Chapter 9

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Differential Equation:- An eq involving independent and dependent variables and the derivatives of the dependent variable with respect to one or more independent variables is called a Diff Eq. In $\frac{dy}{dx}$, $\frac{d^2z}{du^2}$ y, z are dependent variables & x, u are independent variables.

Ordinary Diff Eq:- An eq involving only derivatives of one or more dependent variables, with respect to a single independent variable is called ordinary diff eq. e.g $\frac{d^2y}{dx^2} + xy\left(\frac{dy}{dx}\right)^2 = 0$ is O.D.Eq, but $\frac{d^2y}{dx^2} + xy\left(\frac{dy}{dt}\right)^2 = 0$ is not O.D.Eq

Partial Diff Eq:- An eq involving partial derivatives of one or more dependent variables with respect to two or more independent variables is called partial diff eq.

Order of Diff Eq:- The order of a diff eq is the order of the highest derivative that occurs in the eq.

Degree of Diff Eq:- The degree of a diff eq is the power of highest order derivative (free of fractions) involved in a diff eq.

(i) $\frac{dy}{dx} + y\cos x = \sin x$	Ordinary Diff Eq	order 1	Degree 1
(ii) $\frac{d^2y}{dx^2} + xy\left(\frac{dy}{dx}\right)^2 = 0$	"	order 2	Degree 1
(iii) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = \frac{d^2y}{dx^2}$	"	order 2	Degree 2
(iv) $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$	Partial Diff Eq	order 1	Degree 1
(v) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$	"	order 2	Degree 1
(vi) $\frac{d^2y}{dx^2} + \frac{dv}{dx} = e^x$	Ordinary Diff Eq	order 2	Degree 1

☆ $\left(\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}\right)^2 = \left(\frac{d^2y}{dx^2}\right)^2 \Rightarrow \left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = \left(\frac{d^2y}{dx^2}\right)^2$

Let $(y''')^{2/3} = 4 + y'$

cubing both sides $(y'')^2 = (4+y')^3$. Order 3 degree 2.

$y''' = \sqrt{2x+3y}$ Order 3 degree 1

$6x^2 \frac{d^3y}{dx^3} + \sin x \frac{d^2y}{dx^2} - \cos xy$

Degree is undefined
∵ the unknown fn 'y' is argument of transcendental cosine fn and therefore can not be written as a polynomial in y and its derivatives.
Similarly $y'' + (y')^2 = \log y''$
& $\sin\left(\frac{dy}{dx}\right) = \frac{dy}{dx} + 3x + 2$

Linear Diff Eq.

A diff eq is said to be linear if

i) the dependent variable y and its derivatives are all of degree 'one' only.

ii) No products of 'y' and its derivatives are present.

iii) No transcendental fn of y or its derivatives are present.

e.g $2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 3y = 0$

$\frac{dy}{dx} - x^2y = \cos x.$

A diff eq which is not linear is called Non-Linear Diff Eq:

eg

i) $\frac{d^3y}{dx^3} + 4y^2 = 0$ (∵ power of $y \neq 1$)

ii) $\frac{d^2y}{dx^2} + 7y\frac{dy}{dx} + 12y = 0$ (∵ $7y\frac{dy}{dx}$ involves product of 'y' dependent variable & derivati

iii) $\frac{d^2y}{dx^2} + \sin xy = 0$ (involves transcendental fns of dependent variable.

iv) $5\left(\frac{dy}{dx}\right)^3 + 2\frac{d^2y}{dx^2} + 3y = 0$ (∵ degree of $\frac{dy}{dx}$ is not 1)

Exercise 9.1

① Classify each of the following eqs as ordinary or partial diff eq state the order and degree of each eq and determine whether the eq is linear or non-linear.

(i) $\frac{d^3y}{dx^3} + 4\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 3y = \cos x$

Ordinary Diff Eq, order 3, degree 1,

It is Linear Diff Eq.

