EXERCISE 10.4

Solve.

❖ Question # 1:

\[ x^2 \frac{d^2 y}{dx^2} + 7x \frac{dy}{dx} + 5y = x^5. \]

Solution:

Given equation is

\[ x^2 \frac{d^2 y}{dx^2} + 7x \frac{dy}{dx} + 5y = x^5 \quad \text{--- (i)} \]

Replace \( \frac{d^\, n}{dx^\, n} \) by \( D \) in (i), we have

\[ x^2 D^2 + 7xD + 5y = x^5 \quad \text{--- (ii)} \]

This is Cauchy-Euler equation.

To solve this, we put \( x = e^t \) so that \( t = \ln x \).

Then,

\[ xD = \Delta \]
\[ x^2 D^2 = \Delta(\Delta - 1) = \Delta^2 - \Delta \]

Thus equation (ii) becomes

\[ (\Delta^2 - \Delta + 7\Delta + 5)y = e^{5t} \]
\[ \Rightarrow (\Delta^2 + 6\Delta + 5)y = e^{5t} \quad \text{--- (iii)} \]

The characteristics equation of (iii) is

\[ \Delta^2 + 6\Delta + 5 = 0 \]
\[ \Rightarrow \Delta^2 + \Delta + 5\Delta + 5 = 0 \]
\[ \Rightarrow \Delta(\Delta + 1) + 5(\Delta + 1) = 0 \]
\[ \Rightarrow (\Delta + 1)(\Delta + 5) = 0 \]
\[ \Rightarrow \Delta = -1 \text{ or } \Delta = -5 \]

Therefore, the complementary function will be

\[ y_c = c_1 e^{-t} + c_2 e^{-5t} \]

Now,

\[ y_p = \frac{e^{5t}}{\Delta^2 + 6\Delta + 5} \]
\[ \Rightarrow y_p = \frac{e^{5t}}{(\Delta + 1)(\Delta + 5)} \]
\[ \Rightarrow y_p = \frac{e^{5t}}{(5 + 1)(5 + 5)} \]
\[ \Rightarrow y_p = \frac{e^{5t}}{60} \]

The general solution is

\[ y = y_c + y_p \]
\[ \Rightarrow y = c_1 e^{-t} + c_2 e^{-5t} + \frac{e^{5t}}{60} \]
\[ \Rightarrow y = c_1 x^{-1} + c_2 x^{-5} + \frac{x^5}{60} \quad \because x = e^{-t} \]

is required solution of (i).

❖ Question # 2:

\[ x^2 \frac{d^2 y}{dx^2} - 3x \frac{dy}{dx} + 5y = x^2 \sin(\ln x). \]

Solution:

Given equation is

\[ x^2 \frac{d^2 y}{dx^2} - 3x \frac{dy}{dx} + 5y = x^2 \sin(\ln x) \quad \text{--- (i)} \]

Replace \( \frac{d^\, n}{dx^\, n} \) by \( D \) in (i), we have

\[ x^2 D^2 - 3xD + 5y = x^2 \sin(\ln x) \quad \text{--- (ii)} \]

This is Cauchy-Euler equation.

To solve this, we put \( x = e^t \) so that \( t = \ln x \).

Then,

\[ xD = \Delta \]
\[ x^2 D^2 = \Delta(\Delta - 1) = \Delta^2 - \Delta \]

Thus equation (ii) becomes
\((\Delta^2 - \Delta - 3\Delta + 5)y = e^{2t} \sin t\)
\[\Rightarrow (\Delta^2 - 4\Delta + 5)y = e^{2t} \sin t - - - (iii)\]

The characteristics equation of \((iii)\) is
\[\Delta^2 - 4\Delta + 5 = 0\]
\[\Rightarrow \Delta = \frac{4 \pm \sqrt{16 - 20}}{2}\]
\[\Rightarrow \Delta = \frac{4 \pm \sqrt{-4}}{2}\]
\[\Rightarrow \Delta = \frac{4 \pm 2i}{2}\]
\[\Rightarrow \Delta = 2 \pm i\]

Therefore, the complementary function will be
\[y_c = (c_1 \cos t + c_2 \sin t)e^{2t}\]

Now,
\[y_p = \frac{e^{2t} \sin t}{\Delta^2 - 4\Delta + 5}\]
\[\Rightarrow y_p = \frac{e^{2t} \sin t}{(\Delta + 2)^2 - 4(\Delta + 2) + 5}\]
\[\text{(by exponential shift \(t\))}\]
\[\Rightarrow y_p = \frac{e^{2t} \sin t}{\Delta^2 + 4 + 4\Delta - 4\Delta - 8 + 5}\]
\[\Rightarrow y_p = \frac{e^{2t} \sin t}{\Delta^2 - 3}\]
\[\Rightarrow y_p = \frac{e^{2t} \sin t}{\Delta + i}(\Delta - i)\]
\[\Rightarrow y_p = \frac{te^{2t} \text{Im}(\cos t + i \sin t)}{2i}\]
\[\Rightarrow y_p = -\frac{te^{2t} \text{Im}(\cos t + i \sin t)i}{2}\]
\[\Rightarrow y_p = -\frac{te^{2t} \text{Im}(i \cos t - \sin t)}{2}\]
\[\Rightarrow y_p = \frac{te^{2t} \cos t}{2}\]

