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NOTES OF EXERCISE 9.3

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❖ Homogenous functions:

➢ Definition:

A function $f(x,y)$ is called homogenous of degree "n" if

$$f(tx,ty)=t^n f(x,y)$$

Where t is a nonzero real number.

❖ Homogenous functions:

➢ Definition:

A first order differential equation

$$\frac{dy}{dx}=f(x,y)$$

is said to be homogenous if f is a homogenous function of any degree.

Example:

$\frac{dy}{dx}=\frac{x+y}{x-y}$ *is an example of homogenous equation.*

❖ Theorem:

A homogenous equation $\frac{dy}{dx}=g\left(\frac{y}{x}\right)$ can be transformed into a separable equation by the substitution $y=vx$.

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❖ *EXERCISE 9.3*

Question #1: $(\boldsymbol{x-y})\boldsymbol{dx}+(\boldsymbol{x+y})\boldsymbol{dy}=\mathbf{0}$

Solution:

Given equation is

$$(\boldsymbol{x-y})\boldsymbol{dx}+(\boldsymbol{x+y})\boldsymbol{dy}=\mathbf{0}$$

$$\Rightarrow (x+y)dy=-(x-y)dx$$

$$\Rightarrow \frac{dy}{dx}=\frac{-(x-y)}{(x+y)}---(a)$$

This is a homogenous differential equation in x & y. To solve this, put

$$y=vx$$

$$\Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$$

Thus equation (a) becomes

$$v+x\frac{dv}{dx}=\frac{-(x-vx)}{(x+vx)}$$

$$\Rightarrow v+x\frac{dv}{dx}=\frac{-x(1-v)}{x(1+v)}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{v-1}{v+1}-v$$

$$\Rightarrow x\frac{dv}{dx}=\frac{v-1-v^2-v}{v+1}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{-(1+v^2)}{v+1}$$

$$\Rightarrow \frac{v+1}{1+v^2}dv=-\frac{dx}{x}$$

Integrating both sides, we have

$$\int\frac{v+1}{1+v^2}dv=-\int\frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\int\frac{2v}{1+v^2}dv+\int\frac{dv}{1+v^2}=-\int\frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}ln(1+v^2)+tan^{-1}v=-lnx+lnc$$

$$\Rightarrow \frac{1}{2}ln\left(1+\frac{y^2}{x^2}\right)+tan^{-1}\frac{y}{x}=-lnx+c$$

$$\Rightarrow \frac{1}{2}ln\left(\frac{x^2+y^2}{x^2}\right)+tan^{-1}\frac{y}{x}=-lnx+c$$

$$\Rightarrow \frac{1}{2}ln(x^2+y^2)-\frac{1}{2}ln(x^2)+tan^{-1}\frac{y}{x}=-lnx+c$$

$$\Rightarrow \frac{1}{2}ln(x^2+y^2)-ln(x)+tan^{-1}\frac{y}{x}=-lnx+lnc$$

$$\Rightarrow \frac{1}{2}ln(x^2+y^2)+tan^{-1}\frac{y}{x}=lnc$$

is required solution.

❖ **Question # 2**: $(\boldsymbol{y^2+2xy})\boldsymbol{dx}+\boldsymbol{x^2dy}=\mathbf{0}$

Solution:

Given equation is

$$(\boldsymbol{y^2+2xy})\boldsymbol{dx}+\boldsymbol{x^2dy}=\mathbf{0}$$

$$\Rightarrow \frac{dy}{dx}=-y^2sinx$$

$$\Rightarrow x^2dy=-(y^2+2xy)dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{-(\boldsymbol{y^2 + 2xy})}{x^2} \quad ---(a)$$

This is a homogenous differential equation in x & y. To solve this, put

$$y = vx$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Thus equation (a) becomes

$$v + x\frac{dv}{dx} = \frac{-(v^2x^2 + 2x^2v)}{x^2}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{-x^2(v^2 + 2v)}{x^2}$$

$$\Rightarrow v + x\frac{dv}{dx} = -v^2 - 2v$$

$$\Rightarrow x\frac{dv}{dx} = -v^2 - 2v - v$$

$$\Rightarrow x\frac{dv}{dx} = -v(v+3)$$

$$\Rightarrow \frac{dv}{v(v+3)} = -\frac{dx}{x}$$

Integrating both sides, we have

$$\int \frac{dv}{v(v+3)} = -\int \frac{dx}{x} \quad ---(b)$$

Suppose that

$$\frac{1}{v(v+3)} = \frac{A}{v} + \frac{B}{v+3}$$

$$\Rightarrow 1 = A(v+3) + Bv \quad ---(c)$$

Put $v = 0$ in equation (c), we get

$$1 = A(3)$$

$$\Rightarrow \boldsymbol{A = {}^{1}/_{3}}$$

Put $v + 3 = 0 \Rightarrow v = -3$ in equation (c), we get

$$1 = B(-3)$$

$$\Rightarrow \boldsymbol{B = {}^{-1}/_{3}}$$

Hence,

$$\frac{1}{v(v+3)} = \frac{1}{3v} - \frac{1}{3(v+3)}$$

Thus, equation (b) becomes

$$\int \left[\frac{1}{3v} - \frac{1}{3(v+3)}\right] dv = -\int \frac{dx}{x}$$

$$\Rightarrow \frac{1}{3}\int \frac{dv}{v} - \frac{1}{3}\int \frac{dv}{v+3} = -lnx + lnc$$

$$\Rightarrow \frac{1}{3}lnv - \frac{1}{3}ln(v+3) = ln\frac{c}{x}$$

$$\Rightarrow \frac{1}{3}[lnv - ln(v+3)] = ln\frac{c}{x}$$

$$\Rightarrow ln\frac{v}{v+3} = 3ln\frac{c}{x}$$

$$\Rightarrow ln\frac{v}{v+3} = ln\frac{c^3}{x^3}$$

$$\Rightarrow ln\frac{({}^{y}/_{x})}{({}^{y}/_{x} + 3)} = ln\frac{c^3}{x^3} \quad \because v = \frac{y}{x}$$

$$\Rightarrow ln\left(\frac{y}{y+3x}\right) = ln\frac{c}{x^3} \quad \because c^3 = c\ (a\ constant)$$

$$\Rightarrow \frac{y}{y+3x} = \frac{c}{x^3}$$

$$\Rightarrow y|x^3| = c|y + 3x|$$

is required solution.

