EXERCISE 10.4

Solve.

❖ ***Question # 1:***

$$x^2\frac{d^2y}{dx^2}+7x\frac{dy}{dx}+5y=x^5.$$

Solution:

Given equation is

$$x^2\frac{d^2y}{dx^2}+7x\frac{dy}{dx}+5y=x^5 ---(i)$$

Replace $"\frac{dy}{dx}"$ *by D in* (i)*, we have*

$$x^2D^2+7xD+5y=x^5 ---(ii)$$

This is Cauchy-Euler equation.

To solve this, we put $x=e^t$ *so that* $t=\ln x$.

Then,

$$xD=\Delta$$

$$x^2D^2=\Delta(\Delta-1)=\Delta^2-\Delta$$

Thus equation (ii) **becomes**

$$(\Delta^2-\Delta+7\Delta+5)y=e^{5t}$$

$$\Rightarrow (\Delta^2+6\Delta+5)y=e^{5t} ---(iii)$$

The characteristics equation of (iii) **is**

$$\Delta^2+6\Delta+5=0$$

$$\Rightarrow \Delta^2+\Delta+5\Delta+5=0$$

$$\Rightarrow \Delta(\Delta+1)+5(\Delta+1)=0$$

$$\Rightarrow (\Delta+1)(\Delta+5)=0$$

$$\Rightarrow \Delta+1=0\ or\ \Delta+5=0$$

$$\Rightarrow \Delta=-1\ or\ \Delta=-5$$

Therefore, the complementary function will be

$$y_c=c_1e^{-t}+c_2e^{-5t}$$

Now,

$$y_p=\frac{e^{5t}}{\Delta^2+6\Delta+5}$$

$$\Rightarrow y_p=\frac{e^{5t}}{(\Delta+1)(\Delta+5)}$$

$$\Rightarrow y_p=\frac{e^{5t}}{(5+1)(5+5)}$$

$$\Rightarrow y_p=\frac{e^{5t}}{60}$$

The general solution is

$$y=y_c+y_p$$

$$\Rightarrow y=c_1e^{-t}+c_2e^{-5t}+\frac{e^{5t}}{60}$$

$$\Rightarrow y=c_1x^{-1}+c_2x^{-5}+\frac{x^5}{60}\quad \because x=e^{-t}$$

is required solution of (i).

❖ ***Question # 2:***

$$x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+5y=x^2\sin(\ln x).$$

Solution:

Given equation is

$$x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+5y=x^2\sin(\ln x) ---(i)$$

Replace $"\frac{dy}{dx}"$ *by D in* (i)*, we have*

$$x^2D^2-3xD+5y=x^2\sin(\ln x) ---(ii)$$

This is Cauchy-Euler equation.

To solve this, we put $x=e^t$ *so that* $t=\ln x$.

Then,

$$xD=\Delta$$

$$x^2D^2=\Delta(\Delta-1)=\Delta^2-\Delta$$

Thus equation (ii) **becomes**

$(\Delta^2 - \Delta - 3\Delta + 5)y = e^{2t}\sin t$

$\Rightarrow (\Delta^2 - 4\Delta + 5)y = e^{2t}\sin t - - - (iii)$

The characteristics equation of (iii) is

$\Delta^2 - 4\Delta + 5 = 0$

$$\Rightarrow \Delta = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(5)}}{2(1)}$$

$$\Rightarrow \Delta = \frac{4 \pm \sqrt{16-20}}{2}$$

$$\Rightarrow \Delta = \frac{4 \pm \sqrt{-4}}{2}$$

$$\Rightarrow \Delta = \frac{4 \pm 2i}{2}$$

$\Rightarrow \Delta = 2 \pm i$

Therefore, the complementary function will be

$y_c = (c_1\cos t + c_2 \sin t)e^{2t}$

Now,

$$y_p = \frac{e^{2t}\sin t}{\Delta^2 - 4\Delta + 5}$$

$$\Rightarrow y_p = \frac{e^{2t}\sin t}{(\Delta+2)^2 - 4(\Delta+2) + 5}$$

$(by\ exponential\ shift)$

$$\Rightarrow y_p = \frac{e^{2t}\sin t}{\Delta^2 + 4 + 4\Delta - 4\Delta - 8 + 5}$$

$$\Rightarrow y_p = \frac{e^{2t}\sin t}{\Delta^2 + 1}$$

$$\Rightarrow y_p = \frac{e^{2t}\operatorname{Im} e^{it}}{(\Delta+i)(\Delta-i)}$$

$$\Rightarrow y_p = \frac{te^{2t}\operatorname{Im}(\cos t + i\sin t)}{2i}$$

$$\Rightarrow y_p = -\frac{te^{2t}\operatorname{Im}(\cos t + i\sin t)i}{2}$$

$$\Rightarrow y_p = -\frac{te^{2t}\operatorname{Im}(i\cos t - \sin t)}{2}$$

$$\Rightarrow y_p = -\frac{te^{2t}\cos t}{2}$$

The general solution is

$y = y_c + y_p$

$$\Rightarrow y = (c_1\cos t + c_2\sin t)e^{2t} - \frac{te^{2t}\cos t}{2}$$

$$\Rightarrow y = (c_1\cos(\ln x) + c_2\sin(\ln x))x^2 - \frac{x^2\ln x}{2}\cos(\ln x)$$

$\because x = e^{-t}$

is required solution of (i).

