<u>Orthogonal Trajectory</u> is a curve that intersects every curve of another family at right angle.

If all the curves of a family of curves intersect orthogonally to all the curves of another family of curves then the two families are called Orthogonal Trajectories.

<u>To Solve</u>
1) Differentiate
2) Eliminate const using eq ①
3) Find $\frac{dy}{dx}$ or $r\frac{d\theta}{dr}$ for given curve.
4) Find $\frac{dy}{dx}$ or $r\frac{d\theta}{dr}$ for family of O.Ts
5) Solve by previous methods according i.e separating variables or Homogeneous or LDE or Exact.

Ex 9.7

i. Find the orthogonal trajectories of each of the following curves.

① $x^2 - y^2 = c$ one const.

Diff $2x - 2y\frac{dy}{dx} = 0$ const eliminated.

$$x - y\frac{dy}{dx} = 0$$

$$x = y\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{x}{y}$$ D Eq of given family

$$\frac{dy}{dx} = -\frac{y}{x}$$ D Eq of OTs

$$\int\frac{dy}{y} = -\int\frac{dx}{x}$$ separating variables

$$\ln y = -\ln x + \ln c$$

$$\ln y = \ln\frac{c}{x}$$

$$y = \frac{c}{x}$$

$$xy = c$$ is Required family of orthogonal trajectories.

② $x = cy^2$ — ①

Diff $1 = c\,2y\frac{dy}{dx}$

$$\frac{1}{2cy} = \frac{dy}{dx}$$

$$\frac{1}{2\left(\frac{x}{y\cancel{y}}\right)\cancel{y}} = \frac{dy}{dx}$$ using ① Const eliminated

$$\frac{dy}{dx} = \frac{y}{2x}$$ DEq of given family

$$\frac{dy}{dx} = -\frac{2x}{y}$$ DEq of OTs

$$\int y\,dy = -\int 2x\,dx$$

$$\frac{y^2}{\cancel{2}} = -\frac{\cancel{2}x^2}{\cancel{2}} + c$$

$$y^2 = -2x^2 + 2c$$

$$y^2 = -2x^2 + K$$ Required family of orthogonal trajectories

③ $x^2+y^2=cx$ ——— one const. ①

diff $2x+2y\frac{dy}{dx}=c$

const eliminated using ①

$\therefore\ x^2+y^2=\left(2x+2y\frac{dy}{dx}\right)x$

$x^2+y^2=2x^2+2xy\frac{dy}{dx}$

$y^2-x^2=2xy\frac{dy}{dx}$

$\frac{y^2-x^2}{2xy}=\frac{dy}{dx}$ diff eq for given family

$-\frac{2xy}{y^2-x^2}=\frac{dy}{dx}$ diff eq for orthogonal trajectories

$\frac{2xy}{x^2-y^2}=\frac{dy}{dx}$ Homogeneous Eq.

Put $y=vx$

$\frac{dy}{dx}=v+x\frac{dv}{dx}$

$v+x\frac{dv}{dx}=\frac{2xvx}{x^2-v^2x^2}$

$=\frac{2v\cancel{x^2}}{(1-v^2)\cancel{x^2}}$

$x\frac{dv}{dx}=\frac{2v}{1-v^2}-v$

$x\frac{dv}{dx}=\frac{2v-v+v^3}{1-v^2}$

$x\frac{dv}{dx}=\frac{v+v^3}{1-v^2}$

$\int\frac{1-v^2}{v+v^3}dv=\int\frac{dx}{x}$ separating variables.

$\int\frac{1-v^2}{v(1+v^2)}dv=\int\frac{dx}{x}$ ——— (ii)