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❖ ***Question # 3:***

$$(\boldsymbol{x^2 - 3y^2})\boldsymbol{dx + 2xydy = 0}$$

Solution:

Given equation is

$$(\boldsymbol{x^2 - 3y^2})\boldsymbol{dx + 2xydy = 0}$$

$$\Rightarrow 2xydy = (3y^2 - x^2)dx$$

$$\Rightarrow \frac{dy}{\boldsymbol{dx}} = \frac{3y^2 - x^2}{2xy} - - - (a)$$

This is a homogenous differential equation in x & y. to solve this, put

$$y = vx$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Thus equation (a) becomes

$$v + x\frac{dv}{dx} = \frac{3v^2x^2 - x^2}{2vx^2}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{x^2(3v^2 - 1)}{2vx^2}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{3v^2 - 1}{2v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{3v^2 - 1}{2v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{3v^2 - 1 - 2v^2}{2v} = \frac{v^2 - 1}{2v}$$

$$\Rightarrow \frac{2v}{v^2 - 1}dv = \frac{dx}{x}$$

Integrating both sides, we have

$$\int \frac{2v}{v^2 - 1}dv = \int \frac{dx}{x}$$

$$\Rightarrow ln(v^2 - 1) = lnx + lnc$$

$$\Rightarrow ln(v^2 - 1) = ln\,cx$$

$$\Rightarrow v^2 - 1 = cx$$

$$\Rightarrow \frac{y^2}{x^2} - 1 = cx \quad \because v = \frac{y}{x}$$

$$\Rightarrow |y^2 - x^2| = |cx|x^2$$

is required solution.

❖ ***Question # 4:***

$$\boldsymbol{3xcos\left({}^{y}/_{x}\right)dy = \left[2xsin\left({}^{y}/_{x}\right) + 3ycos\left({}^{y}/_{x}\right)\right]dy}$$

Solution:

Given equation is

$$\boldsymbol{3xcos\left({}^{y}/_{x}\right)dy = \left[2xsin\left({}^{y}/_{x}\right) + 3ycos\left({}^{y}/_{x}\right)\right]dy}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2xsin\left({}^{y}/_{x}\right) + 3ycos\left({}^{y}/_{x}\right)}{3xcos\left({}^{y}/_{x}\right)} - - - (a)$$

This is a homogenous differential equation in x & y. to solve this, put

$$y = vx$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Thus equation (a) becomes

$$v + x\frac{dv}{dx} = \frac{2xsinv + 3vxcosv}{3xcosv}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2xsinv + 3vxcosv}{3xcosv} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2xsinv}{3xcosv}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2}{3}tanv$$

$$\Rightarrow \frac{dv}{tanv} = \frac{2}{3}\frac{dx}{x}$$

Integrating both sides, we have

$$\int \frac{dv}{tanv} = \frac{2}{3}\int \frac{dx}{x}$$

$$\Rightarrow \int cot\, v\, dv = \frac{2}{3}\int \frac{dx}{x}$$

$$\Rightarrow ln\, sin\, v = \frac{2}{3} ln\, x + ln\, c'$$

$$\Rightarrow 3ln\, sin\, v = 2\, ln\, x + 3ln\, c'$$

$$\Rightarrow ln\, sin\, v^3 = ln\, x^2 + ln\, c'^3$$

$$\because\ c' \text{ is a constant.}$$

$$\Rightarrow c'^3 = c\ (we\ say)$$

$$\Rightarrow ln\, sin\, v^3 = ln\, x^2 + ln\, c$$

$$\Rightarrow ln\, sin\, v^3 = ln\, cx^2$$

$$\Rightarrow sin\, v^3 = cx^2$$

$$\Rightarrow \left|Sin\left(\frac{y}{x}\right)\right|^3 = cx^2$$

is required solution.

❖ **Question # 5:**

$$(\boldsymbol{x^2 + xy + y^2})\boldsymbol{dx - x^2 dy = 0}$$

Solution:

Given equation is

$$(\boldsymbol{x^2 + xy + y^2})\boldsymbol{dx - x^2 dy = 0}$$

$$\Rightarrow x^2 dy = (x^2 + xy + y^2)dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2} - - - (a)$$

This is a homogenous differential equation in x & y. to solve this, put

$$y = vx$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Thus equation (a) becomes

$$v + x\frac{dv}{dx} = \frac{x^2 + vx^2 + v^2x^2}{x^2}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{x^2(1 + v + v^2)}{x^2}$$

$$\Rightarrow x\frac{dv}{dx} = 1 + v + v^2 - v$$

$$\Rightarrow x\frac{dv}{dx} = 1 + v^2$$

$$\Rightarrow \frac{dv}{1 + v^2} = \frac{dx}{x}$$

Integrating both sides, we have

$$\int \frac{dv}{1 + v^2} = \int \frac{dx}{x}$$

$$\Rightarrow \tan^{-1} v = \ln x + c$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = \ln x + c$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) - \ln|x| = c$$

is required solution.

❖ **Question # 6:**

$$\boldsymbol{ydy + xdx = \sqrt{x^2 + y^2}dx}$$

Solution:

Given equation is

$$\boldsymbol{ydy + xdx = \sqrt{x^2 + y^2}dx}$$

$$\Rightarrow ydy = \left(\sqrt{x^2+y^2}-x\right)dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{x^2+y^2}-x}{y} --- (a)$$

This is a homogenous differential equation in x & y. to solve this, put

$$y = vx$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Thus equation (a) becomes

$$v + x\frac{dv}{dx} = \frac{\sqrt{x^2+v^2x^2}-x}{vx}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{\sqrt{1+v^2}-1}{v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{\sqrt{1+v^2}-1}{v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{\sqrt{1+v^2}-1-v^2}{v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{\sqrt{1+v^2}-(1+v^2)}{v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{\sqrt{1+v^2}\left(1-\sqrt{1+v^2}\right)}{v}$$

$$\Rightarrow \frac{vdv}{\sqrt{1+v^2}\left(1-\sqrt{1+v^2}\right)} = \frac{dx}{x}$$

Integrating both sides, we have

$$\int \frac{vdv}{\sqrt{1+v^2}\left(1-\sqrt{1+v^2}\right)} = \int \frac{dx}{x} --- (b)$$

Consider

$$I = \int \frac{vdv}{\sqrt{1+v^2}\left(1-\sqrt{1+v^2}\right)}$$

$$Put\ 1-\sqrt{1+v^2} = t$$

$$\Rightarrow -\frac{1}{2}(1+v^2)^{-1/2}\,2vdv = dt$$

$$\Rightarrow \frac{vdv}{\sqrt{1+v^2}} = -dt$$

Therefore,

$$I = -\int \frac{dt}{t}$$

$$\Rightarrow I = -lnt$$

$$\Rightarrow I = -\ln(1-\sqrt{1+v^2})$$

$$\Rightarrow \int \frac{vdv}{\sqrt{1+v^2}\left(1-\sqrt{1+v^2}\right)} = -\ln(1-\sqrt{1+v^2})$$

Hence (b) will becomes

$$-\ln(1-\sqrt{1+v^2}) = lnx + lnc'$$

$$\Rightarrow \ln(1-\sqrt{1+v^2}) = -lnc'x$$

$$\Rightarrow \ln(1-\sqrt{1+v^2}) = \ln(c'x)^{-1}$$

$$\Rightarrow (1-\sqrt{1+v^2}) = (c'x)^{-1}$$

$$\Rightarrow 1-\sqrt{1+\frac{y^2}{x^2}} = (c'x)^{-1}$$

$$\Rightarrow 1-\frac{\sqrt{x^2+y^2}}{x} = \frac{1}{c'x}$$

$$\Rightarrow \frac{x-\sqrt{x^2+y^2}}{x} = \frac{c}{x} \quad \because \frac{1}{c'} = c$$

$$\Rightarrow x-\sqrt{x^2+y^2} = c$$

$\Rightarrow -\sqrt{x^2+y^2} = c - x$

$\Rightarrow \left(-\sqrt{x^2+y^2}\right)^2 = (c-x)^2$

$\Rightarrow x^2+y^2 = c^2 + x^2 - 2cx$

$\Rightarrow y^2 = c^2 - 2cx$

$\Rightarrow y^2 + 2cx - c^2 = 0$

is required solution.

