## Linear Diff Eqs

A diffl eq of the form $\boxed{\dfrac{dy}{dx} + P_{(x)}y = Q_{(x)}}$ —① (P & Q are fn of x only)
is called a linear diffl eq, because it is linear in 'y' and $\frac{dy}{dx}$.

**To Solve** Multiply both sides of eq ① by I.F $e^{\int P dx}$ then L.H.S of ① becomes exact diffl of $y \cdot e^{\int P dx}$ i.e. $d(y e^{\int P dx})$ and then Integrating both sides

∴ Solution is given by $\boxed{\int d(y \times I.F) = \int Q \times I.F\, dx + c}$

**Similarly**

A diffl eq of the form $\boxed{\dfrac{dx}{dy} + P_{(y)}x = Q_{(y)}}$ —(ii) (P, Q are fns of y only)
is called a linear diffl eq, because it is linear in 'x' & $\frac{dx}{dy}$.

**To solve** Multiply both sides of eq (ii) by IF $e^{\int P dy}$, then LHS of (ii) becomes exact diffl of $x \cdot e^{\int P dy}$ i.e. $d(x e^{\int P dy})$ and then Integrating both sides.

∴ solution is given by $\boxed{\int d(x \times I.F) = \int Q \times I.F\, dy + c}$

## Ex 9.6

① $\dfrac{dy}{dx} + \left(\dfrac{2x+1}{x}\right)y = e^{-2x}$ LDE in y

$I.F = e^{\int P dx} = e^{\int \frac{2x+1}{x} dx} = e^{\int (2+\frac{1}{x})dx}$

$= e^{2x+\ln x} = e^{2x} e^{\ln x} = e^{2x} x$

∴ Sol is given by $\int d(y \times I.F) = \int Q \times I.F\, dx + c$

$\Rightarrow \int d(y e^{2x} x) = \int e^{-2x} \cdot e^{2x} x\, dx + c$

$\Rightarrow y e^{2x} x = \int x\, dx + c$

$\Rightarrow x y e^{2x} = \dfrac{x^2}{2} + c$

② $\dfrac{dy}{dx} + \dfrac{3}{x} y = 6x^2$

$I.F = e^{\int P dx} = e^{\int \frac{3}{x} dx} = e^{3\ln x} = e^{\ln x^3} = x^3$

∴ Sol is given by $\int d(y \times I.F) = \int Q \times I.F\, dx + c$

$\Rightarrow \int d(yx^3) = \int 6x^2 \cdot x^3\, dx + c$

$\Rightarrow yx^3 = \int 6x^5\, dx + c$

$\Rightarrow yx^3 = 6\dfrac{x^6}{6} + c$

$\Rightarrow x^3 y = x^6 + c$

(3) $\frac{dy}{dx} + \frac{y}{x\ln x} = \frac{3x^2}{\ln x}$ (LDE in y)

$I.F = e^{\int P dx} = e^{\int \frac{1}{x\ln x}dx} = e^{\int \frac{dx/x}{\ln x}}$

$I.F = e^{\ln(\ln x)} = \boxed{\ln x}$

Sol is given by $\int d(y \times I.F) = \int Q \times I.F\, dx + c$

$\Rightarrow \int d(y\ln x) = \int \frac{3x^2}{\cancel{\ln x}} \cancel{\ln x}\, dx + c$

$y\ln x = \cancel{3}\frac{x^3}{\cancel{3}} + c$

$\Rightarrow y = \frac{x^3+c}{\ln x}$

(4) $\frac{dy}{dx} + 3y = 3x^2 e^{-3x}$ (LDE in y)

$I.F = e^{\int P dx} = e^{\int 3 dx} = \boxed{e^{3x}}$

Sol is given by $\int d(y \times I.F) = \int Q \times I.F\, dx + c$

$\Rightarrow \int d(y e^{3x}) = \int 3x^2 \cancel{e^{-3x}}\, \cancel{e^{3x}}\, dx + c$