By Partial Fractions

$\frac{1-v^2}{v(1+v^2)}=\frac{A}{v}+\frac{Bv+C}{1+v^2}$

$1-v^2=A(1+v^2)+(Bv+C)v$

Put $v=0$ $\boxed{1=A}$

Comparing coeffts of v^2, $-1=A+B$

$-1=1+B$

$\boxed{-2=B}$

Comparing coeffts of v, $\boxed{0=C}$

$\therefore\ \frac{1-v^2}{v(1+v^2)}=\frac{1}{v}+\frac{(-2v)}{1+v^2}$

So (ii) becomes

$\int\left(\frac{1}{v}-\frac{2v}{1+v^2}\right)dv=\int\frac{1}{x}dx$

$\ln v-\ln(1+v^2)=\ln x+\ln k$

$\ln\left(\frac{v}{1+v^2}\right)=\ln kx$

$\frac{v}{1+v^2}=kx$

$\frac{\frac{y}{x}}{1+\frac{y^2}{x^2}}=kx$

$\frac{\frac{y}{\cancel{x}}}{\frac{x^2+y^2}{x^{\cancel{2}}}}=kx$

$\frac{y\cancel{x}}{x^2+y^2}=k\cancel{x}$ $\Rightarrow\boxed{y=k(x^2+y^2)}$ Required family of orthogonal trajectories

Available at www.mathcity.org

④ $y = e^{cx}$ one const. ①

Diff $\frac{dy}{dx} = ce^{cx}$

$\frac{dy}{dx} = cy$

$\frac{1}{y}\frac{dy}{dx} = c$

$y = e^{\frac{x}{y}\frac{dy}{dx}}$ using ① const eliminated

$\ln y = \ln e^{\frac{x}{y}\frac{dy}{dx}}$

$\ln y = \frac{x}{y}\frac{dy}{dx}.\ln e$

$\frac{y}{x}\ln y = \frac{dy}{dx}\cdot 1$, diff eq of given family

$-\frac{x}{y\ln y} = \frac{dy}{dx}$, diff eq of family of O.Ts

$\int -x\,dx = \int \underset{\text{II}}{y}\,\underset{\text{I}}{\ln y}\,dy$ separating variables.

$\Rightarrow \ln y\,\frac{y^2}{2} - \int \frac{1}{y}\frac{y^2}{2}\,dy = -\frac{x^2}{2} + K$

$\Rightarrow \ln y\cdot\frac{y^2}{2} - \frac{1}{2}\left(\frac{y^2}{2}\right) = -\frac{x^2}{2} + K$

$\Rightarrow \frac{y^2}{2}\ln y - \frac{y^2}{4} = -\frac{x^2}{2} + K$

$\Rightarrow \frac{2y^2\ln y - y^2}{4} = -\frac{x^2}{2} + K$

$\Rightarrow y^2\left(\frac{2\ln y - 1}{4}\right) = -\frac{x^2}{2} + K$

$\Rightarrow y^2(\ln y^2 - 1) = 4\left(-\frac{x^2}{2} + K\right)$

$\Rightarrow y^2(\ln y^2 - 1) = -2x^2 + 4K$

$\Rightarrow y^2(\ln y^2 - 1) = 2(K' - x^2)$

where $K' = 2K$.

⑤ $y = x - 1 + ce^{-x}$ one const ①

Diff $\frac{dy}{dx} = 1 - 0 - ce^{-x}$

$= 1 - (y - x + 1)$ const eliminated using ①

$\frac{dy}{dx} = \cancel{1} - y + x - \cancel{1}$

$\frac{dy}{dx} = x - y$ is diff eq of given family

$\frac{dy}{dx} = \frac{-1}{x-y}$ diff eq of family of O.Ts.

$\frac{dy}{dx} = \frac{1}{y-x}$

$\frac{dy}{dx} = y - x$ taking Reciprocal

$\frac{dx}{dy} + x = y$ LDE in x

$\text{I.F} = e^{\int 1.dy} = e^{y}$

Sol is given by $\int d(xe^{y}) = \int \underset{\text{I}}{y}\,\underset{\text{II}}{e^{y}}\,dy + K$

$\Rightarrow xe^{y} = ye^{y} - \int 1.e^{y}dy + K$

$xe^{y} = ye^{y} - e^{y} + K$

$xe^{y} = e^{y}(y-1) + K$

$x = \frac{e^{y}(y-1)}{e^{y}} + \frac{K}{e^{y}}$

$x = (y-1) + K.e^{-y}$

is required family of O.Ts.