❖ ***Question # 7:***

$$\frac{dy}{dx} = \frac{4y-3x}{2x-y}$$

Solution:

Given equation is

$$\frac{dy}{dx} = \frac{4y-3x}{2x-y} --- (a)$$

This is a homogenous differential equation in x & y. to solve this, put

$$y = vx$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Thus equation (a) becomes

$\Rightarrow v + x\frac{dv}{dx} = \frac{4vx-3x}{2x-vx}$

$\Rightarrow v + x\frac{dv}{dx} = \frac{4v-3}{2-v}$

$\Rightarrow x\frac{dv}{dx} = \frac{4v-3}{2-v} - v$

$\Rightarrow x\frac{dv}{dx} = \frac{4v-3-2v+v^2}{2-v}$

$\Rightarrow x\frac{dv}{dx} = \frac{v^2+2v-3}{2-v}$

Integrating both sides, we have

$\int \frac{(2-v)}{v^2+2v-3}dv = \int \frac{dx}{x}$

$\Rightarrow \int \frac{(2-v)}{(v+3)(v-1)}dv = \int \frac{dx}{x} --- (b)$

Suppose

$\frac{2-v}{(v+3)(v-1)} = \frac{A}{v+3} + \frac{B}{v-1}$

$\Rightarrow 2 - v = A(v-1) + B(v+3) --- (i)$

Put $v+3=0 \Rightarrow v=-3$ in (i), we have

$2+3 = A(-4)$

$\Rightarrow \boxed{\boldsymbol{A = -\frac{5}{4}}}$

Put $v-1=0 \quad \Rightarrow v=1$ in (i), we have

$2-1 = B(4)$

$\Rightarrow \boxed{\boldsymbol{B = \frac{1}{4}}}$

Thus equation (i) becomes

$\frac{2-v}{(v+3)(v-1)} = \frac{-5}{4(v+3)} + \frac{1}{4(v-1)}$

Thus equation (b) becomes

$\int \left(\frac{-5}{4(v+3)} + \frac{1}{4(v-1)}\right)dv = \int \frac{dx}{x}$

$\Rightarrow -\frac{5}{4}\int \frac{dv}{v+3} + \frac{1}{4}\int \frac{dv}{v-1} = \int \frac{dx}{x}$

$\Rightarrow -\frac{5}{4}\ln(v+3) + \frac{1}{4}\ln(v-1) = lnx + lnc$

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$$\Rightarrow -5\ln(v+3)+\ln(v-1)=4(lnx+lnc)$$

$$\Rightarrow \ln(v+3)^{-5}+\ln(v-1)=ln(cx)^4$$

$$\Rightarrow ln\frac{(v-1)}{(v+3)^5}=\ln(cx)^4$$

$$\Rightarrow \frac{(v-1)}{(v+3)^5}=(cx)^4$$

$$\Rightarrow \frac{\left(\frac{y}{x}-1\right)}{\left(\frac{y}{x}+3\right)^5}=(cx)^4 \quad \because v=\frac{y}{x}$$

$$\Rightarrow \frac{(y-x)x^4}{(y+3x)^5}=(cx)^4$$

$$\Rightarrow |y-x|=c|y+3x|^5$$

is required solution.

❖ **Question # 8:**

$$\boldsymbol{xsin\left(\frac{y}{x}\right)dy=\left[ysin\left(\frac{y}{x}\right)-x\right]dx}$$

Solution:

Given equation is

$$\boldsymbol{xsin\left(\frac{y}{x}\right)dy=\left[ysin\left(\frac{y}{x}\right)-x\right]dx}$$

$$\Rightarrow \frac{dy}{dx}=\frac{ysin\left(\frac{y}{x}\right)-x}{xsin\left(\frac{y}{x}\right)}$$

This is a homogenous differential equation in x & y. to solve this, put

$$y=vx$$

$$\Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$$

Thus equation (a) becomes

$$v+x\frac{dv}{dx}=\frac{vxsin-x}{xsinv}$$

$$\Rightarrow v+x\frac{dv}{dx}=\frac{vsinv-1}{sinv}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{vsinv-1}{sinv}-v$$

$$\Rightarrow x\frac{dv}{dx}=\frac{vsinv-1-vsinv}{sinv}$$

$$\Rightarrow x\frac{dv}{dx}=-\frac{1}{sinv}$$

$$\Rightarrow -sinvdv=\frac{dx}{x}$$

Integrating both sides, we have

$$-\int sinvdv=\int\frac{dx}{x}$$

$$\Rightarrow cosv=lnx+c$$

$$\Rightarrow cos\frac{y}{x}=lnx+c \quad \because v=\frac{y}{x}$$

is required solution.

❖ **Question # 9:**

$$\boldsymbol{\left(x^2+y^2\sqrt{x^2+y^2}\right)dx-xy\sqrt{x^2+y^2}dy=0}$$

Solution:

Given equation is

$$\boldsymbol{\left(x^2+y^2\sqrt{x^2+y^2}\right)dx-xy\sqrt{x^2+y^2}dy=0}$$

$$\Rightarrow xy\sqrt{x^2+y^2}dy=\left(x^2+y^2\sqrt{x^2+y^2}\right)dx$$

$$\Rightarrow \frac{dy}{dx}=\frac{x^2+y^2\sqrt{x^2+y^2}}{xy\sqrt{x^2+y^2}}$$

This is a homogenous differential equation in x & y. to solve this, put

$$y = vx$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Thus equation (a) becomes

$$\Rightarrow v + x\frac{dv}{dx} = \frac{x^2 + v^2x^2\sqrt{x^2 + v^2x^2}}{x(vx)\sqrt{x^2 + v^2x^2}}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{x^3 + v^2x^3\sqrt{1 + v^2}}{vx^3\sqrt{1 + v^2}}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{1 + v^2\sqrt{1 + v^2}}{v\sqrt{1 + v^2}}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1 + v^2\sqrt{1 + v^2}}{v\sqrt{1 + v^2}} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1 + v^2\sqrt{1 + v^2} - v^2\sqrt{1 + v^2}}{v\sqrt{1 + v^2}}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1}{v\sqrt{1 + v^2}}$$

$$\Rightarrow v\sqrt{1 + v^2}dv = \frac{dx}{x}$$

Integrating both sides, we have

$$\int v\sqrt{1 + v^2}dv = \int \frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\int 2v\sqrt{1 + v^2}dv = lnx + lnc$$

$$\Rightarrow \frac{1}{2}\frac{(1 + v^2)^{\frac{3}{2}}}{\frac{3}{2}} = \ln(cx)$$

$$\Rightarrow \frac{1}{3}\left(1 + \frac{y^2}{x^2}\right)^{\frac{3}{2}} = \ln(cx) \because v = \frac{y}{x}$$

$$\Rightarrow \left(\frac{x^2 + y^2}{x^2}\right)^{\frac{3}{2}} = 3\ln(cx)$$

$$\Rightarrow \frac{(x^2 + y^2)^3}{x^3} = \ln(cx^3).$$

$$\Rightarrow (x^2 + y^2)^3 = x^3\ln(cx^3)$$

is required solution.