❖ ***Question # 3:***

$$x^2\frac{d^2y}{dx^2} - (2m-1)x\frac{dy}{dx} + (m^2+n^2)y = n^2x^m\ln x.$$

Solution:

Given equation is

$$x^2\frac{d^2y}{dx^2} - (2m-1)x\frac{dy}{dx} + (m^2+n^2)y = n^2x^m\ln x - - - (i)$$

Replace "$\frac{dy}{dx}$" by D in (i), we have

$x^2D^2 - (2m-1)xD + (m^2+n^2)y = n^2x^m\ln x - - - (ii)$

This is Cauchy-Euler equation.

To solve this, we put $x = e^t$ so that $t = \ln x$.

Then,

$xD = \Delta$

$x^2D^2 = \Delta(\Delta - 1) = \Delta^2 - \Delta$

Thus equation (ii) becomes

$[\Delta^2 - \Delta - (2m-1)\Delta + (m^2+n^2)]y = n^2e^{mt}t$

$\Rightarrow (\Delta^2 - 2m\Delta + (m^2+n^2))y = n^2te^{mt} - - - (iii)$

The characteristics equation of (iii) is

$\Delta^2 - 2m\Delta + (m^2+n^2) = 0$

$$\Rightarrow \Delta = \frac{-(-2m) \pm \sqrt{(-2m)^2 - 4(1)(m^2+n^2)}}{2(1)}$$

$$\Rightarrow \Delta = \frac{2m \pm \sqrt{4m^2 - 4m^2 - 4n^2}}{2}$$

$$\Rightarrow \Delta = \frac{2m \pm i2n}{2}$$

$$\Rightarrow \Delta = m \pm in$$

Therefore, the complementary function will be

$$y_c = (c_1 \cos nt + c_2 \sin nt)e^{mt}$$

Now,

$$y_p = \frac{n^2 t e^{mt}}{\Delta^2 - 2m\Delta + (m^2 + n^2)}$$

$$\Rightarrow y_p = \frac{n^2 t e^{mt}}{(\Delta + m)^2 - 2m(\Delta + m) + m^2 + n^2}$$

(by exponential shift)

$$\Rightarrow y_p = \frac{n^2 t e^{mt}}{\Delta^2 + 2m\Delta + m^2 - 2m\Delta - 2m^2 + m^2 + n^2}$$

$$\Rightarrow y_p = \frac{n^2 t e^{mt}}{\Delta^2 + n^2}$$

$$\Rightarrow y_p = \frac{n^2 t e^{mt}}{n^2\left(1 + \frac{\Delta^2}{n^2}\right)}$$

$$\Rightarrow y_p = e^{mt}\left(1 + \frac{\Delta^2}{n^2}\right)^{-1} t$$

$$\Rightarrow y_p = e^{mt}\left(1 - \frac{\Delta^2}{n^2} - \cdots\right) t$$

$$\Rightarrow y_p = e^{mt}(1 - neglecting\ terms)t$$

$$\Rightarrow y_p = te^{mt}$$

The general solution is

$$y = y_c + y_p$$

$$\Rightarrow y = (c_1 \cos nt + c_2 \sin nt)e^{mt} + te^{mt}$$

$$\Rightarrow y = (c_1 \cos n(\ln x) + c_2 \sin n(\ln x))x^m + x^m \ln x$$

$$\because x = e^t$$

$$\Rightarrow y = (c_1 \cos(\ln x^n) + c_2 \sin(\ln x^n))x^m + x^m \ln x$$

$$\Rightarrow y = x^m[c_1 \cos(\ln x^n) + c_2 \sin(\ln x^n) + \ln x]$$

is required solution of (i).

❖ ***Question # 4:***

$$4x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 3y = \sin\ln(-x).$$

Solution:

Given equation is

$$4x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 3y = \sin\ln(-x) --- (i)$$

Replace "$\frac{dy}{dx}$" by D in (i), we have

$$4x^2D^2 - 4xD + 3y = \sin\ln(-x) --- (ii)$$

This is Cauchy-Euler equation.

To solve this, we put $-x = e^t$ so that $t = \ln(-x)$.