$y e^{3x} = x^3 + c$

$y = e^{-3x}(x^3+c)$

(7) $(x+1)\frac{dy}{dx} - ny = e^x (x+1)^{n+1}$

$\frac{dy}{dx} - \frac{n}{(x+1)} y = e^x (x+1)^n$ (LDE in y)

$I.F = e^{\int P dx} = e^{-\int \frac{n}{x+1} dx} = e^{-n\ln(x+1)} = e^{\ln(x+1)^{-n}}$

$I.F = (x+1)^{-n} = \boxed{\frac{1}{(x+1)^n}}$

Sol is given by $\int d(y \times I.F) = \int Q \times I.F\, dx + c$

$\Rightarrow \int d\left(y \cdot \frac{1}{(x+1)^n}\right) = \int e^x \cancel{(x+1)^n} \cdot \frac{1}{\cancel{(x+1)^n}} dx + c$

$\frac{y}{(x+1)^n} = e^x + c$

$y = (e^x + c)(x+1)^n$ Ans.

(5) $\cos^3 x \frac{dy}{dx} + y\cos x = \sin x$

$\frac{dy}{dx} + \frac{y\cos x}{\cos^3 x} = \frac{\sin x}{\cos^3 x}$

$\frac{dy}{dx} + \sec^2 x\, y = \sec^2 x \tan x$ (LDE in y)

$I.F = e^{\int P dx} = e^{\int \sec^2 x\, dx} = \boxed{e^{\tan x}}$

Sol is given by $\int d(y \times I.F) = \int Q \times I.F\, dx + c$

$\Rightarrow \int d(y \times e^{\tan x}) = \int \sec^2 x \tan x\, e^{\tan x}\, dx + c$

$\tan x = t$
$\sec^2 x\, dx = dt$

$\Rightarrow y e^{\tan x} = \int \underset{II}{e^t}\, \underset{I}{t}\, dt + c$

$= t e^t - \int 1 \cdot e^t dt + c$

$= t e^t - e^t + c$

$y e^{\tan x} = e^t (t-1) + c$

$y e^{\tan x} = e^{\tan x}(\tan x - 1) + c$

$y = (\tan x - 1) + c\, e^{-\tan x}$

(6) $x\frac{dy}{dx} + (1 + x\cot x) y = x$

$\frac{dy}{dx} + \left(\frac{1}{x} + \cot x\right) y = 1$ (LDE in y)

$I.F = e^{\int P dx} = e^{\int \left(\frac{1}{x} + \cot x\right) dx} = e^{\ln x + \ln \sin x}$

$I.F = e^{\ln(x\sin x)} = \boxed{x\sin x}$

Sol is given by $\int d(y \times I.F) = \int Q \times I.F\, dx + c$

$\Rightarrow \int d(y x\sin x) = \int \underset{I}{x}\, \underset{II}{\sin x}\, dx + c$

$y x\sin x = x(-\cos x) - \int 1 \cdot (-\cos x) dx$

$= x(-\cos x) + \int \cos x\, dx$

$y x\sin x = -x\cos x + \sin x + c$

$y = -\cot x + \frac{1}{x} + \frac{c}{x}\operatorname{cosec} x$

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(8) $(x^2+1)\frac{dy}{dx}+2xy = 4x^2$

$\frac{dy}{dx}+\left(\frac{2x}{x^2+1}\right)y = \frac{4x^2}{x^2+1}$ (LDE in y)

$I.F = e^{\int\left(\frac{2x}{x^2+1}\right)dx} = e^{\ln(x^2+1)} = \boxed{x^2+1}$

Sol is given by $\int d(y.I.F) = \int Q.I.F\,dx + C$

$\Rightarrow \int d\left(y(x^2+1)\right) = \int \frac{4x^2}{\cancel{(x^2+1)}}\cancel{(x^2+1)}\,dx + C$