❖ ***Question # 10:***

$$\left(\sqrt{x + y} + \sqrt{x - y}\right)dx - \left(\sqrt{x + y} - \sqrt{x - y}\right)dy = 0$$

Solution:

Given equation is

$$\left(\sqrt{x + y} + \sqrt{x - y}\right)dx - \left(\sqrt{x + y} - \sqrt{x - y}\right)dy = 0$$

$$\Rightarrow \left(\sqrt{x + y} - \sqrt{x - y}\right)dy = \left(\sqrt{x + y} + \sqrt{x - y}\right)dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{x + y} + \sqrt{x - y}}{\sqrt{x + y} - \sqrt{x - y}} - - - (a)$$

This is a homogenous differential equation in x & y. to solve this, put

$$y = vx$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Thus equation (a) becomes

$$v + x\frac{dv}{dx} = \frac{\sqrt{x + vx} + \sqrt{x - vx}}{\sqrt{x + vx} - \sqrt{x - vx}}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{\sqrt{1 + v} + \sqrt{1 - v}}{\sqrt{1 + v} - \sqrt{1 - v}}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{\sqrt{1+v}+\sqrt{1-v}}{\sqrt{1+v}-\sqrt{1-v}} \times \frac{\sqrt{1+v}+\sqrt{1-v}}{\sqrt{1+v}+\sqrt{1-v}}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{1+v+1-v+2\sqrt{1-v^2}}{1+v-1-v}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{2+2\sqrt{1-v^2}}{2v}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{2(1+\sqrt{1-v^2})}{2v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+\sqrt{1-v^2}}{v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+\sqrt{1-v^2}-v^2}{v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{\sqrt{1-v^2}+(1-v^2)}{v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{\sqrt{1-v^2}\left(1+\sqrt{1-v^2}\right)}{v}$$

$$\Rightarrow \frac{vdv}{\sqrt{1-v^2}\left(1+\sqrt{1-v^2}\right)} = \frac{dx}{x}$$

Integrating both sides, we have

$$\int \frac{vdv}{\sqrt{1-v^2}\left(1+\sqrt{1-v^2}\right)} = \int \frac{dx}{x}$$

$$Put\ 1+\sqrt{1-v^2} = t$$

$$\Rightarrow \frac{1}{2}(1-v^2)^{\frac{-1}{2}}(-2v)dv = dt$$

$$\Rightarrow \frac{vdv}{\sqrt{1-v^2}} = -dt$$

therefore,

$$\int \frac{-dt}{t} = \int \frac{dx}{x}$$

$$\Rightarrow -\int \frac{dt}{t} = \int \frac{dx}{x}$$

$$\Rightarrow -\ln t = lnx + lnc$$

$$\Rightarrow -\ln\left(1+\sqrt{1-v^2}\right) = lnx + lnc$$

$$\Rightarrow -\ln\left(1+\sqrt{1-v^2}\right) = lncx$$

$$\Rightarrow \ln\left(1+\sqrt{1-v^2}\right) = -lncx$$

$$\Rightarrow \ln\left(1+\sqrt{1-v^2}\right) = ln(cx)^{-1}$$

$$\Rightarrow \left(1+\sqrt{1-v^2}\right) = \frac{1}{cx}$$

$$\Rightarrow 1+\sqrt{1-\frac{y^2}{x^2}} = \frac{1}{cx} \quad \because v = \frac{y}{x}$$

$$\Rightarrow 1+\frac{\sqrt{x^2-y^2}}{x} = \frac{1}{cx}$$

$$\Rightarrow \frac{x+\sqrt{x^2-y^2}}{x} = \frac{1}{cx}$$

$$\Rightarrow x+\sqrt{x^2-y^2} = \frac{1}{c} = c\ (\text{a } constant)$$

$$\Rightarrow x+\sqrt{x^2-y^2} = c$$

is required solution.

Solve the initial value problem

❖ **Question # 11:**

$$\frac{dy}{dx} = \frac{x+y}{x} \qquad y(1) = 1$$

Solution:

Given equation is

$$\frac{dy}{dx} = \frac{x+y}{x} - - - (a)$$

This is a homogenous differential equation in x & y. to solve this, put

$$y = vx$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Thus equation (a) becomes

$$v + x\frac{dv}{dx} = \frac{x+vx}{x}$$

$$\Rightarrow v + x\frac{dv}{dx} = 1 + v$$

$$\Rightarrow x\frac{dv}{dx} = 1 + v - v$$

$$\Rightarrow x\frac{dv}{dx} = 1 + v - v$$

$$\Rightarrow dv = \frac{dx}{x}$$

Integrating both sides, we have

$$\int dv = \int \frac{dx}{x}$$

$$\Rightarrow v = lnx + c$$

$$\Rightarrow \frac{y}{x} = \ln x + c - - - (b) \quad \because v = \frac{y}{x}$$

Applying the condition $y(1) = 1$ on (b), we have

$$1 = 0 + c$$

$$\Rightarrow c = 1$$

Therefore,

$$\frac{y}{x} = \ln x + 1$$

$$\Rightarrow y = x\ln x + x$$

is required solution.

❖ **Question # 12:**

$$\left(y + \sqrt{x^2+y^2}\right)dx - xdy = 0 \qquad y(1) = 0$$

Solution:

Given equation is

$$\left(y + \sqrt{x^2+y^2}\right)dx - xdy = 0$$

$$\Rightarrow xdy = \left(y + \sqrt{x^2+y^2}\right)dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{y + \sqrt{x^2+y^2}}{x} - - - (a)$$

This is a homogenous differential equation in x & y. to solve this, put

$$y = vx$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Thus equation (a) becomes

$$v + x\frac{dv}{dx} = \frac{vx + \sqrt{x^2+v^2x^2}}{x}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{vx + x\sqrt{1+v^2}}{x}$$

$$\Rightarrow v + x\frac{dv}{dx} = v + \sqrt{1+v^2}$$

$$\Rightarrow x\frac{dv}{dx} = v + \sqrt{1+v^2} - v$$

$$\Rightarrow x\frac{dv}{dx} = \sqrt{1+v^2}$$

$$\Rightarrow \frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$$

Integrating both sides, we have

$$\int \frac{dv}{\sqrt{1+v^2}} = \int \frac{dx}{x}$$

$$\Rightarrow \text{Sinh}^{-1} v = lnx + c$$

$$\Rightarrow \text{Sinh}^{-1}\left(\frac{y}{x}\right) = lnx + c - - - (b) \because v = \frac{y}{x}$$

Applying the condition $y(1) = 0$ *on* (b), *we have*

$$\text{Sinh}^{-1}\left(\frac{0}{1}\right) = ln1 + c$$

$$\Rightarrow c = 0$$

Therefore (b) *becomes*

$$\text{Sinh}^{-1}\left(\frac{y}{x}\right) = lnx$$

$\because \sinh^{-1} x = ln\left(x + \sqrt{x^2+1}\right)$. Therefore,

$$\Rightarrow \ln\left(\frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}}\right) = lnx$$

$$\Rightarrow \frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}} = x$$

$$\Rightarrow \frac{y+\sqrt{x^2+y^2}}{x} = x$$

$$\Rightarrow y + \sqrt{x^2+y^2} = x^2$$

is required solution.

