Exercise 9.8
Mathematical
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8 Non Linear Diff. Equation of order One

Non Linear Diff. Eq. of order one

An eq. which is not linear, is called Non-Linear. (see ch. 10)

Consider the non-linear diff. eq. of first order

$$x^2\left(\frac{dy}{dx}\right)^2 + x\left(\frac{dy}{dx}\right) - y^2 - y = 0$$

$$\text{or} \quad x^2P^2 + xP - y^2 - y = 0 \qquad \text{where } P = \frac{dy}{dx}$$

$$\text{or} \quad f(x, y, P) = 0$$

Thus, we usually, represents the Non-Linear diff. eq. of the first order by $f(x, y, P) = 0$ where $P = \frac{dy}{dx}$

We shall discuss the four techniques to solve the eq. $f(x,y,P) = 0$

1. Solvable for P
2. Solvable for y
3. Solvable for x
4. Clairaut's eq.

$$xy^2\left(\frac{-dx}{dy}\right)^2 + (x^2y - y^2 - y)\frac{dx}{dy} = xy^2$$

$$\text{by } x\left(\frac{dy}{dx}\right)^2 \quad xy^2 + (x^2y + y^3 - y)\left(\frac{-dx}{dy}\right)\left(\frac{dy}{dx}\right)^2 = xy^2\left(\frac{dy}{dx}\right)^2$$

$$xy^2 - (x^2y - y^2 - y)\frac{dy}{dx} = xy^2\left(\frac{dy}{dx}\right)^2$$

$$xy^2\left(\frac{dy}{dx}\right)^2 + (x^2y - y^3 - y)\frac{dy}{dx} = xy^2$$ which is same $\frac{x^2}{c^2} + \frac{y^2}{c^2-1} = 1$

Solvable for P

The diff. eq. $f(x,y,P)=0$ is said to be solvable for P if it can be reduced into Linear factors.

Example | $x^2P^2+xP-y^2-y=0$.

Sol:-

$$x^2P^2-y^2+xP-y=0$$

$$\Rightarrow (xP+y)(xP-y)+(xP-y)=0$$

$$\Rightarrow (xP-y)[xP+y+1]=0$$

$$\Rightarrow xP-y=0 \quad \text{or} \quad xP+y+1=0$$

$xP-y$

$\Rightarrow x\dfrac{dy}{dx}=y$

$\Rightarrow \dfrac{dy}{y}=\dfrac{dx}{x}$

$\Rightarrow \displaystyle\int\frac{dy}{y}=\int\frac{dx}{x}$

$\Rightarrow \ln y=\ln x+\ln c$

$\Rightarrow \ln y=\ln cx$

$\Rightarrow y=cx$

$\Rightarrow y-cx=0$

$xP+y+1=0$

$\Rightarrow x\dfrac{dy}{dx}=-(y+1)$

$\Rightarrow \dfrac{dy}{y+1}=-\dfrac{dx}{x}$

$\Rightarrow \displaystyle\int\frac{dy}{y+1}=-\int\frac{dx}{x}$

$\Rightarrow \ln(y+1)=-\ln x+\ln c$

$\Rightarrow \ln(y+1)=\ln cx^{-1}$

$\Rightarrow y+1=cx^{-1}$

$\Rightarrow x(y+1)-c=0$

Hence the general sol. is $(y-cx)(xy+x-c)=0$

Example $xP^3-(x^2+x+y)P^2+(x^2+xy+y)P-xy=0$

Sol:-

Since the given eq. is satisfied by $P=1$

1	x	$-x^2-x-y$	x^2+xy+y	$-xy$
		x	$-x^2-y$	xy
	x	$-x^2-y$	xy	0

$\therefore\ (P-1)[xP^2-(x^2+y)P+xy]=0$

$\Rightarrow (P-1)(xP^2-x^2P-yP+xy)=0$

$\Rightarrow (P-1)[xP(P-x)-y(P-x)]=0$

$\Rightarrow (P-1)(P-x)(xP-y)=0$

$\Rightarrow P-1=0$ or $P-x=0$ or $xP-y=0$

$\therefore P-1=0$

$\Rightarrow \dfrac{dy}{dx}=1$

$\Rightarrow dy=dx$

$\Rightarrow \int dy=\int dx$

$\Rightarrow y=x+c$

$\Rightarrow y-x-c=0$

$P-x=0$

$\dfrac{dy}{dx}=x$

$\Rightarrow dy=x\,dx$

$\Rightarrow \int dy=\int x\,dx$

$\Rightarrow y=x^2/2+c$

$\Rightarrow y-x^2/2-c=0$

$xP-y=0$

$x\dfrac{dy}{dx}=y$

$\Rightarrow \dfrac{dy}{y}=\dfrac{dx}{x}$

$\Rightarrow \int\dfrac{dy}{y}=\int\dfrac{dx}{x}$

$\Rightarrow \ln y=\ln x+\ln c$

$\Rightarrow y=cx$

$\Rightarrow y-cx=0$

Hence the general sol. is $(y-x-c)(y-x^2/2-c)(y-cx)=0$

Solvable for Y

The diff. eq. $f(x,y,P)=0$ is said to be solvable for y if it cannot be factorised and can be put in the form

$$y=F(x,P)$$

Example

$$y+Px=P^2x^4$$

Sol:- $y=P^2x^4-Px$ ——— ①

Diff. ① w.r.t. x, we get

$\dfrac{dy}{dx}=4x^3P^2+2x^4P\dfrac{dP}{dx}-P-x\dfrac{dP}{dx}$

$\Rightarrow P=4x^3P^2+2x^4P\dfrac{dP}{dx}-P-x\dfrac{dP}{dx}$

$\Rightarrow 2P-4x^3P^2=x(2x^3P-1)\dfrac{dP}{dx}$

$\Rightarrow 2P(1-2x^3P)-x(1-2x^3P)\dfrac{dP}{dx}=0$

Example

$$y=P^2x+P \text{ ——— ①}$$

Sol:-

Diff. eq. ① w.r.t x we get

$\dfrac{dy}{dx}=P^2+2xP\dfrac{dP}{dx}+\dfrac{dP}{dx}$

$\Rightarrow P=P^2+(2xP+1)\dfrac{dP}{dx}$

$\Rightarrow (2xP+1)\dfrac{dP}{dx}+P^2-P=0$

$\Rightarrow \dfrac{dP}{dx}=\dfrac{P-P^2}{2xP+1}$

$$\Rightarrow (1-2Px^3)(2P+x\frac{dP}{dx}) = 0$$

$$\Rightarrow 1-2Px^3 = 0 \quad \text{or} \quad 2P+x\frac{dP}{dx} = 0$$

Consider,

$$2P + x\frac{dP}{dx} = 0$$

$$\Rightarrow x\frac{dP}{dx} = -2P$$

$$\Rightarrow \frac{dP}{P} = -2\frac{dx}{x}$$

$$\Rightarrow \int\frac{dP}{P} = -2\int\frac{dx}{x}$$

$$\Rightarrow \ln P = -2\ln x + \ln c$$

$$= \ln x^{-2} + \ln c$$

$$= \ln cx^{-2}$$

$$\Rightarrow P = c/x^2 \quad \text{——— ②}$$

Eliminating P from ①, ②

we get, $y = c^2 - c/x$

$$\Rightarrow xy = c^2x - c$$

$$\Rightarrow xy - c^2x + c = 0$$

$$\Rightarrow \frac{dx}{dP} = \frac{2Px+1}{P(1-P)}$$

$$\Rightarrow \frac{dx}{dP} = \frac{2x}{1-P} + \frac{1}{P(1-P)}$$

$$\Rightarrow \frac{dx}{dP} + \left(\frac{2}{P-1}\right)x = \frac{-1}{P(P-1)} \quad \text{——— ②}$$

It is linear in x, $f(P) =$

$$\therefore I\cdot F = e^{\int f(P)\,dP} = e^{\int \frac{2dP}{P-1}} = e^{2\ln(P-1)} = (P-1)^2$$

Multiplying ② by its I.F, we get

$$(P-1)^2\frac{dx}{dP} + 2(P-1)x = -\frac{P-1}{P}$$

$$\Rightarrow (P-1)^2dx + 2(P-1)x\,dP = -\left(\frac{P-1}{P}\right)dP$$

$$\Rightarrow d\left[x(P-1)^2\right] = (-1 + 1/P)\,dP$$

$$\Rightarrow \int d\left[x(P-1)^2\right] = \int(1/P - 1)\,dP$$

$$\Rightarrow x(P-1)^2 = \ln P - P + c$$

$$\Rightarrow x = \frac{c - P + \ln P}{(P-1)^2} \quad \text{——— ③}$$

Putting value of in eq. ①, we get

$$y = P^2\left(\frac{c-P+\ln P}{(P-1)^2}\right) + P \quad \text{——— ④}$$

③, ④ give the parametric sol. of ①

Solvable for X

The diff. eq. $f(x, y, P) = 0$ is said to be solvable for x if it cannot be factorized and can be put in the form

$$x = F(y, P)$$

Example

$$xP = 1 + P^2$$

Sol:-

$$x = \frac{1}{P} + P \quad \text{——— ①}$$

Differentiating eq. ① w.r.t y, we get.

$$\frac{dx}{dy} = 1 - \frac{1}{P^2}\frac{dP}{dy} + \frac{dP}{dy}$$

$$\Rightarrow \quad \frac{1}{P} = (1 - 1/P^2)\frac{dP}{dy}$$

$$\Rightarrow \quad dy = (P - 1/P)\,dP$$

$$\Rightarrow \int dy = \int (P - 1/P)\,dP$$

$$\Rightarrow \quad y = P^2/2 - \ln P + c \quad \text{———} \; ②$$

Thus ①, ② give the general sol. of the given eq. in paramt. form

Clairaut's Eq.

