Power Series

An infinite series of the form

$$\sum_{0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots c_n x^n + \cdots \quad \text{(i)}$$

or $$\sum_{0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots c_n(x-a)^n + \cdots \quad \text{(ii)}$$

is called Power Series in x or $(x-a)$, $c_n \in R$

Series (ii) can be reduced to series (i) by putting $(x-a) = Y$

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Convergence of Power Series (i) i.e. $\sum_{0}^{\infty} c_n x^n$

We use the Ratio Test for Absolute Convergence or Root Test for Absolute convergence to find the values of x for which Power Series (i) converges.

Example. Find the values of x for which the power series $\sum_{0}^{\infty} n^{2n} x^{2n}$ converges.

Sol $\sum a_n = \sum n^{2n} x^{2n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^{2(n+1)} x^{2(n+1)}}{n^{2n} x^{2n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)^{2n}(n+1)^2 x^{2n} x^2}{n^{2n} x^{2n}}\right|$$

$$= \lim_{n\to\infty}\left|\left(\frac{n+1}{n}\right)^{2n}(n+1)^2 x^2\right|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\left(1+\frac{1}{n}\right)^n (n+1)x\right|^2$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0 < 1 \quad \text{, when } x = 0$$

$$= \infty \quad \text{, when } x \neq 0$$

$\therefore$ Series Converges for $x = 0$

$\therefore$ Series diverges for $x \neq 0$

Easy 2nd Method

$$\lim_{n\to\infty}|a_n|^{\frac{1}{n}} = \lim_{n\to\infty}|n^{2n}x^{2n}|^{\frac{1}{n}}$$

$$\lim_{n\to\infty}|a_n|^{\frac{1}{n}} = \lim_{n\to\infty}|n^2x^2|^{n\cdot\frac{1}{n}}$$

$$\lim_{n\to\infty}|a_n|^{\frac{1}{n}} = \lim_{n\to\infty} n^2|x^2|$$

$$= 0 < 1 \quad \text{when } x = 0$$

and $$\lim_{n\to\infty}|a_n|^{\frac{1}{n}} = \infty \quad \text{when } x \neq 0$$

So series converges for $x = 0$ & diverges for $x \neq 0$

Example Find values of x for which Power Series $\sum_{0}^{\infty} \frac{x^n}{2n!}$ converges.

Sol $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{2(n+1)!}}{\frac{x^n}{2n!}}\right| = \left|\frac{x^n \cdot x}{(2n+2)!} \times \frac{2n!}{x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x}{(2n+2)(2n+1)\cancel{2n!}} \times \cancel{2n!}\right| = \lim_{n\to\infty}\left|\frac{x}{(2n+2)(2n+1)}\right| = 0 < 1$$

$\therefore$ given series cgs for all values of x

Example Determine the values of x for which the power series
$$\sum_{n=2}^{\infty} \frac{x^n}{\ln n}$$ converges absolutely, converges conditionally & diverges.

Sol: Ratio Test for Ab. Cgce.

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left|\frac{x^{n+1}}{\ln(n+1)}\right| \times \left|\frac{\ln n}{x^n}\right|$$

$$= \lim_{n\to\infty} \left| \frac{x^n \cdot x \cdot \ln n}{x^n \cdot \ln(n+1)} \right| \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{n\to\infty} |x| \cdot \frac{\frac{1}{n}}{\frac{1}{n+1}} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{n\to\infty} |x| \cdot \frac{1}{n} \cdot \frac{n+1}{1} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{n\to\infty} |x| \cdot \frac{1}{1}$$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = |x| \quad \text{——— (1)}$$

$|a_n| = \left|\frac{x^n}{\ln n}\right|$

$|a_{n+1}| = \left|\frac{x^{n+1}}{\ln(n+1)}\right|$

$|\ln n| = \ln n$

$|\ln(n+1)| = \ln(n+1)$

Now power series Converges Absolutely if $|x| < 1$

power series Diverges if $|x| > 1$

Test fails to determine, if $|x| = 1$

Now $|x| = 1 \Rightarrow x = \pm 1$

For $x = -1$

$$\sum a_n = \sum \frac{x^n}{\ln n} = \sum \frac{(-1)^n}{\ln n} \quad \text{(Alternating Series)}$$

Now Mod

$$\sum |a_n| = \sum \left|\frac{1}{\ln n}\right| < \sum \frac{1}{n}$$ which is dgt. Since $\sum |a_n|$ is dgt by BCT so apply ALST.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\ln n} = \frac{1}{\infty} = \boxed{0}$$

Note $f(x) = \frac{1}{\ln x} = (\ln x)^{-1}$

$f'(x) = (-1)(\ln x)^{-2} \frac{1}{x} < 0$

$f'(x) < 0$ Non Increasing

Also $\frac{1}{\ln(n+1)} < \frac{1}{\ln n}$

$\Rightarrow \boxed{a_{n+1} < a_n}$ Hence non increasing seq

Therefore Alt Series is convergent and

Since $\sum |a_n|$ is dgt So Power Series is Cond Cgt.

For $x = 1$

$$\sum a_n = \sum \frac{x^n}{\ln n} = \sum \frac{1}{\ln n} > \sum \frac{1}{n}$$ which is dgt. So $\sum a_n$ is dgt By BCT.

