Exercise # 9.9
Mathematical Methods

9 The Singular Solutions

Singular Solution

A diff. eq. $f(x,y,P)=0$ may possess a solution which does not involve any arbitrary constant and, in general, is not obtained from the general solution by giving any particular value to the arbitrary constants, is called Singular Solution.

P-discriminant

Consider the non-linear diff. eq. of first order $f(x,y,P)=0$ ——— ①

Differentiating ① partially w.r.t P, i-e $\frac{\partial f(x,y,P)}{\partial P}=0$ ——— ②

If we eliminate P from ① and ② then the eliminant eq. (resulting eq.), is called P-discriminant for eq. ①

Remark

If the eq. ① is quadratic in P i-e is of the shape $AP^2+BP+C=0$ then the P-disc. is given by $B^2-4AC=0$

C-discriminant

Consider the non-linear diff. eq. of first order, $f(x,y,P)=0$ ——— ①

Let the general solution of ①, be $\phi(x,y,c)=0$ ——— ②

Different. ② partially w.r.t. c, i-e $\frac{\partial \phi(x,y,c)}{\partial c}=0$ ——— ③

If, we eliminat c from ② and ①, we get the eliminant eq. (resulting eq.) called c-disc. for ②

Remark

If eq. ② is quadratic in c, i-e of the shape $Ac^2+Bc+C=0$ then the c-disc. is given by $B^2-4AC=0$

Determination of Sing. Sol.

Suppose, we want to find the singular sol. of diff. eq. $f(x,y,P)=0$ —— ①

1. Find the general solution of ①
2. Find the c-discriminant
3. Find the P-discriminant
4. The common part in both the discriminants, that satisfies the diff. eq ①, is the singular solution of ①

The P disc. Method

We can obtain the singular solution of the diff. eq. $f(x,y,P)=0$ —— ① directly from eq. ①, as

Find P-discriminant for ①. The part of this relation that satisfies the diff. eq. ①, is the singular sol. of ①

Example

Solve and find singular sol. of

$$P^2-xP+y=0 \quad \text{——} ①$$

Sol:-

$$y = xP-P^2$$

It is clairaut's eq. so its general sol is $y=cx-c^2$

or $c^2-cx+y=0$ —— ②

Singular sol.

We find the singular sol. of ①, as

Example

Solve and find singular sol. of

$$xP^2-2yP+4x=0 \quad \text{——} ①$$

Sol:-

$$2yP = xP^2+4x$$

$$\Rightarrow 2y = xP+4xP^{-1}$$

Differentiating w.r.t x

$$2\frac{dy}{dx} = x\frac{dP}{dx}+P+4\left(-xP^{-2}\frac{dP}{dx}+P^{-1}\right)$$

Example

Solve and find singular sol. of

$$(x^2-1)P^2-2xyP-x^2=0 \quad \text{——— (1)}$$

Sol:-

$$2xyP = (x^2-1)P^2-x^2$$

$$\Rightarrow 2xyP = x^2P^2-P^2-x^2$$

$$\Rightarrow 2y = xP - x^{-1}P - xP^{-1}$$

Diff. w.r.t x, we get

$$2\frac{dy}{dx} = x\frac{dP}{dx}+P-\left(x^{-1}\frac{dP}{dx}-x^{-2}P\right)-\left(-xP^{-2}\frac{dP}{dx}+P^{-1}\right)$$

$$\Rightarrow 2P = x\frac{dP}{dx}+P-\frac{1}{x}\frac{dP}{dx}+\frac{P}{x^2}+\frac{x}{P^2}\frac{dP}{dx}-\frac{1}{P}$$

$$\Rightarrow P+\frac{1}{P}-\frac{P}{x^2} = \left(x-\frac{1}{x}+\frac{x}{P^2}\right)\frac{dP}{dx}$$

$$\Rightarrow P\left(1+\frac{1}{P^2}-\frac{1}{x^2}\right)-x\left(1-\frac{1}{x^2}+\frac{1}{P^2}\right)\frac{dP}{dx}=0$$

$$\Rightarrow \left(1+\frac{1}{P^2}-\frac{1}{x^2}\right)\left(P-x\frac{dP}{dx}\right)=0$$

$$\Rightarrow 1+\frac{1}{P^2}-\frac{1}{x^2}=0 \quad \text{or} \quad P-x\frac{dP}{dx}=0$$

Consider,

$$P-x\frac{dP}{dx}=0$$

$$\Rightarrow x\frac{dP}{dx}=P$$

$$\Rightarrow \int\frac{dP}{P}=\int\frac{dx}{x}$$

$$\Rightarrow \ln P = \ln x+\ln c$$

$$\Rightarrow P = cx$$

putting in eq. (1), we get

$$(x^2-1)c^2x^2-2cx^2y-x^2=0 \quad \text{——(2)}$$

(req. general sol. of (1))

$$(-2y)^2-4x\cdot 4x=0$$

$$\Rightarrow y^2=4x$$

Since c-disc. and P-disc. are same

$\therefore$ $y^2=4x$ is the singular sol. of (1)

Example

By finding the P-disc. find the singular sol. of

$$x^3P^2+x^2yP+a^3=0 \quad \text{---- (1)}$$

Sol:-

<u>P-discriminant</u>

P-disc. of (1), is given as

$$B^2-4AC=0$$

$$\Rightarrow x^4y^2-4a^3x^3=0$$

$$\Rightarrow x^3(xy^2-4a^3)=0$$

$$\Rightarrow x^3=0 \quad \text{or} \quad xy^2-4a^3=0$$

$$\Rightarrow x=0 \quad \text{or} \quad xy^2-4a^3=0$$

Take $x=0$

Since $x=0$ $\therefore$ $\frac{dx}{dy}=0$

or $\frac{1}{P}=0$

we, first write (1), as

$$x^3+x^2y\cdot\frac{1}{P}+a^3\left(\frac{1}{P}\right)^2=0 \quad \text{----- (2)}$$

L.H.S. of eq. (2)

$$= x^3+x^2y\cdot\frac{1}{P}+a^3\left(\frac{1}{P}\right)^2$$

$$= 0+0+0+0$$

$$= 0 = \text{R.H.S. of eq. 2}$$

$\therefore$ $x=0$ is a singular sol. of (1)

c-discriminant

c-disc for ②, is given by

$$B^2-4AC=0$$

$$\Rightarrow (-x)^2-4y=0$$

$$\Rightarrow x^2=4y$$

P-discriminant

P-disc. for ①, is given by

$$B^2-4AC=0$$

$$\Rightarrow (-x)^2-4y=0$$

$$\Rightarrow x^2=4y$$

Since c-disc. and P-disc. are same

$\therefore$ $x^2=4y$ is singular sol. of ①

Example

Solve $x^2P^2+yP(2x+y)+y^2=0$ —— ①

by making the substitutions

$y=u$, $xy=v$

and find the singular sol.