Then,

$$xD = \Delta$$

$$x^2D^2 = \Delta(\Delta - 1) = \Delta^2 - \Delta$$

Thus equation (ii) becomes

$$[4(\Delta^2 - \Delta) - 4\Delta + 3]y = \sin t$$

$$\Rightarrow (4\Delta^2 - 4\Delta - 4\Delta + 3)y = \sin t$$

$$\Rightarrow (4\Delta^2 - 8\Delta + 3)y = \sin t --- (iii)$$

The characteristics equation of (iii) is

$$4\Delta^2 - 8\Delta + 3 = 0$$

$$\Rightarrow 4\Delta^2 - 2\Delta - 6\Delta + 3 = 0$$

$$\Rightarrow 2\Delta(2\Delta - 1) - 2\Delta(2\Delta - 1) = 0$$

$$\Rightarrow (2\Delta - 1)(2\Delta - 1) = 0$$

$$\Rightarrow 2\Delta - 1 = 0 \ or\ 2\Delta - 1 = 0$$

$$\Rightarrow \Delta = \frac{1}{2} \ or\ \Delta = \frac{1}{2}$$

Therefore, the complementary function will be

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$$y_c = c_1 e^{\frac{t}{2}} + c_2 t e^{\frac{t}{2}}$$

$$\Rightarrow y_c = (c_1 + c_2 t)e^{\frac{t}{2}}$$

Now,

$$y_p = \frac{\sin t}{4\Delta^2 - 8\Delta + 3}$$

$$\Rightarrow y_p = \frac{\text{Im } e^{it}}{4\Delta^2 - 8\Delta + 3}$$

$$\Rightarrow y_p = \frac{\text{Im } e^{it}}{4(i)^2 - 8(i) + 3}$$

$$\Rightarrow y_p = \frac{\text{Im } e^{it}}{-4 - 8i + 3}$$

$$\Rightarrow y_p = \frac{\text{Im}(\cos t + i\sin t)}{-1 - 8i} \times \frac{-1 + 8i}{-1 + 8i}$$

$$\Rightarrow y_p = \frac{\text{Im}(-\cos t - i\sin t + 8i\cos t - 8\sin t)}{65}$$

$$\Rightarrow y_p = \frac{8\cos t - \sin t}{65}$$

The general solution is

$$y = y_c + y_p$$

$$\Rightarrow y = (c_1 + c_2 t)e^{\frac{t}{2}} + \frac{8\cos t - \sin t}{65}$$

$$\Rightarrow y = (c_1 + c_2 \ln(-x))(-x)^{\frac{1}{2}} + \frac{8}{65}\cos\ln(-x) - \frac{1}{65}\sin\ln(-x)$$

$$\because -x = e^t$$

is required solution of (i).

❖ ***Question # 5:*** $x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + 2y = 10x + \frac{10}{x}$.

Solution:

Given equation is

$$x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + 2y = 10x + \frac{10}{x} - - - (i)$$

Replace "$\frac{dy}{dx}$" by D in (i), we have

$$x^3D^3 + 2x^2D^2 + 2y = 10x + \frac{10}{x} - - - (ii)$$

This is Cauchy-Euler equation.

To solve this, we put $x = e^t$ so that $t = \ln x$.

Then,

$$xD = \Delta$$

$$x^2D^2 = \Delta(\Delta - 1) = \Delta^2 - \Delta$$

$$x^3D^3 = \Delta(\Delta - 1)(\Delta - 2) = \Delta^3 - 3\Delta^2 + 2\Delta$$

Thus equation (ii) becomes

$$[\Delta^3 - 3\Delta^2 + 2\Delta + 2(\Delta^2 - \Delta) + 2]y = 10e^t + \frac{10}{e^t}$$

$$\Rightarrow [\Delta^3 - \Delta^2 + 2]y = 10e^t + 10e^{-t} - - - (iii)$$

The characteristics equation of (iii) is

$$\Delta^3 - \Delta^2 + 2 = 0$$

As $\Delta = -1$ is the root of $\Delta^3 - \Delta^2 + 2 = 0$. Therefore, we use synthetic division in order to find the other roots of the characteristics equation.

	1	-1	0	2
-1	0	-1	2	-2
	1	-2	2	0

The residue equation will be

$$\Delta^2 - 2\Delta + 2 = 0$$

$$\Rightarrow \Delta = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)}$$

$$\Rightarrow \Delta = \frac{2 \pm \sqrt{-4}}{2}$$

$$\Rightarrow \Delta = \frac{2 \pm 2i}{2}$$

$\Rightarrow \Delta = 1 \pm i$

Therefore, the complementary function will be

$y_c = c_1 e^{-t} + (c_2 \cos t + c_3 \sin t)e^t$

Now,

$$y_p = \frac{10e^t + 10e^{-t}}{\Delta^3 - \Delta^2 + 2}$$

$$\Rightarrow y_p = \frac{10e^t + 10e^{-t}}{(\Delta+1)(\Delta^2 - 2\Delta + 2)}$$

$$\Rightarrow y_p = \frac{10e^t}{(\Delta+1)(\Delta^2 - 2\Delta + 2)} + \frac{10e^{-t}}{(\Delta+1)(\Delta^2 - 2\Delta + 2)}$$

$$\Rightarrow y_p = \frac{10e^t}{(1+1)(1-2+2)} + \frac{10te^{-t}}{(1+2+2)}$$

$$\Rightarrow y_p = 5e^t + 2te^{-t}$$

The general solution is

$y = y_c + y_p$

$\Rightarrow y = c_1 e^{-t} + (c_2 \cos t + c_3 \sin t)e^t + 5e^t + 2te^{-t}$

$$\Rightarrow y = c_1(x)^{-1} + (c_2 \cos(\ln x) + c_3 \sin(\ln x))x + 5x + 2\ln x.(x)^{-1}$$

$\because\ x = e^t$

$$\Rightarrow y = (c_1 + 2\ln x)x^{-1} + (x)^{-1} + [c_2 \cos(\ln x) + c_3 \sin(\ln x) + 5]x$$

is required solution of (i).

