$\boxed{9.3-1}$

22

Homogeneous Diff Eq (H.DE)

A differential eq of the form $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ is said to be homogeneous diff eq if both fns $f(x,y)$ & $g(x,y)$ are homogeneous of same degree.

Homogeneous Fn:-

A function $f(x,y)$ is said to be homogeneous of degree 'n', if it can be written as $f(tx,ty) = t^n f(x,y)$

e.g $f(x,y) = \sqrt{xy}$

$f(tx,ty) = \sqrt{tx\,ty} = t\sqrt{xy}$

$f(x,y)$ is homogeneous fn of degree '1'

To Solve Put $y = vx$

$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

then by method of separable variable we solve.

Ex 9.3

(1) $(x-y)dx + (x+y)dy = 0$

$x\,dx - y\,dx + x\,dy + y\,dy = 0$
Not separable.

$(x+y)\,dy = -(x-y)\,dx$

$\frac{dy}{dx} = \frac{y-x}{x+y}$ H.DE — (i)

Put $y = vx$ — (ii)

$\frac{dy}{dx} = v + x\frac{dv}{dx}$ — (iii)

using (ii) (iii) in (i)

$v + x\frac{dv}{dx} = \frac{vx - x}{x + vx}$

$x\frac{dv}{dx} = \frac{\cancel{x}(v-1)}{\cancel{x}(1+v)} - v$

$= \frac{\cancel{v} - 1 - \cancel{v} - v^2}{1+v}$

$x\frac{dv}{dx} = -\frac{(v^2+1)}{1+v}$

$\int \frac{v+1}{v^2+1}\,dv = -\int \frac{dx}{x}$

$\frac{1}{2}\int \frac{2v\,dv}{v^2+1} + \int \frac{dv}{v^2+1} = -\int \frac{dx}{x}$

$\frac{1}{2}\ln(v^2+1) + \tan^{-1} v = -\ln x + c$

$\ln(v^2+1)^{\frac{1}{2}} + \tan^{-1} v + \ln x = c$

$\ln\sqrt{\frac{y^2}{x^2}+1} + \tan^{-1}\left(\frac{y}{x}\right) + \ln x = c$

$\ln\sqrt{y^2+x^2} - \cancel{\ln\sqrt{x^2}} + \tan^{-1}\left(\frac{y}{x}\right) + \cancel{\ln x} = c$

$\ln\sqrt{y^2+x^2} + \tan^{-1}\left(\frac{y}{x}\right) = c$

23

② $(y^2+2xy)dx + x^2dy = 0$

$$x^2dy = -(y^2+2xy)dx$$

$$\frac{dy}{dx} = -\frac{(y^2+2xy)}{x^2} \quad \text{H.D.E} \text{ --- (i)}$$

$$\text{Put } y = vx \text{ --- (ii)}$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \text{ --- (iii)}$$

using (ii) & (iii) in (i)

$$v + x\frac{dv}{dx} = -\frac{(v^2x^2+2xvx)}{x^2}$$

$$x\frac{dv}{dx} = -\frac{\cancel{x^2}(v^2+2v)}{\cancel{x^2}} - v$$

$$x\frac{dv}{dx} = -(v^2+3v)$$

$$\int\frac{dv}{v^2+3v} = -\int\frac{dx}{x} \quad \text{separating variables}$$

$$\int\frac{1}{v(v+3)}dv = -\int\frac{dx}{x}$$

$$\frac{1}{3}\int\frac{3}{v(v+3)}dv = -\int\frac{dx}{x}$$

$$\frac{1}{3}\int\frac{v+3-v}{v(v+3)}dv = -\int\frac{dx}{x}$$

$$\frac{1}{3}\int\left(\frac{1}{v} - \frac{1}{v+3}\right)dv = -\int\frac{dx}{x}$$

$$\frac{1}{3}\ln v - \frac{1}{3}\ln(v+3) = -\ln x + \ln c$$

$$\ln\left[\frac{v^{\frac{1}{3}}}{(v+3)^{\frac{1}{3}}}\right] = \ln\frac{c}{x}$$

Antilog

$$\frac{v^{\frac{1}{3}}}{(v+3)^{\frac{1}{3}}} = \frac{c}{x}$$

$$x \cdot v^{\frac{1}{3}} = c(v+3)^{\frac{1}{3}}$$

$$x\left(\frac{y}{x}\right)^{\frac{1}{3}} = c\left(\frac{y}{x}+3\right)^{\frac{1}{3}}$$

$$x\frac{y^{\frac{1}{3}}}{\cancel{x^{\frac{1}{3}}}}\cdot\cancel{x^{\frac{1}{3}}} = c(y+3x)^{\frac{1}{3}}$$

$$x\,y^{\frac{1}{3}} = c(y+3x)^{\frac{1}{3}}$$

$$x^3y = c'(y+3x)$$

Q3 $(x^2-3y^2)dx + 2xydy = 0$

$$2xydy = -(x^2-3y^2)dx$$

$$\frac{dy}{dx} = \frac{3y^2-x^2}{2xy} \quad \text{H.D.E} \text{ --- (i)}$$

$$\text{Put } y = vx \text{ --- (ii)}$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \text{ --- (iii)}$$

using (ii) (iii) in (i)