Sol:-

Here $y=u$, $x=\frac{v}{u}$

$\therefore dy=du$ $\qquad dx=\frac{udv-vdu}{u^2}$

Now $\frac{dy}{dx}=\frac{u^2du}{udv-vdu}$

$$\Rightarrow P=\frac{u^2du}{udv-vdu}$$

Hence eq. ① becomes, as

$$x^2\left(\frac{u^2du}{udv-vdu}\right)^2+y\left(\frac{u^2du}{udv-vdu}\right)+y^2=0$$

$$\Rightarrow 2P=x\frac{dP}{dx}+P-\frac{4x}{P^2}\frac{dP}{dx}+\frac{4}{P}$$

$$\Rightarrow P-\frac{4}{P}=x\left(1-\frac{4}{P^2}\right)\frac{dP}{dx}$$

$$\Rightarrow P\left(1-\frac{4}{P^2}\right)-x\left(1-\frac{4}{P^2}\right)\frac{dP}{dx}=0$$

$$\Rightarrow \left(1-\frac{4}{P^2}\right)\left(P-x\frac{dP}{dx}\right)=0$$

$$\Rightarrow 1-\frac{4}{P^2}=0 \quad \text{or} \quad P-x\frac{dP}{dx}=0$$

Consider,

$$P-x\frac{dP}{dx}=0$$

$$\Rightarrow x\frac{dP}{dx}=P$$

$$\Rightarrow \int\frac{dP}{P}=\int\frac{dx}{x}$$

$$\Rightarrow \ln P=\ln x+\ln c$$

$$\Rightarrow P=cx$$

putting in ①, we get

$$x\cdot c^2x^2-2y\cdot cx+4x=0$$

$$\Rightarrow c^2x^2-2yc+4=0 \quad \text{---} \quad (2)$$

(general sol. of ①)

Singular Sol:-

we find the singular sol of ① as

c-discriminant

c-disc. of ② is given as

$$B^2-4AC=0$$

$$\Rightarrow (-2y)^2-4x^2\cdot 4=0$$

$$\Rightarrow y^2=4x^2$$

P-discriminant

P-disc. of ①, is given by

$$B^2-4AC=0$$

Singular sol:

we find singular sol. of ① as,

C-discriminant

c-disc. of ②, is given by

$$B^2 - 4AC = 0$$

$$\Rightarrow (-2x^2y)^2 - 4x^2(x^2-1)(-x^2) = 0$$

$$\Rightarrow \boxed{y^2 + x^2 - 1 = 0}$$

$$\Rightarrow x^2 + y^2 = 1$$

P-discriminant

P-disc. of ① is given by

$$B^2 - 4AC = 0$$

$$\Rightarrow (-2xy)^2 - 4(x^2-1)(-x^2) = 0$$

$$\Rightarrow y^2 + x^2 - 1 = 0$$

$$\Rightarrow x^2 + y^2 = 1$$

Since c-disc. and p-disc. are same

$\therefore$ $x^2 + y^2 = 1$ is a singular sol. of ①

Take $y^2x - 4a^3 = 0$

$$\Rightarrow x = 4a^3y^{-2}$$

Diff. w.r.t y, we get

$$\frac{dx}{dy} = -8a^3y^{-3}$$

$$\Rightarrow \frac{1}{P} = -\frac{8a^3}{y^3}$$

Now

L.H.S. of eq.②

$$= x^3 + x^2y\cdot\frac{1}{P} + a^3\left(\frac{1}{P}\right)^2$$

$$= \left(\frac{4a^3}{y^2}\right)^3 + \left(\frac{4a^3}{y^2}\right)^2\cdot y\cdot\left(-\frac{8a^3}{y^3}\right) + a^3\left(-\frac{8a^3}{y^3}\right)^2$$

$$= \frac{64a^9}{y^6} - \frac{128a^9}{y^6} + \frac{64a^9}{y^6}$$

$$= \frac{64a^9 - 128a^9 + 64a^9}{y^6}$$

$$= 0$$

$$= \text{R.H.S. of eq.②}$$

i-e eq. ② is satisfied by $y^2x - 4a^3 = 0$

$\therefore$ $y^2x - 4a^3 = 0$ is a singulor sol. of ①

EXERCISE 9.9

(1) Solve and find singular sol. of $y = px + p^n$ ——— ①

Sol:- It is clairaut's eq.

so its general sol. is

$$y = cx + c^n \quad \text{———} ②$$

Singular sol

We find singular sol. of ①, as

C-discriminant

Diff. ② partially w.r.t c, we get

$$0 = x + nc^{n-1}$$

$$\Rightarrow c = (-x/n)^{\frac{1}{n-1}}$$

putting value of c in ②, we get

$$y = \left(-\frac{x}{n}\right)^{\frac{1}{n-1}} x + \left(-\frac{x}{n}\right)^{\frac{n}{n-1}}$$

P-discriminant

Diff. ① partially w.r.t. P, we get

$$0 = x + np^{n-1}$$

$$\Rightarrow p = \left(-\frac{x}{n}\right)^{\frac{1}{n-1}}$$

putting value of p in ①, we get

$$y = \left(-\frac{x}{n}\right)^{\frac{1}{n-1}} x + \left(-\frac{x}{n}\right)^{\frac{n}{n-1}}$$

Since c-disc. and P-disc are same

$$\therefore \quad y = \left(-\frac{x}{n}\right)^{\frac{1}{n-1}} x + \left(-\frac{x}{n}\right)^{\frac{n}{n-1}}$$

is a singular sol. of ①

(2) Solve and find sing sol of

$$p^2(x^2-a^2) - 2pxy + y^2 - b^2 = 0 \quad \text{———} ①$$

Sol:-

$$p^2x^2 - a^2p^2 - 2pxy + y^2 - b^2 = 0$$

It is not solvable for p, x, y

So, we write it as

$$p^2x^2 - 2pxy + y^2 - p^2a^2 - b^2 = 0$$

$$\Rightarrow p^2x^2 - 2pxy + y^2 = a^2p^2 + b^2$$

$$\Rightarrow (px-y)^2 = a^2p^2 + b^2$$

$$\Rightarrow px - y = \pm\sqrt{a^2p^2 + b^2}$$

$$\Rightarrow y = xp \pm \sqrt{a^2p^2+b^2}$$

It is clairaut's eq.