$\Rightarrow$ Hence Power Series

Converges absolutely for $|x| < 1$

Diverges for $|x| > 1$

Diverges for $x = 1$

Converges Conditionally for $x = -1$

$x \longrightarrow x$

Example Find interval of Convergence of p-series $\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$

Sol $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\lfloor n+1 \, x^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{\lfloor n \, x^n}\right|$

$$= \left|\frac{\cancel{(n+1)}\cancel{\lfloor n}\,\cancel{x^n}\cdot x \cdot n^n}{(n+1)^n\cancel{(n+1)}\,\cancel{\lfloor n}\,\cancel{x^n}}\right|$$

$$= \left|\frac{x}{\frac{(n+1)^n}{n^n}}\right| = \left|\frac{x}{\left(\frac{n+1}{n}\right)^n}\right|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x}{\frac{x}{x}\left(1+\frac{1}{n}\right)^n}\right| = \left|\frac{x}{e}\right|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|}{e}$$

Now

If $\frac{|x|}{e} < 1$: P.Series Cgs Abs.

$\frac{|x|}{e} < 1 \Rightarrow |x| < e$

$\Rightarrow -e < x < e$

Radius of Convergence $= e$

If $\frac{|x|}{e} > 1$ Power Series diverges

If $\frac{|x|}{e} = 1$ Test fails to determine

$\frac{|x|}{e} = 1 \Rightarrow |x| = e \Rightarrow x = \pm e$

Now for $x = e$, $\sum_{n=1}^{\infty} \frac{n!\, x^n}{n^n} = \sum_{n=1}^{\infty} \frac{n!}{n^n} e^n$ —— (I)

for $x = -e$, $\sum_{n=1}^{\infty} \frac{n!\, x^n}{n^n} = \sum_{n=1}^{\infty} \frac{n!\,(-1)^n e^n}{n^n}$ —— (II)

Consider the term $\frac{n!\, e^n}{n^n}$ (which is common in (I) & (II) i.e. for $x = e$ or $x = -e$)

$$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\lfloor n \, e^n}{n^n}$$

$$= \lim_{n\to\infty}\left(\sqrt{2\pi n}\,\frac{\cancel{n^n}}{\cancel{e^n}}\right)\cdot\frac{\cancel{e^n}}{\cancel{n^n}}$$

$\lim_{n\to\infty} a_n = \infty \neq 0$ so both (I) & (II) diverges

Therefore Interval of Convergence is $]-e, e[$

If n is very large then by Stirling Formula

$\lfloor n = \sqrt{2\pi n}\,\frac{n^n}{e^n}$

Ist Cond. of AEST is not satisfied Hence dg.

2nd Method for term "$\frac{\lfloor n \, e^n}{n^n}$"

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\lfloor n+1 \, e^{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{\lfloor n \, e^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\cancel{(n+1)}\cancel{\lfloor n}\,\cancel{e^n}\cdot e \; n^n}{(n+1)^n\cdot\cancel{(n+1)}\,\cancel{\lfloor n}\,\cancel{e^n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n^n \cdot e}{(n+1)^n}\right|$$

$$= \lim_{n\to\infty}\left|\left(\frac{n}{n+1}\right)^n\cdot e\right|$$

$$= \lim_{n\to\infty}\left|\left(\frac{n+1}{n}\right)^{-n}\cdot e\right|$$

$$= \lim_{n\to\infty}\left|\left[\left(1+\frac{1}{n}\right)^n\right]^{-1}\cdot e\right|$$

$$= \lim_{n\to\infty}\left|e^{-1}\cdot e\right| = \boxed{1}$$

Test fails

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Imp Note

1) If power series $\sum C_n x^n$ converges for $x = x_1$ then power series is converges for $|x| < |x_1|$

اگر P.S 'x_1' کے لیے cgs ہے تو x_1 سے چھوٹی تمام values کے لیے cgs ہو گی

2) If power series $\sum C_n x^n$ diverges for $x = x_2$ then power series diverges for $|x| > |x_2|$

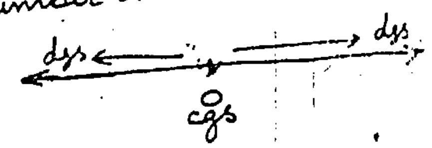
اگر سیریز P.S 'x_2' کے لیے dgs ہے تو x_2 سے بڑی تمام values کے لیے dgs ہے.

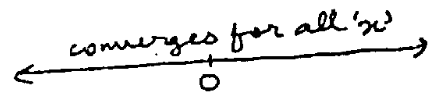
3) For the power series $\sum_0^\infty C_n x^n$ exactly one of the following conditions hold.

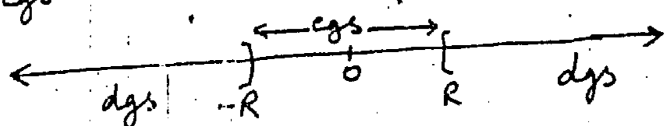
i) The series converges only for $x = 0$
ii) The series converges absolutely for all values of x
iii) The series converges absolutely for all values of x s.t $|x| < R$
iv) The series diverges for all values of x s.t $|x| > R$

where R is +ive number

On number line it can be illustrated as.

i) 

ii) 

iii) 

i.e cgs for $|x| < R$
$-R < x < R$
dgs for $|x| > R$
$-R > x > R$
satisfy

Interval of Convergence & Radius of Convergence

The set of all values of x for which the power series converges is called Interval of convergence of the power series.