The general solution is
\[y = y_c + y_p\]
\[\Rightarrow y = (c_1 \cos t + c_2 \sin t)e^{2t} - \frac{te^{2t} \cos t}{2}\]
\[\Rightarrow y = (c_1 \cos (\ln x) + c_2 \sin (\ln x))x^2 - \frac{x^2 \ln^2 x \cos (\ln x)}{2}\]
\[\Rightarrow x = e^{-t}\]

is required solution of \((i)\)

\[\text{Question # 3:}\]
\[x^2 \frac{d^2 y}{dx^2} - (2m - 1)x \frac{dy}{dx} + (m^2 + n^2)y = n^2 x^m \ln x.\]

\[\text{Solution:}\]

Given equation is
\[x^2 \frac{d^2 y}{dx^2} - (2m - 1)x \frac{dy}{dx} + (m^2 + n^2)y = n^2 x^m \ln x - - - (i)\]

Replace "\(\frac{dy}{dx}\)" by \(D\) in \((i)\), we have
\[x^2 D^2 - (2m - 1)x D + (m^2 + n^2)y = n^2 x^m \ln x - - - (ii)\]

This is Cauchy-Euler equation.

To solve this, we put \(x = e^t\) so that \(t = \ln x\).

Then,
\[xD = \Delta\]
\[x^2 D^2 = \Delta(\Delta - 1) = \Delta^2 - \Delta\]

Thus equation \((ii)\) becomes
\[\begin{aligned} 
\left[\Delta^2 - \Delta - (2m - 1)\Delta + (m^2 + n^2)\right]y &= n^2 e^{mt}t \\
\Rightarrow \left(\Delta^2 - 2m\Delta + (m^2 + n^2)\right) y &= n^2 te^{mt} - -- -(iii) \\
\end{aligned}\]

The characteristics equation of \((iii)\) is
\[\Delta^2 - 2m\Delta + (m^2 + n^2) = 0\]
\[\Rightarrow \Delta = \frac{-(-2m) \pm \sqrt{(-2m)^2 - 4(1)(m^2 + n^2)}}{2(1)}\]
Therefore, the complementary function will be
\[ y_c = (c_1 \cos nt + c_2 \sin nt)e^{nt} \]

Now,
\[ y_p = \frac{n^2 t e^{nt}}{\Delta^2 - 2m\Delta + (m^2 + n^2)} \]
\[ \Rightarrow y_p = \frac{n^2 t e^{nt}}{(\Delta + m)^2 - 2m(\Delta + m) + m^2 + n^2} \]  
(by exponential shift)
\[ \Rightarrow y_p = \frac{n^2 t e^{nt}}{\Delta^2 + n^2} \]
\[ \Rightarrow y_p = \frac{n^2 t e^{nt}}{n^2(1 + \frac{\Delta^2}{n^2})} \]
\[ \Rightarrow y_p = e^{nt} \left(1 + \frac{\Delta^2}{n^2}\right)^{-1} t \]
\[ \Rightarrow y_p = e^{nt} \left(1 - \frac{\Delta^2}{m^2}\right) t \]
\[ \Rightarrow y_p = e^{nt} \left(1 - \text{neglecting terms}\right) t \]
\[ \Rightarrow y_p = te^{nt} \]

The general solution is
\[ y = y_c + y_p \]
\[ \Rightarrow y = (c_1 \cos nt + c_2 \sin nt)e^{nt} + te^{nt} \]
\[ \Rightarrow y = (c_1 \cos n(ln x) + c_2 \sin n(ln x))x^m + x^m \ln x \]
\[ \therefore x = e^t \]
\[ \Rightarrow y = (c_1 \cos (n \ln x) + c_2 \sin (n \ln x))x^m + x^m \ln x \]

\[ \Rightarrow y = x^m[c_1 \cos (n \ln x) + c_2 \sin (n \ln x) + \ln x] \]
is required solution of (i).

\[ \Rightarrow \quad 4x^2 \frac{d^2 y}{dx^2} - 4x \frac{dy}{dx} + 3y = \sin \ln(-x) \]  
\[ \text{Question # 4:} \]

Given equation is
\[ 4x^2 \frac{d^2 y}{dx^2} - 4x \frac{dy}{dx} + 3y = \sin \ln(-x) \]  

Replace \( \frac{dy}{dx} \) by \( D \ln \), we have
\[ 4x^2 D^2 - 4xD + 3y = \sin \ln(-x) \]  