An eq. of the type $y = xP + f(P)$ where $P = \frac{dy}{dx}$ is called Clairaut's Equation

Theorem

General solution of the eq. $y = xP + f(P)$ is $y = cx + f(c)$.

Proof

$$y = xP + f(P) \quad \text{———} \; ①$$

Differentiating ① w.r.t x, we get

$$\frac{dy}{dx} = P + x\frac{dP}{dx} + f'(P)\frac{dP}{dx}$$

$$\cancel{P} + (x + f'(P))\frac{dP}{dx}$$

Remark

In the above theorem, if we consider $x + f'(P) = 0$

if we consider $x + f'(p) = 0$

or $x = -f'(p)$ putting in eq. ① of the above theorem

we get $y = -Pf'(P) + f(P)$

The parametric Eqs.

$$x = -f'(P)$$
$$y = f(P) - Pf'(P)$$

represent the singular sol. of $y = xP + f(P)$

($\because$ This sol. involves no arbitrary constant. called singular sol.)

Example

Find the general sol. and singular sol. of $y = xP + \frac{1}{4}P^4$ ——①

Sol:- It is clairaut's eq.

General Sol:-

$$y = cx + \frac{1}{4}c^4$$

Singular Sol:-

know that,

the clairaut's eq.

form is

Example

Find the general sol. and singular sol. of $x^2(y - Px) = yP^2$ ——①

Sol:- $yP^2 + Px^3 - x^2y = 0$

It is not solvable for P, y, x

we can convert ① into Clairaut's eq. as

Let $u = x^2$, $v = y^2$

$\therefore$ $du = 2x\,dx$, $dv = 2y\,dy$

Now $\frac{2y\,dy}{2x\,dx} = \frac{dv}{du}$

$\rightarrow$ $\frac{dy}{dx} = \frac{x\,dv}{y\,du}$

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We can eliminat P from ③, as

Since $P = (-x)^{1/3}$

$\therefore\ y = -\frac{3}{4}(-x)^{4/3}$

$= -\frac{3}{4}x^{4/3}$

$\Rightarrow\ 4y = -3x^{4/3}$

$\Rightarrow\ 64y^3 = -3x^4$

$\Rightarrow\ 64y^3 + 3x^4 = 0$ req. singul. sol.

It is clairaut's eq.

$\therefore$ its general sol. is

$$v = cu + c^2$$

$\Rightarrow\ y^2 = cx^2 + c^2$ req. general sol.

Singular sol.

since $v = u\frac{dv}{du} + \left(\frac{dv}{du}\right)^2$

$\Rightarrow\ v = uq + q^2\ ,\quad q = \frac{dv}{du}$

$\therefore$ singular sol of above eq. is

$$\left.\begin{aligned} u &= -f'(q) \\ v &= f(q) - q f'(q) \end{aligned}\right] \text{———} ②$$

where

$f(q) = q^2 \quad \therefore\ f'(q) = 2q$

Hence ② becomes,

$$\left.\begin{aligned} u &= -2q \\ v &= q^2 - 2q^2 = -q^2 \end{aligned}\right] \text{———} ③$$

we can eliminate q from ③

since $q = -u/2$

$\therefore\ v = -(-u/2)^2$

$\Rightarrow\ v = -u^2/4$

$\Rightarrow\ y^2 = -\frac{1}{4}x^4$

$\Rightarrow\ 4y^2 + x^4 = 0$ req. sing. sol. of ①

EXERCISE 9.8

1 $P^2 + P - 6 = 0$

Sol:-

$P^2 - 2P + 3P - 6 = 0$

$\Rightarrow P(P-2) + 3(P-2) = 0$

$\Rightarrow (P-2)(P+3) = 0$

$\Rightarrow P-2 = 0$ or $P+3 = 0$

Now

$P - 2 = 0$

$\Rightarrow \frac{dy}{dx} = 2$

$\Rightarrow dy = 2dx$

$\Rightarrow \int dy = 2\int dx$

$\Rightarrow y = 2x + c$

$\Rightarrow y - 2x - c = 0$

$P + 3 = 0$

$\Rightarrow \frac{dy}{dx} + 3 = 0$

$\Rightarrow dy = -3dx$

$\Rightarrow \int dy = -3\int dx$

$\Rightarrow y = -3x + c$

$\Rightarrow 3x + y - c = 0$

Hence the general sol. is $(y-2x-c)(3x+y-c) = 0$

2 $x^2P^2 + xyP - 6y^2 = 0$

Sol:-

$x^2P^2 - 2xyP + 3xyP - 6y^2 = 0$

$\Rightarrow xP(xP-2y) + 3y(xP-2y) = 0$

$\Rightarrow (xP-2y)(xP+3y) = 0$

$\Rightarrow (xP-2y) = 0$ or $(xP+3y) = 0$

Now $xP - 2y = 0$

$\Rightarrow P = 2y/x$

$\Rightarrow \frac{dy}{dx} = \frac{2y}{x}$

$\Rightarrow \frac{dy}{y} = 2\frac{dx}{x}$

$\Rightarrow \ln y = 2\ln x + \ln c$

$\Rightarrow \ln y = \ln x^2 + \ln c$

$xP + 3y = 0$

$\Rightarrow P = -3y/x$

$\Rightarrow \frac{dy}{dx} = -3y/x$

$\Rightarrow \frac{dy}{y} = -\frac{3dx}{x}$

$\Rightarrow \int \frac{dy}{y} = -3\int \frac{dx}{x}$

$$\Rightarrow \ln y = \ln cx^2$$
$$\Rightarrow y = cx^2$$
$$\Rightarrow y - cx^2 = 0$$

$$\Rightarrow \ln y = -3\ln x + \ln c$$
$$\Rightarrow \ln y = \ln x^{-3} + \ln c$$
$$\Rightarrow \ln y = \ln cx^{-3}$$
$$\Rightarrow y = cx^{-3}$$
$$\Rightarrow y - c/x^3 = 0$$

Hence the general sol. is $(y-cx^2)(y-c/x^3) = 0$

(3) $P^2y + (x-y)P - x = 0$

Sol:-

$$P^2y + xP - yP - x = 0$$
$$\Rightarrow P(Py+x) - (Py+x) = 0$$
$$\Rightarrow (Py+x)(P-1) = 0$$
$$\Rightarrow Py + x = 0 \quad \text{or} \quad P-1 = 0$$

$$Py + x = 0$$
$$\Rightarrow P = -x/y$$
$$\Rightarrow \frac{dy}{dx} = -x/y$$
$$\Rightarrow y\,dy = -x\,dx$$
$$\Rightarrow \int y\,dy = -\int x\,dx$$
$$\Rightarrow y^2/2 = -x^2/2 + c_1$$
$$\Rightarrow x^2 + y^2 = c$$
$$\Rightarrow x^2 + y^2 - c = 0$$

$$P - 1 = 0$$
$$\Rightarrow \frac{dy}{dx} = 1$$
$$\Rightarrow dy = dx$$
$$\Rightarrow \int dy = \int dx$$
$$\Rightarrow y = x + c$$
$$\Rightarrow y - x - c = 0$$

Hence the req. sol. is $(x^2+y^2-c)(y-x-c) = 0$

(4) $P^3 - (x^2+xy+y^2)P + x^2y + xy^2 = 0$

Sol:-

Since the given eq. is satisfied by $P = x$

x	1	0	$-x^2-xy-y^2$	x^2y+xy^2
		x	x^2	$-x^2y-xy^2$
	1	x	$-xy-y^2$	0

$$\therefore (P-x)(P^2+xP-xy-y^2) = 0$$
$$\Rightarrow (P-x)[P^2-y^2+x(P-y)] = 0$$
$$\Rightarrow (P-x)[(P+y)(P-y)+x(P-y)] = 0$$
$$\Rightarrow (P-x)(P-y)(P+y+x) = 0$$