(6) $xy = c$

Diff $x\dfrac{dy}{dx} + y = 0$ const eliminated

$\dfrac{dy}{dx} = -\dfrac{y}{x}$ diff eq of given family

$\dfrac{dy}{dx} = \dfrac{x}{y}$ diff eq of family of O.Ts

$\int y\,dy = \int x\,dx$ separating variables

$\dfrac{y^2}{2} = \dfrac{x^2}{2} + K$

$y^2 = x^2 + 2K$

$y^2 = x^2 + K'$

$I.F = e^{\int 2\,dx} = e^{2x}$

sol is given by $\int d(Ve^{2x}) = \int 4xe^{2x}dx + C$

$\Rightarrow Ve^{2x} = \int e^{2x}(2x)\,2dx + C$ Put $2x = t$, $2dx = dt$

$= \int \underset{II}{e^t}\,\underset{I}{t}\,dt + c$

$= te^t - \int 1\, e^t + c$

$= te^t - e^t + c$

$Ve^{2x} = e^t(t-1) + c$

$y^2e^{2x} = e^{2x}(2x-1) + c$

$y^2 = \dfrac{\cancel{e^{2x}}}{\cancel{e^{2x}}}(2x-1) + \dfrac{c}{e^{2x}}$

$y^2 = (2x-1) + Ce^{-2x}$

is required family of O.Ts.

---

(7) $x = \dfrac{y^2}{4} + \dfrac{c}{y^2}$ ——— (1)

Diff $1 = \dfrac{2y}{4}\dfrac{dy}{dx} - \dfrac{2c}{y^3}\dfrac{dy}{dx}$

$1 = \dfrac{dy}{dx}\left(\dfrac{y}{2} - \dfrac{2c}{y^3}\right)$

$\dfrac{dy}{dx} = \dfrac{1}{\dfrac{y}{2} - \dfrac{2c}{y^3}}$

$= \dfrac{1}{\dfrac{y}{2} - \dfrac{2}{y^3}\left(y^2\left(x - \dfrac{y^2}{4}\right)\right)}$ ∵ using (1) Const eliminated

$= \dfrac{1}{\dfrac{y}{2} - \dfrac{2xy^2}{y^3} - \dfrac{2y^4}{4y^3}}$

$= \dfrac{1}{\dfrac{y}{2} - \dfrac{2x}{y} + \dfrac{y}{2}}$

$= \dfrac{1}{\dfrac{y^2 - 4x + y^2}{2y}}$

$= \dfrac{2y}{2y^2 - 4x}$

$\dfrac{dy}{dx} = \dfrac{y}{y^2 - 2x}$ diff eq of given family

$\dfrac{dy}{dx} = -\dfrac{(y^2 - 2x)}{y}$

$\dfrac{dy}{dx} = \dfrac{2x - y^2}{y}$

$\dfrac{dy}{dx} = \dfrac{2x}{y} - y$

$\dfrac{dy}{dx} + y = 2xy^{-1}$ Bernoulli Eq

÷ by y^{-1} $\quad y\dfrac{dy}{dx} + y^2 = 2x$

x by 2 $\quad 2y\dfrac{dy}{dx} + 2y^2 = 4x$

Put $y^2 = V$

$2y\dfrac{dy}{dx} = \dfrac{dV}{dx}$ $\quad \therefore \dfrac{dV}{dx} + 2V = 4x$ LDE in V

see above

63

$y = (x-c)^2$ ——— (1)

$\frac{dy}{dx} = 2(x-c)$

$\frac{dy}{dx} = 2(\pm\sqrt{y})$ const eliminated using (1) D.Eq of given eq

$\frac{dy}{dx} = \frac{-1}{\pm 2\sqrt{y}}$ D.Eq of O.Ts

$\pm\sqrt{y}\,dy = -\frac{dx}{2}$ separating variables

$\int \sqrt{y}\,dy = -\int \frac{dx}{2}$

$\pm \frac{2}{3} y^{3/2} = -\frac{1}{2}x + K$

$\pm \frac{2}{3} y^{3/2} = \frac{-x+2K}{2}$

$\pm \frac{4}{3} y^{3/2} = -x + 2K$

sq $\frac{16}{9} y^3 = (-x + K')^2$

$16y^3 = 9(-x+K')^2$ Required family of O.Ts.

