(1) Reduction of order: If one solution of the second order linear eq.

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + qy = 0 \quad \text{—— (1)}$$

(~~where~~ P, q are functions of x or Constts). is known. Then we can use it to find the general soln. of

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + qy = F(x) \quad \text{—— (2)}$$

This procedure is known as "method of reduction of order".

Suppose it is known that $y = y_1$ is a soln. of (1)

we assume that $y = vy_1$ —(3) is a soln. of (2), where v is some function of x.

from (3), we have

$$\frac{dy}{dx} = v\frac{dy_1}{dx} + y_1\frac{dv}{dx} \quad \text{—— (4)}$$

$$\frac{d^2y}{dx^2} = v\frac{d^2y_1}{dx^2} + \frac{dy_1}{dx}\frac{dv}{dx} + y_1\frac{d^2v}{dx^2} + \frac{dy_1}{dx}\frac{dv}{dx}$$

$$= v\frac{d^2y_1}{dx^2} + 2\frac{dv}{dx}\frac{dy_1}{dx} + y_1\frac{d^2v}{dx^2} \quad \text{—— (5)}$$

Put (3) (4) (5) in (2)

$$v\frac{d^2y_1}{dx^2} + 2\frac{dv}{dx}\frac{dy_1}{dx} + y_1\frac{d^2v}{dx^2} + Pv\frac{dy_1}{dx} + Py_1\frac{dv}{dx} + qvy_1 = F(x)$$

or

$$y_1\frac{d^2v}{dx^2} + \left(2\frac{dy_1}{dx} + Py_1\right)\frac{dv}{dx} + \left(\frac{d^2y_1}{dx^2} + P\frac{dy_1}{dx} + qy_1\right)v = F(x)$$

or

$$y_1\frac{d^2v}{dx^2} + \left(2\frac{dy_1}{dx} + Py_1\right)\frac{dv}{dx} = F(x) \qquad (\because y = y_1 \text{ is a soln. of (1)})$$

Put $\frac{dv}{dx} = u$

So $y_1\frac{du}{dx} + \left(2\frac{dy_1}{dx} + Py_1\right)u = F(x)$

It is a linear diff. eq. in u & can be

(2)

Solved for U

from $\frac{dv}{dx} = U$, we determine V & hence the

Soln. $y_p = V y_1$. general soln. is $y = y_c + y_p$

Note. It is easy to see that

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + q\,y = 0$$

is satisfied by $y = e^{x}$ if $1+p+q=0$

& by $y = x$ if $p+qx=0$

Exercise No. 10.5

Solve:

(1) $\frac{d^2y}{dx^2} + y = \sec^3 x$ const coeffts so $y = y_c + y_p$

Sol. Given

$$\frac{d^2y}{dx^2} + y = \sec^3 x \quad \text{--- (1)}$$

Consider $\frac{d^2y}{dx^2} + y = 0$ --- (2)

$$(D^2+1)y = 0 \qquad \text{where } D = \frac{d}{dx}$$

$$F(D)y = 0$$

where $F(D) = D^2+1$

Characteristic eq. is $F(m) = m^2+1 = 0$

$$\Rightarrow m = \pm i$$

So $\boxed{y_c = C_1\cos x + C_2\sin x}$

Put $C_1 = 1$ & $C_2 = 0$

So $y_1 = \cos x$ is a soln. of (2)

Suppose $y = V y_1 = V\cos x$ is a soln. of (1)

$$\Rightarrow \frac{dy}{dx} = -V\sin x + \frac{dv}{dx}\cos x$$

$$\frac{d^2y}{dx^2} = -V\cos x - \frac{dv}{dx}\sin x - \frac{dv}{dx}\sin x + \frac{d^2v}{dx^2}\cos x$$

$$= -V\cos x - 2\sin x\frac{dv}{dx} + \cos x\frac{d^2v}{dx^2}$$

Put values in (1)

$$-\cancel{V\cos x} - 2\sin x\frac{dv}{dx} + \cos x.\frac{d^2v}{dx^2} + \cancel{V\cos x} = \sec^3 x$$

$$\cos x.\frac{d^2v}{dx^2} - 2\sin x.\frac{dv}{dx} = \sec^3 x$$

or

$$\frac{d^2v}{dx^2} - 2\tan x\frac{dv}{dx} = \sec^4 x$$

Put $\frac{dv}{dx} = U$

So $$\frac{du}{dx} + (-2\tan x)U = \sec^4 x \quad \text{———— (3)}$$

It is a linear diff. eq. in U

$$\therefore I.F. = e^{-2\int\tan x\,dx} = e^{-2\ln\sec x} = e^{\ln\sec^{-2}x} = \frac{1}{\sec^2 x} = \cos^2 x$$

Multiplying both sides of (3) by I.F. $\cos^2 x$

$$\therefore \int d(U\cos^2 x) = \int\sec^2 x\,dx$$

$$U\cos^2 x = \tan x$$

Do 2nd Method

($\because$ No const of Integration since y_p does not contain const)

or $$U = \tan x\sec^2 x$$

$$\frac{dv}{dx} = \tan x\sec^2 x$$

$$\Rightarrow$$

$$\int dv = \int\tan x.\sec^2 x\,dx$$

$$V = \frac{\tan^2 x}{2}$$

So $$y_p = \frac{\tan^2 x}{2}\cos x$$

Hence general soln. is

$$y = y_c + y_p = \underline{C_1\cos x + C_2\sin x + \frac{1}{2}\tan^2 x\cos x}$$

2nd Method

$\because U\cos^2 x = \tan x$

$$U = \frac{\sin x}{\cos x\cos^2 x}$$

$$\frac{dv}{dx} = \cos^{-3}x\sin x$$

$$V = -\int\cos^{-3}x(-\sin x)dx$$

$$= -\frac{\cos^{-2}x}{-2}$$

$$V = \frac{1}{2}\sec^2 x$$

$$\therefore Y_p = V\cos x = \frac{1}{2}\sec^2 x\cos x$$

$$\boxed{Y_p = \frac{1}{2}\sec x}$$

$$Y = C_1\cos x + C_2\sin x + \frac{1}{2}\sec x.$$ Ans

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(2) $\frac{d^2y}{dx^2} + 4y = 4\tan 2x$ (const coeffs so $y = y_c + y_p$)

Sol: Given

$$\frac{d^2y}{dx^2} + 4y = 4\tan 2x \quad \text{---} (1)$$

For C.F.