If power series converges absolutely for x s.t $|x| < R$ then R is called Radius of convergence where R is +ive number.

i) If P.Series cgs for $x = 0$, then Its Interval of convergence $= 0$ & Radius of Cgence $= 0$
ii) If P.Series cgs for all x, then Its Interval of Cgence is $]-\infty, \infty[$ & Radius of convergence $= \infty$
iii) If P.Series cgs for $|x| < R$ and dgs for $|x| > R$ then Its Interval of Convergence is one of $[-R, R]$; $]-R, R[$; $]-R, R]$; $[-R, R[$ & Radius of Convergence is R
v) If P Series $\sum_0^\infty C_n (x-a)^n$ cgs for all x then Interval of Cgce $=]-\infty, \infty[$ & Radius of Cgce $= \infty$
vi) If P Series $\sum_0^\infty C_n (x-a)^n$ cgs only for $x = a$ then Interval of Cgce $= a$ & Radius of Cgce $= 0$
vii) If P Series $\sum_0^\infty C_n (x-a)^n$ cgs for $|x-a| < R$ then Interval of Cgce is one of $]a-R, a+R[$, $[a-R, a+R]$, $[a-R, a+R[$, $]a-R, a+R]$ & Radius of Cgce $= R$

Ex. 8.3

In each of the following, find the radius of convergence and interval of convergence (Problems 1–24):

Q.1 $\sum_{n=0}^{\infty} \frac{x^n}{2n!}$

Note: In all questions n is +ive, integer but x has any real value

SOL. Here $|a_n| = \frac{|x^n|}{2n!}$ and $|a_{n+1}| = \frac{|x|^{n+1}}{(2n+2)!}$

$$\therefore \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x|^{n+1}}{(2n+2)!} \times \frac{2n!}{|x|^n}$$

$$= \lim_{n\to\infty} \frac{\cancel{2n!}\,|x|}{(2n+2)(2n+1)\cancel{2n!}} = \lim_{n\to\infty} \frac{|x|}{(2n+2)(2n+1)}$$

$$= 0 < 1 \quad \text{for all values of } x$$

So by R. test for absolute convergence, the P Series converges absolutely for all values of x. Thus interval of convergence is $]-\infty, \infty[$ and radius of convergence 'R' is ∞.

Q.2 $\sum_{0}^{\infty} \frac{2^n x^n}{Ln(n+2)} \longrightarrow (1)$

SOL. using ratio test for A. convergent when

$$a_n = \frac{2^n x^n}{Ln(n+2)} \quad \text{i.e.}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{2^{n+1} \cdot x^{n+1}}{Ln(n+3)} \times \frac{Ln(n+2)}{2^n \cdot x^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{2\,Ln(n+2)}{Ln(n+3)} \cdot x \right| = \lim_{n\to\infty} 2|x| \frac{Ln(n+2)}{Ln(n+3)} \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{n\to\infty} 2|x| \cdot \frac{\frac{1}{n+2}}{\frac{1}{n+3}} \left(\frac{0}{0}\right) = \lim_{n\to\infty} 2|x| \frac{n+3}{n+2} \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{n\to\infty} 2|x| \frac{\cancel{n}(1+\frac{3}{n})}{\cancel{n}(1+\frac{2}{n})} = 2|x| \cdot 1 \longrightarrow (1)$$

(1) implies that the power series converges for $2|x| < 1$

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$|x| < \frac{1}{2} \rightarrow ②$

and power series diverges for $\because 2|x| > 1$ i.e for $|x| > \frac{1}{2}$ $\rightarrow ③$

and when $2|x| = 1$ i.e $2x = \pm 1 \Rightarrow x = \pm \frac{1}{2}$

for $x = \frac{1}{2}$ Put in ① the Series becomes $\sum_{n=0}^{\infty} \frac{2^n \left(\frac{1}{2}\right)^n}{\ln(n+2)}$ $\boxed{2^n \cdot \left(\frac{1}{2}\right)^n = 1}$

$= \sum_{n=0}^{\infty} \frac{1}{\ln(n+2)} = \sum a_n$

and since $\ln(n+2) < n+2 < 3n$

$\Rightarrow \frac{1}{\ln(n+2)} > \frac{1}{3n} \Rightarrow a_n > b_n$

i.e $\sum a_n = \sum \frac{1}{\ln(n+2)} > \sum \frac{1}{3n} = \sum b_n$

but $\sum b_n = \frac{1}{3} \sum \frac{1}{n}$ is divergent so by C.T. $\sum a_n = \sum_{n=0}^{\infty} \frac{1}{\ln(n+2)}$ is divergent

for $x = -\frac{1}{2}$ Put in ① Series becomes $\sum_{n=0}^{\infty} \frac{2^n \left(-\frac{1}{2}\right)^n}{\ln(n+2)} = \sum_{n=0}^{\infty} (-1)^n / \ln(n+2)$

which is an Alternating Series

ALST

$|a_n| = \frac{1}{\ln(n+2)}$

$\Rightarrow \lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{\ln(n+2)} = 0$

Now $f(x) = \frac{1}{\ln(x+2)} = [\ln(x+2)]^{-1}$

$f'(x) = \frac{-1}{[\ln(x+2)]^2} \cdot \frac{1}{(x+2)} < 0$ Non Increasing Seq.