This is Cauchy-Euler equation.
To solve this, we put \( -x = e^t \) so that \( t = \ln(-x) \).
Then,
\[ xD_x = \Delta \]
\[ x^2 D^2 = \Delta(\Delta - 1) = \Delta^2 - \Delta \]
Thus equation (ii) becomes
\[ [4(\Delta^2 - \Delta) - 4\Delta + 3]y = \sin t \]
\[ \Rightarrow (4\Delta^2 - 4\Delta - 4\Delta + 3)y = \sin t \]
\[ \Rightarrow (4\Delta^2 - 8\Delta + 3)y = \sin t \]  

The characteristics equation of (iii) is
\[ 4\Delta^2 - 8\Delta + 3 = 0 \]
\[ \Rightarrow 4\Delta^2 - 2\Delta - 6\Delta + 3 = 0 \]
\[ \Rightarrow 2\Delta(2\Delta - 1) - 2\Delta(2\Delta - 1) = 0 \]
\[ \Rightarrow (2\Delta - 1)(2\Delta - 1) = 0 \]
\[ \Rightarrow 2\Delta - 1 = 0 \text{ or } 2\Delta - 1 = 0 \]
\[ \Rightarrow \Delta = \frac{1}{2} \text{ or } \Delta = \frac{1}{2} \]

Therefore, the complementary function will be
The general solution is

\[ y = y_c + y_p \]

\[ y = (c_1 + c_2 t)e^t + 8 \cos t - \sin t \]

is required solution of (i).

\[ \therefore -x = e^t \]

Question # 5: \( x^3 \frac{d^3 y}{dx^3} + 2x^2 \frac{d^2 y}{dx^2} + 2y = 10x + \frac{10}{x} \) ——— (i)

Replace \( \frac{dy}{dx} \) by \( D \) in (i), we have

\[ x^3 D^3 + 2x^2 D^2 + 2y = 10x + \frac{10}{x} \] ——— (ii)

This is Cauchy-Euler equation.

To solve this, we put \( x = e^t \) so that \( t = \ln x \).

Then,

\[ xD = \Delta \]

\[ x^2 D^2 = \Delta(\Delta - 1) = \Delta^2 - \Delta \]

\[ x^3 D^3 = \Delta(\Delta - 1)(\Delta - 2) = \Delta^3 - 3\Delta^2 + 2\Delta \]

Thus equation (ii) becomes

\[ [\Delta^3 - 3\Delta^2 + 2\Delta + 2(\Delta^2 - \Delta) + 2]y = 10e^t + \frac{10}{e^t} \]

\[ \Rightarrow [\Delta^3 - 2\Delta + 2]y = 10e^t + 10e^{-t} \] ——— (iii)

The characteristics equation of (iii) is

\[ \Delta^3 - 2\Delta + 2 = 0 \]

As \( \Delta = -1 \) is the root of \( \Delta^3 - 2\Delta + 2 = 0 \). Therefore, we use synthetic division in order to find the other roots of the characteristics equation.

| 1 | -1 | 0 | 2 |
| -1 | 0 | -1 | 2 | -2 |
| 1 | -2 | 2 | 0 |

The residue equation will be

\[ \Delta^2 - 2\Delta + 2 = 0 \]

\[ \Rightarrow \Delta = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)} \]

\[ \Rightarrow \Delta = \frac{2 \pm \sqrt{-4}}{2} \]

\[ \Rightarrow \Delta = \frac{2 \pm 2i}{2} \]
Therefore, the complementary function will be
\[ y_c = c_1e^{-t} + (c_2 \cos t + c_3 \sin t)e^t \]
Now,
\[ y_p = \frac{10e^t + 10e^{-t}}{\Delta^3 - \Delta^2 + 2} \]
\[ \Rightarrow y_p = \frac{10e^t + 10e^{-t}}{(\Delta + 1)(\Delta^2 - 2\Delta + 2)} \]
\[ \Rightarrow y_p = \frac{10e^t}{(1 + 1)(1 - 2 + 2)} + \frac{10e^{-t}}{(1 + 2 + 2)} \]
\[ \Rightarrow y_p = 5e^t + 2te^{-t} \]
The general solution is
\[ y = y_c + y_p \]
\[ \Rightarrow y = c_1e^{-t} + (c_2 \cos t + c_3 \sin t)e^t + 5e^t + 2te^{-t} \]
\[ \Rightarrow y = c_1(x)^{-1} + (c_2 \cos(ln x) + c_3 \sin(ln x))x + 5x + 2 \ln x .(x)^{-1} \]
\[ \Rightarrow y = (c_1 + 2 \ln x)x^{-1} + (x)^{-1} + [c_2 \cos(ln x) + c_3 \sin(ln x) + 5]x \]
is required solution of (i).