D 2nd Method $\frac{dy}{dx} = \frac{-2xy}{y^2+x^2}$

$(y^2+x^2)dy = -2xy\,dx$

$(y^2+x^2)dy + 2xy\,dx = 0$

$M_y = 2x \qquad N_x = 2x$

$\because M_y = N_x \quad \therefore$ Exact Eq

$\therefore \int M dx + \int (\text{terms of } N \text{ free from } x)\,dy = C$

$\int 2xy\,dx + \int y^2 dy = C$

$\cancel{2}\frac{x^2}{\cancel{2}} y + \frac{y^3}{3} = C$

$\frac{3x^2y + y^3}{3} = C$

$y^3 + 3x^2y = 3C$

$y^3 + 3x^2y = C'$

(9) $y^2 = x^2 + Cx$ ——— (1)

$2y\frac{dy}{dx} = 2x + C$

$2y\frac{dy}{dx} - 2x = C$

$y^2 = x^2 + \left(2y\frac{dy}{dx} - 2x\right)x$ const eliminated using (1)

$= x^2 + 2xy\frac{dy}{dx} - 2x^2$

$y^2 + x^2 = 2xy\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{y^2+x^2}{2xy}$ diff Eq of given eq.

$\frac{dy}{dx} = \frac{-2xy}{y^2+x^2}$ D.Eq of O.Ts. Homogeneous Eq.

So Put $y = vx$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

$v + x\frac{dv}{dx} = \frac{-2xvx}{v^2x^2+x^2}$

$x\frac{dv}{dx} = -\frac{2v\cancel{x^2}}{(v^2+1)\cancel{x^2}} - v$

$x\frac{dv}{dx} = \frac{-(2v + v(v^2+1))}{(v^2+1)}$

$x\frac{dv}{dx} = \frac{-(3v+v^3)}{v^2+1}$

$\int \frac{(v^2+1)\,dv}{3v+v^3} = \int -\frac{dx}{x}$

$\frac{1}{3}\int \frac{(3v^2+3)\,dv}{v^3+3v} = -\int\frac{dx}{x}$

$\frac{1}{3}\ln(v^3+3v) = -\ln x + \ln c$

$\ln(v^3+3v)^{\frac{1}{3}} = \ln\frac{c}{x}$

Antilog $\left(\frac{y^3}{x^3} + 3\frac{y}{x}\right)^{\frac{1}{3}} = \frac{c}{x}$

Cubing $\left(\frac{y^3 + 3yx^2}{x^3}\right) = \left(\frac{c}{x}\right)^3$

$y^3 + 3x^2y = \frac{c^3}{\cancel{x^3}}\cancel{x^3}$

$y^3 + 3x^2y = K \qquad K = c^3$

(10) $x^2+y^2 = 1+2cy$ ——— (1)

Diff $2x+2y\frac{dy}{dx} = 2c\frac{dy}{dx}$

$2\left(x+y\frac{dy}{dx}\right) = 2c\frac{dy}{dx}$

$\frac{\cancel{2}}{\cancel{2}\frac{dy}{dx}}\left(x+y\frac{dy}{dx}\right) = c$

$x\cdot\frac{dx}{dy} + y\cancel{\frac{dy}{dx}}\cdot\cancel{\frac{dx}{dy}} = c$

$x^2+y^2 = 1+2\left[x\frac{dx}{dy}+y\right]y$ Const eliminated using (1)

$x^2+y^2 = 1+2xy\frac{dx}{dy}+2y^2$

$x^2-y^2-1 = 2xy\frac{dx}{dy}$

$\frac{x^2-y^2-1}{2xy} = \frac{dx}{dy}$

$\frac{dy}{dx} = \frac{2xy}{x^2-y^2-1}$ D.Eq of given family

$\frac{dy}{dx} = -\frac{(x^2-y^2-1)}{2xy}$ D.Eq of O.Ts

$\frac{dy}{dx} = \frac{1-x^2}{2xy} + \frac{y^{\cancel{2}}}{2x\cancel{y}}$

$\frac{dy}{dx} - \frac{y}{2x} = \frac{1-x^2}{2x}\cdot y^{-1}$ Bernoulli Eq

÷ by y^{-1} $\quad y\frac{dy}{dx} - \frac{y^2}{2x} = \frac{1-x^2}{2x}$

× by 2 $\quad 2y\frac{dy}{dx} - \cancel{2}\frac{y^2}{\cancel{2}x} = \cancel{2}\frac{(1-x^2)}{\cancel{2}x}$

Put $y^2=v$

$2y\frac{dy}{dx}=\frac{dv}{dx}$ $\quad\therefore \frac{dv}{dx}+\left(-\frac{1}{x}\right)v = \frac{1-x^2}{x}$ (LDE in v)