Consider $\frac{d^2y}{dx^2} + 4y = 0$ --- (2)

$(D^2+4)y = 0$

$F(D)y = 0$

where $F(D) = D^2+4$

Characteristic eq. is

$F(m) = m^2+4 = 0$

$\Rightarrow m = \pm 2i$

So $\boxed{y_c = C_1\cos 2x + C_2 \sin 2x}$

Put $C_1 = 1$ & $C_2 = 0$

So $y_1 = \cos 2x$ is also a soln of (2)

Suppose $y = vy_1 = v\cos 2x$ is a soln of (1)

$$\frac{dy}{dx} = -2v\sin 2x + \frac{dv}{dx}\cos 2x$$

$$\frac{d^2y}{dx^2} = -2\left[v(2\cos 2x) + \frac{dv}{dx}\sin 2x\right] + \frac{d^2v}{dx^2}\cos 2x + \frac{dv}{dx}(-2\sin 2x)$$

$$= -4v\cos 2x - 2\sin 2x\frac{dv}{dx} + \cos 2x\frac{d^2v}{dx^2} - 2\sin 2x\frac{dv}{dx}$$

$$\frac{d^2y}{dx^2} = -4v\cos 2x - 4\sin 2x\frac{dv}{dx} + \cos 2x\frac{d^2v}{dx^2}$$

Put values in (1)

$$\cancel{-4v\cos 2x} - 4\sin 2x\frac{dv}{dx} + \cos 2x\frac{d^2v}{dx^2} + \cancel{4v\cos 2x} = 4\tan 2x$$

$$\cos 2x\frac{d^2v}{dx^2} - 4\sin 2x\frac{dv}{dx} = 4\tan 2x$$

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$$\frac{d^2v}{dx^2} - 4\tan 2x \frac{dv}{dx} = \frac{4\tan 2x}{\cos 2x}$$

Put $\frac{dv}{dx} = U$

So $$\frac{du}{dx} - (4\tan 2x)U = \frac{4\tan 2x}{\cos 2x} \qquad (3)$$

It is a linear diff. eq. in U.

$$I.F. = e^{-4\int \tan 2x\, dx} = e^{-4\frac{\ln \sec 2x}{2}} = e^{-2\ln \sec 2x} = e^{\ln \sec^{-2} 2x} = \frac{1}{\sec^2 2x} = \cos^2 2x$$

$\therefore$ I.F. $= \cos^2 2x$

Multiplying both sides of eq. (3) by I.F. $\cos^2 2x$

$$\int d(U\cos^2 2x) = \int \frac{4\tan 2x}{\cancel{\cos 2x}} \cdot \cos^{\cancel{2}} 2x\, dx$$

$$U\cos^2 2x = 4\int \sin 2x\, dx$$

$$= 4\left[-\frac{\cos 2x}{2}\right]$$

$$U\cos^2 2x = -2\cos 2x$$

or $$U = -\frac{2}{\cos 2x}$$

$$\frac{dv}{dx} = -2\sec 2x$$

$$\int dv = -2\int \sec 2x\, dx$$

$$v = -2\left[\frac{\ln(\sec 2x + \tan 2x)}{2}\right] = -\ln[\sec 2x + \tan 2x]$$

or $$= -\ln\left[\frac{1+\sin 2x}{\cos 2x}\right] = -\ln\left[\frac{(\cos x + \sin x)^2}{\cos^2 x - \sin^2 x}\right]$$

$\because (\cos x + \sin x)^2 = \cos^2 x + \sin^2 x + \sin 2x = 1 + \sin 2x$

$$= -\ln\left[\frac{\cos x + \sin x}{\cos x - \sin x}\right]$$

So $$V = -\ln\left[\frac{1+\tan x}{1-\tan x}\right] = -\ln\tan(\pi/4 + x)$$

Hence $y_p = Vy_1 = -\ln\tan(\pi/4 + x) \cdot \cos 2x$

So general soln. is

$$y = y_c + y_p = C_1\cos 2x + C_2\sin 2x - \ln|\tan(\pi/4 + x)|\cos 2x$$

10.5-6

(6)

(i) $(1-x^2)\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 2y = 0$ (Legendre eq. of order one) variable coeffs

Sol. Given

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0 \quad \text{—— (1)}$$

$\dfrac{d^2y}{dx^2} - \dfrac{2x}{1-x^2}\left(\dfrac{dy}{dx}\right) + \dfrac{2y}{1-x^2} = 0$

$P + Qx = 0$

$\dfrac{-2x}{1-x^2} + \left(\dfrac{2}{1-x^2}\right)x = 0$

$0 = 0$

Hence $y_c = x$

We note that $y = x$ is a soln. of (1)

So $\boxed{y_c = x}$ ∴ $y_1 = x$

Suppose $y = vy_1 = vx$ is also a soln. of (1)

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{dv}{dx} + \frac{dv}{dx} + x\frac{d^2v}{dx^2}$$

$$= 2\frac{dv}{dx} + x\frac{d^2v}{dx^2}$$

Put in (1)

$$(1-x^2)\left[2\frac{dv}{dx} + x\frac{d^2v}{dx^2}\right] - 2x\left[v + x\frac{dv}{dx}\right] + 2vx = 0$$

or

$$2(1-x^2)\frac{dv}{dx} + x(1-x^2)\frac{d^2v}{dx^2} - \cancel{2x}v - 2x^2\frac{dv}{dx} + \cancel{2v}x = 0$$

$$x(1-x^2)\frac{d^2v}{dx^2} + 2(1-2x^2)\frac{dv}{dx} = 0$$

or

$$\frac{d^2v}{dx^2} + \frac{2(1-2x^2)}{x(1-x^2)}\frac{dv}{dx} = 0$$

Put $\dfrac{dv}{dx} = u$

So

$$\frac{du}{dx} + \frac{2(1-2x^2)}{x(1-x^2)}u = 0$$

or $$\frac{du}{dx} = \frac{2(2x^2-1)}{x(1-x^2)}u$$

or $$\int\frac{du}{u} = 2\int\frac{2x^2-1}{x(1-x)(1+x)}dx \quad \text{—— (A)}$$

10.5 Now
$$\frac{4x^2-2}{x(1-x)(1+x)} = \frac{A}{x} + \frac{B}{1-x} + \frac{C}{1+x}$$

$$\Rightarrow 4x^2-2 = A(1-x^2) + Bx(1+x) + Cx(1-x) \quad \text{——— I}$$

Put $x = 0$ in I

$$-2 = A \Rightarrow \boxed{A = -2}$$

$1-x=0 \Rightarrow$ Put $x = 1$ in I

$$2 = B(2) \Rightarrow \boxed{B = 1}$$

$1+x=0 \Rightarrow$ Put $x = -1$ in I

$$2 = C(-1)(+2)$$

$$\Rightarrow \boxed{C = -\frac{2}{2} = -1}$$

So
$$\frac{4x^2-2}{x(1-x)(1+x)} = \frac{-2}{x} + \frac{1}{(1-x)} - \frac{1}{(1+x)}$$