(ii) $x^2dy + y^2dx = 0 \Rightarrow \frac{dy}{dx} + \frac{y^2}{x^2} = 0$

Ordinary Diff Eq, order 1, degree 1

It is non linear eq $\because$ power of $y \neq 1$

(iii) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

Partial Diff Eq, order 2, degree 1

It is Linear Diff Eq.

iv) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + u\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y} + u = 0$

It is Partial Diff Eq. order 2, degree 1

Non-linear Diff Eq $\because u\frac{\partial u}{\partial x}, u\frac{\partial u}{\partial y}$ i.e Product.

v) $\left(\frac{dy}{dx}\right)^2 = \left(\frac{d^2y}{dx^2} + y\right)^{3/2}$

Ordinary Diff Eq, order 2, Degree 3

$$\because \left[\left(\frac{dy}{dx}\right)^2\right]^2 = \left[\left(\frac{d^2y}{dx^2} + y\right)^{3/2}\right]^2$$

$$= \left(\frac{d^2y}{dx^2} + y\right)^3$$

Non linear Diff Eq $\because$ Degree $\neq 1$

General Solution or (Integral) or (Complete Primitive) :-

A sol of a diff eq which contains the number of arbitrary constants equal to the order of the diff eq is called General Sol.

Particular Solution :-

A sol obtained from the general sol by giving particular values to the constants is called a particular sol or integral.

Examples. The general Sol of diff eq $\frac{d^2y}{dx^2}=0$ is $y=mx+c$. order 2 const 2, i.e m, c

whereas $y=3x+5$ is obtained by taking particular values $m=3$ & $c=5$.

Singular Sol :- (S.S)

A sol of a diff eq which cannot be obtained from the general sol by any choice of independent arbitrary const is called singular sol.

e.g the general sol of $y'=\sqrt{y}$ is $2\sqrt{y}=x+c$ and S.S is $y=0$

Note The arbitrary constants appearing in the general sol of a diff eq must be independent and to check this we show that they cannot be replaced by or reduced to a smaller number of const.

e.g $y=l\sin(x+\alpha)+m\cos x$ is the sol of $\frac{d^2y}{dx^2}+y=0$

it seems to contain three const l, m, α. But they are not independent as they can be reduced to 'two' only.

$$y = l\sin(x+\alpha)+m\cos x$$
$$= l\sin x\cos\alpha + l\cos x\sin\alpha + m\cos x$$
$$= (l\cos\alpha)\sin x + (m+l\sin\alpha)\cos x$$

or $y = A\sin x + B\cos x$ so two Arbitrary independent Consts A & B.

Initial Value Condition is a condition on the sol of a diff eq at one pt. i.e 'x_0' e.g $y(x_0)=a$, $y'(x_0)=b$ i.e at $x=\boxed{x_0}$ $y=a$ & $y'=b$

Boundary Value Condition is a cond on the sol of a diff eq at more than one pt. i.e x_0, x_1 $y(x_0)=a$, $y(x_1)=b$

<u>Formation of a differential Eq.</u> A diff eq is formed by the elimination of arbitrary constants from a relation of the form $f(x, y) = 0$.

Since to eliminate one const we need two eqs, and to eliminate two constants we need three eqs and so on. N

Now we shall be given one eq of the form $f(x,y)=0$ and the remaining required number of eqs will be formed by differentiating given eq the required number of times.

This also shows that the order of the required diff eq can not exceed the number of constants to be eliminated.

Thus we shall not diff the given eq more than the number of consts. eg to form diff eq from

$$y^2 = cx \quad \text{——— (1)}$$

diff $\quad 2y\dfrac{dy}{dx} = c \quad$ ——— (2)

Required diff eq will be obtained by eliminating 'c' between (1) & (2)

So Put (2) in (1) $\quad y^2 = x\left(2y\dfrac{dy}{dx}\right)$

Note As there is just one const, so the required diff eq is to be of order 'one'. i.e we should not diff (2) again to eliminate c as

Diff (1) $\quad 2y\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx}\cdot\dfrac{dy}{dx} = 0 \qquad$ c eliminated

(6)

Ex 9.1

Q2 (ii)

Form the diff eq of which the given fn is a sol.