$\Rightarrow P-x=0 \quad \text{or} \quad P-y=0 \quad \text{or} \quad P+x+y=0$

$P-x=0$

$\Rightarrow \frac{dy}{dx} = x$

$\Rightarrow dy = x\,dx$

$\Rightarrow \int dy = \int x\,dx$

$\Rightarrow y = x^2/2 + c_1$

$\Rightarrow 2y - x^2 = 2c_1$

$\Rightarrow 2y - x^2 - c = 0$

$P-y=0$

$\Rightarrow \frac{dy}{dx} = y$

$\Rightarrow \frac{dy}{y} = dx$

$\Rightarrow \int \frac{dy}{y} = \int dx$

$\Rightarrow \ln y = x + c_2$

$\Rightarrow \ln y = \ln e^x + \ln c$

$\Rightarrow \ln y = \ln ce^x$

$\Rightarrow y = ce^x$

$\Rightarrow y - ce^x = 0$

$P+x+y=0$

$\Rightarrow \frac{dy}{dx} + y = -x$ (linear in y)

$I.F = e^{\int 1 dx} = e^x$

Multiplying the above eq. by I.F, we get

$e^x \frac{dy}{dx} + ye^x = -xe^x$

$\Rightarrow e^x dy + ye^x dx = -xe^x dx$

$\Rightarrow d(ye^x) = -xe^x dx$

$\Rightarrow \int d(ye^x) = -\int xe^x dx$

$\Rightarrow ye^x = -[x\cdot e^x - \int e^x dx]$ by Parts

$= -xe^x + e^x + c$

$\Rightarrow y = -x + 1 + ce^{-x}$

$\Rightarrow x + y - 1 - ce^{-x} = 0$

Hence the req. sol. is $(2y-x^2-c)(x+y-1-ce^{-x})(y-ce^x) = 0$

5 $xP^2 + (y-1-x^2)P - x(y-1) = 0$

Sol:-

Since the given eq. is satisfied by $P = x$

x	x	$y-1-x^2$	$-xy+x$
		x^2	$xy-x$
	x	$y-1$	0

$\therefore (P-x)(xP+y-1) = 0$

$\Rightarrow P-x=0$, or $xP+y-1=0$

$P-x=0$

$\Rightarrow \frac{dy}{dx} = x$

$\Rightarrow dy = x\,dx$

$\Rightarrow \int dy = \int x\,dx$

$\Rightarrow y = x^2/2 + c_1$

$\Rightarrow 2y - x^2 = 2c_1$

$\Rightarrow 2y - x^2 - c' = 0$

$xP + y - 1 = 0$

$x\frac{dy}{dx} + y = 1$

$\Rightarrow x\,dy + y\,dx = dx$

$\Rightarrow d(xy) = dx$

$\Rightarrow \int d(xy) = \int dx$

$\Rightarrow xy = x + c_2$

$\Rightarrow xy - x - c_2 = 0$

Hence the req. sol. is $(2y-x^2-c)(xy-x-c_2) = 0$

6 $xyP^2 + (x+y)P + 1 = 0$

Sol:-

$$xyP^2 + xP + yP + 1 = 0$$
$$\Rightarrow xP(yP+1) + (yP+1) = 0$$
$$\Rightarrow (yP+1)(xP+1) = 0$$
$$\Rightarrow yP+1 = 0 \quad \text{or} \quad xP+1 = 0$$

$$yP + 1 = 0$$
$$\Rightarrow y\frac{dy}{dx} + 1 = 0$$
$$\Rightarrow y\,dy = -dx$$
$$\Rightarrow \int y\,dy = -\int dx$$
$$\Rightarrow y^2/2 = -x + c_1$$
$$\Rightarrow y^2 = -2x + 2c_1$$
$$\Rightarrow y^2 + 2x - c = 0$$

$$xP + 1 = 0$$
$$\Rightarrow x\frac{dy}{dx} = -1$$
$$\Rightarrow dy = -\frac{dx}{x}$$
$$\Rightarrow \int dy = -\int \frac{dx}{x}$$
$$\Rightarrow y = -\ln x + \ln c$$
$$\Rightarrow y = \ln\frac{c}{x}$$
$$\Rightarrow y - \ln\frac{c}{x} = 0$$

Hence the req. sol. is $(y^2+2x-c)(y-\ln\frac{c}{x}) = 0$

7 $P^2 - (x^2y+3)P + 3x^2y = 0$

Sol:-

$$P^2 - x^2yP - 3P + 3x^2y = 0$$
$$\Rightarrow P(P-x^2y) - 3(P-x^2y) = 0$$
$$\Rightarrow (P-x^2y)(P-3) = 0$$
$$\Rightarrow P-x^2y = 0 \quad \text{or} \quad P-3 = 0$$

$$P - x^2y = 0$$
$$\Rightarrow \frac{dy}{dx} = x^2y$$
$$\Rightarrow \frac{dy}{y} = x^2dx$$
$$\Rightarrow \int\frac{dy}{y} = \int x^2dx$$
$$\Rightarrow \ln y = x^3/3 + \ln c_1$$
$$\Rightarrow \ln y + \ln c = x^3/3$$
$$\Rightarrow \ln cy - x^3/3 = 0$$
$$\Rightarrow 3\ln cy - x^3 = 0$$

$$P - 3 = 0$$
$$\Rightarrow \frac{dy}{dx} = 3$$
$$\Rightarrow dy = 3dx$$
$$\Rightarrow \int dy = 3\int dx$$
$$\Rightarrow y = 3x + c$$
$$\Rightarrow y - 3x - c = 0$$

Hence the req. sol. is $(3\ln cy - x^3)(y-3x-c) = 0$

P

8 $yP^2+(x-y^2)P-xy=0$

Sol:-

$$yP^2+xP-y^2P-xy=0$$

$$\Rightarrow P(yP+x)-y(yP+x)=0$$

$$\Rightarrow (yP+x)(P-y)=0$$

$$\Rightarrow yP+x=0 \quad \text{or} \quad P-y=0$$

$$yP+x=0$$

$$\Rightarrow y\frac{dy}{dx}=-x$$

$$\Rightarrow y\,dy=-x\,dx$$

$$\Rightarrow \int y\,dy=-\int x\,dx$$

$$\Rightarrow y^2/2=-x^2/2+c_1$$

$$\Rightarrow y^2=-x^2+2c_1$$

$$\Rightarrow y^2+x^2-c=0$$

$$P-y=0$$

$$\Rightarrow \frac{dy}{dx}=y$$

$$\Rightarrow \frac{dy}{y}=dx$$

$$\Rightarrow \int\frac{dy}{y}=\int dx$$

$$\Rightarrow \ln y=x+\ln c$$

$$\Rightarrow \ln y=\ln e^x+\ln c$$

$$\Rightarrow \ln y=\ln ce^x$$

$$\Rightarrow y=ce^x \Rightarrow y-ce^x=0$$

Hence the req. sol. is $(y^2+x^2-c)(y-ce^x)=0$

9 $(y+x)^2P^2+(2y^2+xy-x^2)P+y(y-x)=0$

Sol:-

Since $2y^2+xy-x^2=2y^2+2xy-xy-x^2$

$$=2y(y+x)-x(y+x)$$

$$=(x+y)(2y-x)$$

$$\therefore (y+x)^2P^2+(y+x)(2y-x)P+y(y-x)=0$$

$$\Rightarrow (y+x)^2P^2+(y+x)(y+y-x)P+y(y-x)=0$$

$$\Rightarrow (y+x)^2P^2+(y+x)yP+(y+x)(y-x)P+y(y-x)=0$$

$$\Rightarrow (y+x)P[(y+x)P+y]+(y-x)[(y+x)P+y]=0$$

$$\Rightarrow [(y+x)P+y][(y+x)P+y-x]=0$$

$$\Rightarrow (y+x)P+y=0 \quad \text{or} \quad (y+x)P+y-x=0$$

$$(y+x)P + y = 0$$
$$\Rightarrow \frac{dy}{dx} = \frac{-y}{y+x}$$
$$\Rightarrow (y+x)\,dy = -y\,dx$$
$$\Rightarrow y\,dy + x\,dy = -y\,dx$$
$$\Rightarrow x\,dy + y\,dx = -y\,dy$$
$$\Rightarrow \int d(xy) = -\int y\,dy$$
$$\Rightarrow xy = -\frac{y^2}{2} + c$$
$$\Rightarrow xy + \frac{y^2}{2} - c = 0$$

$$(y+x)P + y - x = 0$$
$$\Rightarrow (y+x)\frac{dy}{dx} = x-y$$
$$\Rightarrow (y+x)\,dy = (x-y)\,dx$$
$$\Rightarrow y\,dy + x\,dy = x\,dx - y\,dx$$
$$\Rightarrow y\,dy + x\,dy + y\,dx = x\,dx$$
$$\Rightarrow \int y\,dy + \int d(xy) = \int x\,dx$$
$$\Rightarrow \frac{y^2}{2} + xy = \frac{x^2}{2} + c$$
$$\Rightarrow xy - \frac{x^2}{2} - \frac{y^2}{2} - c = 0$$