$$v + x\frac{dv}{dx} = \frac{3v^2x^2-x^2}{2xvx}$$

$$x\frac{dv}{dx} = \frac{(3v^2-1)\cancel{x^2}}{2v\cancel{x^2}} - v$$

$$x\frac{dv}{dx} = \frac{3v^2-1-2v^2}{2v}$$

$$x\frac{dv}{dx} = \frac{v^2-1}{2v}$$

$$\int\frac{2v}{v^2-1}dv = \int\frac{dx}{x} \quad \text{separating variables}$$

$$\ln(v^2-1) = \ln x + \ln c$$

$$\ln\left(\frac{y^2}{x^2}-1\right) = \ln cx$$

$$\frac{y^2-x^2}{x^2} = cx$$

$$y^2-x^2 = (cx)x^2$$

24

(4) $(x^2+3y^2)dx - 2xy\,dy = 0$

$(x^2+3y^2)\,dx = 2xy\,dy$

$\frac{x^2+3y^2}{2xy} = \frac{dy}{dx}$ HDE —— (I)

Put $y = vx$ —— (II)

$\frac{dy}{dx} = v + x\frac{dv}{dx}$ —— (III)

using (II) (III) in (I)

$v + x\frac{dv}{dx} = \frac{x^2+3v^2x^2}{2xvx}$

$x\frac{dv}{dx} = \frac{\cancel{x^2}(1+3v^2)}{\cancel{x^2}2v} - v$

$x\frac{dv}{dx} = \frac{1+3v^2-2v^2}{2v}$

$x\frac{dv}{dx} = \frac{1+v^2}{2v}$

$\int\frac{2v}{1+v^2}dv = \int\frac{dx}{x}$ separating variables

$\ln(1+v^2) = \ln x + \ln c$

$\ln(1+v^2) = \ln cx$

$\left(1+\frac{y^2}{x^2}\right) = cx$

$\frac{x^2+y^2}{x^2} = cx$

$x^2+y^2 = (cx)\,x^2$

(5) $(x^2+xy+y^2)dx - x^2dy = 0$

$(x^2+xy+y^2)dx = x^2dy$

$\frac{dy}{dx} = \frac{x^2+xy+y^2}{x^2}$ HDE (I)

Put $y = vx$ —— (II)

$\frac{dy}{dx} = v + x\frac{dv}{dx}$ —— (III)

using (II) (III) in (I)

$v + x\frac{dv}{dx} = \frac{x^2+x(vx)+v^2x^2}{x^2}$

$x\frac{dv}{dx} = \frac{(1+v+v^2)\cancel{x^2}}{\cancel{x^2}} - v$

$\int\frac{dv}{1+v^2} = \int\frac{dx}{x}$ separating variables

$\tan^{-1}v = \ln x + c$

$\tan^{-1}\left(\frac{y}{x}\right) = \ln x + c$

(6) $(x^2+3xy+y^2)dx - x^2dy = 0$

$(x^2+3xy+y^2)dx = x^2dy$

$\frac{dy}{dx} = \frac{x^2+3xy+y^2}{x^2}$ HDE —— (I)

Put $y = vx$ —— (II)

$\frac{dy}{dx} = v + x\frac{dv}{dx}$ —— (III)

using (II) (III) in (I)

$v + x\frac{dv}{dx} = \frac{x^2+3xvx+v^2x^2}{x^2}$

$x\frac{dv}{dx} = \frac{\cancel{x^2}(1+3v+v^2)}{\cancel{x^2}} - v$

$x\frac{dv}{dx} = 1+2v+v^2$

$\int\frac{dv}{(1+v)^2} = \int\frac{dx}{x}$ separating variables

$-\frac{1}{(v+1)} = \ln x + c$

$\frac{-1}{\left(\frac{y}{x}+1\right)} = \ln x + c$

$\frac{-1}{\frac{y+x}{x}} = \ln x + c$

$\frac{-x}{(x+y)} = \ln x + c$

(7) $\dfrac{dy}{dx} = \dfrac{4y-3x}{2x-y}$ HDE —— (I)

Put $y = vx$ —— (II)

$\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ —— (III)

using (II) (III) in (I)

$v + x\dfrac{dv}{dx} = \dfrac{4vx-3x}{2x-vx}$

$x\dfrac{dv}{dx} = \dfrac{\cancel{x}(4v-3)}{\cancel{x}(2-v)} - v$

$x\dfrac{dv}{dx} = \dfrac{4v-3-2v+v^2}{2-v}$

$\displaystyle\int \frac{2-v}{v^2+2v-3}\,dv = \int \frac{dx}{x}$ —— (IV) separating variables

By Partial Fractions

$\dfrac{2-v}{(v^2+3)(v-1)} = \dfrac{A}{v+3} + \dfrac{B}{v-1}$

$2-v = A(v-1) + B(v+3)$

Put $v+3=0 \Rightarrow +5 = -4A \Rightarrow \boxed{A = \frac{-5}{4}}$

Put $v-1=0 \Rightarrow 1 = 4B \Rightarrow \boxed{B = \frac{1}{4}}$

$\therefore \dfrac{2-v}{(v+3)(v-1)} = \dfrac{-5}{4(v+3)} + \dfrac{1}{4(v-1)}$

from (IV)

$\dfrac{-5}{4}\displaystyle\int\frac{dv}{v+3} + \frac{1}{4}\int\frac{dv}{v-1} = \int\frac{dx}{x}$

$\dfrac{-5}{4}\ln(v+3) + \dfrac{1}{4}\ln(v-1) = \ln x + \ln c$

$-\ln(v+3)^5 + \ln(v-1) = 4\ln cx$

$\ln\dfrac{(v-1)}{(v+3)^5} = \ln c^4x^4$

Antilog

$\dfrac{(\frac{y}{x}-1)}{(\frac{y}{x}+3)^5} = c^4x^4$

$\dfrac{(y-x)x^5}{x(y+3x)^5 x^4} = c'$

$\dfrac{(y-x)}{(y+3x)^5} = c'$ Ans.

