2$

$\div$ by $\sin x$

$$\frac{dy}{dx}+\frac{1}{\sin x}y=\frac{-\cos x}{\sin x}+\frac{C_1x}{\sin x}+\frac{C_2}{\sin x} \quad \text{L.D.Eq} \ (5)$$

It is a linear diff. eq.

$$I.F. = e^{\int \csc x\,dx} = e^{\ln \tan x/2} = \tan x/2$$

Multiply both sides of above eq (5) by I.F. $\tan\frac{x}{2}$

$$\therefore \int d\left(y\tan\frac{x}{2}\right) = \int \tan\frac{x}{2}\left(-\cot x + c_1 x \csc x + c_2\csc x\right)dx + c_3$$

Imp

(19) $(e^x+2x)\frac{d^4y}{dx^4} + (4e^x+8)\frac{d^3y}{dx^3} + 6e^x\frac{d^2y}{dx^2} + 4e^x\frac{dy}{dx} + e^x y = \frac{1}{x^5}$

Sol. Given that

$$(e^x+2x)\frac{d^4y}{dx^4} + (4e^x+8)\frac{d^3y}{dx^3} + 6e^x\frac{d^2y}{dx^2} + 4e^x\frac{dy}{dx} + e^x y = \frac{1}{x^5} \quad (1)$$

Here $\boxed{n=4}$

$P_0 = e^x+2x,\ P_1 = (4e^x+8),\ P_2 = 6e^x;\ P_3 = 4e^x,\ P_4 = e^x;\ Q(x) = \frac{1}{x^5}$

Also

$$\boxed{P_4 - P_3' + P_2'' - P_1''' + P_0''''} = e^x - 4e^x + 6e^x - 4e^x + e^x = \boxed{0} \text{ Hence exact}$$

and 1st integral is

$$\boxed{R_0\frac{d^3y}{dx^3} + R_1\frac{d^2y}{dx^2} + R_2\frac{dy}{dx} + R_3 y = S(x)} \quad (2)$$

$S(x) = \int Q(x) = \int x^{-5}dx$

where

$R_0 = P_0 = \boxed{e^x+2x}$

$R_1 = P_1 - P_0' = (4e^x+8) - (e^x+2) = \boxed{3e^x+6}$

$R_2 = P_2 - P_1' + P_0'' = 6e^x - 4e^x + e^x = \boxed{3e^x}$

$R_3 = P_3 - P_2' + P_1'' - P_0''' = 4e^x - 6e^x + 4e^x - e^x = \boxed{e^x}$

from (2)

$$(e^x+2x)\frac{d^3y}{dx^3} + (3e^x+6)\frac{d^2y}{dx^2} + 3e^x\frac{dy}{dx} + e^x y = \frac{x^{-4}}{-4} + c_1 \quad (3)$$

Here $n=3$; $P_0 = e^x+2x$; $P_1 = 3e^x+6$; $P_2 = 3e^x$, $P_3 = e^x$

Also $\boxed{P_3 - P_2' + P_1'' - P_0'''} = e^x - 3e^x + 3e^x - e^x = 0$ $\& Q(x) = \frac{x^{-4}}{-4} + c_1$

Hence eq. (3) is also exact & its integral is

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1082 (27)

$$\boxed{R_0\frac{d^2y}{dx^2}+R_1\frac{dy}{dx}+R_2y=S(x)} \quad \text{--- (4)} \qquad S(x)=\int Q(x)\,dx=\int\left(\frac{x^{-4}}{-4}+c_1\right)dx$$

$$S(x) = \frac{x^{-3}}{12}+c_1x+c_2$$

$\boxed{R_0} = P_0 = e^x+2x$

$\boxed{R_1} = P_1-P_0' = (3e^x+6)-(e^x+2) = 2e^x+4$

$\boxed{R_2} = P_2-P_1'+P_0'' = 3e^x-3e^x+e^x = e^x$

from (4) $(e^x+2x)\frac{d^2y}{dx^2}+(2e^x+4)\frac{dy}{dx}+e^xy = \frac{x^{-3}}{12}+c_1x+c_2$ --- (5)

Again

Here $\boxed{n=2}$, $P_0 = e^x+2x$, $P_1 = 2e^x+4$, $P_2 = e^x$, $Q(x) = \frac{x^{-3}}{12}+c_1x+c_2$