Hence the req. sol. is $\left(xy + \frac{y^2}{2} - c\right)\left(xy - \frac{x^2}{2} - \frac{y^2}{2} - c\right) = 0$

Q10 $xy(x^2+y^2)(P^2-1) = P(x^4+x^2y^2+y^4)$

Sol:-

$$xy(x^2+y^2)P^2 - xy(x^2+y^2) - P\left[(x^2+y^2)^2 - x^2y^2\right] = 0$$
$$\Rightarrow xy(x^2+y^2)P^2 - xy(x^2+y^2) - P(x^2+y^2)^2 + Px^2y^2 = 0$$
$$\Rightarrow P(x^2+y^2)\left[xyP - (x^2+y^2)\right] + xy\left[Pxy - (x^2+y^2)\right] = 0$$
$$\Rightarrow \left[xyP - (x^2+y^2)\right]\left[P(x^2+y^2) + xy\right] = 0$$
$$\Rightarrow xyP - (x^2+y^2) = 0 \quad \text{or} \quad P(x^2+y^2) + xy = 0$$

$$xyP - (x^2+y^2) = 0$$
$$\Rightarrow \frac{dy}{dx} = \frac{x^2+y^2}{xy}$$
$$\Rightarrow (x^2+y^2)\,dx - xy\,dy = 0 \quad \text{———} \ (1)$$
$$(M\,dx + N\,dy = 0)$$

Let $M = x^2+y^2$; $N = -xy$

$\therefore \frac{\partial M}{\partial y} = 2y$; $\frac{\partial N}{\partial x} = -y$

(1) is not exact, we find I.F of (1)

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y+y}{-xy} = \frac{3y}{-xy} = -\frac{3}{x} = P(x)$$

$$P(x^2+y^2) + xy = 0$$
$$\Rightarrow \frac{dy}{dx} = \frac{-xy}{x^2+y^2}$$
$$\Rightarrow xy\,dx + (x^2+y^2)\,dy = 0 \quad \text{———} \ (3)$$
$$(M\,dx + N\,dy = 0)$$

Let $M = xy$; $N = x^2+y^2$

$\therefore \frac{\partial M}{\partial y} = x$; $\frac{\partial N}{\partial x} = 2x$

(3) is not exact, we find I.F,

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{x-2x}{x^2+y^2} = \frac{-x}{x^2+y^2} \neq P(x)$$

$I \cdot F = e^{\int -3/x\, dx} = e^{-3\ln x} = e^{\ln x^{-3}} = x^{-3}$

Multiplying ① by its I·F, we get

$(x^{-1} + x^{-3}y^2)\,dx - x^{-2}y\,dy = 0$ ——②

② is exact and here

$M = x^{-1} + x^{-3}y^2, \quad N = -x^{-2}y$

$\int M\,dx = \int (x^{-1} + x^{-3}y^2)\,dx$ (y is const.)

$= \int \frac{dx}{x} + y^2 \int x^{-3}\,dx$

$= \ln x + y^2\, x^{-2}/-2$

Hence sol. of ② is,

$\ln x - \frac{y^2}{2x^2} = c$

$\Rightarrow 2x^2 \ln x - y^2 = 2cx^2$

$\Rightarrow 2x^2 \ln x - y^2 - 2cx^2 = 0$

$\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{2x - x}{xy} = \frac{x}{xy} = \frac{1}{y} = P(y)$

$I \cdot F = e^{\int 1/y\, dy} = e^{\ln y} = y$

Multiplying ③ by its I·F, we get.

$xy^2\,dx + (x^2y + y^3)\,dy = 0$ ——④

④ is exact, and here

$M = xy^2 \quad ; \quad N = x^2y + y^3$

$\int M\,dx = \int xy^2\,dx$, (y is constant)

$= x^2y^2/2$

$\int y^3\,dy = y^4/4$

Hence sol. of ④ is,

$x^2y^2/2 + y^4/4 = c_1$

or $2x^2y^2 + y^4 = 4c_1$

or $2x^2y^2 + y^4 - c = 0$

Hence req. sol. is $(2x^2\ln x - y^2 - 2cx^2)(2x^2y^2 + y^4 - c) = 0$

11 $\quad xP^2 - 3yP + 9x^2 = 0$

Sol:-

$3yP = xP^2 + 9x^2$

$\Rightarrow y = \frac{1}{3}xP + 3x^2P^{-1}$ ——①

Diff. the above eq. w.r.t x

$\frac{dy}{dx} = \frac{1}{3}\left(x\frac{dP}{dx} + P\right) + 3\left(-x^2P^{-2}\frac{dP}{dx} + 2xP^{-1}\right)$

$\Rightarrow P = \frac{1}{3}x\frac{dP}{dx} + \frac{P}{3} - 3x^2P^{-2}\frac{dP}{dx} + 6xP^{-1}$

$\Rightarrow P = x/3\,(1 - 9xP^{-2})\frac{dP}{dx} + \frac{P}{3} + 6xP^{-1}$

$\Rightarrow \frac{2}{3}P - 6xP^{-1} = \frac{x}{3}(1 - 9xP^{-2})\frac{dP}{dx}$

$\Rightarrow \frac{2}{3}P(1 - 9xP^{-2}) - \frac{x}{3}(1 - 9xP^{-2})\frac{dP}{dx} = 0$

$\Rightarrow \frac{1}{3}(1 - 9xP^{-2})\left(2P - x\frac{dP}{dx}\right) = 0$

$2P = x\frac{dP}{dx}$

12 $\quad P^2 + x^3P - 2x^2y = 0$

Sol:-

$2x^2y = P^2 + x^3P$

$\Rightarrow y = \frac{1}{2}x^{-2}P^2 + \frac{xP}{2}$ ——①

Diff. the above eq. w.r.t. x, we get

$\frac{dy}{dx} = \frac{1}{2}\left(x^{-2}\,2P\frac{dP}{dx} - 2x^{-3}P^2\right) + \frac{1}{2}\left(x\frac{dP}{dx} + P\right)$

$\Rightarrow P = \frac{P}{x^2}\frac{dP}{dx} - \frac{P^2}{x^3} + \frac{1}{2}x\frac{dP}{dx} + \frac{P}{2}$

$\Rightarrow (P/x^2 + x/2)\frac{dP}{dx} - P^2/x^3 - P/2 = 0$

$\Rightarrow x(P/x^3 + 1/2)\frac{dP}{dx} - P(P/x^3 + \frac{1}{2}) = 0$

$\Rightarrow (P/x^3 + 1/2)\left(x\frac{dP}{dx} - P\right) = 0$

$\Rightarrow P/x^3 + 1/2 = 0$ or $x\frac{dP}{dx} - P = 0$

$\Rightarrow 1-9xP^{-2}=0 \quad \text{or} \quad 2P-x\frac{dP}{dx}=0$

Consider;

$2P-x\frac{dP}{dx}=0$

$\Rightarrow x\frac{dP}{dx}=2P$

$\Rightarrow \frac{dP}{dx}=\frac{2P}{x}$

$\Rightarrow \int\frac{dP}{P}=\int 2\frac{dx}{x}$

$\Rightarrow \ln P = 2\ln x+\ln c$

$\Rightarrow \ln P = \ln x^2+\ln c$

$\Rightarrow \ln P = \ln cx^2$

$\Rightarrow P = cx^2$

Put this value of P in eq. ①

we get, $y = \frac{1}{3}x\cdot cx^2+\frac{3x^2}{cx^2}$

$\Rightarrow cy = \frac{1}{3}c^2x^3+3$

$\Rightarrow 3cy-c^2x^3-9=0$

$\Rightarrow c^2x^3-3cy+9=0$

13 $P^2+4x^5P-12x^4y=0$

Sol:-

$12x^4y = P^2+4x^5P$

$\Rightarrow y = \frac{P^2}{12x^4}+\frac{xP}{3}$

$\Rightarrow y = \frac{1}{12}x^{-4}P^2+\frac{1}{3}xP$ —— ①

Diff. ① w.r.t x, we get.