$y(x^2+1) = \frac{4x^3}{3} + C$

$3y(x^2+1) = 4x^3 + c'$

(9) $x\frac{dy}{dx}+2y = \sin x$

$\frac{dy}{dx}+\frac{2}{x}y = \frac{\sin x}{x}$ (LDE in y)

$IF = e^{\int P dx} = e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = \boxed{x^2}$

Sol is given by $\int d(y.I.F) = \int Q.I.F\,dx + C$

$\Rightarrow \int d(yx^2) = \int \frac{\sin x}{x}x^2\,dx + C$

$yx^2 = \int \underset{I}{x}\,\underset{II}{\sin x}\,dx + c$

$yx^2 = x(-\cos x) - \int 1.(-\cos x)dx + C$

$y = \frac{1}{x^2}\left(-x\cos x + \sin x + c\right)$

(10) $(1+x^2)\frac{dy}{dx}+4xy = \frac{1}{(1+x^2)^2}$

$\frac{dy}{dx}+\left(\frac{4x}{1+x^2}\right)y = \frac{1}{(1+x^2)^3}$ (LDE in y)

$I.F = e^{\int\frac{4x}{1+x^2}dx} = e^{2\ln(1+x^2)} = e^{\ln(1+x^2)^2} = \boxed{(1+x^2)^2}$

Sol is given by $\int d(y.I.F) = \int Q.I.F\,dx + C$

$\Rightarrow \int d\left(y(1+x^2)^2\right) = \int \frac{1}{(1+x^2)^3}(1+x^2)^2\,dx + C$

$y(1+x^2)^2 = \int \frac{dx}{1+x^2} + C \Rightarrow y = \frac{1}{(1+x^2)^2}\left[\tan^{-1}x + c\right]$ Ans.

(11) $\frac{dy}{dx} = \frac{1}{e^y - x}$

Reciprocal

$\frac{dx}{dy} = e^y - x$

$\frac{dx}{dy} + x = e^y$ (LDE in x)

$I.F = e^{\int 1.dy} = \boxed{e^y}$

Sol is given by $\int d(x.I.F) = \int Q.I.F\,dy + C$

$\Rightarrow \int d(xe^y) = \int e^y e^y\,dy + C$

$\Rightarrow xe^y = \int e^{2y}dy + C$

$x = \frac{1}{e^y}\left(\frac{e^{2y}}{2} + c\right)$

$x = \frac{e^y}{2} + ce^{-y}$

(12) $(x+2y^3)\frac{dy}{dx} = y$

$\left(\frac{1}{x+2y^3}\right)\frac{dx}{dy} = \frac{1}{y}$

$\frac{dx}{dy} = \frac{x+2y^3}{y}$

$\frac{dx}{dy} = \frac{x}{y} + 2y^2$

$\frac{dx}{dy} - \left(\frac{1}{y}\right)x = 2y^2$ (LDE in x)

$I.F = e^{\int\left(-\frac{1}{y}\right)dy} = e^{-\ln y} = e^{\ln y^{-1}} = y^{-1} = \boxed{\frac{1}{y}}$

Sol is given by $\int d(x.I.F) = \int Q.I.F\,dy + C$

$\Rightarrow \int d\left(x.\frac{1}{y}\right) = \int 2y^2.\frac{1}{y}\,dy + C$

$\Rightarrow \frac{x}{y} = \int 2y\,dy + c$

$\Rightarrow x = y\left(2\frac{y^2}{2} + c\right)$

$\Rightarrow x = y^3 + cy$

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**Bernoulli Eq** is the diff eq of the form $\boxed{\frac{dy}{dx} + P(x)y = Q(x)y^n}$ ---- (1)

**To Solve** (1) Divide the eq (1) by $y^n$ $\Rightarrow y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x)$

(2) Multiply both sides by $(1-n)$ $\Rightarrow (1-n)y^{-n}\frac{dy}{dx} + P(x)y^{1-n}(1-n) = (1-n)Q(x)$

(3) Put $y^{1-n} = v$ $\Rightarrow$ $\frac{dv}{dx} + P(x)v(1-n) = (1-n)Q(x)$ which is L.D.Eq in $v$

& Diff $(1-n)y^{-n}\frac{dy}{dx} = \frac{dv}{dx}$ Now solve it easily as before.

