10.4 - 0

original

(15)

The Cauchy Euler Diff Eq

A diff eq of the form $a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots a_{n-1} x \frac{dy}{dx} + a_n y = F(x)$. —— (1)

or $(a_0 x^n D^n + a_1 x^{n-1} D^{n-1} + \dots a_{n-1} x D + a_n) y = F(x)$. $(a_i \in R$

is called Cauchy Euler Diff Eq (variable coeffts).

This eq can be reduced to a linear diff eq with const coeffts as

Put $x = e^t \Rightarrow t = \ln x$

$x = e^t$
$\ln x = \ln e^t$
$\ln x = t \ln e$
$\ln x = t$
$\frac{1}{x} = \frac{dt}{dx}$

$\frac{dx}{dy} = e^t \frac{dt}{dy}$

$\frac{dy}{dx} = \frac{1}{e^t} \frac{dy}{dt}$

$\frac{dy}{dx} = \frac{1}{x} \frac{dy}{dt}$

$D = \frac{d}{dx}$
$\Delta = \frac{d}{dt}$

$x \frac{dy}{dx} = \frac{dy}{dt}$

$xDy = \Delta y \Rightarrow \boxed{xD = \Delta}$

Again $\frac{dy}{dx} = \frac{1}{x} \frac{dy}{dt}$

Diff $\frac{d^2y}{dx^2} = \frac{1}{x} \frac{d^2y}{dt^2} \cdot \frac{dt}{dx} - \frac{1}{x^2} \frac{dy}{dt}$

$= \frac{1}{x} \frac{d^2y}{dt^2} \frac{1}{x} - \frac{1}{x^2} \frac{dy}{dt}$

$\frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$

$x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$

$x^2 D^2 y = \Delta^2 y - \Delta y$

$(x^2 D^2) y = (\Delta^2 - \Delta) y$

$x^2 D^2 = \Delta^2 - \Delta$

$\boxed{x^2 D^2 = \Delta(\Delta - 1)}$

$\boxed{x^3 D^3 = \Delta(\Delta-1)(\Delta-2)}$

$\boxed{x^n D^n = \Delta(\Delta-1)(\Delta-2) \dots \dots (\Delta-(n-1))}$

substituting values of $xD, x^2D^2, x^3D^3, \dots$ in (1) we obtain an eq of nth order with const coeffts having 't' as independent variable. Now it can be solved

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Ex No 10.4

① $(x^2D^2+7xD+5)Y = x^5$ ——— ①

Put $x=e^t$ $\Rightarrow t=\ln x$, $xD=\Delta$, $x^2D^2=\Delta(\Delta-1)$

$(\Delta(\Delta-1)+7\Delta+5)Y = e^{5t}$

$(\Delta^2-\Delta+7\Delta+5)Y = e^{5t}$

$(\Delta^2+6\Delta+5)Y = e^{5t}$

For Characteristic eq

$\Delta^2+6\Delta+5 = 0$

$\Delta^2+1\Delta+5\Delta+5 = 0$

$\Delta(\Delta+1)+5(\Delta+1) = 0$

$(\Delta+1)(\Delta+5) = 0$

$\Delta = -1, -5$

$Y_c = c_1e^{-t}+c_2e^{-5t}$

$= \frac{c_1}{e^t}+\frac{c_2}{e^{5t}}$

$Y_c = \frac{c_1}{x}+\frac{c_2}{x^5}$ ——— ②

For Particular Integral

$Y_p = \frac{1}{\Delta^2+6\Delta+5}e^{5t}$

$= \frac{e^{5t}}{5^2+6(5)+5}$

$= \frac{e^{5t}}{60}$

$Y_p = \frac{x^5}{60}$ ——— ③ $\because x=e^t$

So general Sol is

$Y = Y_c+Y_p$

$= \frac{c_1}{x}+\frac{c_2}{x^5}+\frac{x^5}{60}$

② $x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+5y = x^2\sin(\ln x)$

$(x^2D^2-3xD+5)Y = x^2\sin(\ln x)$

Put $x=e^t$, $t=\ln x$, $xD=\Delta$, $x^2D^2=\Delta(\Delta-1)$

$(\Delta(\Delta-1)-3\Delta+5)Y = e^{2t}\sin t$

$(\Delta^2-\Delta-3\Delta+5)Y = e^{2t}\sin t$

$(\Delta^2-4\Delta+5)Y = e^{2t}\sin t$

For characteristic Eq.