**Question # 6:**
\[
\frac{d^4y}{dx^4} + 2x^2 \frac{d^2y}{dx^2} - x^2 \frac{dy}{dx} + xy = 1.
\]

**Solution:**
Given equation is
\[
\frac{d^4y}{dx^4} + 2x^2 \frac{d^2y}{dx^2} - x^2 \frac{dy}{dx} + xy = 1
\]
Dividing both sides by \(x\), we have
\[
\frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + y = \frac{1}{x} \quad - \quad - \quad - \quad (i)
\]
Replace \(\frac{dy}{dx}\) by \(D\) in (i), we have
\[
x^3D^3 + 2x^2D^2 - xD + y = \frac{1}{x} \quad - \quad - \quad - \quad (ii)
\]
This is Cauchy-Euler equation.
To solve this, we put \(x = e^t\) so that \(t = \ln x\).
Then,
\[
xD = \Delta
\]
\[
x^2D^2 = \Delta(\Delta - 1) = \Delta^2 - \Delta
\]
\[
x^3D^3 = \Delta(\Delta - 1)(\Delta - 2) = \Delta^3 - 3\Delta^2 + 2\Delta
\]
Thus equation (ii) becomes,
\[
[\Delta^3 - 3\Delta^2 + 2\Delta + 2(\Delta^2 - \Delta) - \Delta + 1]y = \frac{1}{e^t}
\]
\[
\Rightarrow [\Delta^3 - \Delta^2 - \Delta + 1]y = e^{-t} \quad - \quad - \quad - \quad (iii)
\]
The characteristics equation of (iii) is
\[
\Delta^3 - \Delta^2 - \Delta + 1 = 0
\]
\[
\Rightarrow \Delta^2(\Delta - 1) - 1(\Delta - 1) = 0
\]
\[
\Rightarrow (\Delta^2 - 1)(\Delta - 1) = 0
\]
\[
\Rightarrow (\Delta + 1)(\Delta - 1)(\Delta - 1) = 0
\]
\[
\Rightarrow \Delta = 1, 1, -1
\]
Therefore, the complementary function will be
\[
y_c = c_1e^t + c_2te^t + c_3e^{-t}
\]
Now,
\[
y_p = \frac{e^{-t}}{\Delta^3 - \Delta^2 - \Delta + 1}
\]
\[
\Rightarrow y_p = \frac{e^{-t}}{(\Delta + 1)(\Delta - 1)(\Delta - 1)}
\]
\[
\Rightarrow y_p = \frac{te^{-t}}{(-1 - 1)(-1 - 1)}
\]
\[
\Rightarrow y_p = \frac{te^{-t}}{4}
\]
The general solution is
\[ y = y_c + y_p \]
\[ \Rightarrow y = y_c + y_p \]
\[ \Rightarrow y = c_1 e^t + c_2 t e^t + c_3 e^{-t} + \frac{t e^{-t}}{4} \]
\[ \Rightarrow y = c_1 x + c_2 x \ln x + c_3 x^{-1} + \frac{\ln x}{4x} \cdot -x = e^t \]
\[ \Rightarrow y = x(c_1 + c_2 \ln x) + c_3 x^{-1} + \frac{\ln x}{4x} \]
is required solution of (i).

**Question # 7:**
\[ x^3 \frac{d^3 y}{dx^3} + 4x^2 \frac{d^2 y}{dx^2} - 5x \frac{dy}{dx} - 15y = x^4. \]

**Solution:**

Given equation is
\[ x^3 \frac{d^3 y}{dx^3} + 4x^2 \frac{d^2 y}{dx^2} - 5x \frac{dy}{dx} - 15y = x^4 \quad ---(i) \]

Replace \( \frac{dy}{dx} \) by \( D \) in (i), we have
\[ x^3 D^3 + 4x^2 D^2 - 5xD - 15y = x^4 \quad ---(ii) \]

This is Cauchy-Euler equation.

To solve this, we put \( x = e^t \) so that \( t = \ln x \).

Then,
\[ xD = \Delta \]
\[ x^2 D^2 = \Delta(\Delta - 1) = \Delta^2 - \Delta \]
\[ x^3 D^3 = \Delta(\Delta - 1)(\Delta - 2) = \Delta^3 - 3\Delta^2 + 2\Delta \]

Thus equation (ii) becomes
\[ [\Delta^3 - 3\Delta^2 + 2\Delta + 4(\Delta^2 - \Delta) - 5\Delta - 15]y = e^{4t} \]
\[ \Rightarrow [\Delta^3 + \Delta^2 - 7\Delta - 15]y = e^{4t} \quad ---(iii) \]

The characteristics equation of (iii) is
\[ \Delta^3 + \Delta^2 - 7\Delta - 15 = 0 \]

As \( \Delta = 3 \) is the root of \( \Delta^3 + \Delta^2 - 7\Delta - 15 = 0 \). Therefore, we use synthetic division in order to find the other roots of the characteristics equation.