Hence At $x = -\frac{1}{2}$, Al. series is Cgt

So. series cgs for $|x| < \frac{1}{2}$ & $x = -\frac{1}{2}$ So

So Interval of Convergence is $\left[-\frac{1}{2}, \frac{1}{2}\right[$

Radius of Convergence $= \frac{1}{2}$

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Q.3 $\sum_{n=2}^{\infty} \frac{(x-5)^n \ln n}{n+1}$ ——— ①

Sol. $\Rightarrow a_n = \frac{(x-5)^n \ln n}{n+1}$

using ratio test for absolute convergence i.e.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-5)^{\cancel{n}+1}\ln(n+1)}{n+2} \times \frac{n+1}{\cancel{(x-5)^n}\ln n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)\ln(n+1)\cdot(x-5)}{(n+2)\ln n}\right|$$

$$= \lim_{n\to\infty}|x-5|\left(\frac{n+1}{n+2}\cdot\frac{\ln(n+1)}{\ln n}\right) = |x-5|\lim_{n\to\infty}\frac{\cancel{n}(1+\frac{1}{n})}{\cancel{n}(1+\frac{2}{n})}\cdot\frac{\ln(n+1)}{\ln n} \quad \frac{\infty}{\infty}$$

$$= |x-5|\lim_{n\to\infty}\frac{\frac{1}{n+1}}{1/n} = |x-5|\lim_{n\to\infty}\left(\frac{n}{n+1}\right)$$

$$= |x-5|\cdot\lim_{n\to\infty}\cancel{n}\left(\frac{\cancel{n}}{1+\frac{1}{n}}\right) = |x-5|\cdot 1 = \boxed{|x-5|}$$

$\Rightarrow$ Power series converges if $|x-5|<1$ and

——— diverges if $|x-5|>1$

and when $|x-5|=1$

$x-5=\pm 1$

$x-5=1$, $x-5=-1$

$x=6$, $x=4$

for $x=4$, Put in ① $\quad \Sigma a_n = \sum_{n=2}^{\infty}\frac{(-1)^n \ln n}{n+1}$, which is an Alt. Series

where $|a_n| = \frac{\ln n}{n+1}$, let $|b_n| = \frac{1}{n}$

$$\therefore \lim_{n\to\infty}\frac{|a_n|}{|b_n|} = \lim_{n\to\infty}\frac{\ln n}{n+1}\times n = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)\ln n = \lim_{n\to\infty}\frac{\cancel{n}}{\cancel{n}(1+\frac{1}{n})}.\ln n$$

$$= \ln\infty = \infty$$

Since $\Sigma|b_n| = \Sigma\frac{1}{n}$ is divergent

$\therefore \Sigma|a_n|$ is divergent (by L.C. test) ——→ ①

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ALTERNA

Note using A Series Test

$$\lim_{n\to\infty} |a_n| = \frac{\ln n}{n+1}\left(\frac{\infty}{\infty}\right) = \lim_{n\to\infty} \frac{1/n}{1} = 0$$

and $|a_n| - |a_{n+1}| = \frac{\ln(n)}{n+1} - \frac{\ln(n+1)}{(n+2)}$

$$= \frac{(n+2)\ln(n) - (n+1)\ln(n+1)}{(n+1)(n+2)} > 0 \text{ for } n \geq 2$$

Note: we can also prove that $|a_n| > |a_{n+1}|$ by taking $f(x) = \frac{\ln x}{x+1}$ and proving
$\Rightarrow f'(x) = \frac{1}{(x+1)^2}\left[(x+1)\frac{1}{x} - 1\ln x\right]$
$= \ldots \left[\frac{(x+1) - x\ln x}{x(x+1)^2}\right] < 0$
$\because x\ln x > x+1$ for $x \geq 4$.

$\Rightarrow |a_n| - |a_{n+1}| > 0 \Rightarrow |a_n| > |a_{n+1}|$

Since both conditions of an A. Series Test are satisfied

So A-Series is convergent $\longrightarrow$ (2)

Combining (1) & (2) for $x = 4$, A-Series is conditionally Convergent.

For $x = 6$, the power series becomes $\sum_{n=2}^{\infty} \frac{\ln n}{n+1} = \sum a_n$

Let $\sum b_n = \sum \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty}\left(\frac{\ln n}{n+1}\right)\cdot\frac{n}{1} = \lim_{n\to\infty}(\ln n)\left(\frac{n}{n+1}\right) = \lim_{n\to\infty} \ln n \left(\frac{1}{1+\frac{1}{n}}\right) = \lim_{n\to\infty} 1\cdot \ln n = \infty$$

By L. Comparision test $\sum a_n = \sum_{n=2}^{\infty}\frac{\ln n}{n+1}$ diverges ($\because \sum b_n = \sum \frac{1}{n}$ dgs)

So Series cgs for $|x-5| < 1$, $-1 < x-5 \leq 1$, $-1+5 < x < 1+5$, $4 < x < 6$

For $x = 4$ cgs
For $x = 6$ dgs

Therefore Interval of Convergence $[4, 6[$
Radius of Convergence $= 1$

(If P. Series converges for $|x-a| < R$ and diverges for $|x-a| > R$ Then Radius of Convergence $= R$)

Q.4 $\sum_{1}^{\infty} \frac{\sin n\pi x}{n^2} = \sum a_n$

Sol. $\Rightarrow \sum |a_n| = \sum_{1}^{\infty}\left|\frac{\sin n\pi x}{n^2}\right| = \frac{|\sin n\pi x|}{n^2}$ $\quad 0 \leq |\sin n\pi x| \leq 1$

$$= \frac{|\sin n\pi x|}{n^2} \leq \frac{1}{n^2} \text{ (cgt)}$$

$$|a_n| \leq b_n$$

$\because \sum b_n = \sum \frac{1}{n^2}$ is a cgt series

So by comparision test $\sum_{1}^{\infty}|a_n| = \sum_{1}^{\infty}\left|\frac{\sin n\pi x}{n^2}\right|$ is convergent. So P. Series (given) is absolutely convergent by A. cgt. Test. $\forall$ values of x. Thus its