so its general sol. is,

$$y = cx \pm \sqrt{a^2c^2+b^2}$$

$$\Rightarrow y - cx = \pm\sqrt{a^2c^2+b^2}$$

$$\Rightarrow (y-cx)^2 = a^2c^2 + b^2$$

$$\Rightarrow y^2 + c^2x^2 - 2cxy = a^2c^2 + b^2$$

$$\Rightarrow c^2x^2 - a^2c^2 - 2cxy + y^2 - b^2 = 0.$$

$$\Rightarrow \underline{(x^2-a^2)}c^2 - \underline{2xy}c + \underline{(y^2-b^2)} = 0 \quad \text{———} ②$$

Singular sol.

We now find the s.sol. of ① as,

C-discriminant.

c-disc. of eq. ② is given by

$$B^2 - 4AC = 0$$

$$\Rightarrow (-2xy)^2 - 4(x^2-a^2)(y^2-b^2) = 0$$

$$\Rightarrow 4x^2y^2 - 4(x^2y^2 - b^2x^2 - a^2y^2 + a^2b^2) = 0$$

3

$$xP^2 + (x-y)P + 1 - y = 0 \quad \text{--- ①}$$

Sol:-

$$xP^2 + xP - yP + 1 - y = 0$$

It is not solvable for x, y, P

So, we write it as,

$$xP(P+1) - y(P+1) + 1 = 0$$

$$\Rightarrow xP - y + \frac{1}{P+1} = 0$$

$$\Rightarrow y = xP + \frac{1}{P+1}$$

It is clairaut's eq. so, its general sol. is $y = cx + \frac{1}{c+1}$

$$\Rightarrow y(c+1) = c(c+1)x + 1$$

$$\Rightarrow cy + y = c^2x + cx + 1$$

$$\Rightarrow c^2x + cx - cy + 1 - y = 0$$

$$\Rightarrow c^2x + c(x-y) + (1-y) = 0 \quad \text{--- ②}$$

Singular Sol:-

We find singular sol of ①, as

C-discriminant

c-disc. of ② is, given by

$$B^2 - 4AC = 0$$

$$\Rightarrow (x-y)^2 - 4x(1-y) = 0$$

$$\Rightarrow x^2 + y^2 - 2xy - 4x + 4xy = 0$$

$$\Rightarrow x^2 + y^2 + 2xy - 4x = 0$$

$$\Rightarrow (x+y)^2 - 4x = 0$$

P-discriminant

P-disc. of ①, is given by

$$B^2 - 4AC = 0$$

$$\Rightarrow (x-y)^2 - 4x(1-y) = 0$$

$$\Rightarrow 4x^2y^2 - 4x^2y^2 + 4b^2x^2 + 4a^2y^2 - 4a^2b^2 = 0$$

$$\Rightarrow b^2x^2 + a^2y^2 = a^2b^2$$

P-discriminant

P-disc. of eq. ① is given by

$$B^2 - 4AC = 0$$

$$\Rightarrow (-2xy)^2 - 4(x^2-a^2)(y^2-b^2) = 0$$

$$\Rightarrow 4x^2y^2 - 4(x^2y^2 - b^2x^2 - a^2y^2 + a^2b^2) = 0$$

$$\Rightarrow \cancel{x^2y^2} - \cancel{x^2y^2} + b^2x^2 + a^2y^2 - a^2b^2 = 0$$

$$\Rightarrow b^2x^2 + a^2y^2 = a^2b^2$$

Since c-disc. and P-disc. are same

$\therefore b^2x^2 + a^2y^2 = a^2b^2$ is the singular sol. of eq. ①

4

$$4P^2 = 9x \quad \text{--- ①}$$

Sol:-

$$x = \frac{4}{9}P^2$$

Diff. it w.r.t y, we get

$$\frac{dx}{dy} = \frac{4}{9}\cdot 2P\frac{dP}{dy}$$

$$\Rightarrow \frac{1}{P} = \frac{8}{9}P\frac{dP}{dy}$$

$$\Rightarrow 9\,dy = 8P^2\,dP$$

$$\Rightarrow \int 8P^2\,dP = \int 9\,dy$$

$$\Rightarrow \frac{8}{3}P^3 = 9y + c_1$$

$$\Rightarrow \frac{8}{27}P^3 = y + c_1/9$$

$$\Rightarrow \left(\frac{2}{3}P\right)^3 = y + c$$

$$\Rightarrow \frac{2}{3}P = (y+c)^{1/3}$$

$$\Rightarrow P = \frac{3}{2}(y+c)^{1/3}$$

$$\Rightarrow x^2+y^2-2xy-4x+4xy=0$$
$$\Rightarrow x^2+y^2+2xy-4x=0$$
$$\Rightarrow (x+y)^2-4x=0$$

Since c-disc. and P-disc. are same $\therefore$ singular sol. of ① is $(x+y)^2-4x=0$

5 $\quad 4xP^2=(3x-1)^2$

Sol:-

$$4xP^2=(3x-1)^2 \quad \text{---①}$$
$$P^2=\frac{(3x-1)^2}{4x}$$
$$\sqrt{P^2}=\sqrt{\frac{(3x-1)^2}{4x}}$$
$$P=\pm\frac{(3x-1)}{2\sqrt{x}}$$
$$dy/dx=\pm\left[\frac{3x}{2\sqrt{x}}-\frac{1}{2\sqrt{x}}\right] \quad \because \frac{1}{P}=\frac{dx}{dy}$$
$$dy=\pm\left[\tfrac{3}{2}x^{1/2}-\tfrac{1}{2}x^{-1/2}\right]dx$$
$$\int dy=\pm\int\left[\tfrac{3}{2}x^{1/2}-\tfrac{1}{2}x^{-1/2}\right]dx$$
$$\Rightarrow y=\pm\left[\int\tfrac{3}{2}x^{1/2}dx-\int\tfrac{1}{2}x^{-1/2}dx\right]$$
$$\Rightarrow y=\pm\left[x^{3/2}-x^{1/2}+c\right]$$
$$\Rightarrow y=\pm\left[x^{3/2}-x^{1/2}\right]-c$$
$$y+c=\pm\left[x^{3/2}-x^{1/2}\right]$$
$$(y+c)^2=\left[x^{3/2}-x^{1/2}\right]^2$$
$$(y+c)^2=x^3+x-2x^2$$
$$(y+c)^2=x(x^2+1-2x)$$
$$y^2+c^2+2cy=x(x-1)^2$$
$$c^2+2cy+y^2-x(x-1)^2=0$$
$$c^2+2yc+\left[y^2-x(x-1)^2\right]=0 \quad \text{---②}$$