**Note** The diff eq of the form $\boxed{\frac{dx}{dy} + P(y)x = Q(y)x^n}$ is also called Bernoulli Eq

(13) $x\frac{dy}{dx} + y = y^2 \ln x$

$\div$ by $x$

$\frac{dy}{dx} + \left(\frac{1}{x}\right)y = \frac{\ln x}{x}y^2$ Bernoulli Eq.

$\div$ by $y^2$

$y^{-2}\frac{dy}{dx} + \left(\frac{1}{x}\right)y^{-1} = \frac{\ln x}{x}$

$\times$ by $-1$

$-y^{-2}\frac{dy}{dx} - \frac{1}{x}y^{-1} = -\frac{\ln x}{x}$

Put $y^{-1} = v$

$-y^{-2}\frac{dy}{dx} = \frac{dv}{dx}$

$\therefore \frac{dv}{dx} + \left(\frac{-1}{x}\right)v = -\frac{\ln x}{x}$ (LDE in $v$)

$\text{I.F} = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = \boxed{x^{-1}}$

Sol is given by $\int d(vx^{-1}) = \int -\frac{\ln x}{x}x^{-1}dx + c$

$\Rightarrow \frac{v}{x} = -\int \underset{I}{\ln x} \cdot \underset{II}{x^{-2}}dx + c$

$\frac{v}{x} = -\left(\ln x\frac{x^{-1}}{-1} - \int \frac{1}{x}\frac{x^{-1}}{-1}dx\right) + c$

$\frac{y^{-1}}{x} = \frac{1}{x}\ln x - \int x^{-2}dx + c$

$\frac{1}{xy} = \frac{1}{x}\ln x - \frac{x^{-1}}{-1} + c$

$\frac{1}{y} = \frac{x\ln x}{x} + x x^{-1} + cx$

$\frac{1}{y} = \ln x + 1 + cx$

(14) $\frac{dy}{dx} + y = xy^3$ Bernoulli Eq

by $y^3$: $y^{-3}\frac{dy}{dx} + y^{-2} = x$

by $(-2)$: $-2y^{-3}\frac{dy}{dx} + (-2)y^{-2} = -2x$

Put $y^{-2} = v$

$-2y^{-3}\frac{dy}{dx} = \frac{dv}{dx}$

$\therefore \frac{dv}{dx} + (-2)v = -2x$ (LDE in v)

$I.F = e^{\int -2dx} = e^{-2x}$

Sol is given by $\int d(ve^{-2x}) = \int (-2x)e^{-2x}dx + c$

Put $-2x = t$, $-2dx = dt$, $dx = \frac{dt}{-2}$

$\Rightarrow ve^{-2x} = \int t e^{t}\frac{dt}{-2} + c$

$\Rightarrow ve^{-2x} = -\frac{1}{2}\int \underset{I}{t}\, \underset{II}{e^{t}}\, dt + c$

$\Rightarrow ve^{-2x} = -\frac{1}{2}\left(t e^{t} - \int 1.e^{t}dt\right) + c$

$\Rightarrow ve^{-2x} = -\frac{1}{2}\left(te^{t} - e^{t}\right) + c$

$\Rightarrow y^{-2}e^{-2x} = -\frac{1}{2}e^{t}(t-1) + c$

$\Rightarrow \frac{1}{y^2}e^{-2x} = -\frac{1}{2}e^{-2x}(-2x-1) + c$

$\Rightarrow \frac{1}{y^2} = -\frac{1}{2}\frac{e^{-2x}}{e^{-2x}}(-2x-1) + \frac{c}{e^{-2x}}$