(i) $y = x + 3e^{-x}$

Diff $y' = 1 - 3e^{-x}$

$= 1 - (y - x)$ $\quad \because y = x + 3e^{-x}$ eliminating e

$y' + y = x + 1$

×————×

(ii) $y = (x^3 + c)e^{-3x}$, c being arbitrary const.

diff $y' = 3x^2 e^{-3x} + (x^3+c)(-3e^{-3x})$

$= 3x^2 e^{-3x} + (-3)y$ $\quad \because y = (x^3+c)e^{-3x}$ c eliminated

$y' + 3y = 3x^2 e^{-3x}$

×————×

(iii) $ax + \ln|y| = y + b$

Diff $a + \frac{1}{y}y' = y'$

Diff $-y^{-2}y'y' + \frac{1}{y}y'' = y''$

$-\frac{(y')^2}{y^2} + \frac{y''}{y} = y''$

$\frac{-(y')^2 + yy''}{y^2} = y''$

$-(y')^2 + y\,y'' = y^2 y''$

$-(y')^2 + y''(y - y^2) = 0$

×————×

(iv) $y = ae^x + b\ln x + cx + d$ $\quad$ four const so diff four times

Diff $y' = ae^x + \frac{b}{x} + c$ —— (i)

Diff $y'' = ae^x - \frac{b}{x^2}$ —— (ii)

Diff $y''' = ae^x + \frac{2b}{x^3}$ —— (iii)

Diff $y^{iv} = ae^x - \frac{6b}{x^4}$ —— (iv)

Eliminating a & b from (ii) (iii) (iv)

$$e^x x^2 \begin{vmatrix} y'' & 1 & -1 \\ y''' & 1 & \frac{2}{x} \\ y^{iv} & 1 & \frac{-6}{x^2} \end{vmatrix} = 0$$

(v) $x^2+y^2+2gx+2fy+c=0$ three const g, f, c. So diff thrice

Diff $2x+2yy'+2g+2fy'=0$

$x+yy'+g+fy'=0$

$(x+g)+(y+f)y'=0$ — (1)

Diff $1+(y+f)y''+y'y'=0$ — (2)

Diff $(y+f)y'''+y'y''+[y'y''+y''y']=0$

$(y+f)y'''+3y'y''=0$ — (3)

$(y+f)=\dfrac{-3y'y''}{y'''}$ Put in (2)

from (2) $1+\left(\dfrac{-3y'y''}{y'''}\right)y''+y'^2=0$

$(1+y'^2)=\dfrac{3y'(y'')^2}{y'''}$

$(1+y'^2)y'''=3y'(y'')^2$

$3y'(y'')^2-(1+y'^2)y'''=0$

(vi) $u=f(x,y,z)=(x^2+y^2+z^2)^{-\frac{1}{2}}$ Diff thrice partially w.r.t x, y, z.

$\dfrac{\partial u}{\partial x}=-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2x)$

$\dfrac{\partial u}{\partial x}=-\dfrac{x}{(x^2+y^2+z^2)^{3/2}}$

$\dfrac{\partial^2 u}{\partial x^2}=-\left(\dfrac{(x^2+y^2+z^2)^{3/2}-x\cdot\frac{3}{2}(x^2+y^2+z^2)^{\frac{1}{2}}2x}{(x^2+y^2+z^2)^3}\right)$

$=-\left(\dfrac{(x^2+y^2+z^2)^{\frac{3}{2}}-3x^2(x^2+y^2+z^2)^{\frac{1}{2}}}{(x^2+y^2+z^2)^3}\right)$

$=-\dfrac{(x^2+y^2+z^2)^{\frac{1}{2}}\{x^2+y^2+z^2-3x^2\}}{(x^2+y^2+z^2)^3}$

$\dfrac{\partial^2 u}{\partial x^2}=-\dfrac{(-2x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}}=\dfrac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}$ — (i)

Similarly

$\dfrac{\partial^2 u}{\partial y^2}=\dfrac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^{5/2}}$ — (ii)

& $\dfrac{\partial^2 u}{\partial z^2}=\dfrac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}$ — (iii)

Adding $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}=0$

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vii) $u = f(x-ay) + g(x+ay)$ f & g are twice diff^{l} fns.