$\Delta^2-4\Delta+5 = 0$

$\Delta = \frac{4\pm\sqrt{16-20}}{2} = \frac{4\pm\sqrt{-4}}{2} = \frac{4\pm2i}{2}$

$= 2\pm i$

$Y_c = e^{2t}(c_1\cos t+c_2\sin t)$

$Y_c = x^2(c_1\cos\ln x+c_2\sin\ln x)$

For Particular Integral

$Y_p = \frac{1}{\Delta^2-4\Delta+5}e^{2t}\sin t$

$= e^{2t}\frac{1}{(\Delta+2)^2-4(\Delta+2)+5}\sin t$

$= e^{2t}\frac{1}{\Delta^2+\cancel{4\Delta}+4-\cancel{4\Delta}-8+5}\sin t$

$= e^{2t}\frac{1}{(\Delta^2+1)}\sin t$ ∵ (failure case $-1^2+1=0$)

$= e^{2t}\frac{t}{2\Delta}\sin t$

$= \frac{te^{2t}}{2}(-\cos t)$ ∵ $\frac{1}{\Delta}\sin t = -\cos t$

$Y_p = -\frac{1}{2}te^{2t}\cos t$

$Y_p = -\frac{1}{2}\ln x\, x^2\cos(\ln x)$

Hence General Sol

$Y = Y_c+Y_p$

$Y = x^2(c_1\cos\ln x+c_2\sin\ln x) - \frac{1}{2}\ln x\, x^2\cos(\ln x)$

16.4-2

(17)

③ $x^2\frac{d^2y}{dx^2}-(2m-1)x\frac{dy}{dx}+(m^2+n^2)y = n^2x^m \ln x$ —— ①

$[x^2D^2-(2m-1)xD+(m^2+n^2)]y = n^2x^m\ln x$

Put $x=e^t$
$t=\ln x$
$xD=\Delta$
$x^2D^2=\Delta(\Delta-1)=\Delta^2-\Delta$

$\therefore [\Delta^2-\Delta-(2m-1)\Delta+(m^2+n^2)]y = n^2e^{mt}.t$

$(\Delta^2-\cancel{\Delta}-2m\Delta+\cancel{\Delta}+m^2+n^2)y = n^2e^{mt}.t$

$(\Delta^2-2m\Delta+m^2+n^2)y = n^2e^{mt}t$ —— ⑪

(LDEq with const coeff)

Characteristic Eq of ⑪ is $\Delta^2-2m\Delta+m^2+n^2=0$

$\Delta = \frac{2m\pm\sqrt{4m^2-4m^2-4n^2}}{2} = \frac{2m\pm\sqrt{-4n^2}}{2}$

$\Delta = \frac{2m\pm 2in}{2} = \frac{\cancel{2}(m\pm in)}{\cancel{2}} = \boxed{m\pm in}$

$\because \left(\begin{matrix}x=e^t\\ x^m=e^{mt}\end{matrix}\right)$ $\begin{cases}t=\ln x\\ nt=n\ln x\\ nt=\ln x^n\end{cases}$

$y_c=e^{mt}(c_1\cos nt+c_2\sin nt)$

$y_c = x^m(c_1\cos\ln x^n + c_2\sin\ln x^n)$

Now $y_p = \frac{1}{\Delta^2-2m\Delta+m^2+n^2}(n^2e^{mt}.t)$

$= n^2e^{mt}\frac{1}{(\Delta+m)^2-2m(\Delta+m)+m^2+n^2}(t)$

$= n^2e^{mt}\frac{1}{\Delta^2+2\cancel{\Delta m}+\cancel{m^2}-2\cancel{\Delta m}-\cancel{2m^2}+\cancel{m^2}+n^2}(t)$

$= n^2e^{mt}\frac{1}{(\Delta^2+n^2)}(t)$

$= \frac{\cancel{n^2}e^{mt}}{\cancel{n^2}(\frac{\Delta^2}{n^2}+1)}(t) = e^{mt}\left(1+\frac{\Delta^2}{n^2}\right)^{-1}(t)$