It is quad. in "c"

Singular Solution: we now find the singular sol. as,

Putting in ①, we get

$$4\cdot\left[\tfrac{3}{2}(y+c)^{1/3}\right]^2=9x$$
$$\Rightarrow 4\cdot\tfrac{9}{4}(y+c)^{2/3}=9x$$
$$\Rightarrow (y+c)^{2/3}=x$$
$$\Rightarrow (y+c)^2=x^3$$
$$\Rightarrow y^2+c^2+2cy=x^3$$
$$\Rightarrow c^2+2yc+y^2-x^3=0 \quad \text{---②}$$

Singular sol.

We now find the singular sol as

C-discriminant

C-disc. of ②, is given by

$$B^2-4AC=0$$
$$\Rightarrow (2y)^2-4(y^2-x^3)=0$$
$$\Rightarrow 4y^2-4(y^2-x^3)=0$$
$$\Rightarrow x^3=0$$

P-discriminant

P-disc. of ①, is given by

$$B^2-4AC=0$$
$$\Rightarrow 0-4(4)(-9x)=0$$
$$\Rightarrow x=0$$

Common in c-disc., P-disc is $x=0$

But ① is not satisfied by $x=0$

$\therefore$ no singular sol. of ① exists

6 $\quad P^2+2Px^3-4x^2y=0$

Sol:-

$$4x^2y=P^2+2Px^3$$

C - Discriminant:

C-discriminant of ② is given by,

$B^2 - 4AC = 0$

$\Rightarrow (2y)^2 - 4(1)[y^2 - x(x-1)^2] = 0$

$\Rightarrow 4y^2 - 4y^2 + 4x(x-1)^2 = 0$

$\Rightarrow 4x(x-1)^2 = 0$

$\Rightarrow x(x-1)^2 = 0$

$\Rightarrow x = 0$ or $(x-1)^2 = 0$

P - Discriminant:

P-discriminant of ① is given by,

$B^2 - 4AC = 0$

$\Rightarrow 0 - 4(4x)(3x-1)^2 = 0$

$\Rightarrow x(3x-1)^2 = 0$

$\Rightarrow x = 0$ or $(3x-1)^2 = 0$

Since $x = 0$ is common in both P-disc. and C-disc.

$\therefore x = 0$ is not singular solution of eq ①.

$\because$ It does not satisfy eq (I).

(7) $6P^2y^2 + 3Px - y = 0$

Sol:-

$3Px = y - 6P^2y^2$

$\Rightarrow 3x = yP^{-1} - 6Py^2$

Diff. w.r.t y, we get,

$$3\frac{dx}{dy} = -yP^{-2}\frac{dP}{dy} + P^{-1} - 6\left(2Py + y^2\frac{dP}{dy}\right)$$

$$\Rightarrow 3\cdot\frac{1}{P} = -\frac{y}{P^2}\frac{dP}{dy} + \frac{1}{P} - 12yP - 6y^2\frac{dP}{dy}$$

$4y = P^2x^{-2} + 2Px$

Differentiating w.r.t x, we get

$$4\frac{dy}{dx} = -2x^{-3}P^2 + 2x^{-2}P\frac{dP}{dx} + 2\left(P + x\frac{dP}{dx}\right)$$

$$\Rightarrow 2P = -x^{-3}P^2 + P + (x^{-2}P + x)\frac{dP}{dx}$$

$$\Rightarrow \left(P + \frac{P^2}{x^3}\right) - \left(\frac{P}{x^2} + x\right)\frac{dP}{dx}$$

$$\Rightarrow P\left(1 + \frac{P}{x^3}\right) - x\left(\frac{P}{x^3} + 1\right)\frac{dP}{dx} = 0$$

$$\Rightarrow \left(1 + \frac{P}{x^3}\right)\left(P - x\frac{dP}{dx}\right) = 0$$

$$\Rightarrow 1 + \frac{P}{x^3} = 0 \quad \text{or} \quad P - x\frac{dP}{dx} = 0$$

Consider,

$$P - x\frac{dP}{dx} = 0$$

$$\Rightarrow x\frac{dP}{dx} = P$$

$$\Rightarrow \int\frac{dP}{P} = \int\frac{dx}{x}$$

$$\Rightarrow \ln P = \ln x + \ln c$$

$$\Rightarrow P = cx$$

Putting in eq. ①, we get

$$c^2x^2 + 2cx^4 - 4x^2y = 0 \quad \text{——②}$$

Singular Sol.

We now find the S.Sol. of ①, as

C - discriminant

C-disc. of ② is given by

$B^2 - 4AC = 0$

$\Rightarrow (2x^4)^2 - 4x^2(-4x^2y)$

$\Rightarrow 4x^8 + 16x^4y = 0$

$\Rightarrow x^4 + 4y = 0$

P - discriminant

P-disc. of ① is given by

$\frac{2}{P} + 12yP = -y\left(\frac{1}{P^2} + 6y\right)\frac{dP}{dy}$

$\Rightarrow 2P\left(\frac{1}{P^2} + 6y\right) + y\left(\frac{1}{P^2} + 6y\right)\frac{dP}{dy} = 0$