So eq. (A) becomes

$$\int \frac{du}{u} = \int \frac{-2\,dx}{x} - \int \frac{(-1)\,dx}{(1-x)} - \int \frac{1}{(1+x)}\,dx$$

$$\ln u = -2\ln x - \ln(1-x) - \ln(1+x) + \ln c_1$$

$$\ln u = \ln\left[\frac{c_1}{x^2(1-x)(1+x)}\right]$$

or
$$\ln u = \ln\left[\frac{c_1}{x^2(1-x^2)}\right]$$

$$\Rightarrow u = \frac{c_1}{x^2(1-x^2)}$$

or

$$\frac{dv}{dx} = \frac{c_1}{x^2(1-x^2)}$$

$$\Rightarrow \int dv = \int \frac{c_1}{x^2(1-x^2)}\,dx$$

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or $V = \int \frac{c_1}{x^2(1-x^2)} dx$ ——— (B)

Now

$$\frac{1}{x^2(1-x^2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{1-x} + \frac{D}{1+x}$$

$\Rightarrow 1 = Ax(1-x^2) + B(1-x^2) + Cx^2(1+x) + Dx^2(1-x)$ ——— II

Put $x=0$ $\quad 1 = B \quad \Rightarrow \boxed{B=1}$

Put $x=1$ $\quad 1 = C(2) \quad \Rightarrow \boxed{C=\frac{1}{2}}$

Put $x=-1$ $\quad 1 = D(2) \quad \Rightarrow \boxed{D=\frac{1}{2}}$

Comparing Coff. of x^3 on both sides in II

$0 = -A + C - D$

or $0 = -A + \frac{1}{2} - \frac{1}{2} \quad \Rightarrow \boxed{A=0}$

So eq. (B) is

$$V = c_1 \int \frac{1}{x^2} + \frac{1}{2(1-x)} + \frac{1}{2(1+x)} dx$$

$$= c_1\left[\frac{-1}{x} - \frac{1}{2}\ln(1-x) + \frac{1}{2}\ln(1+x)\right] + c_2$$

$$V = c_1\left[-\frac{1}{x} + \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\right] + c_2$$

So

$$y = Vx = \left(c_1\left[\frac{-1}{x} + \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\right] + c_2\right)x$$

So gen. soln. is

$$y = c_1\left[\frac{-x}{x} + \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right)\right] + c_2 x$$

or $y = c_1\left[-1 + \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right)\right] + c_2 x$

10.5-9 (4) ~~$(x-1)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 1$~~ ~~variable coeffs~~ $\frac{d^2y}{dx^2} - \frac{x}{x-1}\frac{dy}{dx} + \frac{y}{x-1} = \frac{1}{x-1}$ (9)

Sol. Given

$$(x-1)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 1 \quad (1)$$

$P + Qx = 0$

$\frac{-x}{x-1} + \frac{1\cdot x}{x-1} = 0$

$0 = 0$

Hence $y = x$ is sol of (2)

For C.F.

Consider $(x-1)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0 \quad (2)$

We note that $y = x$ is a soln of (2) $\therefore \boxed{y_1 = x}$

Suppose that $y = vy_1 = vx$ is a soln of (1)

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{dv}{dx} + x\frac{d^2v}{dx^2} + \frac{dv}{dx} = 2\frac{dv}{dx} + x\frac{d^2v}{dx^2}$$

Put values in (1)

$$(x-1)\left[2\frac{dv}{dx} + x\frac{d^2v}{dx^2}\right] - x\left[v + x\frac{dv}{dx}\right] + vx = 1$$

$$2(x-1)\frac{dv}{dx} + x(x-1)\frac{d^2v}{dx^2} - \cancel{xv} - x^2\frac{dv}{dx} + \cancel{vx} = 1$$

or $$x(x-1)\frac{d^2v}{dx^2} + (2x-2-x^2)\frac{dv}{dx} = 1$$

or $$\frac{d^2v}{dx^2} - \frac{x^2-2x+2}{x(x-1)}\frac{dv}{dx} = \frac{1}{x(x-1)}$$

Put $\frac{dv}{dx} = U$

☆ $\frac{x-2}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$

$x-2 = A(x-1) + Bx$

$x = 0 \Rightarrow -2 = -A$

$\boxed{2 = A}$

$x-1 = 0 \Rightarrow x = 1$

$\Rightarrow 1-2 = 0 + B$

$\boxed{-1 = B}$

$\therefore \frac{x-2}{x(x-1)} = \frac{2}{x} - \frac{1}{x-1}$

So

$$\frac{dU}{dx} - \frac{x^2-2x+2}{x(x-1)}U = \frac{1}{x(x-1)} \quad (3)$$

It is a linear diff eq in U

$$\text{I.F.} = e^{-\int \frac{x^2-2x+2}{x^2-x}dx} = e^{-\int \frac{(x^2-x)+(-x+2)}{x^2-x}dx} = e^{-\int 1 - \frac{x-2}{x^2-x}dx}$$

$$= e^{\int\left(\frac{x-2}{x^2-x} - 1\right)dx} = e^{\int \frac{x-2}{x(x-1)}dx} \cdot e^{-x} = e^{\int\left(\frac{2}{x} - \frac{1}{x-1}\right)dx} \cdot e^{-x}$$

☆ see above detail

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(5) $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 8x^3$ variable coeffs

Sol. Given

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 8x^3 \quad \text{--- (1)}$$

For C.F.

$P + Qx = 0$

$\frac{x}{x^2} + \frac{(-1)x}{x^2} = 0$

$0 = 0$

$\Rightarrow y = x$ is a sol.