$\frac{dy}{dx}=\frac{1}{12}(x^{-4}\cdot 2P\frac{dP}{dx}-4P^2x^{-5})+\frac{1}{3}(x\frac{dP}{dx}+P)$

$\Rightarrow P = \frac{1}{6}x^{-4}P\frac{dP}{dx}-\frac{1}{3}P^2x^{-5}+\frac{1}{3}x\frac{dP}{dx}+\frac{1}{3}P$

$\Rightarrow \frac{2}{3}P = \frac{1}{6x^4}P\frac{dP}{dx}+\frac{1}{3}x\frac{dP}{dx}-\frac{1}{3x^5}P^2$

Consider

$x\frac{dP}{dx}=P$

$\Rightarrow \int\frac{dP}{P}=\int\frac{dx}{x}$

$\Rightarrow \ln P = \ln x+\ln c$

$\Rightarrow \ln P = \ln cx$

$\Rightarrow P = cx$

To eliminate P from ① put this value of P in ①, we get

$y = \frac{1}{2}\frac{c^2x^2}{x^2}+\frac{x\cdot cx}{2}$

$\Rightarrow y = c^2/2+cx^2/2$

$\Rightarrow 2y = c^2+cx^2$

14 $x^8P^2+3xP+9y=0$

Sol:-

$y = -\frac{1}{9}(x^8P^2+3xP)$ —— ①

Diff. ① w.r.t x, we get.

$\frac{dy}{dx}=-\frac{1}{9}(8x^7P^2+2x^8P\frac{dP}{dx})-\frac{1}{3}(x\frac{dP}{dx}+P)$

$\Rightarrow P = -\frac{8}{9}P^2x^7-\frac{2}{9}Px^8\frac{dP}{dx}-\frac{x}{3}\frac{dP}{dx}-\frac{P}{3}$

$\Rightarrow \frac{4P}{3}+\frac{8P^2}{9}x^7 = -\frac{x}{3}(1+\frac{2}{3}x^7P)\frac{dP}{dx}$

$\Rightarrow \frac{4}{3}P(1+\frac{2}{3}Px^7)+\frac{x}{3}(1+\frac{2}{3}x^7P)\frac{dP}{dx}=0$

$\Rightarrow (1+\frac{2}{3}Px^7)(\frac{4}{3}P+\frac{x}{3}\frac{dP}{dx})=0$

$\Rightarrow 1+\frac{2}{3}Px^7=0 \quad \text{or} \quad \frac{4}{3}P+\frac{x}{3}\frac{dP}{dx}=0$

Consider,

$\frac{4}{3}P+\frac{x}{3}\frac{dP}{dx}=0$

$\Rightarrow \frac{x}{3}\frac{dP}{dx}=-\frac{4}{3}P$

$$\Rightarrow \frac{2}{3}P + \frac{1}{3x^5}P^2 = \frac{1}{3}\left(\frac{1}{2x^4}P + x\right)\frac{dP}{dx}$$

$$\Rightarrow \frac{2P}{3}\left(1+\frac{P}{2x^5}\right) = \frac{x}{3}\left(1+\frac{P}{2x^5}\right)\frac{dP}{dx}$$

$$\Rightarrow \frac{2P}{3}\left(1+\frac{P}{2x^5}\right) - \frac{x}{3}\left(1+\frac{P}{2x^5}\right)\frac{dP}{dx} = 0$$

$$\Rightarrow \left(1+\frac{P}{2x^5}\right)\left(\frac{2P}{3} - \frac{x}{3}\frac{dP}{dx}\right) = 0$$

$$\Rightarrow 1+\frac{P}{2x^5} = 0 \quad \text{or} \quad \frac{2P}{3} - \frac{x}{3}\frac{dP}{dx} = 0$$

consider

$$\frac{2P}{3} - \frac{x}{3}\frac{dP}{dx} = 0$$

$$\Rightarrow \frac{x}{3}\frac{dP}{dx} = \frac{2}{3}P$$

$$\Rightarrow \int\frac{dP}{P} = 2\int\frac{dx}{x}$$

$$\Rightarrow \ln P = 2\ln x + \ln c$$

$$\Rightarrow \ln P = \ln x^2 + \ln c$$

$$\Rightarrow \ln P = \ln cx^2$$

$$\Rightarrow P = cx^2$$

Put above value of P in ①

we get,

$$y = \frac{1}{12x^4}\cdot c^2x^4 + \frac{1}{3}x\cdot cx^2$$

$$\Rightarrow y = c^2/12 + cx^3/3$$

$$\Rightarrow 12y = c^2 + 4cx^3$$

$$\Rightarrow 12y = c(c+4x^3)$$

15 $P^2 + 3xP - y = 0$

Sol:-

$$y = 3xP + P^2 \quad \text{———} ①$$

Diff. ① w.r.t x, we get

$$\frac{dy}{dx} = 3\left(x\frac{dP}{dx} + P\right) + 2P\frac{dP}{dx}$$

$$\Rightarrow P = 3x\frac{dP}{dx} + 3P + 2P\frac{dP}{dx}$$

$$\Rightarrow \int\frac{dP}{P} = -4\int\frac{dx}{x}$$

$$\Rightarrow \ln P = -4\ln x + \ln c$$

$$\Rightarrow \ln P = \ln x^{-4} + \ln c$$

$$\Rightarrow \ln P = \ln cx^{-4}$$

$$\Rightarrow P = cx^{-4}$$

Put above value of P, in ①,

we get,

$$y = -\frac{1}{9}\left(x^8\cdot c^2x^{-8} + 3x\cdot cx^{-4}\right)$$

$$= -1/9\left(c^2 + 3cx^{-3}\right)$$

$$\Rightarrow -9y = c^2 + \frac{3c}{x^3}$$

$$\Rightarrow -9x^3y = c^2x^3 + 3c$$

16 $y = Px + x^3P^2$ ——— ①

Sol:-

Diff. eq. ①, w.r.t x, we get

$$\frac{dy}{dx} = P + x\frac{dP}{dx} + 2x^3P\frac{dP}{dx} + 3x^2P^2$$

$$\Rightarrow \cancel{P} = \cancel{P} + x(1+2x^2P)\frac{dP}{dx} + 3x^2P^2$$

$$\Rightarrow (1+2x^2P)\frac{dP}{dx} = -3xP^2$$

$$\Rightarrow \frac{dP}{dx} = \frac{-3xP^2}{1+2x^2P}$$

$$\Rightarrow \frac{dx}{dP} = \frac{1+2x^2P}{-3xP^2}$$

$$\Rightarrow \frac{dx}{dP} = -\frac{1}{3xP^2} - \frac{2x}{3P}$$

$$\Rightarrow \frac{dx}{dP} + \frac{2}{3P}x = -\frac{1}{3P^2}x^{-1}$$

(It is Bernoulli eq.)

Multiplying the above eq. by x

i-e $x\frac{dx}{dP} + \frac{2}{3P}x^2 = -\frac{1}{3P^2}$

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$\Rightarrow -2P = (3x+2P)\frac{dP}{dx}$

$\Rightarrow (3x+2P)\frac{dP}{dx} = -2P$

$\Rightarrow \frac{dP}{dx} = \frac{-2P}{3x+2P}$

$\Rightarrow \frac{dx}{dP} = \frac{3x+2P}{-2P}$

$\Rightarrow \frac{dx}{dP} = -\frac{3x}{2P} - 1$

$\Rightarrow \frac{dx}{dP} + \frac{3}{2P}x = -1$

(It is linear in x)

$I \cdot F = e^{\int \frac{3}{2P}dP} = e^{3/2 \ln P} = e^{\ln P^{3/2}} = P^{3/2}$

Multiplying the above eq. by $I \cdot F$

$P^{3/2}\frac{dx}{dP} + \frac{3x}{2P}P^{3/2} = -P^{3/2}$

$\Rightarrow P^{3/2}dx + \frac{3}{2}xP^{1/2}dP = -P^{3/2}dP$

$\Rightarrow d(xP^{3/2}) = -P^{3/2}dP$

$\Rightarrow \int d(xP^{3/2}) = -\int P^{3/2}dP$

$\Rightarrow xP^{3/2} = -\frac{P^{5/2}}{5/2} + c.$

$\Rightarrow x = -\frac{2}{5}P + cP^{-3/2}$ ——— ②

It is difficult to find value of P.

So putting this value of x in eq. ①

we get,

$y = 3P(-2P/5 + cP^{-3/2}) + P^2$ ——— ③

Thus ②, ③, is a parametric sol. of the given eq.