$= e^{mt}\left(1-\frac{\Delta^2}{n^2}\right)t = e^{mt}(t)-\frac{e^{mt}}{n^2}(0)$

$= e^{mt}(t) = e^{m\ln x}(\ln x) = e^{\ln x^m}(\ln x)$

$y_p = x^m\ln x$

So G.Sol. $y=y_c+y_p$
$= x^m(c_1\cos\ln x^n + c_2\sin\ln x^n)+x^m\ln x.$

Ex 10.4-3 (18)

(9) $(4x^2D^2-4xD+3)y=\sin\ln(-x)$ — (1)

Cauchy-Euler Eq

Put $-x=e^t \Rightarrow t=\ln(-x)$

$xD=\Delta$

$x^2D^2=\Delta(\Delta-1)=\Delta^2-\Delta$

Put in (1)

$(4(\Delta^2-\Delta)-4\Delta+3)y=\sin t$

$(4\Delta^2-8\Delta+3)y=\sin t$ — (2)

Characteristic Eq of (2) is

$4\Delta^2-8\Delta+3=0$

$$\Delta=\frac{8\pm\sqrt{64-4\cdot4\cdot3}}{8}=\frac{8\pm\sqrt{64-48}}{8}$$

$$=\frac{8\pm4}{8}=\frac{3}{2},\frac{1}{2}$$

$$y_c=c_1e^{\frac{3}{2}t}+c_2e^{t/2}$$

$$y_p=\frac{\sin t}{4\Delta^2-8\Delta+3}$$

$$=\frac{\sin t}{4(-1^2)-8\Delta+3}=\frac{\sin t}{-(1+8\Delta)}$$

$$=\frac{-(1-8\Delta)\sin t}{(1+8\Delta)(1-8\Delta)}=\frac{-(1-8\Delta)\sin t}{1-64\Delta^2}$$

$$=\frac{-(1-8\Delta)\sin t}{1-64(-1^2)}=\frac{-\sin t+8\cos t}{65}\text{ Ans}$$

2nd Method

$$y_p=\frac{\sin t}{4\Delta^2-8\Delta+3}=\operatorname{Im}\frac{e^{it}}{4\Delta^2-8\Delta+3}$$

$$=\operatorname{Im}\frac{e^{it}}{4(i)^2-8(i)+3}=\operatorname{Im}\frac{(1-8i)}{-(1+8i)(1-8i)}e^{it}$$

$$=\operatorname{Im}\frac{-1}{65}(1-8i)e^{it}=\frac{-1}{65}\operatorname{Im}(1-8i)[\cos t+i\sin t]$$

$$y_p=-\frac{1}{65}[\sin t-8\cos t]=\frac{8\cos t-\sin t}{65}\text{ Ans}$$

$$y=c_1e^{3t/2}+c_2e^{t/2}+\frac{8}{65}\cos t-\frac{1}{65}\sin t$$

(5) $x^3\dfrac{d^3y}{dx^3}+2x^2\dfrac{d^2y}{dx^2}+2y=10x+\dfrac{10}{x}$

$(x^3D^3+2x^2D^2+2)y=10x+\dfrac{10}{x}$ —— (I)

Put $x=e^t \Rightarrow t=\ln x$

$xD=\Delta$

$x^2D^2=\Delta(\Delta-1)$

$x^3D^3=\Delta(\Delta-1)(\Delta-2)$

$(\Delta(\Delta-1)(\Delta-2)+2\Delta(\Delta-1)+2)y=10e^t+\dfrac{10}{e^t}$

$(\Delta^3-3\Delta^2+2\Delta+2(\Delta^2-\Delta)+2)y=10e^t+10e^{-t}$

$(\Delta^3-3\Delta^2+\cancel{2\Delta}+2\Delta^2-\cancel{2\Delta}+2)y=10e^t+10e^{-t}$

$(\Delta^3-\Delta^2+2)y=10e^t+10e^{-t}$ —— (II)

Characteristic Eq

$\Delta^3-\Delta^2+2=0$

$$\begin{array}{r|rrrr} -1 & 1 & -1 & 0 & 2 \\ & \downarrow & -1 & 2 & -2 \\ \hline & 1 & -2 & 2 & \boxed{0} \end{array}$$