Consider $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$

$$(x^2D^2 + xD - 1)y = 0 \quad \text{--- (2)} \quad \text{where } D = \frac{d}{dx}$$

Put $x = e^t \Rightarrow t = \ln x$

Then $xD = \Delta$

$x^2D^2 = \Delta(\Delta - 1) = \Delta^2 - \Delta$

So eq (2) becomes

$(\Delta^2 - \Delta + \Delta - 1)y = 0$

$(\Delta^2 - 1)y = 0$

$F(\Delta)y = 0$

where $F(\Delta) = \Delta^2 - 1$

Characteristic eq. is

$F(m) = m^2 - 1 = 0 \Rightarrow m = \pm 1$

So

$y_c = C_1e^t + C_2e^{-t} = C_1x + \frac{C_2}{x}$

or $\boxed{y_c = C_1x + \frac{C_2}{x}}$

Put $C_1 = 1$ & $C_2 = 0$

No Need.

So $y = x$ is a soln of (2)

Suppose $y = vx$ is a soln. of (1)

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{dv}{dx} + \frac{dv}{dx} + x\frac{d^2v}{dx^2} = 2\frac{dv}{dx} + x\frac{d^2v}{dx^2}$$

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(12)

Put in (1)

$$x^2\left[2\frac{dv}{dx}+x\frac{d^2v}{dx^2}\right]+x\left[V+x\frac{dv}{dx}\right]-Vx=8x^3$$

$$2x^2\frac{dv}{dx}+x^3\frac{d^2v}{dx^2}+\cancel{V}x+x^2\frac{dv}{dx}-\cancel{V}x=8x^3$$

$$x^3\frac{d^2v}{dx^2}+3x^2\frac{dv}{dx}=8x^3$$

$$\frac{d^2v}{dx^2}+\frac{3}{x}\frac{dv}{dx}=8$$

Put $\frac{dv}{dx}=U$

So $\frac{du}{dx}+\frac{3}{x}U=8$

It is a linear diff eq

$I.F=e^{3\int\frac{1}{x}dx}=e^{3\ln x}=e^{\ln x^3}=x^3$

Multiplying both sides by I.F x^3

$$\int d(Ux^3)=\int 8x^3\,dx$$

$$Ux^3=\frac{8}{4}x^4$$

or

$$U=\frac{8}{4}x$$

or $U=2x$

$$\frac{dv}{dx}=2x$$

$$\int dv=\int 2x\,dx$$

$$V=x^2$$

So $y_p=Vx=x^2(x)=x^3$

Note If using $P+Qx=0$, $y=x$ is sol direct

then $\int d(Ux^3)=\int 8x^3\,dx$

$Ux^3=2x^4+C$ ← Note

$U=2x+Cx^{-3}$

$\frac{dv}{dx}=2x+Cx^{-3}$

$\int dv=\int(2x+Cx^{-3})\,dx$

$V=x^2+\frac{Cx^{-2}}{-2}+D$

$\therefore y=Vx$

$=\left[x^2-\frac{1}{2}Cx^{-2}+D\right]x$

$y=x^3-\frac{1}{2}Cx^{-1}+Dx$ G.sol.

So general solu is $y=y_c+y_p=C_1x+\frac{C_2}{x}+x^3$ ✓

variable coeffts

(6) $x^2\frac{d^2y}{dx^2} - (x^2+2x)\frac{dy}{dx} + (x+2)y = x^3e^x$

Soln. Given

~~10.5~~

$$x^2\frac{d^2y}{dx^2} - (x^2+2x)\frac{dy}{dx} + (x+2)y = x^3e^x \quad \text{——— (1)}$$

For C.F.

Consider $x^2\frac{d^2y}{dx^2} - (x^2+2x)\frac{dy}{dx} + (x+2)y = 0$ ——— (2)

$P + Qx = 0$

$-\frac{(x^2+2x)}{x^2} + \frac{(x+2)}{x^2}x = 0$

$-(x^2+2x) + x^2 + 2x = 0$

$-(x^2+2x) + x^2 + 2x = 0$

$-\cancel{x^2} - \cancel{2x} + \cancel{x^2} + \cancel{2x} = 0$

$0 = 0$

We note that $y = x$ is a soln. of (2)

Suppose $y = vx$ is a soln. of (1)

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\frac{d^2y}{dx^2} + 2\frac{dv}{dx} + x\frac{d^2v}{dx^2}$$

Put in (1)

$$x^2\left[2\frac{dv}{dx} + x\frac{d^2v}{dx^2}\right] - (x^2+2x)\left[v + x\frac{dv}{dx}\right] + (x+2)vx = x^3e^x$$

$$\cancel{2x^2\frac{dv}{dx}} + x^3\frac{d^2v}{dx^2} - \cancel{x^2v} - \cancel{2xv} - x^3\frac{dv}{dx} - \cancel{2x^2\frac{dv}{dx}} + \cancel{vx^2} + \cancel{2vx} = x^3e^x$$

$$x^3\frac{d^2v}{dx^2} - x^3\frac{dv}{dx} = x^3e^x$$

or

$$\frac{d^2v}{dx^2} - \frac{dv}{dx} = e^x$$

Put $U = \frac{dv}{dx}$

So $\frac{dU}{dx} - U = e^x$ ——— (3)

It is a linear eq. in U

$$I.F. = e^{-\int 1\,dx} = e^{-x}$$

Multiplying both sides of (3) by I.F. e^{-x}

$$\int d(Ue^{-x}) = \int 1\,dx$$

$$Ue^{-x} = x + c$$

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(4) 100%

or $U = xe^x + ce^x$

$$\frac{dv}{dx} = xe^x + ce^x$$

$$\int dv = \int xe^x\,dx + c\int e^x\,dx$$

$$V = xe^x - \int e^x \cdot 1\,dx + ce^x$$

$$= xe^x - e^x + ce^x + c_2$$

$$V = xe^x + (c-1)e^x + c_2$$

So

$y_p = Vx = x^2e^x + c_1xe^x + c_2x$ is req. gen. soln.

~~$(x\sin x + \cos x)\frac{d^2y}{dx^2} - x\cos x\frac{dy}{dx} + y\cos x = 0$~~

~~Sol:- Given~~

~~$(x\sin x + \cos x)\frac{d^2y}{dx^2} - x\cos x\frac{dy}{dx} + y\cos x = 0$ ——— (1)~~

~~We note that $y = x$ is a soln. of (1)~~

~~Also $y = \cos x$ is another soln. of (1)~~

~~Hence general soln. is~~

~~$y = c_1x + c_2\cos x$~~

---

(8) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = \frac{1}{(1+e^x)^2}$ const coeffs.