Let $v = x^2 \therefore \frac{dv}{dP} = 2x\frac{dx}{dP}$

$\Rightarrow \frac{1}{2}\frac{dv}{dP} = x\frac{dx}{dP}$

Hence the above eq. become:

$\frac{1}{2}\frac{dv}{dP} + \frac{2v}{3P} = -\frac{1}{3P^2}$

$\Rightarrow \frac{dv}{dP} + \frac{4v}{3P} = -\frac{2}{3P^2}$

(It is linear in v)

$I \cdot F = e^{\int \frac{4}{3P}dP} = e^{4/3 \ln P} = e^{\ln P^{4/3}} = P^{4/3}$

Multiplying the above eq. by $I \cdot F$ we get,

$P^{4/3}\frac{dv}{dP} + \frac{4vP^{1/3}}{3} = -\frac{2}{3}P^{-2/3}$

$\Rightarrow P^{4/3}dv + \frac{4}{3}vP^{1/3}dP = -\frac{2}{3}P^{-2/3}dP$

$\Rightarrow d(vP^{4/3}) = -2/3\, P^{-2/3}dP$

$\Rightarrow \int d(vP^{4/3}) = -2/3\int P^{-2/3}dP$

$\Rightarrow vP^{4/3} = -\frac{2}{3}\frac{P^{1/3}}{1/3} + c$

$\Rightarrow vP^{4/3} = -2P^{1/3} + c$

$\Rightarrow x^2P^{4/3} = -2P^{1/3} + c \quad \because v = x^2$

$\Rightarrow x^2 = -2/P + c/P^{4/3}$ ——— ②

It is difficult to find the value of P.

So putting this value of x in ①

we get

$y = P(-\frac{2}{P} + cP^{-4/3})^{1/2} + P^2(-2/P + cP^{-4/3})^{3/2}$ ——— ③

Hence ②, ③ give the parametric sol. of the given eq.

17 $xP^2 - 2yP + ax = 0$

Sol:–

$$2yP = ax + xP^2$$

$$\Rightarrow y = \frac{a}{2}xP^{-1} + \frac{1}{2}xP \quad \text{——} ①$$

Diff. ①, w.r.t x, we get,

$$\frac{dy}{dx} = \frac{a}{2}\left(-xP^{-2}\frac{dP}{dx} + P^{-1}\right) + \frac{1}{2}\left(x\frac{dP}{dx} + P\right)$$

$$\Rightarrow P = -\tfrac{1}{2}axP^{-2}\frac{dP}{dx} + \tfrac{1}{2}aP^{-1} + \tfrac{1}{2}x\frac{dP}{dx} + P/2$$

$$\Rightarrow P/2 = \frac{1}{2}x(1 - aP^{-2})\frac{dP}{dx} + \frac{1}{2}aP^{-1}$$

$$\Rightarrow \frac{1}{2}x(1-aP^{-2})\frac{dP}{dx} + \frac{1}{2}aP^{-1} - \frac{1}{2}P = 0$$

$$\Rightarrow \frac{1}{2}x(1-aP^{-2})\frac{dP}{dx} - \frac{1}{2}P(1-aP^{-2}) = 0$$

$$\Rightarrow \frac{1}{2}(1-aP^{-2})\left(x\frac{dP}{dx} - P\right) = 0$$

$$\Rightarrow 1 - aP^{-2} = 0 \quad \text{or} \quad x\frac{dP}{dx} - P = 0$$

Consider,

$$x\frac{dP}{dx} - P = 0$$

$$\Rightarrow x\frac{dP}{dx} = P$$

$$\Rightarrow \int\frac{dP}{P} = \int\frac{dx}{x}$$

$$\Rightarrow \ln P = \ln x + \ln c$$

$$\Rightarrow \ln P = \ln cx$$

$$\Rightarrow P = cx$$

Put this value of P in ①

We get $y = \frac{ax}{2}(cx)^{-1} + \frac{1}{2}x \cdot cx$

$$\Rightarrow y = \frac{ac^{-1}}{2} + \frac{cx^2}{2}$$

$$\Rightarrow 2cy = a + c^2x^2$$

$$\Rightarrow c^2x^2 - 2cy + a = 0$$

18 $P = \tan\left(x - \frac{P}{1+P^2}\right)$

Sol:–

$$\tan^{-1}P = x - \frac{P}{1+P^2}$$

$$\Rightarrow x = \tan^{-1}P + \frac{P}{1+P^2} \quad \text{——} ①$$

Diff. ① w.r.t y we get

$$\frac{dx}{dy} = \frac{1}{1+P^2}\frac{dP}{dy} + \frac{(1+P^2)\frac{dP}{dy} - P\cdot 2P\frac{dP}{dy}}{(1+P^2)^2}$$

$$\Rightarrow \frac{1}{P} = \frac{1}{1+P^2}\frac{dP}{dy} + \left[\frac{1-P^2}{(1+P^2)^2}\right]\frac{dP}{dy}$$

$$\Rightarrow \frac{1}{P} = \left[\frac{1}{1+P^2} + \frac{1-P^2}{(1+P^2)^2}\right]\frac{dP}{dy}$$

$$\Rightarrow \frac{1}{P} = \left(\frac{1+P^2+1-P^2}{(1+P^2)^2}\right)\frac{dP}{dy}$$

$$\Rightarrow \frac{1}{P} = \frac{2}{(1+P^2)^2}\frac{dP}{dy}$$

$$\Rightarrow dy = \frac{2P}{(1+P^2)^2}dP$$

$$\Rightarrow \int dy = \int (1+P^2)^{-2}\cdot 2P\, dP$$

$$\Rightarrow y = -(1+P^2)^{-1} + c$$

$$\Rightarrow y = -\frac{1}{1+P^2} + c \quad \text{——} ②$$

①, ② give the parametric sol. of the given eq.

20 $aP^2 + Py - x = 0$

Sol:–

$$x = Py + aP^2 \quad \text{——} ①$$

Diff. ① w.r.t. y, we get.

$$\frac{dx}{dy} = P + y\frac{dP}{dy} + 2aP\frac{dP}{dy}$$

$$\Rightarrow \frac{1}{P} = P + (y + 2aP)\frac{dP}{dy}$$

19 $P^3 - 4xyP + 8y^2 = 0$

Sol:-

$4xyP = P^3 + 8y^2$

$\Rightarrow x = \frac{1}{4}P^2y^{-1} + 2P^{-1}y$ —— ①

Diff. ① w.r.t y, we get

$$\frac{dx}{dy} = \frac{1}{4}\left(-P^2y^{-2} + 2y^{-1}P\frac{dP}{dy}\right) + 2\left(-yP^{-2}\frac{dP}{dy} + P^{-1}\right)$$

$$\Rightarrow \frac{1}{P} = -\frac{P^2}{4y^2} + \frac{P}{2y}\frac{dP}{dy} - \frac{2y}{P^2}\frac{dP}{dy} + \frac{2}{P}$$

$$\Rightarrow 4Py^2 = -P^4 + 2yP^3\frac{dP}{dy} - 8y^3\frac{dP}{dy} + 8y^2P$$

$$\Rightarrow -4Py^2 + P^4 = 2yP^3\frac{dP}{dy} - 8y^3\frac{dP}{dy}$$

$$\Rightarrow P(P^3 - 4y^2) = 2y(P^3 - 4y^2)\frac{dP}{dy}$$

$$\Rightarrow 2y(P^3 - 4y^2)\frac{dP}{dy} - P(P^3 - 4y^2) = 0$$

$$\Rightarrow (P^3 - 4y^2)\left(2y\frac{dP}{dy} - P\right) = 0$$

$$\Rightarrow P^3 - 4y^2 = 0 \quad \text{or} \quad 2y\frac{dP}{dy} - P = 0$$

Consider,

$$2y\frac{dP}{dy} - P = 0$$

$$\Rightarrow 2y\frac{dP}{dy} = P$$

$$\Rightarrow 2\int\frac{dP}{P} = \int\frac{dy}{y}$$

$$\Rightarrow 2\ln P = \ln y + \ln c_1$$

$$\Rightarrow \ln P^2 = \ln c_1 y$$

$$\Rightarrow P^2 = c_1 y$$

$$\Rightarrow y = P^2/c_1 \quad \text{or} \quad y = cP^2 \text{ —— ②}$$

Put this value of y in ①, we get

$$x = \frac{1}{4}P^2\cdot c^{-1}P^{-2} + 2P^{-1}\cdot cP^2$$

$$\Rightarrow x = \frac{1}{4c} + 2cP$$

$$\Rightarrow x = (1 + 8c^2P)/4c \text{ —— ③}$$

②, ③ give param. sol. of given eq.