❖ ***Question # 13:***

$$(2x-5y)dx + (4x-y)dy = 0 \quad y(1) = 4$$

Solution:

Given equation is

$$(2x-5y)dx + (4x-y)dy = 0$$

$$\Rightarrow (4x-y)dy = -(2x-5y)dx$$

$$\Rightarrow (4x-y)dy = (5y-2x)dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{5y-2x}{4x-y} - - - (a)$$

This is a homogenous differential equation in x & y. to solve this, put

$$y = vx$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Thus equation (a) *becomes*

$$v + x\frac{dv}{dx} = \frac{5vx-2x}{4x-vx}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{5v-2}{4-v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{5v-2}{4-v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{5v-2-4v+v^2}{4-v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v^2+v-2}{4-v}$$

$$\Rightarrow \frac{4-v}{v^2+v-2}dv = \frac{dx}{x}$$

$$\Rightarrow \frac{4-v}{(v+2)(v-1)}dv = \frac{dx}{x} - - - (b)$$

Consider that

$$\frac{4-v}{(v+2)(v-1)}=\frac{A}{v+2}+\frac{B}{v-1}$$

$$\Rightarrow 4-v=A(v-1)+B(v+2)---(i)$$

Put $v+2=0 \Rightarrow v=-2$ *in* (i), *we have*

$$6=A(-3)$$

$$\boxed{\Rightarrow \boldsymbol{A=-2}}$$

Put $v-1=0 \Rightarrow v=1$ *in* (i), *we have*

$$3=B(3)$$

$$\boxed{\Rightarrow \boldsymbol{B=1}}$$

therefore,

$$\frac{4-v}{(v+2)(v-1)}=\frac{-2}{v+2}+\frac{1}{v-1}$$

Thus equation (b) *becomes*

$$\frac{-2}{v+2}+\frac{1}{v-1}=\frac{dx}{x}$$

Integrating both sides, we have

$$\int\left(\frac{-2}{v+2}+\frac{1}{v-1}\right)dv=\int\frac{dx}{x}$$

$$-2\int\frac{dv}{v+2}+\int\frac{dv}{v-1}=\int\frac{dx}{x}$$

$$\Rightarrow -2\ln(v+2)+\ln(v-1)=lnx+\ln c$$

$$\Rightarrow \ln(v+2)^{-2}+\ln(v-1)=ln\,cx$$

$$\Rightarrow \ln\frac{(v-1)}{(v+2)^2}=ln\,cx$$

$$\Rightarrow \frac{(v-1)}{(v+2)^2}=cx$$

$$\Rightarrow \frac{\frac{y}{x}-1}{\left(\frac{y}{x}+2\right)^2}=cx \quad \because v=\frac{y}{x}$$

$$\Rightarrow \frac{(y-x)/_x}{(y+2x)^2/_{x^2}}=cx$$

$$\Rightarrow \frac{(y-x)x}{(y+2x)^2}=cx$$

$$\Rightarrow y-x=c(y+2x)^2---(c)$$

Applying the condition $y(1)=4$ *on* (b), *we have*

$$4-1=c(4+2)^2$$

$$\boxed{\Rightarrow \boldsymbol{c=\frac{1}{12}}}$$

$$(c)\Rightarrow\ y-x=\frac{1}{12}(y+2x)^2$$

$$\Rightarrow (y+2x)^2=12(y-x)$$

is required solution.

❖ ***Question # 14:***

$\boldsymbol{(3x^2+9xy+5y^2)dx-(6x^2+4xy)dy=0 \quad y(2)=-6}$

Solution:

Given equation is

$$\boldsymbol{(3x^2+9xy+5y^2)dx-(6x^2+4xy)dy=0}$$

$$\Rightarrow (6x^2+4xy)dy=(3x^2+9xy+5y^2)dx$$

$$\Rightarrow \frac{dy}{dx}=\frac{(3x^2+9xy+5y^2)}{(6x^2+4xy)}---(a)$$

This is a homogenous differential equation in x & y. *to solve this, put*

$$y=vx$$

$$\Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$$

Thus equation (a) *becomes*

$$v+x\frac{dv}{dx}=\frac{3x^2+9vx^2+5v^2x^2}{6x^2+4vx^2}$$

$$\Rightarrow v+x\frac{dv}{dx}=\frac{3+9v+5v^2}{6+4v}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{3+9v+5v^2}{6+4v}-v$$

$$\Rightarrow x\frac{dv}{dx}=\frac{3+9v+5v^2-6v-4v^2}{6+4v}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{v^2+3v+3}{6+4v}$$

$$\Rightarrow \frac{6+4v}{v^2+3v+3}dv=\frac{dx}{x}$$

Integrating both sides, we have

$$\int\frac{6+4v}{v^2+3v+3}dv=\int\frac{dx}{x}$$

$$\Rightarrow 2\int\frac{(2v+3)}{v^2+3v+3}dv=\int\frac{dx}{x}$$

$$\Rightarrow 2ln(v^2+3v+3)=lnx+lnc$$

$$\Rightarrow ln(v^2+3v+3)^2=ln\,cx$$

$$\Rightarrow (v^2+3v+3)^2=cx$$

$$\Rightarrow \left(\frac{y^2}{x^2}+3\frac{y}{x}+3\right)^2=cx$$

$$\Rightarrow \left(\frac{y^2+3xy+3x^2}{x^2}\right)^2=cx$$

$$\Rightarrow (y^2+3xy+3x^2)^2=cx^5 ---(b)$$

Applying the condition $y(2)=6$ *on* (b), *we have*

$$(36-36+12)^2=c(8)$$

$$\Rightarrow 144=32c$$

$$\Rightarrow \frac{144}{32}=c$$

$$\Rightarrow c=\frac{9}{2}$$

Therefore (b) *becomes*

$$(y^2+3xy+3x^2)^2=\frac{9}{2}x^5$$

$$\Rightarrow 2(y^2+3xy+3x^2)^2=9x^5$$

is required solution.