Also $\boxed{P_2-P_1'+P_0''} = e^x-2e^x+e^x = \boxed{0}$ Hence exact & its integral is

$$\boxed{R_0\frac{dy}{dx}+R_1y=S(x)} \text{ --- (6)} \qquad S(x)=\int Q(x)=\left(\frac{x^{-3}}{12}+c_1x+c_2\right)dx$$

$$= \frac{x^{-2}}{-24}+c_1\frac{x^2}{2}+c_2x+c_3$$

where

$\boxed{R_0} = P_0 = e^x+2x$

$\boxed{R_1} = P_1-P_0' = (2e^x+4)-(e^x+2) = e^x+2$

from (6) $(e^x+2x)\frac{dy}{dx}+(e^x+2)y = \frac{x^{-2}}{-24}+c_1\frac{x^2}{2}+c_2x+c_3$

$\div$ by (e^x+2x)

$$\frac{dy}{dx}+\left(\frac{e^x+2}{e^x+2x}\right)y = \frac{\left(\frac{-1}{24x^2}+c_1\frac{x^2}{2}+c_2x+c_3\right)}{(e^x+2x)} \quad \text{--- (7)}$$

L.D.Eq

L.D.Eq:

$\therefore$ I.F $= e^{\int\left(\frac{e^x+2}{e^x+2x}\right)dx} = e^{\ln(e^x+2x)} = (e^x+2x)$

Multiply both sides of above eq (7) by I.F. (e^x+2x)

$$\int d\left(y(e^x+2x)\right) = \int \frac{-1}{24x^2}+c_1\frac{x^2}{2}+c_2x+c_3\;dx+c_4$$

$$y(e^x+2x) = \frac{1}{24x}+\frac{c_1}{6}x^3+c_2\frac{x^2}{2}+c_3x+c_4$$

Example 24. ~~$(x^2+1)\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y=2\cos x-2x$~~

Sol. Given that

$$(x^2+1)\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y=2\cos x-2x \quad \text{①}$$

Here $n=2$, $P_0=x^2+1$, $P_1=4x$, $P_2=2$

Also $P_2-P_1'+P_0''=2-4+2=0$

Hence given eq ① is exact & its integral is

$$R_0\frac{dy}{dx}+R_1y=\int(2\cos x-2x)\,dx+C_1$$

where

$R_0=P_0=x^2+1$

$R_1=P_1-P_0'=4x-2x=2x$

So above eq becomes

$$(x^2+1)\frac{dy}{dx}+2xy=2\sin x-x^2+C_1 \quad \text{②}$$

Also it is a linear diff eq.

Again here $n=1$, $P_0=x^2+1$, $P_1=2x$

Also $P_1-P_0'=2x-2x=0$

Hence eq ② is exact & its integral is

$$R_0y=\int(2\sin x-x^2+C_1)\,dx+C_2$$

where

$R_0=P_0=x^2+1$

So above eq becomes

$$(x^2+1)y=-2\cos x-\frac{x^3}{3}+C_1x+C_2$$

10·7−31

(29)

Exmpl 25 $\cancel{(2x-1)\frac{d^3y}{dx^3}+(4+x)\frac{d^2y}{dx^2}+2\frac{dy}{dx}=0}$

Sol. Given that

$$(2x-1)\frac{d^3y}{dx^3}+(4+x)\frac{d^2y}{dx^2}+2\frac{dy}{dx}=0 \quad \text{——— (1)}$$

Here $n = 3$

$P_0 = 2x-1$, $P_1 = 4+x$, $P_2 = 2$, $P_3 = 0$ $Q(x) = 0$.

Also

$\boxed{P_3 - P_2' + P_1'' - P_0'''} = 0 - 0 + 0 - 0 = \boxed{0}$

Hence given eq. (1) is exact & its integral is

$$\boxed{R_0\frac{d^2y}{dx^2}+R_1\frac{dy}{dx}+R_2 y = S(x)} \quad \text{——— (2)}$$

$S(x) = \int Q(x)dx = \int 0\,dx = c_1$

where

$R_0 = P_0 = 2x-1$

$R_1 = P_1 - P_0' = 4+x-2 = 2+x$

$R_2 = P_2 - P_1' + P_0'' = 2-1+0 = 1$

So above eq. becomes

from (2) $$(2x-1)\frac{d^2y}{dx^2}+(2+x)\frac{dy}{dx}+y = c_1 \quad \text{——— (3)}$$