$$\Rightarrow (y + 2aP)\frac{dP}{dy} = \frac{1}{P} - P$$

$$= \frac{1-P^2}{P}$$

$$\Rightarrow \frac{dP}{dy} = \frac{1-P^2}{P(y+2aP)}$$

$$\Rightarrow \frac{dy}{dP} = \frac{Py + 2aP^2}{1-P^2}$$

$$\Rightarrow \frac{dy}{dP} + \frac{P}{P^2-1}y = \frac{-2aP^2}{P^2-1}$$

(It is linear in y)

$$\text{I.F.} = e^{\int\frac{P}{P^2-1}dP} = e^{\frac{1}{2}\int\frac{2P\,dP}{P^2-1}} = e^{\frac{1}{2}\ln(P^2-1)} = e^{\ln(P^2-1)^{1/2}} = \sqrt{P^2-1}$$

Multiplying the above eq. by I.F.

$$\sqrt{P^2-1}\,\frac{dy}{dP} + \frac{P}{\sqrt{P^2-1}}y = \frac{-2aP^2}{\sqrt{P^2-1}}$$

$$\Rightarrow \sqrt{P^2-1}\,dy + \frac{P}{\sqrt{P^2-1}}y\,dP = \frac{-2aP^2}{\sqrt{P^2-1}}dP$$

$$\Rightarrow \int d\left(y\sqrt{P^2-1}\right) = \int\frac{-2aP^2}{\sqrt{P^2-1}}dP$$

$$\Rightarrow y\sqrt{P^2-1} = -2a\int\frac{P^2}{\sqrt{P^2-1}}dP$$

$$= -2a\int\frac{(P^2-1)+1}{\sqrt{P^2-1}}dP$$

$$= -2a\int\sqrt{P^2-1}\,dP - 2a\int\frac{dP}{\sqrt{P^2-1}}$$

$$= -2a\left[\frac{P\sqrt{P^2-1}}{2} - \frac{1}{2}\cosh^{-1}P\right] - 2a\cosh^{-1}P$$

$$= -aP\sqrt{P^2-1} + a\cosh^{-1}P - 2a\cosh^{-1}P$$

$$= -aP\sqrt{P^2-1} - a\cosh^{-1}P + c$$

$$\Rightarrow y = -aP + \frac{c - a\cosh^{-1}P}{\sqrt{P^2-1}}$$

21 $e^{4x}(P-1)+e^{2y}P^2=0$ ——— ①

Sol:-

① is not solvable for P, for y so, we convert ①, in clairot's eq. as,

Let $u=e^{2x}$, $v=e^{2y}$

$\therefore du=2e^{2x}dx$, $dv=2e^{2y}dy$

Now $\dfrac{2e^{2y}dy}{2e^{2x}dx}=\dfrac{dv}{du}$

$\Rightarrow \dfrac{v\,dy}{u\,dx}=\dfrac{dv}{du}$

$\Rightarrow \dfrac{dy}{dx}=\dfrac{u}{v}\dfrac{dv}{du}$

$\Rightarrow P=\dfrac{u}{v}\dfrac{dv}{du}$ $\quad\because P=\dfrac{dy}{dx}$

Hence eq. ① becomes,

$u^2\left(\dfrac{u}{v}\dfrac{dv}{du}-1\right)+v\left(\dfrac{u}{v}\dfrac{dv}{du}\right)^2=0$

$\Rightarrow \dfrac{u}{v}\dfrac{dv}{du}-1+\dfrac{u}{v}\left(\dfrac{dv}{du}\right)^2=0$

$\Rightarrow u\dfrac{dv}{du}-v+\left(\dfrac{dv}{du}\right)^2=0$

$\Rightarrow v=u\left(\dfrac{dv}{du}\right)+\left(\dfrac{dv}{du}\right)^2$

which is clairaut's eq.

Hence its general sol. is,

$v=uc+c^2$

$\Rightarrow e^{2y}=e^{2x}c+c^2$

23

$P^2\cos y+P\sin x\cos x\cos y-\sin y\cos^2 x=0$ ——— ①

It is not solvable for P, y, x

22 $yP^2-2xP+y=0$

Sol:-

It is solvable for x, so we take

$2xP=yP^2+y$

$\Rightarrow x=\frac{1}{2}yP+\frac{1}{2}yP^{-1}$ ——— ①

Diff. ① w.r.t y, we get

$\dfrac{dx}{dy}=\frac{1}{2}\left(y\dfrac{dP}{dy}+P\right)+\frac{1}{2}\left(-yP^{-2}\dfrac{dP}{dy}+P^{-1}\right)$

$\Rightarrow 2\cdot\dfrac{1}{P}=y\dfrac{dP}{dy}+P-\dfrac{y}{P^2}\dfrac{dP}{dy}+\dfrac{1}{P}$

$\Rightarrow \dfrac{2}{P}-\dfrac{1}{P}-P=y\left(1-\dfrac{1}{P^2}\right)\dfrac{dP}{dy}$

$\Rightarrow \dfrac{1}{P}-P=y\left(1-\dfrac{1}{P^2}\right)\dfrac{dP}{dy}$

$\Rightarrow -P\left(1-\dfrac{1}{P^2}\right)-y\left(1-\dfrac{1}{P^2}\right)\dfrac{dP}{dy}=0$

$\Rightarrow P\left(1-\dfrac{1}{P^2}\right)+y\left(1-\dfrac{1}{P^2}\right)\dfrac{dP}{dy}=0$

$\Rightarrow \left(1-\dfrac{1}{P^2}\right)\left(P+y\dfrac{dP}{dy}\right)=0$

$\Rightarrow 1-\dfrac{1}{P^2}=0$ or $P+y\dfrac{dP}{dy}=0$

Consider,

$P+y\dfrac{dP}{dy}=0$

$\Rightarrow y\dfrac{dP}{dy}=-P$

$\Rightarrow \dfrac{dP}{P}=-\dfrac{dy}{y}$

$\Rightarrow \int\dfrac{dP}{P}=-\int\dfrac{dy}{y}$

$\Rightarrow \ln P=-\ln y+\ln c$

$\Rightarrow \ln P=\ln cy^{-1}$

$\Rightarrow P=cy^{-1}$ put in ①

We get $x=\frac{1}{2}c+\frac{1}{2c}y^2$

$\Rightarrow 2cx=c^2+y^2=0$

So, we converts it into clairau's form as,

Let $u = \sin x$, $v = \sin y$

$\therefore$ $du = \cos x\, dx$, $dv = \cos y\, dy$

Now,

$$\frac{\cos y\, dy}{\cos x\, dx} = \frac{dv}{du}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{\cos y}\frac{dv}{du}$$

$$\Rightarrow P = \frac{\cos x}{\cos y}\frac{dv}{du}$$

Hence ① becomes,

$$\cancel{\cos^2 y}\cdot\frac{\cos^2 x}{\cancel{\cos^2 y}}\left(\frac{dv}{du}\right)^2 + \frac{\cos x}{\cancel{\cos y}}\frac{dv}{du}\cdot u\cos x\,\cancel{\cos y} - v\cos^2 x = 0$$

$$\Rightarrow \cos^2 x\left(\frac{dv}{du}\right)^2 + u\cos^2 x\frac{dv}{du} - v\cos^2 x = 0$$

$$\Rightarrow \left(\frac{dv}{du}\right)^2 + u\frac{dv}{du} - v = 0$$

$$\Rightarrow v = u\frac{dv}{du} + \left(\frac{dv}{du}\right)^2 = 0$$

It is clairauts eq. so its general sol. is,

$$v = uc + c^2$$

$$\Rightarrow \sin y = c\sin x + c^2$$

25 $y^2(y - xP) = x^4P^2$ ——— ①

Sol:-

$$y^3 - xy^2P - x^4P^2 = 0$$

It is not solvable for P, x, y

So, we convert it into Clairaut's form as,

24 $(Px - y)(Py + x) = 2P$.