So $\Delta=-1$

Depressed Eq is

$\Delta^2-2\Delta+2=0$

$\Delta=\dfrac{2\pm\sqrt{4-8}}{2}=\dfrac{2\pm2i}{2}=1\pm i$

$\therefore\ \Delta=-1,\ 1\pm i$

Hence $y_c=c_1e^{-t}+e^t(c_2\cos t+c_3\sin t)$

$y_c=\dfrac{c_1}{x}+x(c_2\cos(\ln x)+c_3\sin(\ln x))$

Now $y_p=\dfrac{1}{(\Delta^3-\Delta^2+2)}(10e^t+10e^{-t})$

$=\dfrac{1}{\Delta^3-\Delta^2+2}10e^t+\dfrac{1}{\Delta^3-\Delta^2+2}10e^{-t}$

failure case $\because \dfrac{1}{-1-(-1)^2+2}=\infty$

$=\dfrac{10e^t}{1-1+2}+\dfrac{t}{3\Delta^2-2\Delta+0}(10e^{-t})$

Multiply by t

Take Derivative

$=\dfrac{10e^t}{2}+\dfrac{t\,10e^{-t}}{3(-1)^2-2(-1)}$

$=5e^t+\dfrac{10}{5}te^{-t}$

$y_p=5x+2\ln x\left(\dfrac{1}{x}\right)$

$y=y_c+y_p$

$=\dfrac{c_1}{x}+x(c_2\cos\ln x+c_3\sin\ln x)+5x+2\dfrac{\ln x}{x}$

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(6) $x^4\dfrac{d^3y}{dx^3} + 2x^3\dfrac{d^2y}{dx^2} - x^2\dfrac{dy}{dx} + xy = 1$

$\div$ by x $\quad x^3\dfrac{d^3y}{dx^3} + 2x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = \dfrac{1}{x}$ $\quad$ Na Cauchy Euler Diff Eq.

$(x^3D^3 + 2x^2D^2 - xD + 1)y = \dfrac{1}{x}$

Put $x = e^t \Rightarrow t = \ln x$

$xD = \Delta$

$x^2D^2 = \Delta(\Delta-1)$

$x^3D^3 = \Delta(\Delta-1)(\Delta-2)$

$(\Delta(\Delta-1)(\Delta-2) + 2\Delta(\Delta-1) - \Delta + 1)y = \dfrac{1}{e^t}$

$(\Delta^3 - 3\Delta^2 + \cancel{2\Delta} + 2\Delta^2 - \cancel{2\Delta} - \Delta + 1)y = e^{-t}$

$(\Delta^3 - \Delta^2 - \Delta + 1)y = e^{-t}$

(LDEq with const coeffts)

Characteristic Eq

$\Delta^3 - \Delta^2 - \Delta + 1 = 0$

$\therefore \Delta = 1$

1	1	-1	-1	1
	↓	+1	0	-1
	1	0	-1	0

and Depressed Eq is

$\Delta^2 - 1 = 0$

$\Rightarrow \Delta^2 = 1 \Rightarrow \Delta = \pm 1$

$\therefore \Delta = 1, 1, -1$

$y_c = (c_1 + c_2 t)e^t + c_3 e^{-t}$

$y_c = (c_1 + c_2 \ln x)x + \dfrac{c_3}{x}$

$y_p = \dfrac{1}{\Delta^3 - \Delta^2 - \Delta + 1}(e^{-t})$

$\dfrac{1}{(-1)^3 - (-1)^2 - (-1) + 1} = \dfrac{1}{-1-1+1+1} = \infty$

$= \dfrac{t}{3\Delta^2 - 2\Delta - 1}e^{-t}$

$= \dfrac{te^{-t}}{3(-1)^2 - 2(-1) - 1} = \dfrac{te^{-t}}{3+2-1}$

$y_p = \dfrac{te^{-t}}{4}$

$= \dfrac{\ln x}{4}\cdot\dfrac{1}{x}$

$\because t = \ln x$

$e^t = x$

$e^{-t} = \dfrac{1}{x}$

So $y = y_c + y_p$

$= (c_1 + c_2 \ln x)x + \dfrac{c_3}{x} + \dfrac{\ln x}{4x}$ $\quad$ Ans.

10.3 Q 6

21 (7)

⑦ $(x^3D^3+4x^2D^2-5xD-15)y=x^4$ Cauchy-Euler Eq.