Here

$n = 2$, $P_0 = 2x-1$, $P_1 = (2+x)$ & $P_2 = 1$, $Q(x) = c_1$

Also $\boxed{P_2 - P_1' + P_0''} = 1-1+0 = \boxed{0}$

Hence eq. (3) is exact & its integral is

$$\boxed{R_0\frac{dy}{dx}+R_1 y = S(x)} \quad \text{——— (4)}$$

$S(x) = \int Q(x)dx = \int c_1\,dx = c_1 x$

where

$R_0 = P_0 = (2x-1)$

$R_1 = P_1 - P_0' = 2+x-2 = x$

So above eq. (4) becomes

$(2x-1)\frac{dy}{dx} + xy = c_1x + c_2$ — (5)

Here $n = 1$, $P_0 = 2x-1$, $P_1 = x$

Also $P_1 - P_0' = x - 2 \neq 0$

Hence given eq. is not exact

Now from eq (5)

$$\frac{dy}{dx} + \left(\frac{x}{2x-1}\right)y = \frac{c_1x + c_2}{2x-1} \quad \text{—— (6)} \quad \text{L.D.Eq}$$

$$\text{I.F.} = e^{\int \frac{x}{2x-1}dx} = e^{\frac{1}{2}\int \frac{(2x-1)+1}{2x-1}dx} = e^{\frac{1}{2}\int 1 + \frac{1}{2x-1}dx} = e^{\frac{1}{2}[x + \frac{1}{2}\ln(2x-1)]}$$

$$= e^{x/2} e^{\frac{1}{4}\ln(2x-1)} = e^{x/2} e^{\ln(2x-1)^{1/4}} = e^{x/2}(2x-1)^{1/4}$$

Multiply both sides of above eq (6) by I.F. $e^{x/2}(2x-1)^{1/4}$

$$d\left(y e^{x/2}(2x-1)^{1/4}\right) = \frac{e^{x/2}(c_1x + c_2)}{(2x-1)^{3/4}}$$

$$\int d\left[y e^{x}(2x-1)^{1/4}\right] = \int \frac{e^{x/2}(c_1x + c_2)}{(2x-1)^{3/4}}dx + c_3$$

or $$y e^{x}(2x-1)^{1/4} = \int \frac{e^{x/2}(c_1x + c_2)}{(2x-1)^{3/4}}dx + c_3$$

Ex 10.7

(2) $$2x\frac{d^3y}{dx^3}\frac{d^2y}{dx^2} = \left(\frac{d^2y}{dx^2}\right)^2 - a^2 \quad \text{——(1)}$$

Putting $\frac{d^2y}{dx^2} = p$ in (1) & $\frac{d^3y}{dx^3} = \frac{dp}{dx}$

$$2xp\left(\frac{dp}{dx}\right) = p^2 - a^2$$

$$\frac{2p\,dp}{p^2-a^2} = \frac{dx}{x}$$

Integrating $\ln(p^2-a^2) = \ln x + \ln c_1$

$$p^2 - a^2 = c_1 x \quad \Rightarrow p^2 = c_1 x + a^2$$

$$p = \sqrt{c_1 x + a^2}$$

$$\frac{d^2y}{dx^2} = \sqrt{c_1 x + a^2} \quad \Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = (c_1 x + a^2)^{\frac{1}{2}}$$

Integrating $$\frac{dy}{dx} = \int (c_1 x + a^2)^{\frac{1}{2}} dx + c_2$$

$$= \frac{1}{c_1}\int (c_1 x + a^2)^{\frac{1}{2}} c_1 dx + c_2$$

$$\frac{dy}{dx} = \frac{2}{3c_1}(c_1 x + a^2)^{3/2} + c_2$$

Again Integrating

$$y = \frac{2}{3c_1}\int \frac{1}{c_1}(c_1 x + a^2)^{3/2} c_1 dx + \int c_2 dx$$

$$= \frac{2}{3c_1^2}\frac{2}{5}(c_1 x + a^2)^{5/2} + c_2 x + c_3$$

$$y = \frac{4}{15c_1^2}(c_1 x + a^2)^{5/2} + c_2 x + c_3$$

(13) $$x^5\frac{d^2y}{dx^2} + 3x^3\frac{dy}{dx} + (3-6x)x^2 y = x^4 + 2x - 5 \quad \text{——(1)}$$