Solve:

❖ ***Question # 15:***

$$\boldsymbol{\frac{dy}{dx}=\frac{x+3y-5}{x-y-1}}$$

Solution:

Given equation is

$$\boldsymbol{\frac{dy}{dx}=\frac{x+3y-5}{x-y-1}} ---(\boldsymbol{a})$$

Put

$x=X+h$ & $y=Y+k$

$\Rightarrow dx=dX$ & $dy=dY$

Thus equation (a) *becomes*

$$\frac{dY}{dX}=\frac{X+h+3(Y+k)-5}{X+h-(Y+k)-1}$$

$$\Rightarrow \frac{dY}{dX}=\frac{X+h+3Y+3k-5}{X+h-Y-k-1}$$

$$\Rightarrow \frac{dY}{dX}=\frac{X+3Y+h+3k-5}{X-Y+h-k-1}$$

Put $h+3k-5=0 ---(*)$

& $h-k-1=0 ---(**)$

On solving $(*)$ & $(**)$, *we have*

$h=-2$ & $k=-1$

Therefore,

$1+1=B$

$\boxed{\Rightarrow \boldsymbol{B = 2}}$

To find the $v$alue of A, we ha$v$e to sol$v$e the (i).

Therefore,

$1 - v = Av + A + B$

Comparing the coefficient of v, we ha$v$e

$\boxed{\boldsymbol{A = 1}}$

Therefore,

$$\frac{1-v}{(v+1)^2} = \frac{-1}{v+1} + \frac{2}{(v+1)^2}$$

Now $(i) \Longrightarrow \dfrac{dY}{dX} = \dfrac{X+3Y}{X-Y} - - - (b)$

This is a homogenous differential equation in X & Y. to solve this, put

$$Y = vX$$

$$\Longrightarrow \frac{dY}{dX} = v + X\frac{dv}{dX}$$

Thus equation (b) becomes

$$v + X\frac{dv}{dX} = \frac{X+3vX}{X-vX}$$

$$\Longrightarrow v + X\frac{dv}{dX} = \frac{1+3v}{1-v}$$

$$\Longrightarrow X\frac{dv}{dX} = \frac{1+3v}{1-v} - v$$

$$\Longrightarrow X\frac{dv}{dX} = \frac{1+3v-v+v^2}{1-v}$$

$$\Longrightarrow X\frac{dv}{dX} = \frac{v^2+2v+1}{1-v}$$

$$\Longrightarrow \frac{(1-v)}{v^2+2v+1}dv = \frac{dX}{X}$$

$$\Longrightarrow \frac{(1-v)}{(v+1)^2}dv = \frac{dX}{X}$$

Integrating both sides, we have

$$\int \frac{(1-v)}{(v+1)^2}dv = \int \frac{dX}{X} - - - (i)$$

Suppose that

$$\frac{(1-v)}{(v+1)^2} = \frac{A}{v+1} + \frac{B}{(v+1)^2}$$

$1 - v = A(v+1) + B - - - (ii)$

Put $v + 1 = 0 \Rightarrow v = -1$ in (ii)

$$\int \left[\frac{-1}{v+1} + \frac{2}{v+1^2}\right] dv = \int \frac{dX}{X}$$

$$-\int \frac{dv}{v+1} + 2\int \frac{dv}{(v+1)^2} = \int \frac{dX}{X}$$

$$-\ln(v+1)) + 2\left(\frac{-1}{v+1}\right) = lnX + lnC$$

$$\Longrightarrow -\ln\left(\frac{Y}{X}+1\right) - \frac{2}{\frac{Y}{X}+1} = lnX + lnC$$

$$\Longrightarrow -\ln\left(\frac{X+Y}{X}\right) - \frac{2X}{X+Y} = lnX + lnC$$

Putting the $v$alues of X and Y, we ha$v$e

$$-ln\left(\frac{x-2+y-1}{x-2}\right) - \frac{2(x-2)}{x-2+y-1} = ln(x-2) + \ln C$$

$$\Longrightarrow -ln\left(\frac{x+y-3}{x-2}\right) - \frac{2(x-2)}{x+y-3} = ln(x-2) + \ln C$$

$$\Longrightarrow -ln(x+y-3) + ln(x-2) - \frac{2(x-2)}{x+y-3} = ln(x-2) + \ln C$$

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$$\Rightarrow -ln(x+y-3)-\frac{2(x-2)}{x+y-3}=\ln C$$

$$\Rightarrow -\frac{2(x-2)}{x+y-3}=\ln C+ln(x+y-3)$$

$$\Rightarrow -\frac{2(x-2)}{x+y-3}=\ln C\,(x+y-3)$$

is required solution.

❖ **Question # 16:**

$$\frac{dy}{dx}=-\frac{4x+3y+15}{2x+y+7}$$

Solution:

Given equation is

$$\frac{dy}{dx}=-\frac{4x+3y+15}{2x+y+7}---(a)$$

Put

$$x=X+h \qquad \& \qquad y=Y+k$$

$$\Rightarrow dx=dX \qquad \& \qquad dy=dY$$

Thus equation (a) *becomes*

$$\frac{dY}{dX}=-\left(\frac{4(X+h)+3(Y+k)+15}{2(X+h)+Y+k+7}\right)$$

$$\Rightarrow \frac{dY}{dX}=-\left(\frac{4X+4h+3Y+3k+15}{2X+2h+Y+k+7}\right)$$

$$\Rightarrow \frac{dY}{dX}=\frac{4X+3Y+4h+3k+15}{2X+Y+2h+k+7}$$

Put $4h+3k+15=0---(*)$

& $2h+k+7=0---(**)$

On solving $(*)$ & $(**)$, *we have*

$h=-3$ & $k=-1$

Therefore,

$$\frac{dY}{dX}=\frac{4X+3Y}{2X+Y}---(b)$$

This is a homogenous differential equation in X & Y*. to solve this, put*

$$Y=vX$$

$$\Rightarrow \frac{dY}{dX}=v+X\frac{dv}{dX}$$

Thus equation (b) *becomes*

$$v+X\frac{dv}{dX}=-\left(\frac{4X+3vX}{2X+vX}\right)$$

$$\Rightarrow v+X\frac{dv}{dX}=-\left(\frac{4+3v}{2+v}\right)$$

$$\Rightarrow X\frac{dv}{dX}=-\left(\frac{4+3v}{2+v}\right)-v$$

$$\Rightarrow X\frac{dv}{dX}=\frac{-4-3v-2v-v^2}{2+v}$$

$$\Rightarrow X\frac{dv}{dX}=\frac{-v^2-5v-4}{2+v}$$

$$\Rightarrow X\frac{dv}{dX}=\frac{-(v^2+5v+4)}{2+v}$$

$$\Rightarrow X\frac{dv}{dX}=-\frac{(v+1)(v+4)}{2+v}$$

$$\Rightarrow \frac{2+v}{(v+1)(v+4)}dv=-\frac{dX}{X}$$

Integrating both sides, we have

$$\int\frac{2+v}{(v+1)(v+4)}dv=-\int\frac{dX}{X}$$

Consider,

$$\frac{v+2}{(v+1)(v+4)}=\frac{A}{v+1}+\frac{B}{v+4}$$

$$\Rightarrow v+2=A(v+4)+B(v+1)---(i)$$

Put $v+1=0\Rightarrow v=-1$ in (i)

$-1+2=A(-1+4)$

$\Rightarrow \boldsymbol{A=\frac{1}{3}}$

Put $v+4=0 \Rightarrow v=-4$ in (i)