$I.F = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1} = \frac{1}{x}$ see above

(from below)

Sol is given by $\int d\left(v\frac{1}{x}\right) = \int \frac{1-x^2}{x}\cdot\frac{1}{x}dx + K$

$\Rightarrow \frac{v}{x} = \int\frac{1-x^2}{x^2}dx + K$

$= \int\frac{dx}{x^2} - \int dx + K$

$\frac{v}{x} = \frac{x^{-1}}{-1} - x + K$

$\frac{v}{x} = -\frac{1}{x} - x + K$

$\because v=y^2$ $\quad \frac{y^2}{x} = \frac{-1-x^2+Kx}{x}$

$y^2 = -1-x^2+Kx$ Ans.

$y^2 - Kx + x^2 = -1$

$y^2 - \frac{2}{2}Kx + x^2 = -1$

$y^2-2K'x+x^2=-1 \qquad K'=\frac{K}{2}$

$x^2-2K'x+K'^2-K'^2+y^2=-1$

$(x-K')^2+y^2=-1+K'^2$

$(x-K')^2+y^2 = K'^2-1$

$K'^2>1$

Eq of Required O.Ts.

65

(11) $r = a(1+\sin\theta)$ ——— (1)

Diff $\dfrac{dr}{d\theta} = a\cos\theta$

$\dfrac{dr}{d\theta} = \left(\dfrac{r}{1+\sin\theta}\right)\cos\theta$ using (1) const eliminated

$\dfrac{1+\sin\theta}{\cos\theta} = r\dfrac{d\theta}{dr}$, D.Eq of given family

$\dfrac{-\cos\theta}{1+\sin\theta} = r\dfrac{d\theta}{dr}$, D.Eq of family of OTs

$\displaystyle\int\frac{dr}{r} = -\int\frac{1+\sin\theta}{\cos\theta}\,d\theta$

$\displaystyle\int\frac{dr}{r} = -\int\frac{1}{\cos\theta}\,d\theta - \int\frac{\sin\theta\,d\theta}{\cos\theta}$

$\displaystyle\int\frac{dr}{r} = -\int\sec\theta\,d\theta - \int\tan\theta\,d\theta$

$\ln r = -\ln(\sec\theta+\tan\theta) - (-\ln\cos\theta) + \ln c$

$\ln r = \ln\left(\dfrac{c\cos\theta}{\sec\theta+\tan\theta}\right)$

$r = \dfrac{c\cos\theta}{\frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta}}$

$r = \dfrac{c\cos^2\theta}{1+\sin\theta}$

$= c(1-\sin^2\theta)$

$= c\,\dfrac{(1-\sin\theta)\cancel{(1+\sin\theta)}}{\cancel{(1+\sin\theta)}}$

$r = c(1-\sin\theta)$

is required family of O.Ts.

(12) $r^2 = a\sin 2\theta$ ——— (1)

Diff $2r\dfrac{dr}{d\theta} = 2a\cos 2\theta$

$2r\dfrac{dr}{d\theta} = 2\left(\dfrac{r^2}{\sin 2\theta}\right)\cos 2\theta$ using (1) const eliminated

$\dfrac{\cancel{2}\cancel{r}\frac{dr}{d\theta}}{\cancel{2}r^2} = \dfrac{\cos 2\theta}{\sin 2\theta}$

$\dfrac{r}{dr/d\theta} = \dfrac{\sin 2\theta}{\cos 2\theta}$ Reciprocal

$\dfrac{r\,d\theta}{dr} = \dfrac{\sin 2\theta}{\cos 2\theta}$ D.Eq of given family

$\dfrac{r\,d\theta}{dr} = -\dfrac{\cos 2\theta}{\sin 2\theta}$ D.Eq of family of OTs (Reciprocal)

$\displaystyle\int\frac{dr}{r} = -\int\frac{\sin 2\theta}{\cos 2\theta}\,d\theta$ separating variables

$\displaystyle\int\frac{dr}{r} = \frac{1}{2}\int\frac{-\sin 2\theta\,(2)\,d\theta}{\cos 2\theta}$

$\ln r = \frac{1}{2}\ln\cos 2\theta + \ln c$

$\ln r = \ln(\cos 2\theta)^{\frac{1}{2}} + \ln c$

$\ln r = \ln c\sqrt{\cos 2\theta}$

Antilog $r = c\sqrt{\cos 2\theta}$

$r^2 = c^2\cos 2\theta$

$r^2 = c'\cos 2\theta$. $\quad c^2 = c'$

is required family of O.Ts.