Sol:- Given

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = \frac{1}{(1+e^x)^2} \quad \text{——— (1)}$$

For C.F.

Consider $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$ ——— (2)

$$(D^2 + 2D + 1)y = 0$$

$$F(D)y = 0$$

1003

10-5-15 where $F(D) = (D+1)^2$

Characteristic eq. is

$$F(m) = (m+1)^2 = 0 \Rightarrow \boxed{m = -1, -1}$$

So $\boxed{y_c = (C_1 + C_2 x)e^{-x}}$

Put $c_1 = 1$ & $c_2 = 0$

So $y = e^{-x}$ is a soln. of ②

Suppose $y = ve^{-x}$ is a soln. of ①

$$\frac{dy}{dx} = v(-e^{-x}) + \frac{dv}{dx}(e^{-x})$$

$$\frac{dy}{dx} = e^{-x}\left[\frac{dv}{dx} - v\right]$$

$$\frac{d^2y}{dx^2} = e^{-x}\left[\frac{d^2v}{dx^2} - \frac{dv}{dx}\right] + (-e^{-x})\left[\frac{dv}{dx} - v\right]$$

$$= e^{-x}\left[\frac{d^2v}{dx^2} - \frac{dv}{dx} - \frac{dv}{dx} + v\right] = e^{-x}\left[\frac{d^2v}{dx^2} - 2\frac{dv}{dx} + v\right]$$

Put in ①

$$e^{-x}\left[\frac{d^2v}{dx^2} - 2\frac{dv}{dx} + v\right] + 2e^{-x}\left[\frac{dv}{dx} - v\right] + ve^{-x} = \frac{1}{(1+e^x)^2}$$

$$e^{-x}\frac{d^2v}{dx^2} - \cancel{2e^{-x}\frac{dv}{dx}} + \cancel{ve^{-x}} + \cancel{2e^{-x}\frac{dv}{dx}} - \cancel{2ve^{-x}} + \cancel{ve^{-x}} = \frac{1}{(1+e^x)^2}$$

$$\frac{d^2v}{dx^2} = \frac{e^x}{(1+e^x)^2}$$

$$\int \frac{d^2v}{dx^2}dx = \int (1+e^x)^{-2} e^x \, dx$$

$$\frac{dv}{dx} = \frac{(1+e^x)^{-1}}{-1}$$

$$\frac{dv}{dx} = \frac{-1}{1+e^x}$$

10-5-16

(16)

100%

or $\int dv = -\int (1+e^{x})^{-1} dx$

$$V = -\int \left[e^{x}(e^{-x}+1)\right]^{-1} dx$$

$$= -\int (1+e^{-x})^{-1} e^{-x} dx$$

$$= \int \frac{-e^{-x}}{(1+e^{-x})} dx$$

$$V = \ln(1+e^{-x})$$

$$y_p = Ve^{-x} = \ln(1+e^{-x}) \cdot e^{-x}$$

So general soln. is

$$y = y_c + y_p$$

$$= (C_1+C_2x)e^{-x} + e^{-x}\ln(1+e^{-x})$$

$$y = e^{-x}\left[C_1+C_2x+\ln(1+e^{-x})\right]$$

(17) $\frac{d^2y}{dx^2} - 2\tan x \frac{dy}{dx} + 3y = 2\sec x$

Sol. Given

$$\frac{d^2y}{dx^2} - 2\tan x \frac{dy}{dx} + 3y = 2\sec x \quad (1)$$

To C.F.

Consider $\frac{d^2y}{dx^2} - 2\tan x \frac{dy}{dx} + 3y = 0 \quad (2)$

We note that $y = \sin x$ is a soln. of (2) (given)

Suppose $y = v\sin x$ is a soln. of (1)

$$\frac{dy}{dx} = v\cos x + \sin x \frac{dv}{dx}$$

$$\frac{d^2y}{dx^2} = -V\sin x + \frac{dv}{dx}\cos x + \sin x\frac{d^2V}{dx^2} + \cos x\frac{dv}{dx}$$

Put in (1)

$$\left[-V\sin x + 2\frac{dv}{dx}\cos x + \sin x\frac{d^2v}{dx^2}\right] - 2\tan x\left[V\cos x + \sin x\frac{dv}{dx}\right] + 3V\sin x = 2\sec x$$

$$-V\sin x + 2\frac{dv}{dx}\cos x + \sin x\frac{d^2v}{dx^2} - 2\tan x V\cos x - 2\tan x\sin x\frac{dv}{dx} + 3V\sin x = 2\sec x$$

$$\sin x\frac{d^2v}{dx^2} + 2\left(\cos x - \frac{\sin^2 x}{\cos x}\right)\frac{dv}{dx} - \cancel{2V\sin x} + \cancel{2V\sin x} = 2\sec x$$

or $$\frac{d^2v}{dx^2} + 2\left[\frac{\cos^2 x - \sin^2 x}{\sin x\cos x}\right]\frac{dv}{dx} = \frac{2}{\sin x\cos x}$$

$$\frac{d^2v}{dx^2} + 4\frac{\cos 2x}{\sin 2x}\frac{dv}{dx} = \frac{4}{\sin 2x}$$

Put $\frac{dv}{dx} = U$

So

$$\frac{du}{dx} + 4\frac{\cos 2x}{\sin 2x}U = \frac{4}{\sin 2x} \quad (3)$$

It is a linear diff. eq. in U.

$$I.F. = e^{2\int\frac{2\cos 2x}{\sin 2x}dx} = e^{2\ln\sin 2x} = e^{\ln(\sin 2x)^2} = \sin^2 2x$$