❖ ***Question # 6:***

$x^4 \frac{d^3y}{dx^3} + 2x^3 \frac{d^2y}{dx^2} - x^2 \frac{dy}{dx} + xy = 1.$

Solution:

Given equation is

$$x^4 \frac{d^3y}{dx^3} + 2x^3 \frac{d^2y}{dx^2} - x^2 \frac{dy}{dx} + xy = 1$$

Dividing both sides by x, we have

$$x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + y = \frac{1}{x} - - - (i)$$

Replace "$\frac{dy}{dx}$" by D in (i), we have

$$x^3D^3 + 2x^2D^2 - xD + y = \frac{1}{x} - - - (ii)$$

This is Cauchy-Euler equation.

To solve this, we put $x = e^t$ so that $t = \ln x$.

Then,

$xD = \Delta$

$x^2D^2 = \Delta(\Delta - 1) = \Delta^2 - \Delta$

$x^3D^3 = \Delta(\Delta-1)(\Delta-2) = \Delta^3 - 3\Delta^2 + 2\Delta$

Thus equation (ii) becomes

$$[\Delta^3 - 3\Delta^2 + 2\Delta + 2(\Delta^2 - \Delta) - \Delta + 1]y = \frac{1}{e^t}$$

$$\Rightarrow [\Delta^3 - \Delta^2 - \Delta + 1]y = e^{-t} - - - (iii)$$

The characteristics equation of (iii) is

$\Delta^3 - \Delta^2 - \Delta + 1 = 0$

$\Rightarrow \Delta^2(\Delta - 1) - 1(\Delta - 1) = 0$

$\Rightarrow (\Delta^2 - 1)(\Delta - 1) = 0$

$\Rightarrow (\Delta + 1)(\Delta - 1)(\Delta - 1) = 0$

$\Rightarrow \Delta = 1, 1, -1$

Therefore, the complementary function will be

$y_c = c_1 e^t + c_2 te^t + c_3 e^{-t}$

Now,

$$y_p = \frac{e^{-t}}{\Delta^3 - \Delta^2 - \Delta + 1}$$

$$\Rightarrow y_p = \frac{e^{-t}}{(\Delta+1)(\Delta-1)(\Delta-1)}$$

$$\Rightarrow y_p = \frac{te^{-t}}{(-1-1)(-1-1)}$$

$$\Rightarrow y_p = \frac{te^{-t}}{4}$$

The general solution is

$$y = y_c + y_p$$

$$\Rightarrow y = c_1 e^t + c_2 t e^t + c_3 e^{-t} + \frac{te^{-t}}{4}$$

$$\Rightarrow y = c_1 x + c_2 x \ln x + c_3 x^{-1} + \frac{\ln x . x^{-1}}{4} \because -x = e^t$$

$$\Rightarrow y = x(c_1 + c_2 \ln x) + c_3 x^{-1} + \frac{\ln x}{4x}$$

is required solution of (i).

❖ ***Question # 7:***

$$x^3 \frac{d^3y}{dx^3} + 4x^2 \frac{d^2y}{dx^2} - 5x \frac{dy}{dx} - 15y = x^4.$$

Solution:

Given equation is

$$x^3 \frac{d^3y}{dx^3} + 4x^2 \frac{d^2y}{dx^2} - 5x \frac{dy}{dx} - 15y = x^4 - - - (i)$$

Replace "$\frac{dy}{dx}$" ***by D in*** (i), ***we have***

$$x^3D^3 + 4x^2D^2 - 5xD - 15y = x^4 - - - (ii)$$

This is Cauchy-Euler equation.

To solve this, we put $x = e^t$ ***so that*** $t = \ln x$.