<table>
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<tr>
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<td>4</td>
<td>5</td>
<td>0</td>
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Now, the residual equation is
\[ \Delta^2 + 4\Delta + 5 = 0 \]
\[ \Rightarrow \Delta = \frac{-4 \pm \sqrt{16 - 20}}{2(1)} \]
\[ \Rightarrow \Delta = \frac{-4 \pm 2i}{2} \]
\[ \Rightarrow \Delta = -2 \pm i \]

Therefore, the complementary function will be
\[ y_c = c_1 e^{3t} + (c_2 \cos t + c_3 \sin t)e^{-2t} \]

Now,
\[ y_p = \frac{e^{4t}}{\Delta^3 + \Delta^2 - 7\Delta - 15} \]
\[ \Rightarrow y_p = \frac{e^{4t}}{(\Delta - 3)(\Delta^2 + 4\Delta + 5)} \]
\[ \Rightarrow y_p = \frac{e^{4t}}{(4 - 3)(16 + 16 + 5)} \]
\[ \Rightarrow y_p = \frac{e^{4t}}{37} \]

The general solution is
\[ y = y_c + y_p \]
\[ \Rightarrow y = c_1 e^{3t} + (c_2 \cos t + c_3 \sin t)e^{-2t} + \frac{e^{4t}}{37} \]
\[ \Rightarrow y = c_1 x^3 + (c_2 \cos (\ln x) + c_3 \sin (\ln x))x^{-2} + \frac{x^4}{37} \]
\( (x + 1)^2 \frac{d^2 y}{dx^2} + (x + 1) \frac{dy}{dx} + y = 4[\cos \ln(x + 1)]^2 \)

is required solution of \((i)\).

\[ \therefore -x = e^t \]

**Question # 8:**

\[ (x + 1)^2 \frac{d^2 y}{dx^2} + (x + 1) \frac{dy}{dx} + y = 4[\cos \ln(x + 1)]^2 \]

\[ y_p = \frac{4(1 + \cos 2t)}{\Delta^2 + 1} \]

**Solution:**

Given equation is

\[ (x + 1)^2 \frac{d^2 y}{dx^2} + (x + 1) \frac{dy}{dx} + y = 4[\cos \ln(x + 1)]^2 \quad \text{--- (i)} \]

Replace \( \frac{dy}{dx} \) by \( D \) in \((i)\), we have

\[ (x + 1)^2 D^2 + (x + 1)D + y = 4[\cos \ln(x + 1)]^2 \quad \text{--- (ii)} \]

This is Cauchy-Euler equation.

To solve this, we put \( x + 1 = e^t \) so that \( t = \ln(x + 1) \).

Then,

\[ (x + 1)D = \Delta \]

\[ (x + 1)^2 D^2 = \Delta(\Delta - 1) = \Delta^2 - \Delta \]

Thus equation \((ii)\) becomes

\[ \Delta^2 - \Delta + \Delta + 1 \]

\[ y = 4 \cos^2 t \quad \text{--- (iii)} \]

The characteristics equation of \((iii)\) is

\[ \Delta^2 + 1 = 0 \]

\[ \Delta = \pm i \]

Therefore, the complementary function will be

\[ y_c = c_1 \cos t + c_2 \sin t \]

Now,

\[ y_p = \frac{4 \cos^2 t}{\Delta^2 + 1} \]

\[ y_p = \frac{2 + 2 \cos 2t}{\Delta^2 + 1} \]

\[ y_p = \frac{2}{\Delta^2 + 1} \]

\[ y_p = \frac{2}{\Delta^2 + 1} + 2 \frac{\cos 2t}{\Delta^2 + 1} \]

\[ y_p = 2(1 + \Delta)^{-1} + \frac{2 \Re e^{2it}}{(\Delta + i)(\Delta - i)} \]

\[ y_p = 2(1) + \frac{2 \Re(\cos t + i \sin t)}{(2i + i)(2i - i)} \]

\[ y_p = 2 - \frac{2}{3} \cos t \]

The general solution is

\[ y = y_c + y_p \]

\[ y = c_1 \cos t + c_2 \sin t + 2 - \frac{2}{3} \cos t \]

\[ y = c_1 \cos \ln(x + 1) + c_2 \sin \ln(x + 1) + 2 \]

\[ - \frac{2}{3} \cos \ln(x + 1) \therefore -x = e^t \]

**Question # 9:**

\[ (2x + 1)^2 \frac{d^2 y}{dx^2} - 6(2x + 1) \frac{dy}{dx} + 16y = 8(2x + 1)^2 \]

\[ (2x + 1)^2 \frac{d^2 y}{dx^2} - 6(2x + 1) \frac{dy}{dx} + 16y = 8(2x + 1)^2 \quad \text{--- (i)} \]

**Solution:**

Given equation is

\[ (2x + 1)^2 \frac{d^2 y}{dx^2} - 6(2x + 1) \frac{dy}{dx} + 16y = 8(2x + 1)^2 \quad \text{--- (i)} \]

Replace \( \frac{dy}{dx} \) by \( D \) in \((i)\), we have

\[ (2x + 1)^2 D^2 - 6(2x + 1)D + 16y = 8(2x + 1)^2 \quad \text{--- (ii)} \]

This is Cauchy-Euler equation.