$P_0 = x^5$

$P_1 = 3x^3$

$P_2 = 3x^2 - 6x^3$

$$P_2 - P_1' + P_0'' = 3x^2 - 6x^3 - 9x^2 + 20x^3 = 14x^3 - 6x^2 \neq 0$$

$\therefore$ (1) is not exact

We Multiply (1) by x^m and choose 'm' so as to make it exact.

$$x^{m+5}\frac{d^2y}{dx^2} + 3x^{m+3}\frac{dy}{dx} + (3-6x)x^{m+2}y = x^m(x^4 + 2x + 5) \quad \text{——(2)}$$

Now $P_0 = x^{m+5}$

$P_1 = 3x^{m+3}$

$P_2 = (3-6x)x^{m+2}$

$$P_2 - P_1' + P_0'' = 3x^{m+2} - 6x^{m+3} - 3(m+3)x^{m+2} + (m+5)(m+4)x^{m+3}$$

$$= 3x^{m+2} - 6x^{m+3} - 9x^{m+2} - 3mx^{m+2} + m^2x^{m+3} + 9mx^{m+3} + 20x^{m+3}$$

$= m^2x^{m+3} + 9mx^{m+3} + 14x^{m+3} - 6x^{m+2} - 3mx^{m+2}$

To make ② exact put $= 0$.

$x^{m+3}(m^2+9m+14) - x^{m+2}(6x+3m) = 0$

$x^{m+3}(m^2+7m+2m+14) - x^{m+2}\,3(2x+m) = 0$

$x^{m+3}[m(m+7)+2(m+7)] - x^{m+2}\,3(2+m) = 0$

$x^{m+3}(m+2)(m+7) - x^{m+2}\,3(2+m) = 0$

$(m+2)\left[x^{m+3}(m+7) - 3x^{m+2}\right] = 0$

$m = -2$

Put $m=-2$ in ②

$$x^3\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + (3-6x)y = x^{-2}(x^4+2x-5)$$

$$x^3\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + (3-6x)y = x^2 + \frac{2}{x} - \frac{5}{x^2}$$

This is an exact. $\because \begin{cases} P_0 = x^3 \\ P_1 = 3x \\ P_2 = 3-6x \end{cases}$ $\quad P_2 - P_1' + P_0'' = 3-6x-3+6x = 0$

Now first integral is $R_0\frac{dy}{dx} + R_1 y = \int S(x)\,dx$ $\quad S(x) = \int x^2 + \frac{2}{x} - \frac{5}{x^2}$

$R_0 = x^3$

$R_1 = P_1 - P_0'$

$R_1 = 3x - 3x^2$

$$x^3\frac{dy}{dx} + (3x-3x^2)y = \int x^2 + \frac{2}{x} - \frac{5}{x^2}\,dx$$

$$x^3\frac{dy}{dx} + 3x(1-x)y = \frac{x^3}{3} + 2\ln x + \frac{5}{x} + c_1$$

$$\frac{dy}{dx} + \frac{3}{x^2}(1-x)y = \frac{1}{3} + \frac{2\ln x}{x^3} + \frac{5}{x^4} + \frac{c_1}{x^3} \qquad \text{L.D.E}$$

$$I.F = e^{\int\left(\frac{3}{x^2} - \frac{3}{x}\right)dx} = e^{\int 3x^{-2}dx - 3\int\frac{dx}{x}} = e^{-\frac{3}{x} - 3\ln x}$$

$$I.F = e^{\ln\left(\frac{e^{-3/x}}{x^3}\right)} = \frac{1}{x^3}e^{-3/x}$$

$-\frac{3}{x} - 3\ln x$
$= -\frac{3}{x}\ln e - \ln x^3$
$= \ln e^{-\frac{3}{x}} - \ln x^3$
$= \ln\left(\frac{e^{-3/x}}{x^3}\right)$

$$\therefore \int d\left(y\cdot\frac{e^{-3/x}}{x^3}\right) = \int\frac{e^{-3/x}}{x^3}\left(\frac{1}{3} + \frac{2\ln x}{x^3} + \frac{5}{x^4} + \frac{c_1}{x^3}\right)dx$$

$$\frac{y}{x^3e^{3/x}} = \int\frac{1}{x^3e^{3/x}}\left(\frac{1}{3} + \frac{2\ln x}{x^3} + \frac{5}{x^4} + \frac{c_1}{x^3}\right)dx$$

$$\Rightarrow y = x^3e^{\frac{3}{x}}\int\left(\quad\right)dx$$

is the required solution

10.7 (33) 10.7-35

(14) (i) $$\frac{d^3y}{dx^3} = \ln x$$ (Integrate thrice ∵ order 3)