(8) $x\sin\left(\frac{y}{x}\right)dy = \left(y\sin\frac{y}{x} - x\right)dx$

$\dfrac{dy}{dx} = \dfrac{y\sin\frac{y}{x} - x}{x\sin(\frac{y}{x})}$ HDE —— (I)

Put $y = vx$ —— (II)

$\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ —— (III)

using (II) (III) in (I)

$v + x\dfrac{dv}{dx} = \dfrac{vx\sin\frac{vx}{x} - x}{x\sin(\frac{vx}{x})}$

$x\dfrac{dv}{dx} = \dfrac{\cancel{x}(v\sin v - 1)}{\cancel{x}\sin v} - v$

$x\dfrac{dv}{dx} = \dfrac{v\cancel{\sin}v - 1 - v\cancel{\sin v}}{\sin v}$

$\displaystyle\int \sin v\,dv = \int -\frac{dx}{x}$ separating variables

$-\cos v = -\ln x + c$

$\cos v = \ln x - c$

$\cos\dfrac{y}{x} = \ln x - c$

Available at www.mathcity.org

(9) $(x^3+y^2\sqrt{x^2+y^2})\,dx - xy\sqrt{x^2+y^2}\,dy = 0$

$$x^3+y^2\sqrt{x^2+y^2}\,dx = xy\sqrt{x^2+y^2}\,dy$$

$$\frac{dy}{dx} = \frac{x^3+y^2\sqrt{x^2+y^2}}{xy\sqrt{x^2+y^2}} \quad \text{HDE} \text{ ——— (i)}$$

$$\left\{\frac{dy}{dx} = \frac{t^3x^3+t^2y^2\sqrt{t^2x^2+t^2y^2}}{tx\,ty\sqrt{t^2x^2+t^2y^2}}\right.$$

$$= \frac{t^3x^3+y^2t^3\sqrt{x^2+y^2}}{t^2xy\,t\sqrt{x^2+y^2}}$$

$$= \frac{t^3(x^3+y^2\sqrt{x^2+y^2})}{t^3(xy\sqrt{x^2+y^2})}$$

Degree of N & D is same '3'
So Homogeneous.

Put $y = vx$ ——— (ii)

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \text{ ——— (iii)}$$

using (ii) & (iii) in (i)

$$v + x\frac{dv}{dx} = \frac{x^3+v^2x^2\sqrt{x^2+v^2x^2}}{xvx\sqrt{x^2+v^2x^2}}$$

$$= \frac{\cancel{x^3}(1+v^2\sqrt{1+v^2})}{\cancel{x^3}v\sqrt{1+v^2}} - v$$

$$x\frac{dv}{dx} = \frac{1+v^2\cancel{\sqrt{1+v^2}} - v^2\cancel{\sqrt{1+v^2}}}{v\sqrt{1+v^2}}$$

$$x\frac{dv}{dx} = \frac{1}{v\sqrt{1+v^2}}$$

$$\int v\sqrt{1+v^2}\,dv = \int\frac{dx}{x} \quad \text{separating variables}$$

$$\frac{1}{2}\int\sqrt{1+v^2}\,(2v)\,dv = \int\frac{dx}{x}$$

$$\frac{1}{2}\,\frac{(1+v^2)^{3/2}}{3/2} = \ln x + c$$

$$\frac{1}{\cancel{2}}\,\frac{\cancel{2}}{3}(1+v^2)^{3/2}$$

$$\left(1+\frac{y^2}{x^2}\right)^{3/2} = 3\ln x + 3c$$

$$\left(\frac{x^2+y^2}{x^2}\right)^{3/2} = \ln x^3 + c'$$

$$\frac{(x^2+y^2)^{3/2}}{x^3} = \ln x^3 + c'$$

$$(x^2+y^2)^{3/2} = x^3\ln x^3 + c'x^3$$

27

Imp

(10) $\left(\sqrt{x+y}+\sqrt{x-y}\right)dx-\left(\sqrt{x+y}-\sqrt{x-y}\right)dy=0$

$$\left(\sqrt{x+y}+\sqrt{x-y}\right)dx=\left(\sqrt{x+y}-\sqrt{x-y}\right)dy$$

$$\frac{\sqrt{x+y}+\sqrt{x-y}}{\sqrt{x+y}-\sqrt{x-y}}=\frac{dy}{dx} \qquad \text{HDE} \quad \text{(I)}$$