"cgt" mean Convergent

Interval of convergence is $]-\infty, \infty[$

and radius of convergence is ∞

Q.5 $\sum_{n=1}^{\infty} n^2 (x-2)^n$ ——— ①

SOL: Here $a_n = n^2 (x-2)^n$

using R-T. for absolute cgt.

i.e $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^2 (x-2)^{\cancel{n}+1}}{n^2 \cancel{(x-2)^n}}\right| = \lim_{n\to\infty} \frac{(n+1)^2}{n^2} |x-2|$

$= \lim_{n\to\infty} \frac{\cancel{n^2}\left(1+\frac{1}{n}\right)^2}{\cancel{n^2}} |x-2| = |x-2|$

Equivalent to

$\Rightarrow$ P-Series is cgt for $|x-2| < 1$,

——— dgt for $|x-2| > 1$

$|x-2| = 1 \Rightarrow x-2 = \pm 1$ or $x = 2 \pm 1 = 3, 1$

For $x=1$ Put in ① Power Series becomes $\sum n^2 (-1)^n$ (i.e A. Series)

$\Rightarrow \sum |a_n| = \sum n^2 \quad \because \lim_{n\to\infty} |a_n| = \lim_{n\to\infty} n^2 = \infty \neq 0$

$\lim_{n\to\infty} a_n \neq 0$ So by divergent test $\therefore \sum |a_n|$ diverges.

Now for Al.S.T $f(x) = x^2 \Rightarrow f'(x) = 2x \nless 0$ $\therefore$ Not Non-Increasing

Hence Al. Series is divergent.

For $x=3$ Put in ① Power Series becomes $\sum n^2$ which is divergent (already proved)

Thus interval of convergence is $]1, 3[$

and radius of convergence is 1.

Q.6 $\sum_{n=1}^{\infty} \frac{n!\, x^n}{2n!}$

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SOL. Here $a_n = \frac{n!\, x^n}{2n!}$

using R. Test for A. cgt. i.e

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$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,x^{n+1}}{(2n+2)!} \times \frac{2n!}{n!\,x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)\cancel{n!}\,x\,\cancel{x^n}}{(2n+2)(2n+1)\cancel{2n!}} \times \frac{\cancel{2n!}}{\cancel{n!}\,\cancel{x^n}}\right|$$

$$= \lim_{n\to\infty}\frac{1+\frac{1}{n}}{n\left(2+\frac{2}{n}\right)\left(2+\frac{1}{n}\right)}|x| = 0 < 1$$

So by ratio test for A-Convergent, Power Series

Converges for all x

So Radius of Convergence is $=\infty$

and interval of convergence $]-\infty, \infty[$

Q.7 $\sum_{n=1}^{\infty} \frac{n \cdot 2^n (x-1)^n}{n+1}$ ——— (1)

Sol: $a_n = \frac{n \cdot 2^n (x-1)^n}{n+1}$

using Ratio test for absolute convergent i.e

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)\,2^{n+1}\,(x-1)^{n+1}}{n+2} \times \frac{n+1}{n\cdot\cancel{2^n}\,\cancel{(x-1)^n}}\right|$$

$$= \lim_{n\to\infty}\frac{(n+1)^2}{n(n+2)} \cdot 2|x-1| = \lim_{n\to\infty}\frac{\cancel{n}\left(1+\frac{1}{n}\right)^2}{\cancel{n}\left(1+\frac{2}{n}\right)} 2|x-1|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 2|x-1|$$

By Ratio Test Power Series Converges for $2|x-1|<1 \Rightarrow |x-1|<\frac{1}{2}$

Power Series diverges for $2|x-1|>1 \Rightarrow |x-1|>\frac{1}{2}$

$2|x-1|=1 \Rightarrow |x-1|=\frac{1}{2}$

$x-1=\pm\frac{1}{2}$

$x=1\pm\frac{1}{2}$

$=\frac{3}{2}, \frac{1}{2}$

For $x=\frac{3}{2}$ (Put in (1)) $\sum \frac{n\,2^n\left(\frac{3}{2}-1\right)^n}{n+1} = \sum \frac{n\,\cancel{2^n}\left(\frac{1}{2}\right)^{\cancel{n}}}{n+1} = \sum \frac{n}{n+1} = 1$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{\cancel{n}}{\cancel{n}\left(1+\frac{1}{n}\right)} = 1 \neq 0$ so dgs by Divergent Test.

For $x=\frac{1}{2}$ (Put in (1)) Power series becomes $\sum_{n=1}^{\infty}\frac{n\cdot 2^n\left(-\frac{1}{2}\right)^n}{n+1}$

or $= \sum_{n=1}^{\infty} \dfrac{n \cdot \cancel{2^n} (-1)^n \frac{1}{\cancel{2^n}}}{n+1} = \sum_{n=1}^{\infty} (-1)^n \dfrac{n}{n+1}$ (Al. Series)

Mod

$\Rightarrow |a_n| = \dfrac{n}{n+1}$

then $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \dfrac{n}{n+1} = \lim_{n\to\infty} \dfrac{\cancel{n}}{\cancel{n}(1+\frac{1}{n})} = 1 \neq 0$

So by Divergent test $\Sigma |a_n|$ diverges i.e $\lim_{n\to\infty} a_n \neq 0$

Now by ALST

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{n}{n+1} = \lim_{n\to\infty} \dfrac{\cancel{n}}{\cancel{n}(1+\frac{1}{n})} = 1 \neq 0,$

$\because$ Ist Condition of ALST is not satisfied Hence given series is Dgt by DEST

Hence given series is Divergent for $x = \frac{1}{2}$

Therefore Power Series cgs for $|x-1| < \frac{1}{2}$

$-\frac{1}{2} < x-1 < \frac{1}{2}$

$1-\frac{1}{2} < x < \frac{1}{2}+1$

$\frac{1}{2} < x < \frac{3}{2}$

$\therefore$ Interval of Convergence is $]\frac{1}{2}, \frac{3}{2}[$

Radius of Convergence is $R = \frac{1}{2}$.