66

(13) $r^n = a^n \cos n\theta$ —— (1)

Diff $\quad n r^{n-1}\dfrac{dr}{d\theta} = -a^n \sin n\theta\,(n)$

$r^{n-1}\dfrac{dr}{d\theta} = -\left(\dfrac{r^n}{\cos n\theta}\right)\sin n\theta$ using (1) const eliminated

$\dfrac{r^{n-1}}{r^n}\dfrac{dr}{d\theta} = -\dfrac{\sin n\theta}{\cos n\theta}$

$\dfrac{1}{r}\dfrac{dr}{d\theta} = -\tan n\theta$

$r\dfrac{d\theta}{dr} = -\cot n\theta$ (Reciprocal) DEq of given family

$r\dfrac{d\theta}{dr} = -(-\tan n\theta)$ DEq of family of O.Ts.

$\dfrac{d\theta}{\tan n\theta} = \dfrac{dr}{r}$ separating variables

$\int\dfrac{dr}{r} = \int \cot n\theta\, d\theta$

$\int\dfrac{dr}{r} = \dfrac{1}{n}\int \cot n\theta\; n\,d\theta$

$\ln r = \dfrac{1}{n}\ln \sin(n\theta) + \ln c$

$\ln r = \ln(\sin n\theta)^{\frac{1}{n}} + \ln c$

$\ln r = \ln c(\sin n\theta)^{\frac{1}{n}}$

$r = c(\sin n\theta)^{\frac{1}{n}}$

$r^n = c^n \sin n\theta$

(14) $r = \dfrac{a}{2+\cos\theta}$ —— (1)

Diff $\quad \dfrac{dr}{d\theta} = -a(2+\cos\theta)^{-2}(-\sin\theta)$

$\dfrac{dr}{d\theta} = \dfrac{a\sin\theta}{(2+\cos\theta)^2}$

$\dfrac{dr}{d\theta} = \dfrac{[r(2+\cos\theta)]\sin\theta}{(2+\cos\theta)^2}$ using (1) Const eliminated

$\dfrac{dr}{d\theta} = \dfrac{r\sin\theta}{2+\cos\theta}$

$\dfrac{d\theta}{dr} = \dfrac{2+\cos\theta}{r\sin\theta}$

$r\dfrac{d\theta}{dr} = \dfrac{2+\cos\theta}{\sin\theta}$ D.Eq of given family

$r\dfrac{d\theta}{dr} = \dfrac{-\sin\theta}{2+\cos\theta}$ D.Eq of family of OTs

$\int\dfrac{dr}{r} = \int\dfrac{2+\cos\theta}{-\sin\theta}\,d\theta$ separating variables

$\int\dfrac{dr}{r} = -\int 2\operatorname{cosec}\theta\, d\theta - \int\cot\theta\, d\theta$

$\ln r = -2\ln(\operatorname{cosec}\theta - \cot\theta) - \ln\sin\theta + \ln C$

$\ln r = \ln(\operatorname{cosec}\theta - \cot\theta)^{-2} + \ln\sin\theta^{-1} + \ln c$

$\ln r = \ln\dfrac{c}{\sin\theta(\operatorname{cosec}\theta - \cot\theta)^2}$

$r = \dfrac{c}{\sin\theta\left(\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}\right)^2}$

$r = \dfrac{c\sin^{\cancel{2}}\theta}{\cancel{\sin\theta}(1-\cos\theta)^2}$

$r\dfrac{\sin^3\theta}{\sin^3\theta} = \dfrac{c\sin\theta}{(1-\cos\theta)^2}$

$r\sin^3\theta = \dfrac{c\sin^4\theta}{(1-\cos\theta)^2}$

$r\sin^3\theta = \dfrac{c(\sin^2\theta)^2}{(1-\cos\theta)^2}$

$r\sin^3\theta = \dfrac{c(1-\cos^2\theta)^2}{(1-\cos\theta)^2}$

$r\sin^3\theta = \dfrac{c[(1-\cos\theta)(1+\cos\theta)]^2}{(1-\cos\theta)^2}$

$r\sin^3\theta = c(1+\cos\theta)^2$ Ans

(22)