Integrating $\int \frac{d^3y}{dx^3} = \int \ln x\, dx$

I.B.P $\frac{d^2y}{dx^2} = \ln x \cdot x - \int \frac{1}{x}\cdot x\, dx$

$$\frac{d^2y}{dx^2} = x\ln x - x + c_1$$

Integrating $\int \frac{d^2y}{dx^2} = \int \underset{II}{x}\underset{I}{(\ln x - 1)}\,dx + \int c_1\, dx$

$$\frac{dy}{dx} = (\ln x - 1)\frac{x^2}{2} - \int \frac{1}{x}\cdot\frac{x^2}{2}\,dx + c_1 x$$

$$\frac{dy}{dx} = \frac{x^2}{2}(\ln x - 1) - \frac{x^2}{4} + c_1 x + c_2$$

Integrating

$$\int \frac{dy}{dx} = \frac{1}{2}\int \underset{II}{x^2}\underset{I}{(\ln x - 1)}\,dx - \int \frac{x^2}{4}dx + c_1\int x\,dx + \int c_2\, dx$$

$$y = \frac{1}{2}\left(\frac{x^3}{3}(\ln x - 1) - \int \frac{1}{x}\,\frac{x^3}{3}dx\right) - \frac{x^3}{12} + c_1\frac{x^2}{2} + c_2 x + c_3$$

$$y = \frac{x^3}{6}(\ln x - 1) - \frac{x^3}{18} - \frac{x^3}{12} + c_1\frac{x^2}{2} + c_2 x + c_3$$

$$36y = 6x^3(\ln x - 1) - 2x^3 - 3x^3 + 18c_1 x^2 + 36c_2 x + 36c_3$$

$$36y = 6x^3\ln x - 11x^3 + c_1' x^2 + c_2' x + c_3'$$

(14) (ii) $$\frac{d^2y}{dx^2} = \underset{I}{x^2}\underset{II}{\sin x}$$ (Integrate twice ∵ order 2)

Integrating $\frac{dy}{dx} = x^2(-\cos x) - \int \underset{I}{2x}\cdot\underset{II}{(-\cos x)}\,dx + c_1$

$$\frac{dy}{dx} = -x^2\cos x + 2\left(x\sin x - \int 1\cdot\sin x\, dx\right) + c_1$$

$$\frac{dy}{dx} = -x^2\cos x + 2x\sin x + 2\cos x + c_1$$

Integrating

$$\int \frac{dy}{dx} = -\int \underset{I}{x^2}\underset{II}{\cos x}\,dx + 2\int \underset{I}{x}\underset{II}{\sin x}\,dx + 2\int \cos x\,dx + \int c_1$$

$$y = -\left(x^2\sin x - \int 2x(+\sin x)dx\right) + 2\left(x(-\cos x) - \int -\cos x\, dx\right) + 2\sin x + c_1 x$$

$$y = -x^2\sin x + 2\int \underset{I}{x}\underset{II}{\sin x}\,dx - 2x\cos x + 2\int \cos x\,dx + 2\sin x + c_1 x$$

$$y = -x^2\sin x + 2(x(-\cos x) - \int -\cos x\, dx) - 2x\cos x + 2\sin x + 2\sin x + c_1 x$$

$$y = -x^2\sin x - 4x\cos x + 2\sin x + 4\sin x + c_1 x + c_2$$

$$y = -x^2\sin x - 4x\cos x + 6\sin x + c_1 x + c_2$$

(15) (i) $$\frac{d^2y}{dx^2} = -\cot y\,\operatorname{cosec}^2 y \quad \text{—①} \qquad y'(0) = 1 \quad y(0) = \frac{\pi}{2}$$ (Integrate twice ∵ order 2.)