$\Rightarrow \left(\frac{1}{P^2} + 6y\right)\left(2P + \frac{dP}{dy}\right) = 0$

$\Rightarrow \frac{1}{P^2} + 6y = 0$ or $2P + y\frac{dP}{dy} = 0$

Consider,

$2P + y\frac{dP}{dy} = 0$

$\Rightarrow y\frac{dP}{dy} = -2P$

$\Rightarrow \int \frac{dP}{P} = -2\int \frac{dy}{y}$

$\Rightarrow \ln P = -2\ln y + \ln c$

$\Rightarrow P = c/y^2$

putting in ①, we get

$6 \cdot \frac{c^2}{y^4} \cdot y^2 + 3 \cdot \frac{c}{y^2} \cdot x - y = 0$

$\Rightarrow 6c^2 + 3cx - y^3 = 0$ —— ②

Singular Sol:-

we now find the s.sol. of ①, as

C-discriminant

C-disc. of ②, is given by

$B^2 - 4AC = 0$

$\Rightarrow 9x^2 - 4(6)(-y^3) = 0$

$\Rightarrow 9x^2 + 24y^3 = 0$

$\Rightarrow 3x^2 + 8y^3 = 0$

P-discriminant

P-disc. of ①, is given by

$B^2 - 4AC = 0$

$\Rightarrow (2x^3)^2 - 4(-4x^2y) = 0$

$\Rightarrow 4x^6 + 16x^2y = 0$

$\Rightarrow x^4 + 4y = 0$

Since c-disc. and P-disc. are same.

$\therefore x^4 + 4y = 0$ is s.sol. of ①

8 $x^3P^2 + x^2yP + 1 = 0$ ...... ①

Sol:-

$x^2yP = -x^3P^2 - 1$

$\Rightarrow y = -xP - x^{-2}P^{-1}$

Diff. w.r.t x, we get

$\frac{dy}{dx} = -\left(x\frac{dP}{dx} + P\right) - \left(-x^{-2}P^{-2}\frac{dP}{dx} - 2x^{-3}P^{-1}\right)$

$\Rightarrow P = -x\frac{dP}{dx} - P + \frac{1}{x^2P^2}\frac{dP}{dx} + \frac{2}{x^3P}$

$\Rightarrow 2P - \frac{2}{x^3P} = -\left(x - \frac{1}{x^2P^2}\right)\frac{dP}{dx}$

$\Rightarrow 2P\left(1 - \frac{1}{x^3P^2}\right) + x\left(1 - \frac{1}{x^3P^2}\right)\frac{dP}{dx} = 0$

$\Rightarrow \left(1 - \frac{1}{x^3P^2}\right)\left(2P + x\frac{dP}{dx}\right) = 0$

$\Rightarrow 1 - \frac{1}{x^3P^2} = 0$ or $2P + x\frac{dP}{dx} = 0$

Consider,

$2P + x\frac{dP}{dx} = 0$

$\Rightarrow x\frac{dP}{dx} = -2P$

$\Rightarrow \int \frac{dP}{P} = -2\int \frac{dx}{x}$

$\Rightarrow \ln P = -2\ln x + \ln c$

$$B^2 - 4AC = 0$$

$$\Rightarrow \quad 9x^2 - 4\,6y^2(-y) = 0$$

$$\Rightarrow \quad 9x^2 + 24y^3 = 0$$

$$\Rightarrow \quad 3x^2 + 8y^3 = 0$$

Since c-disc. is equal to p-disc. $\therefore$ $3x^2 + 8y^3 = 0$ is a singular sol. of ①

Q9 $xp^4 - 2yp^3 + 12x^3 = 0$ ——— ①

Sol:-

$$2yp^3 = xp^4 + 12x^3$$

$$\Rightarrow 2y = xp + 12x^3p^{-3}$$

Diff. it w.r.t x, we get

$$2\frac{dy}{dx} = x\frac{dP}{dx} + P + 12\left(-3x^3P^{-4}\frac{dP}{dx} + 3x^2P^{-3}\right)$$

$$\Rightarrow 2P = x\frac{dP}{dx} + P - 36x^3P^{-4}\frac{dP}{dx} + 36x^2P^{-3}$$

$$\Rightarrow P - \frac{36x^2}{P^3} = x\frac{dP}{dx} - \frac{36x^3}{P^4}\frac{dP}{dx}$$

$$\Rightarrow P - \frac{36x^2}{P^3} = x\left(1 - \frac{36x^2}{P^4}\right)\frac{dP}{dx}$$

$$\Rightarrow P\left(1 - \frac{36x^2}{P^4}\right) - x\left(1 - \frac{36x^2}{P^4}\right)\frac{dP}{dx} = 0$$

$$\Rightarrow \left(1 - \frac{36x^2}{P^4}\right)\left(P - x\frac{dP}{dx}\right) = 0$$

$$\Rightarrow 1 - \frac{36x^2}{P^4} = 0 \quad \text{or} \quad P - x\frac{dP}{dx} = 0$$

Consider

$$P - x\frac{dP}{dx} = 0$$

$$\Rightarrow x\frac{dP}{dx} = P$$

$$\Rightarrow \int\frac{dP}{P} = \int\frac{dx}{x}$$

$$\Rightarrow \ln P = \ln x^{-2} + \ln c$$

$$\Rightarrow P = c/x^2$$

putting in eq. ①, we get,

$$x^3\cdot\frac{c^2}{x^4} + x^2y\cdot\frac{c}{x^2} + 1 = 0$$

$$\Rightarrow c^2x + cxy + x = 0 \quad \text{———} ②$$

Singular sol.

we now find the s.sol. of ① as

C-discriminant

c-disc. of ② is given by

$$B^2 - 4AC = 0$$

$$\Rightarrow \quad x^2y^2 - 4x = 0$$

$$\Rightarrow \quad x(xy^2 - 4) = 0$$

$$\Rightarrow \quad x = 0 \;,\; xy^2 - 4 = 0$$

P-discriminant

p-disc. of ①, is given by

$$B^2 - 4AC = 0$$

$$\Rightarrow \quad x^4y^2 - 4x^3 = 0$$

$$\Rightarrow \quad x^3(xy^2 - 4) = 0$$

$$\Rightarrow \quad x\cdot x^2(xy^2 - 4) = 0$$

$$\Rightarrow \quad x = 0,\; x^2 = 0,\; xy^2 - 4 = 0$$

Since $x = 0$, $xy^2 - 4 = 0$ are common in both c-disc. and P-disc.