Let $x=e^t \Rightarrow t=\ln x$

$$xD=\Delta$$
$$x^2D^2=\Delta^2-\Delta$$
$$x^3D^3=\Delta^3-3\Delta^2+2\Delta$$

Put in ① $(\Delta^3-3\Delta^2+2\Delta+4\Delta^2-4\Delta-5\Delta-15)y=e^{4t}$

$$(\Delta^3+\Delta^2-7\Delta-15)y=e^{4t}$$

Characteristic Eq is $\Delta^3+\Delta^2-7\Delta-15=0$

3	1	1	-7	-15
	↓	3	12	15
	1	4	5	0

$$\Delta^2+4\Delta+5=0$$
$$\Delta=\frac{-4\pm\sqrt{16-4\cdot1\cdot5}}{2}=\frac{-4\pm\sqrt{-4}}{2}$$
$$=\frac{-4\pm2i}{2}=-2\pm i$$

$$y_c=c_1e^{3t}+e^{-2t}(c_2\cos t+c_3\sin t)$$

$$y_p=\frac{1}{\Delta^3+\Delta^2-7\Delta-15}e^{4t}$$
$$=\frac{e^{4t}}{64+16-28-15}=\frac{e^{4t}}{37}$$

$$y=y_c+y_p=c_1e^{3t}+e^{-2t}(c_2\cos t+c_3\sin t)+\frac{e^{4t}}{37}$$

Replacing t by $\ln x$

$$y=c_1x^3+x^{-2}\left[c_2\cos(\ln x)+c_3\sin(\ln x)\right]+\frac{x^4}{37}$$

⑧ $((x+1)^2D^2+(x+1)D+1)y=4\{\cos\ln(x+1)\}^2$

Let $x+1=e^t$
$\Rightarrow t=\ln(x+1)$

$$(x+1)D=\Delta$$
$$(x+1)^2D^2=\Delta^2-\Delta$$

putting values

$$(\Delta^2-\Delta+\Delta+1)y=4(\cos t)^2$$
$$\Delta^2+1=4\cos^2t$$
$$\Delta^2+1=2(1+\cos2t)$$

Characteristic Eq is $\Delta^2+1=0$
$\Delta=\pm i$

$$y_c=e^{0t}(c_1\cos t+c_2\sin t)$$

$$y_p=\frac{2(1+\cos2t)}{\Delta^2+1}$$
$$=\frac{2}{\Delta^2+1}+\frac{2\cos2t}{\Delta^2+1}$$
$$=(1+\Delta^2)^{-1}2+\frac{2\cos2t}{(-2^2)+1}$$
$$=2+\frac{2\cos2t}{-3}$$
$$y_p=2-\frac{2\cos2t}{3}$$

$$y=c_1\cos t+c_2\sin t+2-\frac{2\cos2t}{3}$$

Replacing t by $\ln(x+1)$

$$y=c_1\cos(\ln(x+1))+c_2\sin(\ln(x+1))+2-\frac{2\cos(2\ln(x+1))}{3}$$

$\boxed{10.4-7}$

(10) $x^2\dfrac{d^2y}{dx^2}+2x\dfrac{dy}{dx}-6y=10x^2$ where $y(1)=1$, $y'(1)=-6$

$(x^2D^2+2xD-6)y=10x^2$ —— (I) Cauchy Euler Diff Eq

Put $x=e^t \Rightarrow t=\ln x$

$xD=\Delta$

$x^2D^2=\Delta(\Delta-1)$

$(\Delta(\Delta-1)+2\Delta-6)y=10e^{2t}$

$(\Delta^2-\Delta+2\Delta-6)y=10e^{2t}$

$(\Delta^2+\Delta-6)y=10e^{2t}$

$\Delta^2+\Delta-6=0$

$\Delta^2+3\Delta-2\Delta-6=0$

$\Delta(\Delta+3)-2(\Delta+3)=0$

$(\Delta+3)(\Delta-2)=0 \Rightarrow \boxed{\Delta=-3,+2}$

$y_c=c_1e^{2t}+c_2e^{-3t}$

$\boxed{y_c=c_1x^2+c_2\frac{1}{x^3}}$

$y_p=\dfrac{1}{\Delta^2+\Delta-6}10e^{2t}$ $\quad\because \dfrac{1}{2^2+2-6}=\infty$ Failure Case.