Put $y = vx$ (II)

$$\frac{dy}{dx}=v+x\frac{dv}{dx} \quad \text{(III)}$$

using (II) (III) in (I)

$$v+x\frac{dv}{dx}=\frac{\sqrt{x+vx}+\sqrt{x-vx}}{\sqrt{x+vx}-\sqrt{x-vx}}$$

$$v+x\frac{dv}{dx}=\frac{\cancel{\sqrt{x}}}{\cancel{\sqrt{x}}}\left(\frac{\sqrt{1+v}+\sqrt{1-v}}{\sqrt{1+v}-\sqrt{1-v}}\right)$$

Rationalize

$$v+x\frac{dv}{dx}=\frac{\sqrt{1+v}+\sqrt{1-v}}{\sqrt{1+v}-\sqrt{1-v}}\times\frac{\sqrt{1+v}+\sqrt{1-v}}{\sqrt{1+v}+\sqrt{1-v}}$$

$$=\frac{(1+v)+(1-v)+2\sqrt{1+v}\sqrt{1-v}}{(1+v)-(1-v)}$$

$$=\frac{2+2\sqrt{1-v^2}}{2v}$$

$$x\frac{dv}{dx}=\frac{1+\sqrt{1-v^2}}{v}-v$$

$$x\frac{dv}{dx}=\frac{1+\sqrt{1-v^2}-v^2}{v}$$

$$\int\frac{v\,dv}{(1-v^2)+\sqrt{1-v^2}}=\int\frac{dx}{x} \quad \text{(IV)}$$

separate variables

$$\int\frac{v\,dv}{(1-v^2)+\sqrt{1-v^2}}$$

Put $v=\sin x$
$dv=\cos x\,dx$

$$\int\frac{\sin x\cos x\,dx}{(1-\sin^2 x)+\sqrt{1-\sin^2 x}}$$

$$\int\frac{\sin x\cos x\,dx}{\cos^2 x+\cos x}$$

$$=\int\frac{\sin x\cancel{\cos x}\,dx}{\cancel{\cos x}(\cos x+1)}$$

$$=-\int\frac{-\sin x\,dx}{(\cos x+1)}$$

$$=-\ln(\cos x+1) \quad \text{Put in (IV)}$$

$$-\ln(\cos x+1)=\ln x+\ln c$$

$$-\ln\left(\sqrt{1-\sin^2 x}+1\right)=\ln cx$$

$$-\ln\left(\sqrt{1-v^2}+1\right)=\ln cx$$

$$-\ln\left(\sqrt{1-\frac{y^2}{x^2}}+1\right)=\ln cx$$

$$-\ln\left(\frac{\sqrt{x^2-y^2}}{x}+1\right)=\ln(cx)$$

$$\ln\left(\frac{\sqrt{x^2-y^2}+x}{x}\right)^{-1}=\ln(cx)$$

Antilog

$$\frac{\cancel{x}}{\sqrt{x^2-y^2}+x}=c\cancel{x}$$

$$c'=\sqrt{x^2-y^2}+x$$

where $c'=\frac{1}{c}$

28

(11) $\dfrac{dy}{dx} = \dfrac{x+y}{x}$ ——— (i) , $y(1)=1$

Put $y = vx$ ——— (ii)

$\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ ——— (iii)

using (ii) (iii) in (i)

$v + x\dfrac{dv}{dx} = \dfrac{x+vx}{x}$

$x\dfrac{dv}{dx} = \dfrac{\cancel{x}(1+v)}{\cancel{x}} - v$

$x\dfrac{dv}{dx} = 1$

$\int dv = \int \dfrac{dx}{x}$ separating variables

$v = \ln x + c$

$\dfrac{y}{x} = \ln x + c$

$\because y(1)=1$

$\therefore \dfrac{1}{1} = \ln 1 + c$

$\boxed{1 = 0+c}$

So $\dfrac{y}{x} = \ln x + 1$

$y = x\ln x + x = x[\ln x + 1]$ Ans

(12) $(y + \sqrt{x^2+y^2})dx - x\,dy = 0$, $y(1)=0$

$\because$ $(y + \sqrt{x^2+y^2})dx = x\,dy$

$\dfrac{dy}{dx} = \dfrac{y+\sqrt{x^2+y^2}}{x}$ HDE ——— (i)

Put $y = vx$ ——— (ii)

$\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ ——— (iii)

using (ii) (iii) in (i)

$v + x\dfrac{dv}{dx} = \dfrac{vx + \sqrt{x^2+v^2x^2}}{x}$

$= \dfrac{x(v+\sqrt{1+v^2})}{x}$

$x\dfrac{dv}{dx} = \cancel{v} + \sqrt{1+v^2} - \cancel{v}$

$\int \dfrac{dv}{\sqrt{1+v^2}} = \int \dfrac{dx}{x}$ separating variables

$\sinh^{-1} v = \ln x + c$

$\sinh^{-1}\left(\dfrac{y}{x}\right) = \ln x + c$

$\because y(1)=0$

$\therefore \sinh^{-1}\left(\dfrac{0}{1}\right) = \ln 1 + c$

$\boxed{0 = c}$

$\therefore \sinh^{-1}\left(\dfrac{y}{x}\right) = \ln x$

$\because \sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$

$\Rightarrow \ln\left(\dfrac{y}{x} + \sqrt{1+\dfrac{y^2}{x^2}}\right) = \ln x$

antilog

$\Rightarrow \dfrac{y + \sqrt{x^2+y^2}}{x} = x$

$\Rightarrow y + \sqrt{x^2+y^2} = x^2$

$y = x^2 - \sqrt{x^2+y^2}$

29

(13) $(2x-5y)dx + (4x-y)dy = 0 \quad , \quad y(1) = 4$

$$(4x-y)dy = -(2x-5y)dx$$

$$\frac{dy}{dx} = \frac{5y-2x}{4x-y} \quad \text{HDE} \quad \text{(i)}$$

$$\text{Put } y = vx \quad \text{(ii)}$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \quad \text{(iii)}$$