To solve this, we put \( 2x + 1 = e^t \) so that \( t = \ln(2x + 1) \).

Then,

\[ y_p = \frac{4(1 + \cos 2t)}{\Delta^2 + 1} \]
Thus equation \((ii)\) becomes
\[
[4\Delta^2 - 4\Delta - 12\Delta + 16]y = 8e^{2t}
\]
\[
\Rightarrow [4\Delta^2 - 16\Delta + 16]y = 8e^{2t}
\]
\[
\Rightarrow [\Delta^2 - 4\Delta + 4]y = 2e^{2t} \quad --- (iii)
\]

The characteristics equation of \((iii)\) is
\[
\Delta^2 - 4\Delta + 4 = 0
\]
\[
\Rightarrow (\Delta - 2)^2 = 0
\]
\[
\Rightarrow \Delta = 2 \text{ or } \Delta = 2
\]

Therefore, the complementary function will be
\[
y_c = c_1e^{2t} + c_2te^{2t}
\]
\[
\Rightarrow y_c = (c_1 + c_2t)e^{2t}
\]

Now,
\[
y_p = \frac{2e^{2t}}{\Delta^2 - 4\Delta + 4}
\]
\[
\Rightarrow y_p = \frac{2e^{2t}}{(\Delta - 2)^2}
\]
\[
\Rightarrow y_p = 2te^{2t}
\]

The general solution is
\[
y = y_c + y_p
\]
\[
\Rightarrow y = (c_1 + c_2t)e^{2t} + 2te^{2t}
\]
\[
y = [c_1 + c_2\ln((2x + 1))][2(2x + 1) + 2(2\ln((2x + 1))^2(2x + 1)^2
\]

is required solution of \((i)\).

**Question # 10:**

\[
x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - 6y = 10x^2 \quad y(1) = 1 \quad y'(1) = -6
\]

**Solution:**

**Given equation is**
\[
x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - 6y = 10x^2 \quad --- (i)
\]

Replace \(\frac{dy}{dx}\) by \(D\) in \((i)\), we have
\[
x^2D^2 + 2xD - 6y = 10x^2 \quad --- (ii)
\]

This is Cauchy-Euler equation.

To solve this, we put \(x = e^t\) so that \(t = \ln x\).

Then,
\[
xD = \Delta
\]
\[
x^2D^2 = \Delta(\Delta - 1) = \Delta^2 - \Delta
\]

Thus equation \((ii)\) becomes
\[
(\Delta^2 - \Delta + 2\Delta - 6)y = 10e^{2t}
\]
\[
\Rightarrow (\Delta^2 + \Delta - 6)y = 10e^{2t} \quad --- (iii)
\]

The characteristics equation of \((iii)\) is
\[
\Delta^2 + \Delta - 6 = 0
\]
\[
\Rightarrow \Delta^2 + 3\Delta - 2\Delta - 6 = 0
\]
\[
\Rightarrow \Delta(\Delta + 3) - 2(\Delta + 3) = 0
\]
\[
\Rightarrow (\Delta + 3)(\Delta - 2) = 0
\]
\[
\Rightarrow \Delta = 2 \text{ or } \Delta = -3
\]

Therefore, the complementary function will be
\[
y_c = c_1e^{2t} + c_2e^{-3t}
\]

Now,
\[
y_p = \frac{10e^{2t}}{\Delta^2 + \Delta - 6}
\]
\[
\Rightarrow y_p = \frac{10e^{2t}}{(\Delta + 3)(\Delta - 2)}
\]
\[
\Rightarrow y_p = \frac{10e^{2t}}{(2 + 3)}
\]
\[ y_p = 2te^{2t} \]

The general solution is
\[ y = c_1e^{2t} + c_2e^{-3t} + 2te^{2t} \]
\[ y = c_1x^2 + c_2x^{-3} + 2(lnx)x^2 - - - (iv) \]

To find the constants \( c_1 \) & \( c_2 \), we will use the initial values.