Multiplying both sides of eq. (3) by I.F. $\sin^2 2x$

$$\int d(U\sin^2 2x) = 4\int \sin 2x\, dx$$

$$U\sin^2 2x = 4\left[-\frac{\cos 2x}{2}\right] + C_1$$

or

$$U = \frac{-2\cos 2x}{\sin^2 2x} + C_1\sin^{-2}2x$$

$$\frac{dv}{dx} = \frac{2\cos 2x}{\sin^2 2x} + C_1\sin^{-2}2x$$

$\boxed{10.5-18}$

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(18) $\int dV = \int \sin^{-2} 2x \,(2\cos 2x)\, dx + C_1 \int \operatorname{cosec}^2 2x \, dx$

$$V = -\frac{(\sin 2x)^{-1}}{-1} - \frac{C_1}{2}\cot 2x + C_2$$

or $V = \dfrac{1}{\sin 2x} - \dfrac{C_1}{2}\cot 2x + C_2$

So general soln. is

$$y = V \sin x$$

$$= \left[\frac{1}{\sin 2x} - \frac{C_1}{2}\cot 2x + C_2\right]\sin x$$

$$= \frac{\cancel{\sin} x}{2\cancel{\sin} x \cos x} - \frac{C_1}{2}\,\frac{\cos 2x}{2\sin x \cos x}(\sin x) + C_2 \sin x$$

$$= \frac{1}{2}\sec x - \frac{C_1}{4}\,\frac{\cos 2x}{\cos x} + C_2 \sin x$$

$$= \frac{1}{2}\sec x - \frac{C_1}{4}\left[(2\cos^2 x - 1)\sec x\right] + C_2 \sin x$$

$$= \frac{1}{2}\sec x - \frac{C_1}{4}\left(2\cos x - \sec x\right) + C_2 \sin x$$

$$= \frac{1}{2}\sec x - \frac{C_1}{2}\cos x + \frac{C_1}{4}\sec x + C_2 \sin x$$

$$= \frac{1}{2}\sec x - \frac{C_1}{2}\left[\cos x - \frac{1}{2}\sec x\right] + C_2 \sin x$$

$$y = \frac{1}{2}\sec x + C_1'\left(\cos x - \frac{1}{2}\sec x\right) + C_2 \sin x$$

where $C_1' = -\dfrac{C_1}{2}$

$$x\frac{d^2y}{dx^2}-(2x+1)\frac{dy}{dx}+(x+1)y$$

10.5.19

Sol. Given

$$x\frac{d^2y}{dx^2}-(2x+1)\frac{dy}{dx}+(x+1)y=(x^2+x-1)e^x \quad \text{—— (1)}$$

For C.F

Consider $$x\frac{d^2y}{dx^2}-(2x+1)\frac{dy}{dx}+(x+1)y=0 \quad \text{—— (2)}$$

we note that $y=e^x$ is a soln. of (2)

$\because P+Qx \neq 0$

where $P=\frac{-(2x+1)}{x}$ & $Q=\frac{x+1}{x}$

$\therefore 1+P+Q=1-\frac{(2x+1)}{x}+\frac{x+1}{x}$

$=\frac{\cancel{x}-2x-1+x+1}{x}$

$=0$

$\therefore y=e^x$ is sol of (2)

Suppose that $y=ve^x$ is a soln. of (1)

$$\frac{dy}{dx}=ve^x+\frac{dv}{dx}e^x$$

$$\frac{d^2y}{dx^2}=ve^x+\frac{dv}{dx}e^x+\frac{dv}{dx}e^x+\frac{d^2v}{dx^2}e^x$$

$$\frac{d^2y}{dx^2}=ve^x+2\frac{dv}{dx}e^x+\frac{d^2v}{dx^2}e^x$$

Put in (1)

$$x\left[ve^x+2\frac{dv}{dx}e^x+\frac{d^2v}{dx^2}e^x\right]-(2x+1)\left[ve^x+\frac{dv}{dx}e^x\right]+(x+1)ve^x=(x^2+x-1)e^x$$

$$vxe^x+2x\cancel{\frac{dv}{dx}}e^x+xe^x\frac{d^2v}{dx^2}-2vxe^x-2xe^x\cancel{\frac{dv}{dx}}-\cancel{v}e^x-\frac{dv}{dx}e^x+vxe^x+\cancel{v}e^x=(x^2+x-1)e^x$$

$$v\cancel{x}+x\frac{d^2v}{dx^2}-2\cancel{v}x-\frac{dv}{dx}+v\cancel{x}=(x^2+x-1)$$

or $$x\frac{d^2v}{dx^2}-\frac{dv}{dx}=x^2+x-1$$

or

$$\frac{d^2v}{dx^2}-\frac{1}{x}\frac{dv}{dx}=x+1-\frac{1}{x}$$

Put $\frac{dv}{dx}=U$

So $$\frac{du}{dx}-\frac{1}{x}U=x-\frac{1}{x}+1 \quad \text{—— (3)}$$

It is a linear diff eq

$\boxed{10.5\text{-}20}$

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(20) I.F. $= e^{-\int \frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = \frac{1}{x}$

Multiplying both sides of (3) by I.F. $\frac{1}{x}$

$$\int d\left(U\frac{1}{x}\right) = \int\left(1 - \frac{1}{x^2} + \frac{1}{x}\right)dx$$

$$\frac{U}{x} = x + \frac{1}{x} + \ln x + c_1$$

or $U = x^2 + 1 + x\ln x + c_1 x$

$$\frac{dv}{dx} = x^2 + 1 + x\ln x + c_1 x$$

$$\int dV = \int x^2 + 1 + x\ln x + c_1 x \, dx$$

$$V = \frac{x^3}{3} + x + \ln x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x}dx + c_1\frac{x^2}{2} + C_2$$

$$= \frac{x^3}{3} + x + \frac{x^2\ln x}{2} - \left[\frac{x^2}{4}\right] + C_1\frac{x^2}{2} + C_2$$

or

$$V = \frac{x^3}{3} + \frac{x^2\ln x}{2} - \frac{x^2}{4} + \frac{x^2}{2}C_1 + C_2$$

So general Soln. is

$$y = Ve^x$$

$$= e^x\left[\frac{x^3}{3} + \frac{x^2\ln x}{2} - \frac{x^2}{4} + \frac{C_1x^2}{2} + C_2\right]$$

(9) $$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = (1+x+x^2+\dots+x^{25})e^{2x}$$

Sol. Given

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = (1+x+x^2+\dots+x^{25})e^{2x} \quad \text{(1)}$$

For C.F.