using (ii) (iii) in (i)

$$v + x\frac{dv}{dx} = \frac{5vx-2x}{4x-vx}$$

$$= \frac{\cancel{x}(5v-2)}{\cancel{x}(4-v)} - v$$

$$= \frac{5v-2-4v+v^2}{4-v}$$

$$x\frac{dv}{dx} = \frac{v^2+v-2}{4-v}$$

$$\int \frac{4-v}{v^2+v-2}\,dv = \int \frac{dx}{x} \quad \text{separating variables.}$$

By Partial Fractions

$$\int \frac{(4-v)\,dv}{(v-1)(v+2)} = \int \frac{dx}{x}$$

$$\int \left[\frac{dv}{v-1} - \frac{2}{(v+2)}\right]dv = \int \frac{dx}{x}$$

$$\Rightarrow \ln(v-1) - 2\ln(v+2) = \ln x + \ln c$$

$$\Rightarrow \ln\left[\frac{(v-1)}{(v+2)^2}\right] = \ln cx$$

Antilog

$$\Rightarrow \frac{\left(\frac{y}{x}-1\right)}{\left(\frac{y}{x}+2\right)^2} = cx$$

$$\Rightarrow \frac{y-x}{\cancel{x}\frac{(y+2x)^2}{\cancel{x^2}}} \cdot \frac{1}{\cancel{x}} = c$$

$$\because y(1) = 4$$

$$\therefore \frac{4-1}{(4+2)^2} = c$$

$$c = \frac{3}{36} = \frac{1}{12}$$

$$\Rightarrow \frac{y-x}{(y+2x)^2} = \frac{1}{12} \quad \Rightarrow \quad 12(y-x) = (y+2x)^2$$

Partial Fractions.

$$\frac{4-v}{(v-1)(v+2)} = \frac{A}{v-1} + \frac{B}{v+2}$$

$$4-v = A(v+2) + B(v-1)$$

$$v+2=0 \Rightarrow v=-2$$
$$\Rightarrow \boxed{B=-2}$$

$$v-1=0 \Rightarrow v=1$$
$$\boxed{A=1}$$

$$\text{So } \frac{4-v}{(v-1)(v+2)} = \frac{1}{v-1} - \frac{2}{v+2}$$

(14) $(3x^2+9xy+5y^2)dx - (6x^2+4xy)dy = 0$, $y(2)=-6$

$(6x^2+4xy)dy = (3x^2+9xy+5y^2)dx$

$\frac{dy}{dx} = \frac{3x^2+9xy+5y^2}{6x^2+4xy}$ HDE (I)

Put $y = vx$ ——— (II)

$\frac{dy}{dx} = v + x\frac{dv}{dx}$ ——— (III)

using (II) (III) in (I)

$v + x\frac{dv}{dx} = \frac{3x^2+9xvx+5v^2x^2}{6x^2+4xvx}$

$x\frac{dv}{dx} = \frac{\cancel{x^2}(3+9v+5v^2)}{\cancel{x^2}(6+4v)} - v$

$x\frac{dv}{dx} = \frac{3+9v+5v^2-6v-4v^2}{6+4v}$

$x\frac{dv}{dx} = \frac{v^2+3v+3}{6+4v}$

$\int \frac{(4v+6)dv}{v^2+3v+3} = \int \frac{dx}{x}$ separating variables

$2\int \frac{(2v+3)dv}{v^2+3v+3} = \int \frac{dx}{x}$

$2\ln(v^2+3v+3) = \ln x + \ln c$

$\ln(v^2+3v+3)^2 = \ln cx$

$(v^2+3v+3)^2 = cx$

$\left(\frac{y^2}{x^2}+3\frac{y}{x}+3\right)^2 = cx$

$\left(\frac{y^2+3yx+3x^2}{x^2}\right)^2 = cx$

$\frac{y^2+3yx+3x^2}{x^2} = \sqrt{cx}$

$y^2+3yx+3x^2 = x^2\sqrt{cx}$

$\because y(2) = -6$

$\therefore 36-36+12 = 4\sqrt{2c}$

$\frac{12}{4} = \sqrt{2c}$

$(3)^2 = 2c \Rightarrow \boxed{c = \frac{9}{2}}$

Therefore

$3x^2+3xy+y^2 = x^2\sqrt{\frac{9}{2}x}$

$3x^2+3xy+y^2 = \frac{3}{\sqrt{2}}x^{2+\frac{1}{2}}$

$\sqrt{2}(3x^2+3xy+y^2) = 3x^{5/2}$

$2(3x^2+3xy+y^2)^2 = 9x^5$

Non Homogeneous Diff Eq :- is of the form $\dfrac{dy}{dx} = \dfrac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$

Non Homogeneous Diff Eq are of two types.