Then,

$$xD = \Delta$$

$$x^2D^2 = \Delta(\Delta - 1) = \Delta^2 - \Delta$$

$$x^3D^3 = \Delta(\Delta - 1)(\Delta - 2) = \Delta^3 - 3\Delta^2 + 2\Delta$$

Thus equation (ii) **becomes**

$$[\Delta^3 - 3\Delta^2 + 2\Delta + 4(\Delta^2 - \Delta) - 5\Delta - 15]y = e^{4t}$$

$$\Rightarrow [\Delta^3 + \Delta^2 - 7\Delta - 15]y = e^{4t} - - - (iii)$$

The characteristics equation of (iii) **is**

$$\Delta^3 + \Delta^2 - 7\Delta - 15 = 0$$

As $\Delta = 3$ ***is the root of*** $\Delta^3 + \Delta^2 - 7\Delta - 15 = 0$. ***Therefore, we use synthetic division in order to find the other roots of the characteristics equation.***

	1	1	-7	-15
3	0	3	12	15
	1	4	5	0

Now, the residual equation is

$$\Delta^2 + 4\Delta + 5 = 0$$

$$\Rightarrow \Delta = \frac{-4 \pm \sqrt{16 - 20}}{2(1)}$$

$$\Rightarrow \Delta = \frac{-4 \pm 2i}{2}$$

$$\Rightarrow \Delta = -2 \pm i$$

Therefore, the complementary function will be

$$y_c = c_1 e^{3t} + (c_2 \cos t + c_3 \sin t)e^{-2t}$$

Now,

$$y_p = \frac{e^{4t}}{\Delta^3 + \Delta^2 - 7\Delta - 15}$$

$$\Rightarrow y_p = \frac{e^{4t}}{(\Delta - 3)(\Delta^2 + 4\Delta + 5)}$$

$$\Rightarrow y_p = \frac{e^{4t}}{(4 - 3)(16 + 16 + 5)}$$

$$\Rightarrow y_p = \frac{e^{4t}}{(4 - 3)(16 + 16 + 5)}$$

$$\Rightarrow y_p = \frac{e^{4t}}{37}$$

The general solution is

$$y = y_c + y_p$$

$$\Rightarrow y = c_1 e^{3t} + (c_2 \cos t + c_3 \sin t)e^{-2t} + \frac{e^{4t}}{37}$$

$$\Rightarrow y = c_1 x^3 + +(c_2 \cos(\ln x) + c_3 \sin(\ln x))x^{-2} + \frac{x^4}{37}$$

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$$\because -x = e^t$$

is required solution of (i).

❖ ***Question # 8:***

$$(x+1)^2\frac{d^2y}{dx^2}+(x+1)\frac{dy}{dx}+y=4[\cos\ln(x+1)]^2$$

Solution:

Given equation is

$$(x+1)^2\frac{d^2y}{dx^2}+(x+1)\frac{dy}{dx}+y = 4[\cos\ln(x+1)]^2 ---(i)$$

Replace "$\frac{dy}{dx}$" *by D in* (i), *we have*

$$(x+1)^2D^2+(x+1)D+y = 4[\cos\ln(x+1)]^2 ---(ii)$$

This is Cauchy-Euler equation.

To solve this, we put $x+1=e^t$ *so that* $t = \ln(x+1)$.

Then,

$$(x+1)D=\Delta$$

$$(x+1)^2D^2=\Delta(\Delta-1)=\Delta^2-\Delta$$

Thus equation (ii) becomes

$$[\Delta^2-\Delta+\Delta+1]y=4[\cos t]^2$$

$$\Rightarrow [\Delta^2+1]y=4\cos^2 t ---(iii)$$

The characteristics equation of (iii) is

$$\Delta^2+1=0$$

$$\Rightarrow \Delta=\pm i$$

Therefore, the complementary function will be

$$y_c=c_1\cos t+c_2\sin t$$

Now,

$$y_p=\frac{4\cos^2 t}{\Delta^2+1}$$

$$\Rightarrow y_p=\frac{4\left(\frac{1+\cos 2t}{2}\right)}{\Delta^2+1}$$

$$\Rightarrow y_p=\frac{2+2\cos 2t}{\Delta^2+1}$$

$$\Rightarrow y_p=\frac{2}{\Delta^2+1}+\frac{2\cos 2t}{\Delta^2+1}$$

$$\Rightarrow y_p=2(1+\Delta^2)^{-1}+\frac{2\,\mathrm{Re}\,e^{2it}}{(\Delta+i)(\Delta-i)}$$

$$\Rightarrow y_p=2(1)+\frac{2\,\mathrm{Re}(\cos t+i\sin t)}{(2i+i)(2i-i)}$$

$$\Rightarrow y_p=2-\frac{2}{3}\cos t$$

The general solution is

$$y=y_c+y_p$$

$$\Rightarrow y=c_1\cos t+c_2\sin t+2-\frac{2}{3}\cos t$$

$$\Rightarrow y=c_1\cos\ln(x+1)+c_2\sin\ln(x+1)+2-\frac{2}{3}\cos\ln(x+1) \because -x=e^t$$

is required solution of (i).

❖ ***Question # 9:***

$$(2x+1)^2\frac{d^2y}{dx^2}-6(2x+1)\frac{dy}{dx}+16y=8(2x+1)^2$$

Solution:

Given equation is

$$(2x+1)^2\frac{d^2y}{dx^2}-6(2x+1)\frac{dy}{dx}+16y = 8(2x+1)^2 ---(i)$$

Replace "$\frac{dy}{dx}$" *by D in* (i), *we have*

$$(2x+1)^2D^2-6(2x+1)D+16y=8(2x+1)^2 ---(ii)$$

This is Cauchy-Euler equation.