$-4+2=B(-4+1)$

$\Rightarrow \boldsymbol{B=\frac{2}{3}}$

Thus,

$$\frac{v+2}{(v+1)(v+4)}=\frac{1}{3(v+1)}+\frac{2}{3(v+4)}$$

Putting $v$alues in (3)

$$\int\left[\frac{1}{3(v+1)}+\frac{2}{3(v+4)}\right]dv=-\int\frac{dX}{X}$$

$$\Longrightarrow \frac{1}{3}\int\frac{dv}{v+1}+\frac{2}{3}\int\frac{dv}{v+4}=-\int\frac{dX}{X}$$

$$\Longrightarrow \frac{1}{3}\ln(v+1)+\frac{2}{3}\ln(v+4)=-lnX+lnC$$

$$\Longrightarrow \ln(v+1)^{\frac{1}{3}}+\ln(v+4)^{\frac{2}{3}}=ln\frac{C}{X}$$

$$\Longrightarrow ln(v+1)^{\frac{1}{3}}(v+4)^{\frac{2}{3}}=ln\frac{C}{X}$$

$$\Longrightarrow (v+1)^{\frac{1}{3}}(v+4)^{\frac{2}{3}}=\frac{C}{X}$$

$$\Longrightarrow \left(\frac{Y}{X}+1\right)^{\frac{1}{3}}\left(\frac{Y}{X}+4\right)^{\frac{2}{3}}=\frac{C}{X}$$

$$\Longrightarrow \left(\frac{Y+X}{X}\right)^{\frac{1}{3}}\left(\frac{Y+4X}{X}\right)^{\frac{2}{3}}=\frac{C}{X}$$

$$\Longrightarrow (Y+X)^{\frac{1}{3}}(Y+4X)^{\frac{2}{3}}=C$$

Putting the $v$alues of X and Y

$$(y+1+x+3)^{\frac{1}{3}}(y+1+4x+12)^{\frac{2}{3}}=c$$

$$\Longrightarrow (x+y+4)^{\frac{1}{3}}(y+4x+13)^{\frac{2}{3}}=c$$

$$\Longrightarrow (x+y+4)(y+4x+13)^{2}=c$$

is required solution.

❖ **Question # 17:**

$$\boldsymbol{(3y-7x-3)dx+(7y-3x-7)dy=0}$$

Solution:

Given equation is

$\boldsymbol{(3y-7x-3)dx+(7y-3x-7)dy=0}$

$$\frac{dy}{dx}=-\frac{(3y-7x-3)}{(7y-3x-7)}---(a)$$

Put

$x=X+h \quad \& \quad y=Y+k$

$\Longrightarrow dx=dX \quad \& \quad dy=dY$

Thus equation (a) *becomes*

$$\frac{dY}{dX}=-\frac{3(Y+k)-7(X+h)-3}{7(Y+k)-3(X+h)-7}$$

$$\Longrightarrow \frac{dY}{dX}=-\frac{3Y+3k-7X-7h-3}{7Y+7k-3X-3h-7}$$

$$\Longrightarrow \frac{dY}{dX}=-\frac{3Y-7X+3k-7h-3}{7Y-3X+7k-3h-7}$$

Put $3k-7h-3=0---(*)$

& $7k-3h-7=0---(**)$

On solving $(*)$ & $(**)$, *we have*

$h=0$ & $k=1$

Therefore,

$$\frac{dY}{dX}=\frac{4X+3Y}{2X+Y}---(b)$$

This is a homogenous differential equation in X & Y. *to solve this, put*

$$Y=vX$$

$$\Rightarrow \frac{dY}{dX} = v + X\frac{dv}{dX}$$

Thus equation (b) becomes

$$v + X\frac{dv}{dX} = \frac{7X - 3vX}{-3X + 7vX}$$

$$\Rightarrow v + X\frac{dv}{dX} = \frac{7 - 3v}{-3 + 7v}$$

$$\Rightarrow X\frac{dv}{dX} = \frac{7 - 3v}{-3 + 7v} - v$$

$$\Rightarrow X\frac{dv}{dX} = \frac{7 - 3v + 3v - 7v^2}{-3 + 7v}$$

$$\Rightarrow X\frac{dv}{dX} = \frac{7(1 - v^2)}{-3 + 7v}$$

$$\Rightarrow \frac{(-3 + 7v)}{7(1 - v^2)}dv = \frac{dX}{X}$$

$$\Rightarrow \frac{7\left(\frac{-3}{7} + v\right)}{7(1 - v^2)}dv = \frac{dX}{X} - - - (c)$$

Consider,

$$\Rightarrow \frac{v - \frac{3}{7}}{1 - v^2} = \frac{A}{1 - v} + \frac{B}{1 + v}$$

$$\Rightarrow v - \frac{3}{7} = A(1 + v) + B(1 - v) - - - (i)$$

Put $1 - V = 0 \Rightarrow V = 1$ in (i), *we have*

$$1 - \frac{3}{7} = A(2)$$

$$\Rightarrow \frac{4}{7} = A(2)$$

$$\boxed{\Rightarrow \boldsymbol{A} = \frac{\mathbf{2}}{\mathbf{7}}}$$

Put $1 + V = 0 \Rightarrow V = -1$ in (i)

$$\Rightarrow -1 - \frac{3}{7} = B(2)$$

$$\boxed{\Rightarrow \boldsymbol{B} = \frac{\mathbf{-5}}{\mathbf{7}}}$$

So,

$$\frac{v - \frac{3}{7}}{1 - v^2} = \frac{2}{7(1 - v)} + \frac{-5}{7(1 + v)}$$

Equation (c) will be

$$\left[\frac{2}{7(1 - v)} - \frac{5}{7(1 + v)}\right]dv = \frac{dX}{X}$$

Integrating both sides, we have

$$\frac{2}{7}\int \frac{dv}{1 - v} - \frac{5}{7}\int \frac{dv}{1 + v} = \int \frac{dX}{X}$$

$$\frac{-2}{7}ln(1 - v) - \frac{5}{7}\ln(1 + v) = lnX + lnC$$

$$(1 - v)^{\frac{-2}{7}}(1 + v)^{\frac{-5}{7}} = CX$$

$$\Rightarrow \left(1 - \frac{Y}{X}\right)^{\frac{-2}{7}}\left(1 + \frac{Y}{X}\right)^{\frac{-5}{7}} = CX \quad \because v = \frac{Y}{X}$$

$$\left(\frac{X - Y}{X}\right)^{\frac{-2}{7}}\left(\frac{X + Y}{X}\right)^{\frac{-5}{7}} = \frac{1}{CX}$$

$$\left(\frac{1}{C}\right)^7 = (X - Y)^2(X + Y)^5$$

Putting values of X and Y

$$(x - y + 1)^2(x + y - 1)^5 = c$$

is required solution.