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Q.8 $\sum_{0}^{\infty} \dfrac{(x-2)^n}{2^n \sqrt{n+1}}$ ——— (1)

Sol. Here $|a_n| = \left| \dfrac{(x-2)^n}{2^n \sqrt{n+1}} \right| = \dfrac{|x-2|^n}{2^n \sqrt{n+1}}$

using ratio test for absolute convergent i.e

$\lim_{n\to\infty} \dfrac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \dfrac{|x-2|^{n+1}}{2^{n+1} \sqrt{n+2}} \times \dfrac{\cancel{2^n} \sqrt{n+1}}{\cancel{|x-2|^n}}$

$= \lim_{n\to\infty} \dfrac{|x-2| \sqrt{n+1}}{2 \sqrt{n+2}} = \lim_{n\to\infty} \dfrac{\sqrt{1+\frac{1}{n}}}{\sqrt{1+\frac{2}{n}}} \left(\dfrac{|x-2|}{2} \right)$

$= \dfrac{|x-2|}{2}$

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$\Rightarrow$ Power series converges if $\frac{|x-2|}{2} < 1 \Rightarrow |x-2| < 2$

and Power series diverges if $\frac{|x-2|}{2} > 1 \Rightarrow |x-2| > 2$

$\Rightarrow$ $\frac{|x-2|}{2} = 1 \Rightarrow |x-2| = 2$

$x-2 = \pm 2$

$x = 2 \pm 2$

$x = 4, 0$

For $x=4$ Put in ① $\sum_0^\infty a_n = \sum \frac{(4-2)^n}{2^n\sqrt{n+1}} = \sum \frac{\cancel{2^n}}{\cancel{2^n}\sqrt{n+1}} = \sum \frac{1}{\sqrt{n+1}}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0$ so may be cgt may be dgt.

BCT $\because \sqrt{n+1} \leq n$

$\therefore \frac{1}{\sqrt{n+1}} \geq \frac{1}{n}$ So $\sum \frac{1}{b_n} = \sum \frac{1}{n}$ is Dgt we know already

Hence by BCT $\sum \frac{1}{\sqrt{n+1}}$ is Dgt.

For $x=0$ Put in ① Power Series becomes $\sum \frac{(-2)^n}{2^n\sqrt{n+1}} = \sum \frac{(-1)^n 2^n}{2^n\sqrt{n+1}} = \sum \frac{(-1)^n}{\sqrt{n+1}}$ which is an alternating Series.

where $\sum a_n = \sum \frac{(-1)^n}{\sqrt{n+1}}$

$\Rightarrow |a_n| = \frac{1}{\sqrt{n+1}}$

$\because \sqrt{n+1} \leq n$ for $n = 0, 1, 2, \ldots \infty$

$\Rightarrow \frac{1}{\sqrt{n+1}} \geq \frac{1}{n}$ So let $\sum b_n = \sum \frac{1}{n}$ which is divergent

also $|a_n| > b_n$ and $\sum b_n$ is divergent. So by B.C.T. $\sum |a_n|$ is divergent

LCT

① ALTERNATE.

$\because |a_n| = \frac{1}{\sqrt{n+1}}$

let $|b_n| = \frac{1}{n}$

Then $\lim_{n\to\infty} \frac{|a_n|}{|b_n|} = \lim_{n\to\infty} \frac{n}{\sqrt{n+1}}$

by L.C. Test

$= \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}\sqrt{1+\frac{1}{n}}} = 1 \neq 0$

② Alternate

$\lim_{n\to\infty} \frac{|a_n|}{|b_n|} = \frac{\frac{1}{\sqrt{n+1}}}{\frac{1}{\sqrt{n}}}$

$= \lim_{n\to\infty} \frac{1}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{1}$

$= \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}\sqrt{1+\frac{1}{n}}} = 1 \neq 0$

So depending on b_n

$b_n = \frac{1}{\sqrt{n}}$, is Dgt

$\therefore \sum a_n$ is Dgt

using A. Series test:

$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0$

also $f(x) = \frac{1}{\sqrt{x+1}} = (x+1)^{-\frac{1}{2}}$

$f'(x) = -\frac{1}{2}(x+1)^{-3/2} < 0$. $\therefore$ Nonincreasing

So by alternating Series (A.S.) test

A.S. is conditionally cgt for $x = 0$.

$\therefore$ Series is cgt for $|x-2| < 2$ and $x=0$

$-2 < x-2 < 2$ and $x=0$

$2-2 < x < 2+2$ and $x=0$ So $[0, 4[$

$0 < x < 4$.

Thus interval of convergence is

$$[0, 4[$$ & Radius of Cgce = 2.

Q.9 $\sum_{1}^{\infty} x^{2n}$

Sol. Here $a_n = x^{2n}$

using R.T. for A. convergent i.e $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{2n+2}}{x^{2n}}\right| = |x^2| = |x|^2$

$\Rightarrow$ P. Series converge if $|x|^2<1$ i.e $|x|<$

and Power Series diverges if $|x|>1$.