× ① by $\frac{dy}{dx}$ $$\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right) = +\operatorname{cosec} y\,(-\operatorname{cosec} y\cot y)\frac{dy}{dx}$$

$\frac{d}{dx}(\operatorname{cosec} y) = -\operatorname{cosec} y\cot y\,\frac{dy}{dx}$

Integrating $$\frac{1}{2}\left(\frac{dy}{dx}\right)^2 = \frac{\operatorname{cosec}^2 y}{2} + c_1$$

$$\left(\frac{dy}{dx}\right)^2 = +\operatorname{cosec}^2 y + c_1$$

$y'(0) = 1,\ y(0) = \frac{\pi}{2}$ $\Rightarrow 1 = 1 + c_1$ or $\boxed{c_1 = 0}$

$$\frac{dy}{dx} = \operatorname{cosec} y \Rightarrow \frac{dy}{\operatorname{cosec} y} = dx$$

Integrating $\Rightarrow \int \sin y\, dy = \int dx$

$\Rightarrow -\cos y = x + c_2$

$y(0) = \frac{\pi}{2} \Rightarrow -\cos\frac{\pi}{2} = 0 + c_2$

$\boxed{0 = c_2}$

∴ $-\cos y = x$

$\cos y = -x$ is required sol.

K 120 R.S

(ii) $\dfrac{d^2y}{dx^2} = \dfrac{-a^2}{y^2}$

Multiply by $\dfrac{dy}{dx}$ $\quad \left(\dfrac{dy}{dx}\right)\dfrac{d^2y}{dx^2} = -\dfrac{a^2}{y^2}\cdot\dfrac{dy}{dx} = -a^2(y)^{-2}\dfrac{dy}{dx}$

Integrating $\dfrac{\left(\frac{dy}{dx}\right)^2}{2} = -\dfrac{y^{-1}}{-1}a^2 + c \Rightarrow \left(\dfrac{dy}{dx}\right)^2 = \dfrac{2a^2}{y} + c$

$$= \frac{2a^2 + yc}{y}$$

$$= \frac{2a^2\left(1 + \frac{c}{2a^2}y\right)}{y}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{2a^2(1 + c_1 y)}{y}$$

$$\left(\frac{dy}{dx}\right) = \sqrt{2}\,a\frac{\sqrt{1+c_1 y}}{\sqrt{y}}$$

Let $\sqrt{1+c_1y} = t \Rightarrow 1 + c_1 y = t^2$

$c_1\,dy = 2t\,dt$

$dy = \frac{2}{c_1}t\,dt$

$1 + c_1 y = t^2$

$c_1 y = t^2 - 1$

$y = \frac{t^2-1}{c_1}$

$$\int \frac{\sqrt{y}}{\sqrt{1+c_1y}}\,dy = \sqrt{2}\,a\int dx$$

$$\int \frac{\sqrt{t^2-1}}{\sqrt{c_1}\,\cancel{t}}\,\frac{2}{c_1}\cancel{t}\,dt = \sqrt{2}\,ax + c_2$$

$$\frac{2}{c_1^{3/2}}\int \sqrt{t^2-1}\,dt = \sqrt{2}\,ax + c_2$$

$$\frac{2}{c_1^{3/2}}\left[\frac{t\sqrt{t^2-1}}{2} - \frac{1}{2}\ln\left(t + \sqrt{t^2-1}\right)\right] = \sqrt{2}\,ax + c_2$$

Replace t by $\sqrt{1+c_1y}$

$$\frac{2}{c_1}\left[\frac{\sqrt{1+c_1y}\,\sqrt{c_1y}}{2\sqrt{c_1}} - \frac{1}{2\sqrt{c_1}}\ln\left(\sqrt{1+c_1y} + \sqrt{c_1y}\right)\right] = \sqrt{2}\,ax + c_2$$

$$\frac{1}{c_1}\left[\frac{\sqrt{c_1y + c_1^2y^2}\,\cancel{\sqrt{c_1}}}{\cancel{\sqrt{c_1}}} - \frac{1}{\sqrt{c_1}}\ln\left(\sqrt{1+c_1y} + \sqrt{c_1y}\right)\right] = \sqrt{2}\,ax + c_2$$

$$\sqrt{y + c_1y^2} - \frac{1}{\sqrt{c_1}}\ln\left(\sqrt{1+c_1y} + \sqrt{c_1y}\right) = \sqrt{2}\,axc_1 + c_2'$$

is general sol.