$=\dfrac{t}{2\Delta+1}10e^{2t}$

$y_p=\dfrac{10te^{2t}}{2(2)+1}=2te^{2t}=\boxed{2(\ln x)(x^2)}$

$y=y_c+y_p=c_1x^2+c_2\frac{1}{x^3}+2(\ln x)x^2$ —— (II)

$y'=2c_1x-3c_2x^{-4}+2\left[\frac{1}{x}\cdot x^2+\ln x(2x)\right]$

$y'=2c_1x-\dfrac{3c_2}{x^4}+2x+2\ln x(2x)$

$y'=2c_1x-\dfrac{3c_2}{x^4}+2x+4x\ln x$ —— (III)

$\because \ln 1=0$

$y(1)=1 \Rightarrow$ from (II) $1=c_1+c_2+0$ —— (IV)

$y'(1)=-6 \Rightarrow$ from (III) $-6=2c_1-3c_2+2+0$

$-8=2c_1-3c_2$ —— (V)

from (IV) & (V)

$2=2c_1+2c_2$

$-8=2c_1-3c_2$

$10=5c_2 \Rightarrow c_2=\frac{10}{5}=\boxed{2}$

Put c_2 in (IV) $1=c_1+2 \Rightarrow c_1=\boxed{-1}$

$\therefore$ (II) becomes $y=-1x^2+\dfrac{2}{x^3}+2\ln x(x^2)$ Ans.

Ex 10.4-8 (23) (9)

(11) $x^2y'' - 2xy' + 2y = x\ln x \; ; \; y(1)=1, \; y'(1)=0$

$(x^2D^2 - 2xD + 2)y = x\ln x$

Let $x = e^t$ $\quad t = \ln x$

$xD = \Delta$

$x^2D^2 = \Delta(\Delta-1) = \Delta^2 - \Delta$

Substituting we get.

$(\Delta^2 - \Delta - 2\Delta + 2)y = e^t . t$

$(\Delta^2 - 3\Delta + 2)y = te^t \quad \text{———} (1)$

Characteristic Eq of (1) is

$\Delta^2 - 3\Delta + 2 = 0$

$\Delta^2 - \Delta - 2\Delta + 2 = 0$

$\Delta(\Delta-1) - 2(\Delta-1) = 0$

$(\Delta-2)(\Delta-1) = 0$

$\Delta = 1, 2.$

$y_c = c_1 e^t + c_2 e^{2t}$

$y_p = \dfrac{te^t}{\Delta^2 - 3\Delta + 2} = e^t \dfrac{t}{(\Delta+1)^2 - 3(\Delta+1) + 2}$ (Shift Theorem)

$= e^t \dfrac{t}{\Delta^2 + 2\Delta + 1 - 3\Delta - 3 + 2} = e^t \dfrac{. t}{\Delta^2 - \Delta} = e^t \dfrac{. t}{\Delta(\Delta-1)}$

$= -e^t \dfrac{t}{\Delta(1-\Delta)} = -e^t \dfrac{1}{\Delta}(1-\Delta)^{-1} t$

$= -e^t \dfrac{1}{\Delta}(1-(-1)\Delta)t = -e^t \dfrac{1}{\Delta}(1+\Delta)t = -e^t \dfrac{1}{\Delta}(t + \Delta t)$

$= -e^t \int (t+1)\, dt = -e^t \left(\dfrac{t^2}{2} + t\right) = -\dfrac{e^t}{2}(t^2 + 2t)$

General Sol is $y = c_1 e^t + c_2 e^{2t} - \dfrac{e^t}{2}(t^2 + 2t)$

Replace t by $\ln x$ $\quad y = c_1 e^{\ln x} + c_2 e^{2\ln x} - \dfrac{e^{\ln x}}{2}((\ln x)^2 + 2\ln x)$

$y = c_1 x + c_2 x^2 - \frac{1}{2} x[(\ln x)^2 + 2\ln x]$ → see above. *

$\ldots \; x[2\ln x . \frac{1}{x} + 2 . \frac{1}{x}]$

* From below.