To solve this, we put $2x+1=e^t$ *so that* $t = \ln(2x+1)$.

Then,

$$(2x+1)D = 2\left(x+\frac{1}{2}\right)D = 2\Delta$$

$$(2x+1)^2D^2 = 4\left(x+\frac{1}{2}\right)^2 D^2 = 4\Delta(\Delta-1)$$
$$= 4\Delta^2 - 4\Delta$$

Thus equation (ii) becomes

$$[4\Delta^2 - 4\Delta - 12\Delta + 16]y = 8e^{2t}$$

$$\Rightarrow [4\Delta^2 - 16\Delta + 16]y = 8e^{2t}$$

$$\Rightarrow [\Delta^2 - 4\Delta + 4]y = 2e^{2t} - - - (iii)$$

The characteristics equation of (iii) is

$$\Delta^2 - 4\Delta + 4 = 0$$

$$\Rightarrow (\Delta-2)^2 = 0$$

$$\Rightarrow \Delta = 2 \text{ or } \Delta = 2$$

Therefore, the complementary function will be

$$y_c = c_1e^{2t} + c_2te^{2t}$$

$$\Rightarrow y_c = (c_1 + c_2t)e^{2t}$$

Now,

$$y_p = \frac{2e^{2t}}{\Delta^2 - 4\Delta + 4}$$

$$\Rightarrow y_p = \frac{2e^{2t}}{(\Delta-2)^2}$$

$$\Rightarrow y_p = 2t^2e^{2t}$$

The general solution is

$$y = y_c + y_p$$

$$\Rightarrow y = (c_1 + c_2t)e^{2t} + 2t^2e^{2t}$$

$$\Rightarrow y = [c_1 + c_2(\ln(2x+1))](2x+1)^2 + 2(\ln(2x+1))^2(2x+1)^2$$

is required solution of (i).

❖ ***Question # 10:***

$$x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 6y = 10x^2 \quad y(1) = 1 \quad y'(1) = -6$$

Solution:

Given equation is

$$x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 6y = 10x^2 - - - (i)$$

Replace "$\frac{dy}{dx}$" by D in (i), we have

$$x^2D^2 + 2xD - 6y = 10x^2 - - - (ii)$$

This is Cauchy-Euler equation.

To solve this, we put $x = e^t$ so that $t = \ln x$.

Then,

$$xD = \Delta$$

$$x^2D^2 = \Delta(\Delta-1) = \Delta^2 - \Delta$$

Thus equation (ii) becomes

$$(\Delta^2 - \Delta + 2\Delta - 6)y = 10e^{2t}$$

$$\Rightarrow (\Delta^2 + \Delta - 6)y = 10e^{2t} - - - (iii)$$

The characteristics equation of (iii) is

$$\Delta^2 + \Delta - 6 = 0$$

$$\Rightarrow \Delta^2 + 3\Delta - 2\Delta - 6 = 0$$

$$\Rightarrow \Delta(\Delta+3) - 2(\Delta+3) = 0$$

$$\Rightarrow (\Delta+3)(\Delta-2) = 0$$

$$\Rightarrow \Delta = 2 \text{ or } \Delta = -3$$

Therefore, the complementary function will be

$$y_c = c_1e^{2t} + c_2e^{-3t}$$

Now,

$$y_p = \frac{10e^{2t}}{\Delta^2 + \Delta - 6}$$

$$\Rightarrow y_p = \frac{10e^{2t}}{(\Delta+3)(\Delta-2)}$$

$$\Rightarrow y_p = \frac{10te^{2t}}{(2+3)}$$

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$\Rightarrow y_p = 2te^{2t}$

The general solution is

$y = c_1e^{2t} + c_2e^{-3t} + 2te^{2t}$

$\Rightarrow y = c_1x^2 + c_2x^{-3} + 2(\ln x)x^2 - - - (iv)$

To find the constants c_1 & c_2, we will use the initial values.

Applying $y(1) = 1$ in equation (iv), we have

$1 = c_1 + c_2 + 2(\ln 1)1$

$\Rightarrow 1 = c_1 + c_2 + 0 \quad \because \ln 1 = 0$

$\Rightarrow c_1 + c_2 = 1 - - - (a)$

Differentiating (iv) w.r.t x, we have

$y' = 2c_1x - 3c_2x^{-4} + 4(\ln x)x + 2\left(\frac{1}{x}\right)x^2 - - - (v)$

Applying $y'(1) = -6$ in equation (v), we have

$-6 = 2c_1 - 3c_2 + 0 + 2$

$\Rightarrow -8 = 2c_1 - 3c_2$

$\Rightarrow 2c_1 - 3c_2 = -8 - - - (b)$

From (a), we have

$c_1 = 1 - c_2 - - - (c)$

Using $c_1 = 1 - c_2$ in equation (b), we have

$2(1 - c_2) - 3c_2 = -8$

$\Rightarrow 2 - 2c_2 - 3c_2 = -8$

$\Rightarrow -5c_2 = -10$

$\Rightarrow c_2 = 2$

Now $(c) \Rightarrow$

$c_1 = -1$

Hence,

$y = -x^2 + 2x^{-3} + 2(\ln x)x^2$

is required solution.