Applying \( y(1) = 1 \) in equation \((iv)\), we have
\[ 1 = c_1 + c_2 + 2(\ln 1)1 \]
\[ \Rightarrow 1 = c_1 + c_2 + 0 \cdot \ln 1 = 0 \]
\[ \Rightarrow c_1 + c_2 = 1 - - - (a) \]

Differentiating \((iv)\) w.r.t \( x \), we have
\[ y' = 2c_1x - 3c_2x^{-4} + 4(\ln x)x + 2\left(\frac{1}{x}\right)x^2 - - - (v) \]

Applying \( y'(1) = -6 \) in equation \((v)\), we have
\[ -6 = 2c_1 - 3c_2 + 0 + 2 \]
\[ \Rightarrow -8 = 2c_1 - 3c_2 \]
\[ \Rightarrow 2c_1 - 3c_2 = -8 - - - (b) \]

From \((a)\), we have
\[ c_1 = 1 - c_2 - - - (c) \]

Using \( c_1 = 1 - c_2 \) in equation \((b)\), we have
\[ 2(1 - c_2) - 3c_2 = -8 \]
\[ \Rightarrow 2 - 2c_2 - 3c_2 = -8 \]
\[ \Rightarrow -5c_2 = -10 \]
\[ \Rightarrow c_2 = 2 \]

Now \((c)\) \( \Rightarrow \)
\[ c_1 = -1 \]

Hence,
\[ y = -x^2 + 2x^{-3} + 2(lnx)x^2 \]
is required solution.

**Question #11:**
\[ x^2y'' - 2xy' + 2y = x \ln x \quad y(1) = 1 \quad y'(1) = 0 \]

**Solution:**

Given equation is
\[ x^2y'' - 2xy' + 2y = x \ln x - - - (i) \]

Replace "y" by \( D \) in \((i)\), we have
\[ x^2D^2 - 2xD + 2y = x \ln x - - - (ii) \]

This is Cauchy-Euler equation.

To solve this, we put \( x = e^t \) so that \( t = \ln x \).

Then,
\[ xD = \Delta \]
\[ x^2D^2 - \Delta(\Delta - 1) = \Delta^2 - \Delta \]

Thus equation \((ii)\) becomes
\[ (\Delta^2 - \Delta - 2\Delta + 2)y = te^t \]
\[ \Rightarrow (\Delta^2 - 3\Delta + 2)y = te^t - - - (iii) \]

The characteristics equation of \((iii)\) is
\[ \Delta^2 - 3\Delta + 2 = 0 \]
\[ \Rightarrow \Delta^2 - \Delta - 2\Delta + 2 = 0 \]
\[ \Rightarrow \Delta(\Delta - 1) - 2(\Delta - 1) = 0 \]
\[ \Rightarrow (\Delta - 1)(\Delta - 2) = 0 \]
\[ \Rightarrow \Delta = 1 \ or \ \Delta = 2 \]

Therefore, the complementary function will be
\[ y_c = c_1e^t + c_2e^{2t} \]

Now,
\[ y_p = \frac{te^t}{\Delta^2 - 3\Delta + 2} \]
\[ \Rightarrow y_p = \frac{te^t}{(\Delta + 1)^2 - 3(\Delta + 1) + 2} \]

(by exponential shift)
The general solution is
\[ y = c_1 e^t + c_2 e^{2t} - \frac{e^t}{\Delta} \left( t^2 + 2t \right) \]

To find the constants \( c_1 \) & \( c_2 \), we will use the initial values.

Applying \( y(1) = 1 \) in equation (iv), we have
\[ 1 = c_1 + c_2 + 0 \]
\[ \Rightarrow c_1 + c_2 = 1 \] \( \quad \) (a)

Differentiating (iv) w.r.t \( x \), we have
\[ y' = c_1 e^t + 2c_2 x - \frac{1}{2} \left( \ln x \right)^2 + 2 \left( \ln x \right) \]
\[ - \frac{x}{2} \left( \frac{2 \ln x}{x} - \frac{1}{x} \right) \] \( \quad \) (v)

Applying \( y'(1) = 0 \) in equation (v), we have
\[ 0 = c_1 + 2c_2 - \frac{1}{2} \left( 0 + \frac{2}{1} \right) \]
\[ \Rightarrow 0 = c_1 + 2c_2 - 1 \]

\[ \Rightarrow c_1 + 2c_2 = 1 \] \( \quad \) (b)

From (a), we have
\[ c_1 = 1 - c_2 \] \( \quad \) (c)

Using \( c_1 = 1 - c_2 \) in equation (b), we have
\[ 1 - c_2 + 2c_2 = 1 \]
\[ \Rightarrow c_2 = 0 \]

Now (c) \( \Rightarrow \)
\[ c_1 = 1 \]

Hence,
\[ y = x - \frac{1}{2} \left( \ln x \right)^2 + 2 \left( \ln x \right) \]
\[ y = x - \frac{x}{2} \left( \ln x \right)^2 + x \ln x \]

is required solution.

\[ \text{Question # 12:} \]
\[ x^3 y''' + 2x^2 y'' + xy' - y = 15 \cos (2 \ln x) \]
\[ y(1) = 2 \text{ } y'(1) = -3 \text{ } y''(1) = 0 \]

\[ \text{Solution:} \]

Given equation is
\[ x^3 y''' + 2x^2 y'' + xy' - y = 15 \cos (2 \ln x) \] \( \quad \) (i)

Replace "y'''" by \( D \) in (i), we have
\[ x^3 D^3 + 2x^2 D^2 + xD - y = 15 \cos (2 \ln x) \] \( \quad \) (ii)

This is Cauchy-Euler equation.