$y(1) = 1 \Rightarrow 1 = c_1 . 1 + c_2 . 1 - \frac{1}{2} . 1 (\ln 1 + 2\ln 1)$

$1 = c_1 + c_2 \quad \because \ln 1 = 0$

$y'(1) = 0 \Rightarrow 0 = c_1 + 2c_2 - 1$

from $c_1 + c_2 = 1 \Rightarrow c_1 = 1 - c_2$

So $0 = 1 - c_2 + 2c_2 - 1 \Rightarrow \boxed{c_2 = 0}$

$\therefore 1 = c_1 + 0 \Rightarrow \boxed{c_1 = 1}$

Required Sol. $y = x - \frac{1}{2} x[(\ln x)^2 + 2\ln x]$

$= x - \frac{x}{2}(\ln x)^2 - x\ln x.$

Ans.

10.4 - 9 ⑳ (19)

(12) $(x^3D^3+2x^2D^2+xD-1)y=15\cos(2\ln x)$ $\quad y(1)=2$, $y'(1)=-3$, $y''(1)=0$

Let $x=e^t$ $\quad t=\ln x$

$xD=\Delta$

$x^2D^2=\Delta^2-\Delta$

$x^3D^3=\Delta^3-3\Delta^2+2\Delta$

Substituting we get

$(\Delta^3-3\Delta^2+2\Delta+2\Delta^2-2\Delta+\Delta-1)y=15\cos 2t$

$(\Delta^3-\Delta^2+\Delta-1)y=15\cos 2t$ —— ①

Characteristic Eq of ①

$\Delta^3-\Delta^2+\Delta-1=0$

$(\Delta-1)(\Delta^2+1)=0$

$\Delta=1,\ \pm i$

	1	−1	1	−1
1	↓	1	0	1
	1	0	1	$\lfloor 0$

$\Delta^2+1=0$

$y_c=c_1e^t+c_2\cos t+c_3\sin t$

$y_p=\dfrac{15\cos 2t}{\Delta^3-\Delta^2+\Delta-1}=15\dfrac{\cos 2t}{\Delta(\Delta^2)-\Delta^2+\Delta-1}=15\dfrac{\cos 2t}{\Delta(-2^2)-(-2^2)+\Delta-1}=15\dfrac{\cos 2t}{-4\Delta+4+\Delta-1}$

$=15\dfrac{\cos 2t}{-3\Delta+3}=\dfrac{15}{3}\dfrac{\cos 2t}{(-\Delta+1)}=5\dfrac{(1+\Delta)\cos 2t}{(1+\Delta)(1-\Delta)}=\dfrac{5(1+\Delta)\cos 2t}{1-\Delta^2}$

$=\dfrac{5}{1-(-4)}(1+\Delta)\cos 2t=\dfrac{5}{5}(\cos 2t+2(-\sin 2t))=\cos 2t-2\sin 2t$

General Sol $\quad y=c_1e^t+c_2\cos t+c_3\sin t+\cos 2t-2\sin 2t$

Replace t by '$\ln x$' $\quad y=c_1x+c_2\cos(\ln x)+c_3\sin(\ln x)+\cos 2(\ln x)-2\sin(2\ln x)$

$y'=c_1-c_2\dfrac{\sin(\ln x)}{x}+c_3\dfrac{\cos\ln x}{x}-2\dfrac{\sin(2\ln x)}{x}-4\dfrac{\cos(2\ln x)}{x}$

$y''=-c_2\dfrac{\cos(\ln x)}{x^2}+c_2\dfrac{\sin\ln x}{x^2}-c_3\dfrac{\sin(\ln x)}{x^2}-c_3\dfrac{\cos\ln x}{x^2}-4\dfrac{\cos(2\ln x)}{x^2}$

$\qquad+2\dfrac{\sin(2\ln x)}{x^2}+8\dfrac{\sin(2\ln x)}{x^2}+4\dfrac{\cos(2\ln x)}{x^2}$

$y(1)=2 \Rightarrow 2=c_1+c_2+1 \Rightarrow c_1+c_2=1$ —— ②

$y'(1)=-3 \Rightarrow -3=c_1+c_3-4 \Rightarrow c_1+c_3=1$ —— ③

$y''(1)=0 \Rightarrow 0=-c_2-c_3-4+4 \Rightarrow c_2+c_3=0 \Rightarrow c_2=-c_3$ —— ④

from ② $\quad c_1+(-c_3)=1$

& $\quad c_1+c_3=1$

$\overline{2c_1=2} \Rightarrow \boxed{c_1=1}\ \therefore \boxed{c_2=0}$ $\boxed{c_3=0}$

Hence $y=x+\cos 2(\ln x)-2\sin(2\ln x)$

10-4-10.