$\therefore$ $x = 0$, $xy^2 - 4 = 0$ are the singular sols. of ①

$\Rightarrow \ln P = \ln x + \ln c$

$\Rightarrow P = cx$

putting in eq. ①, we get,

$c^4x^5 - 2c^3x^3y + 12x^3 = 0$

$\Rightarrow c^4x^2 - 2c^3y + 12 = 0$ —— ②

Singular sol:-

we now, find singular sol. as

C-discriminant

Since ② is not quadratic

$\therefore$ c-disc. of ② is found as,

Diff. ②, partially w.r.t c, we get

$4c^3x^5 - 6c^2x^3y = 0$

$\Rightarrow 2cx^2 - 3y = 0$

$\Rightarrow c = \frac{3y}{2x^2}$ put in ②, we get.

$\left(\frac{3y}{2x^2}\right)^4 x^2 - 2\left(\frac{3y}{2x^2}\right)^3 y + 12 = 0$

$\Rightarrow \frac{81y^4}{16x^6} - \frac{27y^4}{4x^6} + 12 = 0$

$\Rightarrow 81y^4 - 108y^4 + 192x^6 = 0$

$\Rightarrow -27y^4 + 192x^6 = 0$

$\Rightarrow -9y^4 + 64x^6 = 0$

$\Rightarrow 9y^4 = 64x^6$

P-discriminant

Diff. ①, partially w.r.t P, we get

$4xP^3 - 6yP^2 = 0$

$\Rightarrow 2xP - 3y = 0$

$\Rightarrow P = \frac{3y}{2x}$ putting in ①

10 $8P^3x - 12P^2y - 27x = 0$ —— ①

Sol:

$12P^2y = -27x + 8P^3x$

$\Rightarrow 12y = 8Px - 27P^{-2}x$

Diff. w.r.t x, we get

$12\frac{dy}{dx} = 8\left(P + x\frac{dP}{dx}\right) - 27\left(P^{-2} - 2xP^{-3}\frac{dP}{dx}\right)$

$\Rightarrow 12P = 8P + 8x\frac{dP}{dx} - \frac{27}{P^2} + \frac{54x}{P^3}\frac{dP}{dx}$

$\Rightarrow 4P + \frac{27}{P^2} = 2x\left(4 + \frac{27}{P^3}\right)\frac{dP}{dx}$

$\Rightarrow P\left(4 + \frac{27}{P^3}\right) - 2x\left(4 + \frac{27}{P^3}\right)\frac{dP}{dx} = 0$

$\Rightarrow \left(4 + \frac{27}{P^3}\right)\left(P - 2x\frac{dP}{dx}\right) = 0$

$\Rightarrow 4 + \frac{27}{P^3} = 0$ or $P - 2x\frac{dP}{dx} = 0$

Consider,

$P - 2x\frac{dP}{dx} = 0$

$\Rightarrow 2x\frac{dP}{dx} = P$

$\Rightarrow \int\frac{dP}{P} = \frac{1}{2}\int\frac{dx}{x}$

$\Rightarrow \ln P = \frac{1}{2}\ln x + \ln c$

$\Rightarrow P = c\sqrt{x}$

putting in ①, we get

$8(c\sqrt{x})^3x - 12(c\sqrt{x})^2y - 27x = 0$

$\Rightarrow 8c^3x^{3/2}x - 12c^2xy - 27x = 0$

$\Rightarrow 8c^3x^{3/2} - 12c^2y - 27 = 0$ —— ②

Singular sol:-

we now find the s.sol. of ① as,

$$x\left(\frac{3y}{2x}\right)^4 - 2y\left(\frac{3y}{2x}\right)^3 + 12x^3 = 0$$

$$\Rightarrow \frac{81y^4}{16x^3} - \frac{27y^4}{4x^3} + 12x^3 = 0$$

$$\Rightarrow 81y^4 - 108y^4 + 192x^6 = 0$$

$$\Rightarrow -27y^4 + 192x^6$$

$$\Rightarrow -9y^4 + 64x^6$$

$$\Rightarrow 9y^4 = 64x^6$$

Since C-disc. and P-disc. are same

$\therefore$ $9y^4 = 64x^6$ is the s.sol.

11 Investigate for singular sol. by finding P-disc.

$$e^P - P + xy - x - 1 = 0 \quad \text{——} (1)$$

Sol:-

P-discriminant

Diff. (1), partially w.r.t P, we get.

$$e^P - 1 = 0$$

$$\Rightarrow e^P = 1$$

$$\Rightarrow \ln e^P = \ln 1$$

$$\Rightarrow P \ln e = \ln 1$$

$$\Rightarrow P = 0 \quad (\because \ln e = 1, \ln 1 = 0)$$

putting in eq. (1), we get

$$xy - x = 0$$

$$\Rightarrow x(y-1) = 0$$

$$\Rightarrow x = 0, \quad y - 1 = 0$$

Take $x = 0$

Since $x = 0 \quad \therefore \frac{dx}{dy} = 0$

or $\frac{1}{P} = 0$

Since eq. (1) is not satisfied

C-discriminant

Diff. (2) partially w.r.t c, we get

$$24c^2x^{3/2} - 24cy = 0$$

$$\Rightarrow cx^{3/2} - y = 0$$

$$\Rightarrow c = \frac{y}{x^{3/2}} \quad \text{putting in (2), we get}$$

$$8\left(\frac{y}{x^{3/2}}\right)^3 x^{3/2} - 12\left(\frac{y}{x^{3/2}}\right)^2 y - 27 = 0$$

$$\Rightarrow 8\frac{y^3}{x^3} - 12\frac{y^3}{x^3} - 27 = 0$$

$$\Rightarrow 8y^3 - 12y^3 - 27x^3 = 0$$

$$\Rightarrow -4y^3 - 27x^3 = 0$$

$$\Rightarrow 27x^3 + 4y^3 = 0$$

P-discriminant

Diff. (1) partially w.r.t P, we get

$$24P^2x - 24Py = 0$$

$$\Rightarrow Px = y$$

$$\Rightarrow P = y/x \quad \text{putting in (1) we get}$$

$$8\frac{y^3}{x^3}\cdot x - 12\frac{y^2}{x^2}\cdot y - 27x = 0$$

$$\Rightarrow 8\frac{y^3}{x^2} - 12\frac{y^3}{x^2} - 27x = 0$$

$$\Rightarrow 8y^3 - 12y^3 - 27x^3 = 0$$

$$\Rightarrow -4y^3 - 27x^3 = 0$$

$$\Rightarrow 27x^3 + 4y^3 = 0$$

Since c-disc. and P-disc. are same

$\therefore$ $27x^3 + 4y^3 = 0$ is the s.sol. of (1)