So Radius of convergence is 1.

also $|x|<1 \Rightarrow -1<x<1$

For $|x|^2 = 1 \Rightarrow |x| = 1 \Rightarrow x = \pm 1$

So for $x = \pm 1$ Power Series becomes $\sum_{1}^{\infty} (\pm 1)^{2n} = \sum_{1}^{\infty} 1$ which is +ve term series, so is divergent. for

$x = \pm 1$.

Thus interval of convergence is $]-1, 1[$

Q.10 $\sum x^n/(\ln n)^n$

Sol. Here $|a_n|^{\frac{1}{n}} = \left|\frac{x^n}{(\ln n)^n}\right|^{\frac{1}{n}} = \left|\frac{x}{\ln n}\right| = \frac{|x|}{\ln n}$

using root test for absolute convergent

i.e $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{|x|}{\ln n} = 0 < 1$

So by Root test $\sum x^n/(\ln n)^n$ is absolutely converge for all values of x.

Thus Interval of convergence $]-\infty, \infty[$,

Radius of convergence $= \infty$

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Q.11 $\sum_{1}^{\infty} n^n (x+1)^n$

SOL Here $|a_n| = n^n (x+1)^n$

using Root test for absolute convergence i.e

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left| n^n (x+1)^n \right|^{1/n} = \lim_{n\to\infty} n|x+1|$$

or $\lim_{n\to\infty} |a_n|^{1/n} = 0$ for $x = -1$

$= \infty \quad \forall x$ (other than $x=-1$)

So by root test given power series converges absolutely for $x = -1$, and diverges for $\forall x$ (other than $x = -1$)

Since given power series converges only for $x = -1$ therefore its interval of convergence is a point $= -1$ or $[-1,-1]$ and its radius of convergence is $= '0'$.

If Power Series of the form $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges at a pt $x = a$' Then interval of convergence is $= a$ i.e one point and radius of convergence is zero.

Q.12 $\sum_{n=1}^{\infty} \frac{2^n (x-3)^n}{n^2}$

SOL. Here $|a_n| = \left| \frac{2^n (x-3)^n}{n^2} \right| = \frac{2^n |(x-3)|^n}{n^2}$

using root test for absolute convergent i.e

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{2^n |(x-3)|^n}{n^2} \right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{2|x-3|}{(n^2)^{\frac{1}{n}}}$$

$$= \lim_{n\to\infty} \frac{2|x-3|}{\left(n^{\frac{1}{n}}\right)^2} = 2|x-3|$$

$\because \lim_{n\to\infty} n^{\frac{1}{n}} = 1$ already proved

implies that power series converges if $2|x-3| < 1$ i.e $|x-3| < \frac{1}{2}$

and Power series diverges if $|x-3| > \frac{1}{2}$

$\Rightarrow$ radius of convergence 'R' $= \frac{1}{2}$

also $|x-3| < \frac{1}{2}$ implies that, $-\frac{1}{2} < x-3 < \frac{1}{2}$

or $-\frac{1}{2}+3 < x < \frac{1}{2}+3$

or $\frac{5}{2} < x < \frac{7}{2}$

For $|x-3| = \frac{1}{2} \Rightarrow x-3 = \pm\frac{1}{2}$

or $x = 3+\frac{1}{2}, \; 3-\frac{1}{2} = \frac{7}{2}, \frac{5}{2}$

So for $x = \frac{5}{2}$, given power series becomes $\sum_{n=1}^{\infty} \frac{2^n\left(\frac{5}{2}-3\right)^n}{n^2}$

$$= \sum_{n=1}^{\infty} \frac{2^n\left(-\frac{1}{2}\right)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

which is an alternating Series

where $\sum |a_n| = \sum \frac{1}{n^2}$ which is convergent, since $p=2>1$

So power series is absolutely convergent for $x = \frac{5}{2}$

Now for $x = \frac{7}{2}$, power series becomes $\sum_{n=1}^{\infty} \frac{2^n\left(\frac{7}{2}-3\right)^n}{n^2}$

$$= \sum_{n=1}^{\infty} \frac{2^n\left(\frac{1}{2}\right)^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$ a +ive term series

which is convergent

Thus interval of convergence is $\left[\frac{5}{2} : \frac{7}{2}\right]$

x———————x

Q. NO. 13 $\sum_{1}^{\infty} \frac{1+(-1)^n}{n} x^n$

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SOL: Given power series can be written as

$$\sum_{1}^{\infty} \frac{x^n}{n} + \sum_{1}^{\infty} \frac{(-1)^n x^n}{n}$$

$$= \sum a_n + \sum b_n \quad (\text{Say})$$

then by ratio test for absolute convergent

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n}\right| = \lim_{n\to\infty} \left|\frac{n}{n+1} x\right|$$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} |x| = |x|$$

$\Rightarrow$ The power series $\sum_{1}^{\infty} \frac{x^n}{n}$ converges for $|x|<1$

and " " " " diverges for $|x|>1$

also since $\sum b_n = \sum \frac{(-1)^n x^n}{n}$

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then $\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}}{n+1}\times\frac{n}{(-1)^n x^n}\right|$

$= \lim_{n\to\infty}\left|\frac{-n\,x}{n+1}\right| = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)|x|$

$= \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)|x| = |x|$

$\Rightarrow$ The Power series $\sum_1^\infty \frac{(-1)^n x^n}{n}$ is convergent if $|x|<1$

and " " " " " divergent if $|x|>1$

$\Rightarrow$ both series converges for $|x|<1$

and " " diverges " $|x|>1$

Thus radius of convergence "R" $= 1$

In both series if $|x| = 1$ i.e. $x = \pm 1$

So **for $x=1$**, $\sum a_n + \sum b_n = \sum \frac{1}{n} + \sum \frac{(-1)^n}{n} \longrightarrow ①$

and **for $x=-1$**, $\sum a_n + \sum b_n = \sum \frac{(-1)^n}{n} + \sum \frac{(-1)^n(-1)^n}{n}$