$$\frac{\partial u}{\partial x} = f'(x-ay) + g'(x+ay)$$

$$\frac{\partial^2 u}{\partial x^2} = f''(x-ay) + g''(x+ay) \quad \text{---} (1)$$

$$\frac{\partial u}{\partial y} = f'(x-ay)(-a) + g'(x+ay)(a)$$

$$\frac{\partial^2 u}{\partial y^2} = f''(x-ay)(-a)(-a) + g''(x+ay)(a)(a)$$

$$= a^2\left[f''(x-ay) + g''(x+ay)\right]$$

$$\frac{\partial^2 u}{\partial y^2} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{using } (1)$$

(3) MT (1) Find the differential eq of all circles of radius 'a'. (a is fixed)

Eq of circles of radius 'a': $(x-h)^2 + (y-k)^2 = a^2$

a is fixed given (h, k two arbitrary const so differentiate twice)

Diff $2(x-h) + 2(y-k)y' = 0$

$$(x-h) + (y-k)y' = 0 \quad \text{---} (1)$$

Diff $1 + (y-k)y'' + y'y' = 0$

$$(y-k)y'' = -1 - y'^2$$

$$(y-k) = \frac{-(1+y'^2)}{y''} \quad \text{---} (2)$$

Put (2) in (1) $(x-h) - \frac{(1+y'^2)}{y''}y' = 0$

$$(x-h) = \frac{(1+y'^2)}{y''}y' \quad \text{---} (3)$$

Squaring & Adding (2) & (3) to eliminate consts.

$$(x-h)^2 + (y-k)^2 = \left(\frac{1+y'^2}{y''}\right)^2 y'^2 + \left(\frac{1+y'^2}{y''}\right)^2$$

$$a^2 = \left(\frac{1+y'^2}{y''}\right)^2\left[y'^2 + 1\right]$$

$$a^2(y'')^2 = (1+y'^2)^2(y'^2+1)$$

$$a^2(y'')^2 = (1+y'^2)^3$$

(9)

(ii) Find the diff eq of circles that pass through origin.

Eq of all circles passing through origin is

$$x^2+y^2+2gx+2fy = 0 \quad \text{—— (1)}$$ Two Const f, g So diff twice

Diff $2x+2yy'+2g+2fy' = 0$

$$x+yy'+g+fy' = 0$$

$$(x+g)+y'(y+f) = 0 \quad \text{—— (2)}$$

Diff $1+(y+f)y''+y'y' = 0$

$$(y+f) = -\frac{(1+y'^2)}{y''} \quad \text{—— (3)}$$

Put (3) in (2) $(x+g)+y'\left(-\frac{(1+y'^2)}{y''}\right) = 0$

$$(x+g) = y'\left(\frac{1+y'^2}{y''}\right) \quad \text{—— (4)}$$

Multiply (4) by x & (3) by y and adding

$$x(x+g)+y(y+f) = xy'\left(\frac{1+y'^2}{y''}\right)-y\frac{(1+y'^2)}{y''}$$

$$x^2+gx+y^2+fy = (xy'-y)\left(\frac{1+y'^2}{y''}\right)$$

$$x^2+y^2+(gx+fy) = (xy'-y)\left(\frac{1+y'^2}{y''}\right)$$

using (1)

$$x^2+y^2+\left(\frac{x^2+y^2}{-2}\right) = (xy'-y)\left(\frac{1+y'^2}{y''}\right)$$

$$\frac{x^2+y^2}{2} = (xy'-y)\left(\frac{1+y'^2}{y''}\right)$$

$$(x^2+y^2)y'' = 2(xy'-y)(1+y'^2)$$

(iii) Find the diff eq of ellipses in standard form.