❖ ***Question # 18:***

$$\frac{dy}{dx} = \frac{3x - 4y - 2}{3x - 4y - 3}$$

Solution:

Given equation is

$$\boldsymbol{\frac{dy}{dx} = \frac{3x-4y-2}{3x-4y-3}} ---(\boldsymbol{a})$$

Put

$t = 3x - 4y$

$$\Longrightarrow \frac{dt}{dx} = 3 - 4\frac{dy}{dx}$$

$$\Longrightarrow 4\frac{dy}{dx} = 3 - \frac{dt}{dx}$$

$$\Longrightarrow \frac{dy}{dx} = \frac{3}{4} - \frac{1}{4}\frac{dt}{dx}$$

Thus equation (a) *becomes*

$$\frac{3}{4} - \frac{1}{4}\frac{dt}{dx} = \frac{t-2}{t-3}$$

$$\Longrightarrow \frac{1}{4}\frac{dt}{dx} = \frac{3}{4} - \frac{t-2}{t-3}$$

$$\Longrightarrow \frac{1}{4}\frac{dt}{dx} = \frac{3(t-3)-4(t-2)}{4(t-3)}$$

$$\Longrightarrow \frac{1}{4}\frac{dt}{dx} = \frac{3t-9-4t+8}{4(t-3)}$$

$$\Longrightarrow \frac{dt}{dx} = \frac{-t-1}{t-3}$$

$$\Longrightarrow \frac{dt}{dx} = \frac{-(t+1)}{t-3}$$

$$\Longrightarrow \frac{t-3}{t+1}dt = dx$$

Integrating both sides, we have

$$\int \frac{t-3}{t+1}dt = \int dx$$

$$\Longrightarrow \int \frac{t+1-1-3}{t+1}dt = \int dx$$

$$\Longrightarrow \int \frac{t+1}{t+1}dt - 4\int \frac{dt}{t+1} = \int dx$$

$$\Longrightarrow t - 4\ln(t+1) = x + c$$

$$\Longrightarrow 3x - 4y - 4\ln(3x-4y+1) = x + c$$

is required solution.

❖ ***Question # 19:***

$$\boldsymbol{\frac{dy}{dx} = \frac{y-x+1}{y-x+5}}$$

Solution:

Given equation is

$$\boldsymbol{\frac{dy}{dx} = \frac{y-x+1}{y-x+5}} ---(\boldsymbol{a})$$

Put

$y - x = t$

$$\Longrightarrow \frac{dy}{dx} - 1 = \frac{dt}{dx}$$

$$\Longrightarrow \frac{dy}{dx} = \frac{dt}{dx} + 1$$

Thus equation (a) *becomes*

$$\Longrightarrow \frac{dt}{dx} + 1 = \frac{t+1}{t+5}$$

$$\Longrightarrow \frac{dt}{dx} = \frac{t+1}{t+5} - 1$$

$$\Longrightarrow \frac{dt}{dx} = \frac{t+1-t-5}{t+5}$$

$$\Longrightarrow \frac{dt}{dx} = \frac{-4}{t+5}$$

$$\Longrightarrow (t+5)dt = -4dx$$

Integrating both sides, we have

$$\Rightarrow \int (t+5)dt = -4\int dx$$

$$\Rightarrow \frac{t^2}{2} + 5t = -4x + c$$

$$\Rightarrow \frac{1}{2}(y-x)^2 + 5(y-x) = -4x + c \quad \because t = y - x$$

$$\Rightarrow \frac{1}{2}(y-x)^2 + 5y - 5x = -4x + c$$

$$\Rightarrow \frac{1}{2}(y-x)^2 + 5y - x = c$$

$$\Rightarrow (y-x)^2 - 2x + 10y = 2c$$

$$\Rightarrow (y-x)^2 - 2x + 10y = c'$$

is required solution.

❖ ***Question # 20:***

$$\boldsymbol{\frac{dy}{dx} = \frac{x-2y+5}{2x+y-1}}$$

Solution:

Given equation is

$$\boldsymbol{\frac{dy}{dx} = \frac{x-2y+5}{2x+y-1}} - - - (\boldsymbol{i})$$

Put

$$x = X + h \quad \& \quad y = Y + k$$

$$\Rightarrow dx = dX \quad \& \quad dy = dY$$

Thus equation (a) *becomes*

$$\frac{dY}{dX} = \frac{(X+h) - 2(Y+k) + 5}{2(X+h) + (Y+k) - 1}$$

$$\Rightarrow \frac{dY}{dX} = \frac{X + h - 2Y - 2k + 5}{2X + 2h + Y + k - 1}$$

$$\Rightarrow \frac{dY}{dX} = \frac{X - 2Y + h - 2k + 5}{2X + Y + 2h + k - 1}$$

Put $h - 2k + 5 = 0 - - - (*)$

& $2h + k - 1 = 0 - - - (**)$

On solving $(*)$ & $(**)$, *we have*

$$h = -\frac{3}{5} \;\&\; k = +\frac{11}{5}$$

Therefore,

$$\frac{dY}{dX} = \frac{X - 2Y}{2X + Y} - - - (b)$$

This is a homogenous differential equation in X & Y. *to solve this, put*

$$Y = vX$$

$$\Rightarrow \frac{dY}{dX} = v + X\frac{dv}{dX}$$

Thus equation (b) *becomes*

$$v + X\frac{dv}{dX} = \frac{X - 2vX}{2X + vX}$$

$$\Rightarrow v + X\frac{dv}{dX} = \frac{1-2v}{2+v}$$

$$\Rightarrow X\frac{dv}{dX} = \frac{1-2v}{2+v} - v$$

$$\Rightarrow X\frac{dv}{dX} = \frac{1 - 2v - 2v - v^2}{2+v}$$

$$\Rightarrow X\frac{dv}{dX} = \frac{-v^2 - 4v + 1}{2+v}$$

$$\Rightarrow X\frac{dv}{dX} = \frac{-(v^2 + 4v - 1)}{2+v}$$

$$\Rightarrow \frac{v+2}{v^2 + 4v - 1}dv = -\frac{dX}{X}$$

Integrating both sides, we have

$$\int \frac{v+2}{v^2+4v-1}dv = -\int \frac{dX}{X}$$

$$\frac{1}{2}\int \frac{2v+4}{v^2+4v-1}\,dv = -\int \frac{dX}{X}$$

$$\frac{1}{2}\ln(v^2+4v-1) = -lnX + lnC$$

$$(v^2+4v-1)^{\frac{1}{2}} = \frac{c}{X}$$

$$\sqrt{(v^2+4v-1)} = \frac{C}{X}$$

$$\sqrt{\frac{Y^2}{X^2}+4\frac{Y}{X}-1} = \frac{C}{X} \quad \because v = \frac{Y}{X}$$

$$\frac{\sqrt{Y^2+4XY-X^2}}{X} = \frac{C}{X}$$

$$\sqrt{Y^2+4XY-X^2} = C$$

Putting the values of Y and X

$$\sqrt{\left(y-\frac{11}{5}\right)^2 + 4\left(x+\frac{3}{5}\right)\left(y-\frac{11}{5}\right) - \left(x+\frac{3}{5}\right)^2} = C$$

On solving this by simple algebraic formulae and operation, we will finally obtain the answer as follow:

$$x^2 - y^2 - 4xy + 10x + 2y = c$$

is required solution.