$\Rightarrow \frac{1}{y^2} = -\frac{1}{2}(-2x-1) + ce^{2x}$

(15) $x\frac{dy}{dx} - 2x^2y = y\ln y$

$\frac{dy}{dx} - 2xy = \frac{y\ln y}{x}$

$\frac{1}{y}\frac{dy}{dx} - 2x = \frac{\ln y}{x}$

Put $\ln y = v$

$\frac{1}{y}\frac{dy}{dx} = \frac{dv}{dx}$

$\therefore \frac{dv}{dx} - 2x = \frac{v}{x}$

$\frac{dv}{dx} - \frac{v}{x} = 2x$ (LDE in v)

$I.F = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = \boxed{x^{-1}}$

Sol is given by $\int d(vx^{-1}) = \int 2x\, x^{-1}dx + c$

$\Rightarrow vx^{-1} = \int 2dx + c$

$\Rightarrow \frac{v}{x} = 2x + c$

$\Rightarrow \frac{\ln y}{x} = 2x + c$

$\Rightarrow \ln y = 2x^2 + cx$

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(16) $(x^2+1)\dfrac{dy}{dx}+4xy=x$ , $y(2)=1$

$$\frac{dy}{dx}+\left[\frac{4x}{x^2+1}\right]y=\frac{x}{(x^2+1)} \quad (\text{LDE in } y)$$

$$I.F = e^{\int\frac{4x}{(x^2+1)}dx}=e^{2\ln(x^2+1)}=e^{\ln(x^2+1)^2}=\boxed{(x^2+1)^2}$$

Sol is given by $\int d\left(y(x^2+1)^2\right)=\int\frac{x}{(x^2+1)}(x^2+1)^2dx+c$

$$\Rightarrow y(x^2+1)^2=\int x(x^2+1)\,dx+c$$

$$\Rightarrow y(x^2+1)^2=\int(x^3+x)\,dx+c$$

$$\Rightarrow y(x^2+1)^2=\frac{x^4}{4}+\frac{x^2}{2}+c$$

$\because y(2)=1$

$\therefore 1(25)=6+c$

$\boxed{19=c}$

$$\therefore y(x^2+1)^2=\frac{x^4}{4}+\frac{x^2}{2}+19$$

(17) $e^x\left(y-3(e^x+1)^2\right)dx+(e^x+1)dy=0$ , $y(0)=4$

$$(e^x+1)dy=-e^x\left(y-3(e^x+1)^2\right)dx$$

$$\frac{dy}{dx}=\frac{-e^x}{e^x+1}\left[y-3(e^x+1)^2\right]$$

$$\frac{dy}{dx}=-\left(\frac{e^x}{e^x+1}\right)y+\frac{3e^x(e^x+1)^2}{(e^x+1)}$$

$$\frac{dy}{dx}+\left(\frac{e^x}{e^x+1}\right)y=3e^x(e^x+1) \quad (\text{LDE in } y)$$

$$I.F=e^{\int\frac{e^x}{e^x+1}dx}=e^{\ln(e^x+1)}=\boxed{e^x+1}$$

Sol is given by $\int d\left[y(e^x+1)\right]=\int 3e^x(e^x+1)^2dx+c$

$$\Rightarrow y(e^x+1)=\cancel{3}\frac{(e^x+1)^3}{\cancel{3}}+c$$

$$\Rightarrow y=(e^x+1)^2+c$$

$\because y(0)=4$

$4=(1+1)^2+c$

$\boxed{0=c}$

$$\Rightarrow y=(e^x+1)^2 \quad \text{Ans}$$

(19) $x(2+x)\dfrac{dy}{dx}+2(1+x)y=1+3x^2$, $y(-1)=1$

$$\frac{dy}{dx}+\left[\frac{2(1+x)}{x(2+x)}\right]y=\frac{1+3x^2}{x(2+x)} \quad (\text{LDE in } y)$$