Ellipses in standard form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ —— ①

Diff twice because two const a, b.

$$\frac{2x}{a^2}+\frac{2y}{b^2}y'=0$$

$$\frac{x}{a^2}+\frac{yy'}{b^2}=0 \quad —— ②$$

Diff $\frac{1}{a^2}+\frac{yy''+y'^2}{b^2}=0$

$\times$ by x

$$\Rightarrow \frac{x}{a^2}+\frac{x(yy''+y'^2)}{b^2}=0$$

$$\Rightarrow \frac{x}{a^2}=-\frac{x(yy''+y'^2)}{b^2} \quad —— ③$$

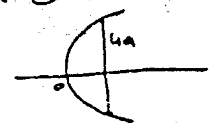
Put ③ in ②

$$\Rightarrow -\frac{x(yy''+y'^2)}{b^2}+\frac{yy'}{b^2}=0$$

$$\Rightarrow -xyy''-xy'^2+yy'=0$$

$$xy'^2+xyy''-yy'=0$$

(iv) Find the diff eq of Parabolas each of which has a latus rectum $4a$ and whose axes are // to x-axis.



$y^2=4ax$

$(y-k)^2=4a(x-h)$

$\therefore$ axis // to x-axis.

Eq of given parabola is $(y-k)^2=4a(x-h)$ —— ①

Diff twice because two const h, k.

$$2(y-k)y'=4a$$

$$(y-k)y'=2a \quad —— ②$$

$$y'y'+(y-k)y''=0$$

$$y'^2+(y-k)y''=0$$

$$(y-k)=-\frac{y'^2}{y''} \quad —— ③$$

Put ③ in ②

$$\left[-\frac{y'^2}{y''}\right]y'=2a$$

$$-y'^3=2ay''$$

$$0=2ay''+y'^3$$

(v) Find diff eq of Hyperbolas in standard form.

standard eq of Hyperbolas is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ —(1)

Diff twice ∵ two const a, b

Diff $\frac{2x}{a^2}-\frac{2y}{b^2}y'=0$

$\frac{x}{a^2}-\frac{yy'}{b^2}=0$ —(2)

Diff $\frac{1}{a^2}-\frac{(yy''+y'^2)}{b^2}=0$

× by x ⇒ $\frac{x}{a^2}-\frac{x(yy''+y'^2)}{b^2}=0$ —

⇒ $\frac{x}{a^2}=\frac{x(yy''+y'^2)}{b^2}$ —(3)

Put (3) in (2)

$\frac{x(yy''+y'^2)}{b^2}-\frac{yy'}{b^2}=0$

$xyy''+xy'^2-yy'=0$

vi) Find diff eq of conics which coincide with the axes of coordinates.

Find diff eq of conics whose axes coincide with axes of coord.

$ax^2+by^2=1$ —(1) is Eq of conics whose axes coincide with axes of coord.

Diff twice because two const a, b

Diff $2ax+2byy'=0$

$ax+byy'=0$ —(2)

Diff $a+b(yy''+y'^2)=0$ —(3)

Eliminating a, b from (1) (2), (3)

$$\begin{vmatrix} x^2 & y^2 & 1 \\ x & yy' & 0 \\ 1 & yy''+y'^2 & 0 \end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix} x & yy' \\ 1 & yy''+y'^2 \end{vmatrix}=0$$

$$\Rightarrow x(yy''+y'^2)-yy'=0$$

(4) Solve the following initial value problems. (at one value of x)

(i) $\frac{dy}{dx} = -\frac{x}{y}$, $y(3) = 4$

G.Sol is $x^2 + y^2 = c^2$

$3^2 + 4^2 = c^2 \Rightarrow \boxed{c = 5}$

$\therefore x^2 + y^2 = 25$ is req sol.

(ii) $\frac{dy}{dx} + y = 2xe^{-x}$, $y(-1) = e+3$

G.Sol is $y = (x^2 + c)e^{-x}$

$e+3 = (1+c)e^{-(-1)}$ $\quad \because y(-1) = e+3$

$\cancel{e}+3 = \cancel{e} + ce \Rightarrow \boxed{c = \frac{3}{e}}$

$\therefore y = (x^2 + \frac{3}{e})e^{-x}$ is Particular Sol

(iii) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 12 = 0$, $y(0) = -2$, $y'(0) = 6$

G.Sol is $y = Ae^{4x} + Be^{-3x}$ —— (I)

$\therefore (-2) = Ae^0 + Be^0$ $\quad \because y(0) = -2$

$-2 = A + B$ —— (II)

diff (I) $y' = 4Ae^{4x} - 3Be^{-3x}$

$6 = 4Ae^0 - 3Be^0$ $\quad \because y'(0) = 6$

$6 = 4A - 3B$ —— (III)

solving (II) & (III)

× (II) by 4. $\quad -8 = \cancel{4A} + 4B$

$\quad 6 = \cancel{4A} - 3B$

$-14 = 7B \quad \boxed{B = -2}$

using (II) $\boxed{0 = A}$. $\therefore y = -2e^{-3x}$ is P.Sol.