12 Investigate for singular sol. by finding P-disc.

$$4x(x-1)(x-2)P^2 - (3x^2 - 6x + 2)^2 = 0$$

by $x = 0$ and $1/p = 0$

$\therefore$ $x = 0$ is not singul. sol. of ①

Take $y-1=0$

$y = 1$ and $\frac{dy}{dx} = 0$

or $p = 0$

Since ① is satisfied by $y=1$ and $P=0$

$\therefore$ $y-1=0$ is a S. Sol. of ①

13 $P^4 - 2P^2 + 1 - y^4 = 0$ —— ①

Sol:-

P- discriminant

Diff. ① partially w.r.t P, we get

$4P^3 - 4P = 0$

$\Rightarrow 4P(P^2-1) = 0$

$\Rightarrow P(P-1)(P+1) = 0$

$\Rightarrow P=0, P=1, P=-1$

Putting in eq. ①, we get

$1-y^4 = 0$, $y^4 = 0$

or $(1-y^2)(1+y^2) = 0$, $y = 0$

or $(1-y)(1+y)(1+y^2) = 0$, $y = 0$

o $1-y = 0$, $1+y = 0$, $y = 0$

($1+y^2 = 0$ is rejected $\because$ y is imaginary)

Take $1-y=0$

$y = 1$ $\therefore \frac{dy}{dx} = 0$ i-e $P = 0$

Since ① is satisfied by $y=1$ and $\frac{dy}{dx} = 0$

Sol:-

P- discriminant

P-disc. of ① is given by

$B^2 - 4AC = 0$

$\Rightarrow 0 - 4.4x(x-1)(x-2)[-(3x^2-6x+2)^2] = 0$

$\Rightarrow x(x-1)(x-2)(3x^2-6x+2)^2 = 0$

$\Rightarrow x=0, x-1=0, x-2=0$

$3x^2-6x+2=0$

First, we write ①, as

$4x(x-1)(x-2) = (3x^2-6x+2)^2 \cdot \frac{1}{P^2}$ —— ②

Take $x=0$

$x = 0$ $\therefore \frac{dx}{dy} = 0$ i-e $\frac{1}{P} = 0$

Since ② is satisfied by $x=0$ and $\frac{dx}{dy} = 0$

$\therefore$ $x=0$ is a S. sol. of ①

Take $x-1=0$

$x = 1$ $\therefore \frac{dx}{dy} = 0$ i-e $\frac{1}{P} = 0$

Since ② is satisfied by $x=1$ and $\frac{dx}{dy} = 0$

$\therefore$ $x-1=0$ is a S. sol. of ①

Take $x-2=0$

$x = 2$ $\therefore \frac{dx}{dy} = 0$ i-e $\frac{1}{P} = 0$

Since ② is satisfied by $x=2$ and $\frac{dx}{dy} = 0$

$\therefore$ $x-2=0$ is a S. sol. of ①

$\therefore$ $y-1=0$ is s.sol. of ①

Take $1+y=0$

$y=-1$ $\therefore \frac{dy}{dx}=0$ i-e $p=0$

Since ① is satisfied by

$y=-1$ and $\frac{dy}{dx}=0$

$\therefore$ $y+1=0$ is s.sol. of ①

Take $y=0$

$y=0$ $\therefore \frac{dy}{dx}=0$ i-e $P=0$

Since ① is not satisfied by

$y=0$ and $\frac{dy}{dx}=0$

$\therefore$ $y=0$ is not a s.sol. of ①

15 $4P^3+3xP-y=0$ ——①

Sol:-

P-discriminant

Diff. ① partially w.r.t P, we get

$12P^2+3x=0$

$\Rightarrow 4P^2=-x$

$\Rightarrow P^2=-x/4$

$\Rightarrow P=\pm\sqrt{-x/4}$

putting in eq. ①, we get

$4(\pm(-x/4)^{3/2})+3x(\pm(-x/4)^{1/2})-y=0$

$\Rightarrow \pm(-x/4)^{1/2}\left(4(-x/4)+3x\right)-y=0$

$\Rightarrow \pm(-x/4)^{1/2}(-2x)-y=0$

$\Rightarrow y=\pm 2(-x/4)^{1/2}x$

$\Rightarrow y^2=4(-x/4)\cdot x^2$

$\Rightarrow y^2=-x^3$

$\Rightarrow x^3+y^2=0$

Take $3x^2-6x+2=0$

$x=\frac{6\pm\sqrt{36-24}}{6}=\frac{6\pm\sqrt{12}}{6}=\frac{3\pm\sqrt{3}}{3}$

Thus $x=1\pm\frac{1}{\sqrt{3}}$ $\therefore \frac{dx}{dy}=0$ i-e $1/p=0$

Since ② is not satisfied by $x=1\pm\frac{1}{\sqrt{3}}$ and $\frac{dx}{dy}=0$

$\therefore$ $3x^2-6x+2=0$ is not s.sol. of ①

14 $P^3-4xyP+8y^2=0$ ——①

Sol:-

P-discriminant

Diff. ① partially w.r.t. P, we get

$3P^2-4xy=0$

$\Rightarrow 3P^2=4xy$

$\Rightarrow P^2=\frac{4}{3}xy$

$\Rightarrow P=\pm\left(\frac{4}{3}xy\right)^{1/2}$

putting in eq. ①, we get,

$\pm\left(\frac{4}{3}xy\right)^{3/2}-4xy\left[\pm\left(\frac{4}{3}xy\right)^{1/2}\right]+8y^2=0$

$\Rightarrow \pm\left(\frac{4}{3}xy\right)^{1/2}\cdot\left[\frac{4}{3}xy-4xy\right]+8y^2=0$

$\Rightarrow \pm\left(\frac{4}{3}xy\right)^{1/2}\cdot\left(-\frac{8}{3}xy\right)+8y^2=0$

$\Rightarrow -2\left[\pm\left(\frac{4}{3}xy\right)^{1/2}\cdot\left(\frac{4}{3}xy\right)\right]+8y^2=0$