To solve this, we put \( x = e^t \) so that \( t = \ln x \).

Then,
\[ xD = \Delta \]
\[ x^2 D^2 = \Delta (\Delta - 1) = \Delta^2 - \Delta \]
\[ x^3 D^3 = \Delta (\Delta - 1)(\Delta - 2) = \Delta^3 - 3\Delta^2 + 2\Delta \]
Thus equation \((ii)\) becomes
\[
(\Delta^3 - 3\Delta^2 + 2\Delta + 2\Delta^2 - 2\Delta + \Delta - 1)y = 15 \cos 2t
\]
\[
\Rightarrow (\Delta^3 - \Delta^2 + \Delta - 1)y = 15 \cos 2t \quad ---(iii)
\]
The characteristics equation of \((iii)\) is
\[
\Delta^3 - \Delta^2 + \Delta - 1 = 0
\]
\[
\Rightarrow \Delta^2(\Delta - 1) + 1(\Delta - 1) = 0
\]
\[
\Rightarrow (\Delta - 1)(\Delta^2 + 1) = 0
\]
\[
\Rightarrow \Delta = 1 \text{ or } \Delta = \pm i
\]
Therefore, the complementary function will be
\[
y_c = c_1e^t + c_2 \cos x + c_3 \sin x
\]
Now,
\[
y_p = \frac{15 \cos 2t}{\Delta^3 - \Delta^2 + \Delta - 1}
\]
\[
\Rightarrow y_p = \frac{15 \Re e^{2it}}{(2i)^3 - (2i)^2 + 2i - 1}
\]
\[
\Rightarrow y_p = \frac{15 \Re e^{2it}}{-8i + 4 + 2i - 1}
\]
\[
\Rightarrow y_p = \frac{15 \Re e^{2it}}{3 - 6i}
\]
\[
\Rightarrow y_p = \frac{5 \Re (\cos 2t + i \sin 2t)}{1 - 2i} \times \frac{1 + 2i}{1 + 2i}
\]
\[
\Rightarrow y_p = \frac{5 \Re (\cos 2t + 2i \cos 2t + i \sin 2t - 2 \sin 2t)}{5}
\]
\[
\Rightarrow y_p = \cos 2t - 2 \sin 2t
\]
The general solution is
\[
y = c_1e^t + c_2 \cos t + c_3 \sin t + \cos 2t - 2 \sin 2t
\]
\[\Rightarrow y = c_1x + c_2 \cos(\ln x) + c_3 \sin(\ln x)
\]
\[+ \cos 2(\ln x) - 2 \sin 2(\ln x)
\]
\[--- (iv)
\]
To find the constants \(c_1, c_2 \text{ and } c_3\), we will use the initial values.
Applying \(y(1) = 2\) in equation \((iv)\), we have
\[
2 = c_1 + c_2 + 1
\]
\[
\Rightarrow c_1 + c_2 = 1 \quad ---(a)
\]
Differentiating \((iv)\) w.r.t \(x\), we have
\[
y' = c_1 - c_2 \frac{\sin(\ln x)}{x} + c_3 \frac{\cos(\ln x)}{x} - \frac{4 \cos 2(\ln x)}{x}
\]
\[--- (v)
\]
Applying \(y'(1) = -3\) in equation \((v)\), we have
\[
-3 = c_1 + c_3 - 4
\]
\[
\Rightarrow c_1 + c_3 = 1 \quad ---(b)
\]
Differentiating \((v)\) w.r.t \(x\), we have
\[
y'' = -c_2 \left[ - \frac{\sin(\ln x)}{x^2} + \frac{\cos(\ln x)}{x^2} \right]
\]
\[+ c_3 \left[ - \frac{\cos(\ln x)}{x^2} - \frac{\sin(\ln x)}{x^2} \right]
\]
\[+ \left[ - \frac{2 \sin 2(\ln x)}{x^2} \right]
\]
\[+ \frac{4 \cos 2(\ln x)}{x^2}
\]
\[+ \left[ - \frac{8 \sin 2(\ln x)}{x^2} \right]
\]
\[--- (vi)
\]
Applying \(y''(1) = 0\) in equation \((vi)\), we have
\[
0 = -c_2 - c_3
\]
\[
\Rightarrow c_2 + c_3 = 0 \quad ---(c)
\]
From \((c)\), we have
\[
c_2 = -c_3 \quad ---(d)
Using $c_2 = -c_3$ in equation (a), we have

$$c_1 - c_3 = 1 - - - (e)$$

Now $(b) + (e) \implies$

$$2c_1 = 2$$

$$\implies c_1 = 1$$

$$(e) \implies 1 - c_3 = 1$$

$$\implies c_3 = 0$$

$$\implies c_2 = 0$$

Hence,

$$\implies y = x + \cos 2(\ln x) - 2 \sin 2(\ln x)$$

is required solution.