Type 1 if $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$ To solve Put $\left.\begin{aligned} x &= X+h \\ y &= Y+K \end{aligned}\right] \Rightarrow \begin{aligned} dx &= dX \\ dy &= dY \end{aligned}$ so $\dfrac{dy}{dx} = \dfrac{dY}{dX}$

find values of h & K.

Type 2 if $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2}$ To Solve Put $z = a_1x+b_1y$ and solve by separable variable in x & z

(15) $\dfrac{dy}{dx} = \dfrac{x+3y-5}{x-y-1}$ ... NHDEq $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$ $\because \dfrac{1}{1} \neq \dfrac{3}{-1}$

Put $\left.\begin{aligned} x &= X+h \\ y &= Y+K \end{aligned}\right] \Rightarrow \dfrac{dy}{dx} = \dfrac{dY}{dX}$

$$\frac{dY}{dX} = \frac{X+h+3(Y+K)-5}{X+h-(Y+K)-1}$$

where $h+3K-5=0$, & $h-K-1=0$

Subtract

$$\begin{aligned} h+3K-5&=0 \\ h-K-1&=0 \\ \hline 4K-4&=0 \\ \boxed{K=1} \end{aligned}$$

Add

$$\begin{aligned} h+3K-5&=0 \\ 3h-3K-3&=0 \\ \hline 4h-8&=0 \\ \boxed{h=2} \end{aligned}$$

$x = X+2$
$y = Y+1$

$$\frac{dY}{dX} = \frac{X+3Y}{X-Y} \quad \text{HDE} \quad (i)$$

Put $Y = VX$ ——— (ii)

$$\frac{dY}{dX} = V+X\frac{dV}{dX} \quad \text{———} \quad (iii)$$

using (ii) (iii) in (i)

$$V+X\frac{dV}{dX} = \frac{X+3VX}{X-VX}$$

$$X\frac{dV}{dX} = \frac{\cancel{X}(1+3V)}{\cancel{X}(1-V)} - V$$

$$X\frac{dV}{dX} = \frac{1+3V-V+V^2}{1-V}$$

$$X\frac{dV}{dX} = \frac{(1+V)^2}{1-V}$$

$$\int\frac{1-V}{(1+V)^2}dV = \int\frac{dX}{X} \quad \text{separating variables}$$

$$\int\frac{1\,dV}{(1+V)^2} - \int\frac{V\,dV}{(1+V)^2} = \int\frac{dX}{X}$$

$$\int(1+V)^{-2}dV - \int\frac{1+V-1}{(1+V)^2}dV = \int\frac{dX}{X}$$

$$\Rightarrow \int(1+V)^{-2}dV - \int\frac{1}{1+V}dV + \int\frac{1\,dV}{(1+V)^2} = \int\frac{dX}{X}$$

$$\Rightarrow \frac{-1}{(1+V)} - \ln(1+V) - \frac{1}{(1+V)} = \ln X + \ln c$$

$$\Rightarrow \frac{-2}{1+V} = \ln(1+V) + \ln X + \ln c$$

$$\Rightarrow \frac{-2}{1+V} = \ln cX(1+V)$$

$$\Rightarrow \frac{-2}{1+\frac{Y}{X}} = \ln cX\left(1+\frac{Y}{\cancel{X}}\right)$$

$$\Rightarrow \frac{-2X}{X+Y} = \ln c(X+Y)$$

$$\Rightarrow \frac{-2(x-2)}{x-2+y-1} = \ln c(x-2+y-1)$$

$$\Rightarrow \frac{-2x+4}{x+y-3} = \ln c'(x+y-3) \quad \text{Ans.}$$

(16) $\dfrac{dy}{dx} = -\dfrac{(4x+3y+15)}{2x+y+7}$ NHDEq $\qquad \dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2} \because \dfrac{4}{2} \neq \dfrac{3}{1}$

Put $\left.\begin{array}{l} x = X+h \\ y = Y+K \end{array}\right\} \Rightarrow \dfrac{dy}{dx} = \dfrac{dY}{dX}$

$\therefore \dfrac{dY}{dX} = -\dfrac{(4X+4h+3Y+3K+15)}{2X+2h+Y+K+7}$

where $-4h-3K-15=0$
& $2h+K+7=0$

Add
$-4h-3K-15=0$
$4h+2K+14=0$
$-K-1=0$
$\boxed{-1=K}$

$-4h-3(-1)-15=0$
$-12=4h$
$\boxed{-3=h}$

$x = X-3$
$y = Y-1$

$\dfrac{dY}{dX} = -\dfrac{4X-3Y}{2X+Y}$ —— HDE (I)

Put $Y = VX$ —— (II)

$\dfrac{dY}{dX} = V + X\dfrac{dV}{dX}$ —— (III)

using (II) (III) in (I)

$V + X\dfrac{dV}{dX} = -\dfrac{4X-3VX}{2X+VX}$

$X\dfrac{dV}{dX} = -\dfrac{\cancel{X}(4+3V)}{\cancel{X}(2+V)} - V$

$= -\dfrac{4-3V-2V-V^2}{2+V}$

$X\dfrac{dV}{dX} = -\left(\dfrac{V^2+5V+4}{2+V}\right)$

$\displaystyle\int \frac{(V+2)\,dV}{V^2+5V+4} = -\int \frac{dX}{X}$ separating variables.