❖ *Question # 11:*

$x^2y'' - 2xy' + 2y = x\ln x \qquad y(1) = 1 \;\; y'(1) = 0$

Solution:

Given equation is

$x^2y'' - 2xy' + 2y = x\ln x - - - (i)$

Replace "y'" by D in (i), we have

$x^2D^2 - 2xD + 2y = x\ln x - - - (ii)$

This is Cauchy-Euler equation.

To solve this, we put $x = e^t$ so that $t = \ln x$.

Then,

$xD = \Delta$

$x^2D^2 = \Delta(\Delta - 1) = \Delta^2 - \Delta$

Thus equation (ii) becomes

$(\Delta^2 - \Delta - 2\Delta + 2)y = te^t$

$\Rightarrow (\Delta^2 - 3\Delta + 2)y = te^t - - - (iii)$

The characteristics equation of (iii) is

$\Delta^2 - 3\Delta + 2 = 0$

$\Rightarrow \Delta^2 - \Delta - 2\Delta + 2 = 0$

$\Rightarrow \Delta(\Delta - 1) - 2(\Delta - 1) = 0$

$\Rightarrow (\Delta - 1)(\Delta - 2) = 0$

$\Rightarrow \Delta = 1 \text{ or } \Delta = 2$

Therefore, the complementary function will be

$y_c = c_1e^t + c_2e^{2t}$

Now,

$$y_p = \frac{te^t}{\Delta^2 - 3\Delta + 2}$$

$$\Rightarrow y_p = \frac{te^t}{(\Delta + 1)^2 - 3(\Delta + 1) + 2}$$

$(by\ exponential\ shift)$

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$$\Rightarrow y_p = \frac{te^t}{\Delta^2 + 2\Delta + 1 - 3\Delta - 3 + 2}$$

$$\Rightarrow y_p = \frac{te^t}{\Delta^2 - \Delta}$$

$$\Rightarrow y_p = \frac{te^t}{-\Delta(1-\Delta)}$$

$$\Rightarrow y_p = -\frac{e^t}{\Delta}(1-\Delta)^{-1}t$$

$$\Rightarrow y_p = -\frac{e^t}{\Delta}(1+\Delta)t$$

$$\Rightarrow y_p = -\frac{e^t}{\Delta}(t+1)$$

$$\Rightarrow y_p = -e^t\left(\frac{t^2}{2} + t\right)$$

$$\Rightarrow y_p = -\frac{e^t}{2}(t^2 + 2t)$$

The general solution is

$$y = c_1 e^t + c_2 e^{2t} - \frac{e^t}{2}(t^2 + 2t)$$

$$\Rightarrow y = c_1 x + c_2 x^2 - \frac{x}{2}\left((\ln x)^2 + 2(\ln x)\right) - - - (iv)$$

To find the constants c_1 & c_2, we will use the initial values.

Applying $y(1) = 1$ in equation (iv), we have

$$1 = c_1 + c_2 + 0$$

$$\Rightarrow c_1 + c_2 = 1 - - - (a)$$

Differentiating (iv) w.r.t x, we have

$$y' = c_1 + 2c_2 x - \frac{1}{2}\left((\ln x)^2 + 2(\ln x)\right) - \frac{x}{2}\left(\frac{2\ln x}{x} + \frac{2}{x}\right) - - - (v)$$

Applying $y'(1) = 0$ in equation (v), we have

$$0 = c_1 + 2c_2 - \frac{1}{2}\left(0 + \frac{2}{1}\right)$$

$$\Rightarrow 0 = c_1 + 2c_2 - 1$$

$$\Rightarrow c_1 + 2c_2 = 1 - - - (b)$$

From (a), we have

$$c_1 = 1 - c_2 - - - (c)$$

Using $c_1 = 1 - c_2$ in equation (b), we have

$$1 - c_2 + 2c_2 = 1$$

$$\Rightarrow c_2 = 0$$

Now $(c) \Rightarrow$

$$c_1 = 1$$

Hence,

$$\Rightarrow y = x - \frac{x}{2}\left((\ln x)^2 + 2(\ln x)\right)$$

$$\Rightarrow y = x - \frac{x}{2}(\ln x)^2 - x\ln x$$

is required solution.

❖ ***Question # 12:***

$$x^3 y''' + 2x^2 y'' + xy' - y = 15\cos(2\ln x)$$

$$y(1) = 2 \quad y'(1) = -3 \;\&\; y''(1) = 0$$

Solution:

Given equation is

$$x^3 y''' + 2x^2 y'' + xy' - y = 15\cos(2\ln x) - - - (i)$$

Replace "y'" by D in (i), we have

$$x^3 D^3 + 2x^2 D^2 + xD - y = 15\cos(2\ln x) - - - (ii)$$

This is Cauchy-Euler equation.