(iv) $x\frac{dy}{dx} + 2y = 4x^2$ $\quad y(1) = 2$

G.Sol is $y = x^2 + \frac{c}{x^2}$

$2 = 1^2 + \frac{c}{1^2}$ $\quad \because y(1) = 2$

$\boxed{1 = c}$

$\therefore$ P. Sol is $y = x^2 + \frac{1}{x^2}$

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(v) $x^3\frac{d^3y}{dx^3}-3x^2\frac{d^2y}{dx^2}+6x\frac{dy}{dx}-6y=0$, $y(2)=0$, $y'(2)=2$, $y''(2)=6$

G.Sol is $y=c_1x+c_2x^2+c_3x^3$ —(I)

$y'=c_1+c_2(2x)+c_3(3x^2)$ —(II)

$y''=2c_2+c_3 6x$ —(III)

from (I) $0=2c_1+4c_2+8c_3$ (IV) $\because y(2)=0$

from (II) $2=c_1+4c_2+12c_3$ (V) $\because y'(2)=2$

from (III) $6=2c_2+12c_3$ (VI) $\because y''(2)=6$

from (IV) $0=c_1+2c_2+4c_3$

from (V) $2=c_1+4c_2+12c_3$ subtracting

$-2=-2c_2-8c_3$

from (VI) $6=2c_2+12c_3$ Adding

$4=0+4c_3$

$\boxed{1=c_3}$

from (VI) $\boxed{-3=c_2}$

from (V) $\boxed{2=c_1}$

$\therefore$ P.Sol is $y=2x-3x^2+x^3$

5 (i) Solve boundary value Problem (values of x are more than one)

$\frac{d^2y}{dx^2}+y=0$, $y(0)=1$ $y'(\frac{\pi}{2})=-1$

G.Sol is $y=c_1\sin x+c_2\cos x$ —(I)

$y'=c_1\cos x-c_2\sin x$ —(II)

from (I) $\boxed{1=c_2}$ $\because y(0)=1$

from (II) $\boxed{-1=-c_2}$ $\because y'(\frac{\pi}{2})=-1$

$y=c_1\sin x+\cos x$ is P.Sol.

(ii) $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} + 2y = 0$, $y(0) = 0$ $y(1) = 1$

$y = c_1 e^x + c_2 e^{3x}$ is the G. Sol — (i)

$0 = c_1 e^0 + c_2 e^0$ $\because y(0) = 0$

$0 = c_1 + c_2$ — (ii) $\Rightarrow c_1 = -c_2$

$1 = c_1 e + c_2 e^3$ $\because y(1) = 1$

$\dfrac{1}{e} = c_1 + c_2 e^2$ — (iii) $\div$ by e

Subtract (iii) from (ii)

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_1 + c_2 e^2 &= \tfrac{1}{e} \\ \hline c_2 - c_2 e^2 &= -\tfrac{1}{e} \end{aligned}$$

$c_2(1-e^2) = -\dfrac{1}{e}$ $\Rightarrow c_2 = \dfrac{-1/e}{1-e^2}$

$\Rightarrow c_2 = \dfrac{-1}{e(1-e^2)}$

using (ii) $\therefore$ $c_1 = \dfrac{1}{e(1-e^2)}$

$y = \dfrac{1}{e(1-e^2)} e^x + \left(\dfrac{-1}{e(1-e^2)}\right) e^{3x}$

$= \dfrac{1}{e(1-e^2)}\left[e^x - e^{3x}\right]$ Ans.

If c_1 or c_2 has two different values as $c_1 = -1$ & $c_1 = 2$ then we can not determine c_2, hence No Solution exist.

See Example 7.