(17) $y^2 = 4cx + 4c^2$ ——— (1)

Diff $2y\frac{dy}{dx} = 4c$

$\frac{2y}{4}\frac{dy}{dx} = c$

$y^2 = 4\left(\frac{y}{2}\frac{dy}{dx}\right)x + 4\left(\frac{y}{2}\frac{dy}{dx}\right)^2$ using (1) Const eliminated

$y^2 = 2xy\frac{dy}{dx} + y^2\left(\frac{dy}{dx}\right)^2$

$y = 2x\frac{dy}{dx} + y\left(\frac{dy}{dx}\right)^2$ ——— (2) D.Eq of given family

Put $\frac{dy}{dx} = \frac{-1}{dy/dx}$ For D.Eq of family of O.Ts.

$\therefore y = 2x\left(\frac{-1}{dy/dx}\right) + y\left(\frac{-1}{dy/dx}\right)^2$

$y = -2x\frac{1}{(dy/dx)} + y\frac{1}{(dy/dx)^2}$

LCM $y\left(\frac{dy}{dx}\right)^2 = -2x\left(\frac{dy}{dx}\right) + y$

$y\left(\frac{dy}{dx}\right)^2 + 2x\left(\frac{dy}{dx}\right) = y$ same as (2)

Hence $y^2 = 4cx + 4c^2$ is self orthogonal

---

(18) $\frac{x^2}{c^2} + \frac{y^2}{c^2-1} = 1$ ——— (1)

Diff $\frac{2x}{c^2} + \frac{2y}{c^2-1}\left(\frac{dy}{dx}\right) = 0$

$\frac{2y}{c^2-1}\cdot\frac{dy}{dx} = -\frac{2x}{c^2}$

$\frac{dy}{dx} = -\frac{\cancel{2}x}{c^2}\cdot\frac{(c^2-1)}{\cancel{2}y}$

$y' = -\frac{x}{y}\frac{(c^2-1)}{c^2}$

$= -\frac{x}{y}\left(\frac{c^2}{c^2} - \frac{1}{c^2}\right)$

$y' = -\frac{x}{y}\left(1 - \frac{1}{c^2}\right)$

$\frac{yy'}{x} = -1 + \frac{1}{c^2}$

$\frac{1+yy'}{x} = \frac{1}{c^2}$

$\frac{x+yy'}{x} = \frac{1}{c^2}$

$c^2 = \frac{x}{x+yy'}$, Put in (1) to eliminate const.

$\frac{x^2}{\frac{x}{x+yy'}} + \frac{y^2}{\frac{x}{x+yy'} - 1} = 1$ using (1) Const eliminated

$x^{\cancel{2}}\left(\frac{x+yy'}{\cancel{x}}\right) + \left(\frac{y^2}{\frac{\cancel{x} - \cancel{x} - yy'}{x+yy'}}\right) = 1$

$x^2 + xyy' + y^{\cancel{2}}\left(\frac{x+yy'}{-\cancel{y}y'}\right) = 1$

$\frac{x^2y' + xyy'^2 - xy - y^2y'}{y'} = 1$

$x^2y' + xyy'^2 - xy - y^2y' = y'$

$x^2y' + xyy'^2 - xy - y^2y' - y' = 0$

$y'(x^2 + xyy' - y^2 - 1) - xy = 0$ (2) D.Eq of given family

Put $y' = -\frac{1}{y'}$ For D.Eq of family of O.Ts

$\therefore -\frac{1}{y'}\left[x^2 + xy\left(-\frac{1}{y'}\right) - y^2 - 1\right] - xy = 0$

LCM $\frac{-\left(x^2 - \frac{xy}{y'} - y^2 - 1\right) - xyy'}{y'} = 0$

LCM $\frac{-x^2y' + xy + y^2y' + y' - xyy'^2}{y'^2} = 0$

$-x^2y' + xy + y^2y' + y' - xyy'^2 = 0$

$x^2y' - xy - y^2y' - y' + xyy'^2 = 0$

$y'(x^2 + xyy' - y^2 - 1) - xy = 0$ same as (2) Hence (1) is self orthogonal.