To solve this, we put $x = e^t$ so that $t = \ln x$.

Then,

$$xD = \Delta$$

$$x^2 D^2 = \Delta(\Delta - 1) = \Delta^2 - \Delta$$

$$x^3 D^3 = \Delta(\Delta - 1)(\Delta - 2) = \Delta^3 - 3\Delta^2 + 2\Delta$$

Thus equation (ii) becomes

$$(\Delta^3-3\Delta^2+2\Delta+2\Delta^2-2\Delta+\Delta-1)y = 15\cos 2t$$

$$\Rightarrow (\Delta^3-\Delta^2+\Delta-1)y = 15\cos 2t \; ---(iii)$$

The characteristics equation of (iii) is

$$\Delta^3-\Delta^2+\Delta-1=0$$

$$\Rightarrow \Delta^2(\Delta-1)+1(\Delta-1)=0$$

$$\Rightarrow (\Delta-1)(\Delta^2+1)=0$$

$$\Rightarrow \Delta=1 \; or \; \Delta=\pm i$$

Therefore, the complementary function will be

$$y_c = c_1e^t + c_2\cos x + c_3\sin x$$

Now,

$$y_p = \frac{15\cos 2t}{\Delta^3-\Delta^2+\Delta-1}$$

$$\Rightarrow y_p = \frac{15\,\mathrm{Re}\,e^{2it}}{\Delta^3-\Delta^2+\Delta-1}$$

$$\Rightarrow y_p = \frac{15\,\mathrm{Re}\,e^{2it}}{(2i)^3-(2i)^2+2i-1}$$

$$\Rightarrow y_p = \frac{15\,\mathrm{Re}\,e^{2it}}{-8i+4+2i-1}$$

$$\Rightarrow y_p = \frac{15\,\mathrm{Re}\,e^{2it}}{3-6i}$$

$$\Rightarrow y_p = \frac{5\,\mathrm{Re}(\cos 2t + i\sin 2t)}{1-2i}\times\frac{1+2i}{1+2i}$$

$$\Rightarrow y_p = \frac{5\,\mathrm{Re}(\cos 2t + 2i\cos 2t + i\sin 2t - 2\sin 2t)}{5}$$

$$\Rightarrow y_p = \cos 2t - 2\sin 2t$$

The general solution is

$$y = c_1e^t + c_2\cos t + c_3\sin t + \cos 2t - 2\sin 2t$$

$$\Rightarrow y = c_1x + c_2\cos(\ln x) + c_3\sin(\ln x) + \cos 2(\ln x) - 2\sin 2(\ln x) \; ---(iv)$$

To find the constants c_1, c_2 & c_3, we will use the initial values.

Applying $y(1)=2$ in equation (iv), we have

$$2 = c_1+c_2+1$$

$$\Rightarrow c_1+c_2 = 1 \; ---(a)$$

Differentiating (iv) w.r.t x, we have

$$y' = c_1 - c_2\frac{\sin(\ln x)}{x} + c_3\frac{\cos(\ln x)}{x} - \frac{2\sin 2(\ln x)}{x} - \frac{4\cos 2(\ln x)}{x} \; ---(v)$$

Applying $y'(1)=-3$ in equation (v), we have

$$-3 = c_1 + c_3 - 4$$

$$\Rightarrow c_1+c_3 = 1 \; ---(b)$$

Differentiating (v) w.r.t x, we have

$$y'' = -c_2\left[-\frac{\sin(\ln x)}{x^2}+\frac{\cos(\ln x)}{x^2}\right] + c_3\left[-\frac{\cos(\ln x)}{x^2}-\frac{\sin(\ln x)}{x^2}\right] - \left[-\frac{2\sin 2(\ln x)}{x^2}+\frac{4\cos 2(\ln x)}{x^2}\right] - \left[-\frac{4\cos 2(\ln x)}{x^2}-\frac{8\sin 2(\ln x)}{x^2}\right] \; ---(vi)$$

Applying $y''(1)=0$ in equation (vi), we have

$$\Rightarrow 0 = -c_2 - c_3$$

$$\Rightarrow c_2 + c_3 = 0 \; ---(c)$$

From (c), we have

$$c_2 = -c_3 \; ---(d)$$

Using $c_2 = -c_3$ in equation (a), we have

$c_1 - c_3 = 1 - - - (e)$

Now $(b) + (e) \Rightarrow$

$2c_1 = 2$

$\Rightarrow c_1 = 1$

$(e) \Rightarrow 1 - c_3 = 1$

$\Rightarrow c_3 = 0$

$\Rightarrow c_2 = 0$

Hence,

$\Rightarrow y = x + \cos 2(\ln x) - 2 \sin 2(\ln x)$

is required solution.

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