$\Rightarrow \pm\left(\frac{4}{3}xy\right)^{3/2}-4y^2=0$

$\Rightarrow \pm\left(\frac{4}{3}xy\right)^{3/2}=4y^2$

$\Rightarrow \left(\frac{4}{3}xy\right)^3=16y^4$

$\Rightarrow \frac{64}{27}x^3y^3-16y^4=0$

$\Rightarrow \frac{4}{27}x^3y^3-y^4=0$

Take $x^3+y^2=0$

$y^2=-x^3$

$\therefore\ 2y\frac{dy}{dx}=-3x^2$

$\Rightarrow\ 2yP=-3x^2$

$\Rightarrow\ P=-\frac{3x^2}{2y}$

Putting in L.H.S of eq. ①, we get

L.H.S of eq. ①

$=4P^3+3xP-y$

$=4\left(-\frac{3x^2}{2y}\right)^3+3x\left(-\frac{3x^2}{2y}\right)-y$

$=4\left(-\frac{27x^6}{8y^3}\right)-\frac{9x^3}{2y}-y$

$=-\frac{27x^6}{2y^3}-\frac{9x^3}{2y}-y$

$=\frac{27(x^3)^2}{2y^3}-\frac{9x^3}{2y}-y$

$=\frac{27(-y^2)^2}{2y^3}-\frac{9(-y^2)}{2y}-y \quad \because x^3=-y^2$

$=\frac{27y^4}{2y^3}+\frac{9y^2}{2y}-y$

$=\frac{27y}{2}+\frac{9y}{2}-y$

$=\frac{27y+9y-2y}{2}=17y\neq 0=R.H.S$

16 $P^3-2x^2P-4xy=0$ ——— ①

Sol:-

P-discriminant

Diff. ① partially w.r.t P, we get

$3P^2-2x^2=0$

$\Rightarrow\ 3P^2=2x^2$

$\Rightarrow\ P=\pm\left(\frac{2}{3}\right)^{1/2}x$

$4x^3y^3-27y^4=0$

$\Rightarrow\ y^3(4x^3-27y)=0$

$\Rightarrow\ y^3=0\ ,\ 4x^3-27y=0$

$\Rightarrow\ y=0\ ,\ 4x^3-27y=0$

Take $y=0$

$y=0\ \therefore\ \frac{dy}{dx}=0$ i.e $P=0$

Since ① is satisfied by

$y=0$ and $\frac{dy}{dx}=0$

$\therefore\ y=0$ is a S.Sol. of ①

Take $4x^3-27y=0$

$4x^3-27y=0$

$\Rightarrow\ 27y=4x^3$

$\Rightarrow\ y=\frac{4}{27}x^3\quad\therefore\ \frac{dy}{dx}=\frac{4}{9}x^2$

or $P=4/9\,x^2$

putting in L.H.S of eq. ①

L.H.S. of eq. ①

$=P^3-4xyP+8y^2$

$=\left(\frac{4}{9}x^2\right)^3-4x\left(\frac{4}{27}x^3\right)\left(\frac{4}{9}x^2\right)+8\left(\frac{4}{27}x^3\right)^2$

$=\frac{64}{729}x^6-\frac{64}{243}x^6+\frac{128}{729}x^6$

$=\frac{64x^6-192x^6+128x^6}{729}=0=R.H.S$

Since eq. ① is satisfied by

$y=\frac{4}{27}x^3$ and $\frac{dy}{dx}=\frac{4}{9}x^2$

$\therefore\ 4x^3-27y=0$ is a S.Sol. of ① putting in eq. ①, we get

$\left(\pm\left(\frac{2}{3}\right)^{1/2}x\right)^3 - 2x^2\left(\pm\left(\frac{2}{3}\right)^{1/2}x\right) - 4xy = 0$

$\Rightarrow \pm\left(\frac{2}{3}\right)^{3/2}x^3 - 2\left(\pm(2/3)^{1/2}x^3\right) - 4xy = 0$

$\Rightarrow \pm\left(\frac{2}{3}\right)^{1/2}x^3\left(\frac{2}{3} - 2\right) - 4xy = 0$

$\Rightarrow \pm\left(\frac{2}{3}\right)^{1/2}x^3(-4/3) = 4xy$

$\Rightarrow \pm\left(\frac{2}{3}\right)^{1/2}x^3 = -3xy$

$\Rightarrow \frac{2}{3}x^6 = 9x^2y^2$

$\Rightarrow 2x^6 - 27x^2y^2 = 0$

$\Rightarrow x^2(2x^4 - 27y^2) = 0$

$\Rightarrow x^2 = 0, \quad 2x^4 - 27y^2 = 0$

$\Rightarrow x = 0 \quad 2x^4 - 27y^2 = 0$

Take $x = 0$

$x = 0 \quad \therefore \frac{dx}{dy} = 0 \quad$ i-e $\frac{1}{P} = 0$

first, we write ①, as,

$1 - 2x^2\cdot\left(\frac{1}{P}\right)^2 - 4xy\cdot\left(\frac{1}{P}\right)^3 = 0$ —②

since ②, is not satisfied by $x = 0$ and $\frac{dx}{dy} = 0$

$\therefore x = 0$ is not a s. sol. of ①

Take $2x^4 - 27y^2 = 0$

$2x^4 - 27y^2 = 0$

$\Rightarrow 27y^2 = 2x^4$

$\Rightarrow y^2 = \frac{2}{27}x^4$

$\Rightarrow y = \pm\left(\frac{2}{27}\right)^{1/2}x^2$

$\therefore \frac{dy}{dx} = \pm 2\left(\frac{2}{27}\right)^{1/2}x$

$\Rightarrow P = \pm 2\left(\frac{2}{27}\right)^{1/2}x$

putting in L.H.S of eq. ①

L.H.S of eq. ①

$= P^3 - 2x^2P - 4xy$

$= \left[\pm 2\left(\frac{2}{27}\right)^{1/2}x\right]^3 - 2x^2\left[\pm 2\left(\frac{2}{27}\right)^{1/2}x\right] - 4x\left[\pm\left(\frac{2}{27}\right)^{1/2}x^2\right]$

$= \pm 8\left(\frac{2}{27}\right)^{3/2}x^3 - \left[\pm 4\left(\frac{2}{27}\right)^{1/2}x^3\right] - \left[\pm 4\left(\frac{2}{27}\right)^{1/2}x^3\right]$

$= \pm 4x^3\left(\frac{2}{27}\right)^{1/2}\left[2\cdot\left(\frac{2}{27}\right) - 1 - 1\right]$

THE END.