Eqs Reducible to Cauchy's form

$$a_0 (a+bx)^n \frac{d^n y}{dx^n} + a_1 (a+bx)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots a_{n-1}(a+bx)\frac{dy}{dx} + a_n y = \phi(a+bx)$$

such a diff eq is reducible to cauchy's form. In order to solve it We first reduce it to Cauchy's diff eq

Put $a+bx = z$

Diff w.r.t 'x'

$$b = \frac{dz}{dx}$$

$$b = \frac{dz}{dy}\cdot\frac{dy}{dx}$$

$$\boxed{b\frac{dy}{dz} = \frac{dy}{dx}}$$

Diff $\frac{d^2y}{dx^2} = b\frac{d^2y}{dz^2}\cdot\frac{dz}{dx}$

$$= b\frac{d^2y}{dz^2}\cdot b \quad \because z = a+bx \Rightarrow \frac{dz}{dx} = b$$

$$\boxed{\frac{d^2y}{dx^2} = b^2\frac{d^2y}{dz^2}}$$

Similarly $\boxed{\frac{d^3y}{dx^3} = b^3\frac{d^3y}{dz^3}}$

So $\boxed{\frac{d^ny}{dx^n} = b^n\frac{d^ny}{dz^n}}$

So above eq becomes

$$a_0 z^n b^n \frac{d^n y}{dz^n} + a_1 z^{n-1} b^{n-1} \frac{d^{n-1} y}{dz^{n-1}} + \cdots a_{n-1} z b \frac{dy}{dz} + a_n y = \phi(z)$$

which is Cauchy-Euler Eq

Note If we put $\boxed{a=0,\ b=1}$ in eq reducible to Cauchy it becomes Cauchy's Diff Eq.

Note 'b' is coeff of x in $(a+bx)$

9 $(2x+1)^2\frac{d^2y}{dx^2} - 6(2x+1)\frac{dy}{dx} + 16y = 8(2x+1)^2$

Put $2x+1 = z$

$\frac{dy}{dx} = 2\frac{dy}{dz}$

$\frac{d^2y}{dx^2} = 2^2\frac{d^2y}{dz^2}$

$\therefore z^2\left(2^2\frac{d^2y}{dz^2}\right) - 6z\left(2\frac{dy}{dz}\right) + 16y = 8z^2$

$4z^2\frac{d^2y}{dz^2} - 12z\frac{dy}{dz} + 16y = 8z^2$

$z^2\frac{d^2y}{dz^2} - 3z\frac{dy}{dz} + 4y = 2z^2$

$(z^2D^2 - 3zD + 4)y = 2z^2$ — Cauchy E.D. Eq (1)

$z = e^t \Rightarrow t = \ln z$

$zD = \Delta$

$z^2D^2 = \Delta(\Delta-1)$

$(\Delta(\Delta-1) - 3\Delta + 4)y = 2e^{2t}$

$(\Delta^2 - \Delta - 3\Delta + 4)y = 2e^{2t}$

$(\Delta^2 - 4\Delta + 4)y = 2e^{2t}$ — (11)

$\Delta^2 - 4\Delta + 4 = 0$

$(\Delta-2)^2 = 0 \Rightarrow \boxed{\Delta = 2,2}$

$\therefore y_c = (c_1 + c_2 t)e^{2t} \Rightarrow y_c = (c_1 + c_2\ln z)z^2$

$y_c = [c_1 + c_2\ln(2x+1)](2x+1)^2$

from (11)

$y_p = \frac{1}{\Delta^2 - 4\Delta + 4}(2e^{2t})$ $\qquad \frac{1}{8-8} = \frac{1}{0}$ Failure case

$= \frac{t}{2\Delta - 4}2e^{2t}$ $\qquad \frac{t}{2(2)-4} = \infty$ Failure case

$= \frac{t^2}{2}\,2e^{2t}$

$= t^2e^{2t}$

$= z^2(\ln z)^2$

$y_p = (2x+1)^2[\ln(2x+1)]^2$

So $y = y_c + y_p$

$y = [c_1 + c_2\ln(2x+1)](2x+1)^2 + (2x+1)^2(\ln(2x+1))^2$

$= (2x+1)^2[c_1 + c_2\ln(2x+1) + \ln(2x+1)$ …

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