Consider $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$ —— (2)

$$(D^2 - 4D + 4)y = 0$$

or $F(D)y = 0$

10.5.21

where $F(D) = D^2 - 4D + 4$

Characteristic eq. is

$$F(m) = m^2 - 4m + 4 = 0$$

$$(m-2)^2 = 0 \implies \boxed{m = 2, 2}$$

So $\boxed{y_c = (C_1 + C_2 x) e^{2x}}$

Put $C_1 = 1$ & $C_2 = 0$

So $y = e^{2x}$ is a soln. of ②

Suppose that $y = V e^{2x}$ is a soln. of ①

$$\frac{dy}{dx} = V(2e^{2x}) + \frac{dv}{dx} e^{2x}$$

$$= e^{2x}\left[2V + \frac{dv}{dx}\right]$$

$$\frac{d^2y}{dx^2} = e^{2x}\left[2\frac{dv}{dx} + \frac{d^2v}{dx^2}\right] + 2e^{2x}\left[2V + \frac{dv}{dx}\right]$$

$$= e^{2x}\left[2\frac{dv}{dx} + \frac{d^2v}{dx^2} + 4V + 2\frac{dv}{dx}\right]$$

or

$$\frac{d^2y}{dx^2} = e^{2x}\left[\frac{d^2v}{dx^2} + 4\frac{dv}{dx} + 4V\right]$$

Put Values in ①

$$e^{2x}\left[\frac{d^2v}{dx^2} + 4\frac{dv}{dx} + 4V\right] - e^{2x} 4\left[2V + \frac{dv}{dx}\right] + 4Ve^{2x} = (1 + x + x^2 + x^{25}) e^{2x}$$

$$\frac{d^2v}{dx^2} + \cancel{4\frac{dv}{dx}} + \cancel{4V} - \cancel{8V} - \cancel{4\frac{dv}{dx}} + \cancel{4V} = (1 + x + \cdots + x^{25})$$

$$\frac{d^2v}{dx^2} = (1 + x + \cdots + x^{25})$$

Integ. w.r.t. x

$$\int \frac{d^2v}{dx^2}\, dx = \int (1 + x + x^2 + \cdots + x^{25})\, dx$$

10-5-22

22

$$\frac{dy}{dx} = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^{26}}{26}$$

$$V = \frac{x^2}{1\cdot 2} + \frac{x^3}{2\cdot 3} + \frac{x^4}{3\cdot 4} + \cdots + \frac{x^{27}}{26\cdot 27}$$

So

$$y_p = V e^{2x}$$

$$\therefore y_p = \left[\frac{x^2}{1\cdot 2} + \frac{x^3}{2\cdot 3} + \frac{x^4}{3\cdot 4} + \cdots + \frac{x^{27}}{26\cdot 27}\right] e^{2x}$$

So general soln is

$$y = y_c + y_p$$

$$= (c_1 + c_2 x)e^{2x} + \left[\frac{x^2}{1\cdot 2} + \frac{x^3}{2\cdot 3} + \frac{x^4}{3\cdot 4} + \cdots + \frac{x^{27}}{26\cdot 27}\right] e^{2x}$$

or $$y = e^{2x}\left[c_1 + c_2 x + \frac{x^2}{1\cdot 2} + \frac{x^3}{2\cdot 3} + \frac{x^4}{3\cdot 4} + \cdots + \frac{x^{27}}{26\cdot 27}\right]$$

Solved examples

Example (1) $$\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$$

Sol:- Given

$$\frac{d^2y}{dx^2} + y = \operatorname{cosec} x \quad \text{--- (1)}$$

For C.F.

Consider $$\frac{d^2y}{dx^2} + y = 0 \quad \text{--- (2)}$$

$$(D^2+1)y = 0$$

$$F(D)y = 0$$

where $F(D) = D^2+1$

Characteristic eq. is

$$F(m) = m^2 + 1 = 0$$

$$\Rightarrow \boxed{m = \pm i}$$

So $\boxed{y_c = C_1 \cos x + C_2 \sin x}$

Put $C_1 = 1$ & $C_2 = 0$

In books, we used $C_1 = 0$ & $C_2 = 1$ then question will become more easy than this method

So $y = \cos x$ is a soln of ②

Suppose that $\boxed{y = V\cos x}$ is a soln. of ①

$$\frac{dy}{dx} = -V\sin x + \cos x \frac{dv}{dx}$$

$$\frac{d^2y}{dx^2} = -V\cos x - \sin x\frac{dv}{dx} - \sin x\frac{dv}{dx} + \cos x\frac{d^2v}{dx^2}$$

$$\frac{d^2y}{dx^2} = -V\cos x - 2\sin x\frac{dv}{dx} + \cos x\frac{d^2v}{dx^2}$$

Put values in ①

$$-V\cancel{\cos x} - 2\sin x\frac{dv}{dx} + \cos x\frac{d^2v}{dx^2} + V\cancel{\cos x} = \operatorname{cosec} x$$

$$\cos x\frac{d^2v}{dx^2} - 2\sin x\frac{dv}{dx} = \operatorname{cosec} x$$

Put $\frac{dv}{dx} = U$

So $\frac{dU}{dx} - 2\tan x\, U = \frac{1}{\sin x\cos x}$ ——— ③

It is a linear diff eq in U

I.F. $= e^{-2\int \tan x\,dx} = e^{-2\ln\sec x} = e^{\ln\sec^{-2}x} = \frac{1}{\sec^2 x} = \cos^2 x$

Multiplying both sides of ③ by I.F. $= \cos^2 x$

$$\int d(U\cos^2 x) = \int \cot x\, dx$$

$$U\cos^2 x = \ln\sin x$$

$$U = \ln\sin x \cdot \sec^2 x$$

$$\frac{dv}{dx} = \ln\sin x\cdot\sec^2 x$$

$$\int dv = \int \ln\sin x\cdot\sec^2 x\, dx$$

$$= \ln\sin x\cdot\tan x - \int \tan x\cdot\frac{\cos x}{\sin x}\,dx$$

$$V = \tan x\cdot\ln\sin x - x$$

10.5-24

(22)

So $y_p = V\cos x$

$= \left[\tan x \ln \sin x - x\right] \cos x$

$y_p = \sin x \ln \sin x - x\cos x$

So general soln is

$y = y_c + y_p$

$= c_1 \cos x + c_2 \sin x - x\cos x + \sin x \ln|\sin x|$

Example (18) $(x+2)\dfrac{d^2y}{dx^2} - (2x+5)\dfrac{dy}{dx} + 2y = (x+1)e^x$

Sol:- Given

$$(x+2)\frac{d^2y}{dx^2} - (2x+5)\frac{dy}{dx} + 2y = (x+1)e^x \quad \text{--- (1)}$$

For C.F.