Partial Fractions

$\dfrac{V+2}{V^2+5V+4} = \dfrac{V+2}{(V+1)(V+4)} = \dfrac{A}{V+1} + \dfrac{B}{V+4}$

$V+2 = A(V+4) + B(V+1)$

Put $V+4=0 \quad V=-4 \Rightarrow \boxed{B=\frac{2}{3}}$

$V+1=0 \quad V=-1 \Rightarrow \boxed{A=\frac{1}{3}}$

$\therefore \dfrac{V+2}{(V+1)(V+4)} = \dfrac{1}{3(V+1)} + \dfrac{2}{3(V+4)}$

By P.F

$\dfrac{1}{3}\displaystyle\int \frac{dV}{(V+1)} + \frac{2}{3}\int \frac{dV}{V+4} = -\int \frac{dX}{X}$

$\dfrac{1}{3}\ln(V+1) + \dfrac{2}{3}\ln(V+4) = -\ln X + \ln c$

$\dfrac{1}{3}\ln\left[(V+1)(V+4)^2\right] = \ln\left(\dfrac{c}{X}\right)$

$\ln\left[\left(\dfrac{Y}{X}+1\right)\left(\dfrac{Y}{X}+4\right)^2\right]^{\frac{1}{3}} = \ln\left(\dfrac{c}{X}\right)$

Antilog
Cube $\left(\dfrac{Y+X}{X}\right)\left(\dfrac{Y+4X}{X}\right)^2 = \dfrac{c^3}{X^3}$

$(Y+X)(Y+4X)^2 \dfrac{\cancel{X^3}}{\cancel{X^3}} = c'$

$(y+1+x+3)(y+1+4x+12) = c' \Rightarrow (x+y+4)(4x+y+13) = c'$ Ans

(17) $(3y-7x-3)dx+(7y-3x-7)dy=0$

$(7y-3x-7)dy = -(3y-7x-3)dx$

$\dfrac{dy}{dx} = \dfrac{-3y+7x+3}{7y-3x-7}$ NHDEq

$\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2} \because \dfrac{7}{-3} \neq \dfrac{-3}{7}$

Put $\left.\begin{aligned} x &= X+h \\ y &= Y+K \end{aligned}\right\} \Rightarrow \dfrac{dy}{dx} = \dfrac{dY}{dX}$

where $-3K+7h+3=0$
& $7K-3h-7=0$

Add $\cancel{-21K}+49h+21=0$
$\cancel{21K}-9h-21=0$

$40h=0$

$\boxed{h=0}$

$-3K+3=0$

$\boxed{K=\dfrac{-3}{-3}=1}$

$x = X+0$
$y = Y+1$

$\therefore \dfrac{dY}{dX} = \dfrac{-3(Y+K)+7(X+h)+3}{7(Y+K)-3(X+h)-7}$

$\dfrac{dY}{dX} = \dfrac{-3Y+7X}{7Y-3X}$ HDG ——— (i)

Put $Y = VX$ ——— (ii)

$\dfrac{dY}{dX} = V+X\dfrac{dV}{dX}$ ——— (iii)

using (ii) & (iii) in (i)

$V+X\dfrac{dV}{dX} = \dfrac{-3VX+7X}{7VX-3X}$

$X\dfrac{dV}{dX} = \dfrac{\cancel{X}(-3V+7)}{\cancel{X}(7V-3)} - V$

$X\dfrac{dV}{dX} = \dfrac{-\cancel{3V}+7-7V^2+\cancel{3V}}{7V-3}$

$\displaystyle\int \frac{7V-3}{7(1-V^2)}\,dV = \int \frac{dX}{X}$ separating variables

Partial Fractions

$\dfrac{7V-3}{7(1-V^2)} = \dfrac{V-\frac{3}{7}}{(1-V)(1+V)} = \dfrac{A}{1-V} + \dfrac{B}{1+V}$

$V-\dfrac{3}{7} = A(1+V)+B(1-V)$

Put $1+V=0 \quad \Rightarrow -1-\dfrac{3}{7} = 2B \quad \boxed{B=-\dfrac{5}{7}}$

Put $1-V=0 \quad \Rightarrow 1-\dfrac{3}{7} = 2A \quad \boxed{A=\dfrac{2}{7}}$

By PF $\displaystyle \frac{2}{7}\int \frac{dV}{1-V} - \frac{5}{7}\int \frac{dV}{1+V} = \int \frac{dX}{X}$

$-\dfrac{2}{7}\ln(1-V) - \dfrac{5}{7}\ln(1+V) = \ln X + \ln c$

$\dfrac{1}{7}\ln\left[(1-V)^{-2}(1+V)^{-5}\right] = \ln cX$

$\ln\left[\left(1-\dfrac{Y}{X}\right)^{-2}\left(1+\dfrac{Y}{X}\right)^{-5}\right] = 7\ln cX$

$\ln\left[\left(\dfrac{X-Y}{X}\right)^{-2}\left(\dfrac{X+Y}{X}\right)^{-5}\right] = \ln c^7X^7$

Antilog

$\dfrac{X^2}{(X-Y)^2}\,\dfrac{X^5}{(X+Y)^5} = c^7X^7$

$\dfrac{\cancel{X^7}}{\cancel{X^7}c^7} = (X-Y)^2(X+Y)^5$

$c' = (X-Y)^2(X+Y)^5$

$\therefore c' = (x-(y-1))^2(x+y-1)^5$

$c' = (x-y+1)^2(x+y-1)^5$

$\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \because \frac{1}{2} \neq \frac{-2}{1}$