$$I.F=e^{\int\frac{2+2x}{2x+x^2}dx}=e^{\ln(2x+x^2)}=\boxed{2x+x^2}$$

Sol is given by $\int d\left(y(2x+x^2)\right)=\int\frac{1+3x^2}{x(2+x)}(2x+x^2)\,dx+c$

$$\Rightarrow y(2x+x^2)=x+\cancel{3}\frac{x^3}{\cancel{3}}+c$$

$$\Rightarrow y=\frac{x+x^3+c}{2x+x^2}$$

$\because y(-1)=1$

$\therefore 1=\dfrac{-1-1+c}{-2+1}$

$-1=-2+c$

$\boxed{1=c}$

$$\therefore y=\frac{x+x^3+1}{2x+x^2} \quad \text{Ans.}$$

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**(18)** $\dfrac{dy}{dx} + \dfrac{y}{2x} = \dfrac{x}{y^3}$ , $y(1) = 2$

$\dfrac{dy}{dx} + \dfrac{1}{2x} y = x y^{-3}$ Bernoulli Eq.

÷ by $y^{-3}$ $\quad y^3\dfrac{dy}{dx} + \dfrac{1}{2x} y^4 = x$

× by 4 $\quad 4y^3\dfrac{dy}{dx} + \dfrac{2}{x} y^4 = 4x$

Put $y^4 = v$

$4y^3\dfrac{dy}{dx} = \dfrac{dv}{dx}$

$\therefore \dfrac{dv}{dx} + \dfrac{2}{x} v = 4x$ (LDE in v)

$I.F = e^{\int \frac{2}{x} dx} = e^{2\ln x} = e^{\ln x^2} = \boxed{x^2}$

Sol is given by $\int d(vx^2) = \int (4x)x^2 dx + c$

$\Rightarrow vx^2 = \int 4x^3 dx + c$

$y^4 x^2 = \cancel{4}\dfrac{x^4}{\cancel{4}} + c$

$\because y(1) = 2$

$(2)^4 1^2 = 1^4 + c$

$16 - 1 = c$

$\boxed{15 = c}$

$y^4 x^2 = x^4 + 15$

**(20)** $\dfrac{dy}{dx} + \dfrac{3y}{x} = x^2 y^2$ , $y(1) = 2$ Bernoulli Eq

÷ by $y^2$ $\quad y^{-2}\dfrac{dy}{dx} + \dfrac{3}{x} y^{-1} = x^2$

× by (−1) $\quad -y^{-2}\dfrac{dy}{dx} - \dfrac{3}{x} y^{-1} = -x^2$

Put $y^{-1} = v$

$-y^{-2}\dfrac{dy}{dx} = \dfrac{dv}{dx}$

$\therefore \dfrac{dv}{dx} - \dfrac{3}{x} v = -x^2$ (LDE in v)

$I.F = e^{\int \frac{-3}{x} dx} = e^{-3\ln x} = e^{\ln x^{-3}} = \boxed{x^{-3}}$

Sol is given by $\int d(v x^{-3}) = \int (-x^2) x^{-3} dx + c$

$\Rightarrow v x^{-3} = -\int x^{-1} dx + c$

$\Rightarrow \dfrac{v}{x^3} = -\int \dfrac{dx}{x} + c$

$\Rightarrow \dfrac{1}{y x^3} = -\ln x + c$

$\Rightarrow \dfrac{1}{y} = x^3(\ln x^{-1} + c)$

$\because y(1) = 2$

$\therefore \dfrac{1}{2} = 1(\ln 1^{-1} + c)$

$\boxed{\dfrac{1}{2} = c}$

$\therefore \dfrac{1}{y} = x^3\left(\ln x^{-1} + \dfrac{1}{2}\right)$ Ans