Consider $(x+2)\dfrac{d^2y}{dx^2} - (2x+5)\dfrac{dy}{dx} + 2y = 0$ --- (2)

We note that $y = e^{2x}$ is a soln of (2)

MT484

$Y = e^{ax}$ is a sol of $\dfrac{d^2y}{dx^2} + P\dfrac{dy}{dx} + QY = 0$ if $a^2 + Pa + Q = 0$

$2^2 + \left(\dfrac{-2x-5}{x+2}\right)2 + \dfrac{2}{x+2} = 0$

$\dfrac{4(x+2) - 4x - 10 + 2}{x+2} = 0$

$4x + 8 - 4x - 10 + 2 = 0$

$0 = 0$

Suppose $y = Ve^{2x}$ is a soln of (1)

$$\frac{dy}{dx} = V(2e^{2x}) + \frac{dV}{dx}e^{2x}$$

$$= e^{2x}\left[2V + \frac{dV}{dx}\right]$$

$$\frac{d^2y}{dx^2} = e^{2x}\left[2\frac{dV}{dx} + \frac{d^2V}{dx^2}\right] + (2e^{2x})\left[2V + \frac{dV}{dx}\right]$$

$$= e^{2x}\left[2\frac{dV}{dx} + \frac{d^2V}{dx^2} + 4V + 2\frac{dV}{dx}\right]$$

$$\frac{d^2y}{dx^2} = e^{2x}\left[\frac{d^2V}{dx^2} + 4\frac{dV}{dx} + 4V\right]$$

Put values in (1)

$$(x+2)e^{2x}\left[\frac{d^2V}{dx^2} + 4\frac{dV}{dx} + 4V\right] - (2x+5)e^{2x}\left[2V + \frac{dV}{dx}\right] + 2Ve^{2x} = (x+1)e^x$$

10.5-25

$$(x+2)\left[\frac{d^2v}{dx^2}+4\frac{dv}{dx}+4v\right]-(2x+5)\left[2v+\frac{dv}{dx}\right]+2v=(x+1)e^{-x}$$

$$(x+2)\frac{d^2v}{dx^2}+\left[4(x+2)-(2x+5)\right]\frac{dv}{dx}+4v(x+2)-2v(2x+5)+2v=(x+1)e^{-x}$$

$$(x+2)\frac{d^2v}{dx^2}+(2x+3)\frac{dv}{dx}+(\cancel{4x}+\cancel{8}-\cancel{4x}-\cancel{16}+\cancel{2})v=(x+1)e^{-x}$$

$$(x+2)\frac{d^2v}{dx^2}+(2x+3)\frac{dv}{dx}=(x+1)e^{-x}$$

or

$$\frac{d^2v}{dx^2}+\left(\frac{2x+3}{x+2}\right)\frac{dv}{dx}=\frac{(x+1)e^{-x}}{x+2}$$

Put $\frac{dv}{dx}=u$

So

$$\frac{du}{dx}+\left(\frac{2x+3}{x+2}\right)u=\frac{(x+1)e^{-x}}{x+2} \qquad (3)$$

It is a linear diff eq in u

$$I.F.=\exp\int\frac{2x+3}{x+2}dx=\exp\int 2-\frac{1}{(x+2)}dx=\exp\left[2x-\ln(x+2)\right]$$

$$=e^{2x}\cdot e^{-\ln(x+2)}=e^{2x}\cdot e^{\ln(x+2)^{-1}}=\frac{e^{2x}}{x+2}$$

Multiplying both sides of (3) by I.F. $\frac{e^{2x}}{x+2}$

$$\int d\left(u\cdot\frac{e^{2x}}{x+2}\right)=\int\frac{x+1}{(x+2)^2}e^{x}dx$$

$$\frac{ue^{2x}}{x+2}=\int\frac{(x+2)-1}{(x+2)^2}e^{x}dx$$

$$=\int\frac{e^{x}}{x+2}dx-\int\frac{e^{x}}{(x+2)^2}dx$$

$$=\frac{1}{(x+2)}e^{x}-\int e^{x}\cdot\cancel{\frac{-1}{(x+2)^2}}dx-\int\cancel{\frac{e^{x}}{(x+2)^2}}dx$$

$$u\frac{e^{2x}}{x+2}=\frac{e^{x}}{x+2}+c_1$$

10.5-26

26 1014

$\cancel{V = e^{-x} + c_1(x+2)e^{-2x}}$

$$\frac{dv}{dx} = e^{-x} + c_1(x+2)e^{-2x}$$

$$\int dv = \int e^{-x}\,dx + c_1\int e^{-2x}(x+2)\,dx$$

$$V = -e^{-x} + c_1\left\{ (x+2)\cdot\frac{e^{-2x}}{-2} - \int \frac{e^{-2x}}{-2}\cdot 1\,dx \right\} + c_2$$

$$= -e^{-x} - \frac{c_1(x+2)}{2}e^{-2x} + \frac{c_1}{2}\left[\frac{e^{-2x}}{-2}\right] + c_2$$

$$V = -e^{-x} - \frac{c_1(x+2)}{2}e^{-2x} - \frac{c_1}{4}\left[e^{-2x}\right] + c_2$$

$$= -e^{-x} - c_1 e^{-2x}\left[\frac{x+2}{2} + \frac{1}{4}\right] + c_2$$

$$= -e^{-x} - c_1 e^{-2x}\left[\frac{2x+4+1}{4}\right] + c_2$$

$$V = -e^{-x} - \frac{1}{4}c_1(2x+5)e^{-2x} + c_2$$

So general soln is

$$y = Ve^{2x} = -e^{x} - \frac{1}{4}c_1(2x+5) + c_2 e^{+2x}$$

☆ Method of Variation of Parameters ☆

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + Qy = F(x) \quad -①$$

Steps

① $y_c = c_1y_1 + c_2y_2$

② $y_p = u_1y_1 + u_2y_2$

③ $y_1 = ?$ $y_2 = ?$

④ $F(x) = \text{RHS of } ①$

⑤ $W = y_1y_2' - y_1'y_2$

⑥ $u_1 = \int \frac{-y_2F(x)}{W}dx$

⑦ $u_2 = \int \frac{y_1F(x)}{[illegible]}$