(20) $\frac{dy}{dx} = \frac{x-2y+5}{2x+y-1}$ NHDEq

Put $\left.\begin{array}{l} x = X+h \\ y = Y+K \end{array}\right] \Rightarrow \frac{dy}{dx} = \frac{dY}{dX}$

$\therefore \frac{dY}{dX} = \frac{X+h-2Y-2K+5}{2X+2h+Y+K-1}$.

where $h-2K+5=0$
& $2h+K-1=0$

Add
$2h-4K+10=0$
$\cancel{2h}+K-1=0$
$-5K+11=0$
$\boxed{K=+\frac{11}{5}}$

$h-2K+5=0$
$+4h+2K-2=0$
$5h+3=0$
$\boxed{h=\frac{-3}{5}}$

$x = X-\frac{3}{5}$ $\quad y = Y+\frac{11}{5}$

$\frac{dY}{dX} = \frac{X-2Y}{2X+Y}$ HDE ——— (I)

Put $Y = VX$ ——— (II)

$\frac{dY}{dX} = V + X\frac{dV}{dX}$ ——— (III)

using (II) (III) in (I)

$V + X\frac{dV}{dX} = \frac{X-2VX}{2X+VX}$

$X\frac{dV}{dX} = \frac{\cancel{X}(1-2V)}{\cancel{X}(2+V)} - V$

$X\frac{dV}{dX} = \frac{1-2V-2V-V^2}{2+V}$

$\frac{2+V}{1-4V-V^2}\,dV = \frac{dX}{X}$

$\int \frac{2+V}{V^2+4V-1}\,dV = -\int \frac{dX}{X}$

$\frac{1}{2}\left[\ln(V^2+4V-1)\right] = -\ln X + \ln c$

$\ln\sqrt{V^2+4V-1} = \ln\frac{c}{X}$

Antilog $\sqrt{\frac{Y^2}{X^2}+\frac{4Y}{X}-1} = \frac{c}{X}$

$\frac{Y^2+4XY-X^2}{X^2} = \frac{c^2}{X^2}$

$Y^2+4XY-X^2 = c^2$

$(y-\frac{11}{5})^2 + 4(x+\frac{3}{5})(y-\frac{11}{5}) - (x+\frac{3}{5})^2 = c^2$

Type 2 $\quad \frac{a_1}{a_2} = \frac{b_1}{b_2} \quad \because \frac{3}{3} = \frac{-4}{-4}, \quad 1 = 1$

(18) $\frac{dy}{dx} = \frac{3x-4y-2}{3x-4y-3}$ NHDE ——— (i)

Put $3x-4y = z$ ——— (ii)

$3 - 4\frac{dy}{dx} = \frac{dz}{dx}$

$3 - \frac{dz}{dx} = 4\frac{dy}{dx}$

$\frac{1}{4}\left(3 - \frac{dz}{dx}\right) = \frac{dy}{dx}$ ——— (iii)

using (ii) & (iii) in (i)

$\frac{1}{4}\left(3 - \frac{dz}{dx}\right) = \frac{z-2}{z-3}$

$3 - \frac{dz}{dx} = \frac{4z-8}{z-3}$

$3 - \frac{(4z-8)}{z-3} = \frac{dz}{dx}$

$\frac{3z-9-4z+8}{z-3} = \frac{dz}{dx}$

$-\left(\frac{1+z}{z-3}\right) = \frac{dz}{dx}$

$-dx = \frac{(z-3)}{1+z}dz$ separately variables

$\frac{z-3-1+1}{1+z} = -dx$

$\int \frac{(z+1)-4}{1+z}dz = -\int dx$

$\int \left(1 - \frac{4}{1+z}\right)dz = -\int dx$

$z - 4\ln(1+z) = -x + c$

$(3x-4y) - 4\ln(1+3x-4y) = -x + c$

$4x - 4y - 4\ln(1+3x-4y) = c$

$x - y - \ln(1+3x-4y) = \frac{c}{4}$

$x - y - \ln(1+3x-4y) = c'$

Type 2 $\frac{1}{1} = \frac{-1}{-1}$

(19) $\frac{dy}{dx} = \frac{y-x+1}{y-x+5}$ ——— (i)

Put $y - x = z$ ——— (ii)

$\frac{dy}{dx} - 1 = \frac{dz}{dx}$

$\frac{dy}{dx} = 1 + \frac{dz}{dx}$ ——— (iii)

using (ii) & (iii) in (i)

$1 + \frac{dz}{dx} = \frac{z+1}{z+5}$

$\frac{dz}{dx} = \frac{z+1}{z+5} - 1$

$= \frac{z+1-z-5}{z+5}$

$\frac{dz}{dx} = \frac{-4}{z+5}$

$\int (z+5)dz = -4\int dx$ separately variables

$\frac{z^2}{2} + 5z = -4x + c$

$\frac{z^2 + 10z}{2} = -4x + c$

$z^2 + 10z = -8x + 2c$

$(y-x)^2 + 10(y-x) = -8x + c'$

$(y-x)^2 + 10(y-x) + 8x = c'$

$(y-x)^2 + 10y - 10x + 8x = c'$

$(y-x)^2 + 10y - 2x = c'$