Sol:-

$$P^2xy + Px^2 - Py^2 - xy - 2P = 0 \text{ ——— ①}$$

It is not solvable for P, x, y

So, we convert it into clairaut's eq. as,

Let $u = x^2$, $v = y^2$

$\therefore$ $du = 2x\, dx$, $dv = 2y\, dy$

Now $$\frac{2y\, dy}{2x\, dx} = \frac{dv}{du}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{y}\frac{dv}{du}$$

$$\Rightarrow P = \frac{x}{y}\frac{dv}{du}$$

Putting in the given eq., we get

$$\left(\frac{x}{y}\frac{dv}{du}\cdot x - y\right)\left(\frac{x}{\cancel{y}}\frac{dv}{du}\cdot\cancel{y} + x\right) = 2\frac{x}{y}\frac{dv}{du}$$

$$\Rightarrow \left(\frac{x^2}{y}\frac{dv}{du} - y\right)\left(x\frac{dv}{du} + x\right) = 2\frac{x}{y}\frac{dv}{du}$$

$$\Rightarrow \cancel{x}\left(x^2\frac{dv}{du} - y^2\right)\left(\frac{dv}{du} + 1\right) = 2\cancel{x}\frac{dv}{du}$$

$$\Rightarrow \left(u\frac{dv}{du} - v\right)\left(\frac{dv}{du} + 1\right) = 2\frac{dv}{du}$$

$$\Rightarrow u\frac{dv}{du} - v = \frac{2\frac{dv}{du}}{\frac{dv}{du} + 1}$$

$$\Rightarrow v = u\frac{dv}{du} - \frac{2\frac{dv}{du}}{1 + \frac{dv}{du}}$$

which is clairaut's form and its sol. is,

$$v = u\cdot c - \frac{2c}{1+c}$$

$$\Rightarrow y^2 = cx^2 - \frac{2c}{1+c}$$

Let $u = 1/x$, $v = 1/y$

$\therefore\ du = -\frac{dx}{x^2}$, $dv = -\frac{dy}{y^2}$

Now $\frac{dy}{y^2} \Big/ \frac{dx}{x^2} = dv/du$

$\Rightarrow \frac{dy}{y^2} \cdot \frac{x^2}{dx} = \frac{dv}{du}$

$\Rightarrow \frac{dy}{dx} = \frac{y^2}{x^2}\frac{dv}{du}$

$\Rightarrow P = \frac{u^2}{v^2}\frac{dv}{du}$

Hence ① becomes as,

$$\frac{1}{v^2}\left(\frac{1}{v} - \frac{1}{u}\cdot\frac{u^2}{v^2}\frac{dv}{du}\right) = \frac{1}{u^4}\cdot\frac{u^4}{v^4}\left(\frac{dv}{du}\right)^2$$

$$\Rightarrow \frac{1}{v^2}\left(\frac{1}{v} - \frac{u}{v^2}\frac{dv}{du}\right) = \frac{1}{v^4}\left(\frac{dv}{du}\right)^2$$

$$\Rightarrow \frac{1}{v^4}\left(v - u\frac{dv}{du}\right) = \frac{1}{v^4}\left(\frac{dv}{du}\right)^2$$

$$\Rightarrow v - u\frac{dv}{du} = \left(\frac{dv}{du}\right)^2$$

$$\Rightarrow v = u\frac{dv}{du} + \left(\frac{dv}{du}\right)^2$$

which is clairauts form,
Hence its general sol. is,

$$v = u\cdot c + c^2$$

$$\Rightarrow 1/y = c/x + c^2$$

$$\Rightarrow x = cy + c^2xy$$

27 $y = xP - e^P$ ——— ①

It is clairauts eq.

General sol. of ①

$$y = cx - e^c$$

Singular sol. of ①

26 Find the general sol. and singular sol. of the diff. eqs. from 26 to 30

$y = xP + \ln P$ ——— ①

Sol:-

It is Clairaut's eq.

General sol. of ①

$$y = cx + \ln c$$

Singular sol. of ①

We know that,
the singular sol. of the Clairaut's eq. $y = xP + f(P)$ in parametric form, is

$$\left.\begin{aligned} x &= -f'(P) \\ y &= f(P) - Pf'(P) \end{aligned}\right\} \text{———②}$$

Where.

$f(P) = -\ln P$ $\therefore f'(P) = -\frac{1}{P}$

Hence ② becomes, as

$$\left.\begin{aligned} x &= 1/P \\ y &= -\ln P - P\cdot -1/P = -\ln P + 1 \end{aligned}\right\} \text{———③}$$

we can eliminate P in eqs. ③, as
since $P = 1/x$

$\therefore\ y = -\ln(1/x) + 1$

$= -\ln x^{-1} + 1$

$= \ln x + 1$ req. s. sol. of ①

28 $y = xP + a\sqrt{1+P^2}$ ——— ①

Sol:- It is clairaut's eq.

General sol:-

$$y = cx + a\sqrt{1+c^2}$$

We know that,
singular sol. of the clairaut's eq. $y = xP + f(P)$ in param. is,

$$\left.\begin{aligned} x &= -f'(P) \\ y &= f(P) - Pf'(P) \end{aligned}\right] \text{——②}$$

Where,

$$f(P) = -e^{P} \quad \therefore \quad f'(P) = -e^{P}$$

Hence ② becomes, as.

$$\left.\begin{aligned} x &= e^{P} \\ y &= -e^{P} - P(-e^{P}) = -e^{P} + Pe^{P} \end{aligned}\right] -③$$

we can eliminat P from ③

Since $x = e^{P}$ or $\ln x = P$

$$\therefore \quad y = -x + \ln x \cdot x$$

$$= x(\ln x - 1) \quad \text{req. sol. of ①}$$

29 $\quad y = xP - \sqrt{P}$ ——①

Sol:- It is clairaut's eq.

General sol:-

$$y = cx - \sqrt{c}$$

Singular sol:-

We know that,
singular sol. of the clairaut's eq. $y = xP + f(P)$ in param. is

$$\left.\begin{aligned} x &= -f'(P) \\ y &= f(P) - Pf'(P) \end{aligned}\right] -②$$

Where

$$f(P) = -\sqrt{P} \quad \therefore \quad f'(P) = -\frac{1}{2\sqrt{P}}$$

Hence ② becomes,

$$\left.\begin{aligned} x &= \frac{1}{2\sqrt{P}} \\ y &= -\sqrt{P} + P\cdot\frac{1}{2\sqrt{P}} = -\sqrt{P} + \frac{\sqrt{P}}{2} \end{aligned}\right] -③$$

Singular sol:-

we know that,
singular sol. of the clairaut's eq. $y = xP + f(P)$ in paramet. is,

$$\left.\begin{aligned} x &= -f'(P) \\ y &= f(P) - Pf'(P) \end{aligned}\right] \text{——②}$$

where

$$f(P) = a\sqrt{1+P^2} \quad \therefore \quad f'(P) = \frac{aP}{\sqrt{1+P^2}}$$

Hence ② becomes, as

$$\left.\begin{aligned} x &= -\frac{aP}{\sqrt{1+P^2}} \\ y &= a\sqrt{1+P^2} - \frac{aP^2}{\sqrt{1+P^2}} = \frac{a}{\sqrt{1+P^2}} \end{aligned}\right] \text{——③}$$

We can eliminat P from ③, as squaring and adding two eqs, we get,

$$x^2 + y^2 = \frac{a^2P^2}{1+P^2} + \frac{a^2}{1+P^2}$$

$$= \frac{a^2P^2 + a^2}{1+P^2}$$

$$= \frac{a^2(1+P^2)}{(1+P^2)}$$

$$= a^2 \quad \text{req. s. sol. of ①}$$

30 $\quad y = xP + P^3$ ——①

Sol:- It is clairaut's eq.

General sol:-

$$y = cx + c^3$$

Singular sol:-

We know that,
Singular sol. of the clairaut's eq. $y = xP + f(P)$ in paramet. is

we can eliminat P from ③

Since $\sqrt{P} = \frac{1}{2x}$

$\therefore \quad y = -\frac{1}{2x} + \frac{1}{2x}\cdot\frac{1}{2}$

$= -\frac{1}{2x} + \frac{1}{4x}$

$= -\frac{1}{4x}$ req. s.sol. of ①

$$\left.\begin{aligned} x &= -f'(P) \\ y &= f(P) - P f'(P) \end{aligned}\right\} \text{———} ②$$

Where

$f(P) = P^3 \quad \therefore \quad f'(P) = 3P^2$

Hence ② becomes, as

$$\left.\begin{aligned} x &= -3P^2 \\ y &= P^3 - 3P^3 = -2P^3 \end{aligned}\right\} \text{---} ③$$

we can eliminate P from ③ as,

Since $x = -3P^2$

or $P = \pm\sqrt{\frac{-x}{3}}$

$\therefore \quad y = -2\left(\pm\sqrt{\frac{-x}{3}}\right)^3$

$= -2\left(\pm\sqrt{\frac{-x}{3}}\right)^2\left(\pm\sqrt{\frac{-x}{3}}\right)$

$= -2\left(\frac{-x}{3}\right)\left(\pm\sqrt{\frac{-x}{3}}\right)$

$= \frac{2x}{3}\left(\pm\sqrt{\frac{-x}{3}}\right)$

$\therefore \quad y^2 = \frac{4x^2}{9}\left(-\frac{x}{3}\right)$

$\Rightarrow 27y^2 = -4x^3$ req. s.sol. of ①