$= \sum \frac{(-1)^n}{n} + \sum \frac{(-1)^{2n}}{n}$

$= \sum \frac{(-1)^n}{n} + \sum \frac{1}{n} \longrightarrow ②$

from ① and ② we see that for $x = \pm 1$, $\sum a_n + \sum b_n$ becomes

both power series

$= \sum_1^\infty \frac{1}{n} + \sum_1^\infty \frac{(-1)^n}{n}$ = +ive Series + Al Series

= Dgt Series + [Dgt or Cgt or Cond cgt]

$\because \sum\frac{1}{n}$ is dgt series, = Dgt. Series (By Theorem Page 262.)

$\therefore$ Interval of Convergence is $]-1, 1[$

Radius of Convergence $= 1$.

No Need for all aimed:

also $\sum_1^\infty |b_n| = \sum_1^\infty \left|\frac{(-1)^n}{n}\right| = \sum_1^\infty \frac{1}{n}$ is divergent Series $\longrightarrow ③$

also $\frac{|b_{n+1}|}{|b_n|} = \frac{1}{n+1}\times\frac{n}{1} = \frac{n}{n+1} < 1 \Rightarrow \frac{|b_{n+1}|}{|b_n|} < 1$

$\Rightarrow |b_{n+1}| < |b_n|$ or $|b_n| > |b_{n+1}|$

$\Rightarrow$ radius of convergence is $= 3$

if $|x| = 3 \Rightarrow x = \pm 3$

Then either $x = 3$ or $x = -3$. power series will become

$$\sum_{n=0}^{\infty}\left(\frac{9+3}{12}\right)^n = \sum_{n=0}^{\infty}(1)^n = \sum_{n=0}^{\infty} 1$$

which is a divergent series

$\boxed{\because \sum_{}^{\infty} 1 = 1+1+1+\cdots\infty = \infty \quad \therefore \sum 1 \text{ is divergent}}$

therefore interval of convergence is $]-3, 3[$

Q. 15 $\sum_{n=2}^{\infty} \frac{(-1)^n x^n}{n(\ln n)^2} \Rightarrow |a_n| = \frac{|x|^n}{n(\ln n)^2}$

SOL. using ratio test for absolute convergent i.e

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{|x|^{n+1}}{(n+1)(\ln(n+1))^2} \times \frac{n(\ln n)^2}{|x|^n}$$

$$= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)\left(\frac{\ln n}{\ln(n+1)}\right)^2|x|$$

$$= \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)\frac{\ln n}{\ln(n+1)}|x|$$

$$= \lim_{n\to\infty} 1 \cdot 1 \cdot |x| = |x|$$

$\because \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} = 1$

and $\lim_{n\to\infty}\frac{\ln n}{\ln n+1} \; \frac{\infty}{\infty}$

$= \lim_{n\to\infty}\frac{1/n}{1/n+1} = \lim_{n\to\infty}\frac{n+1}{n}$

$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1$

So by ratio test power series converges if $|x|<1$

and power series diverges if $|x|>1$

$\therefore$ radius of convergence $'R' = 1$

if $|x| = 1$ i.e $x = \pm 1$

So for $x = 1$, Power Series becomes $\sum_{2}^{\infty}\frac{(-1)^n}{n(\ln n)^2}$ which is

an alternating series $\Rightarrow |a_n| = \frac{1}{n(\ln n)^2}$

$$\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\frac{1}{n(\ln n)^2} = 0$$

and $|a_n| = \frac{1}{n(\ln n)^2} > \frac{1}{(n+1)\ln(n+1)} = |a_{n+1}|$

So by Alternating series test $\sum_{2}^{\infty}\frac{(-1)^n}{n(\ln n)^2}$ converges for $x=1$

For $x = -1$, Power series becomes $\sum \frac{(-1)^n (-1)^n}{n(\ln n)^2}$

$$= \sum \frac{(-1)^{2n}}{n(\ln n)^2} = \sum_{2}^{\infty} \frac{1}{n(\ln n)^2}$$

which is +ive term series and is convergent as already proved. Thus

interval of convergence is $[-1, 1]$.

x ———————— x

Q.16 $\sum_{n=0}^{\infty} \frac{n x^n}{(n+1)(n+2)2^n}$

SOL. using ratio test for absolute convergent i.e

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)\, x^{n+1}}{(n+\cancel{2})(n+3)\, 2^{\cancel{n}+1}} \times \frac{(n+1)(\cancel{n+2})\, 2^{\cancel{n}}}{n\, \cancel{x^n}}\right|$$

$$= \lim_{n\to\infty} \left|\frac{(n+1)^2 x}{2\, n(n+3)}\right| = \lim_{n\to\infty} \left|\frac{(1+\frac{1}{n})^2 x}{2\,(1+\frac{3}{n})}\right|$$

$$= \lim_{n\to\infty} \frac{1}{2}|x| = \frac{|x|}{2}$$

$\Rightarrow$ Power series converges for $\frac{|x|}{2} < 1$ i.e $|x| < 2$

and " " diverges " $\frac{|x|}{2} > 1$ i.e $|x| > 2$

so radius of convergence is 'R' = 2.

Now when $|x| = 2$ i.e $x = \pm 2$. Then

for $x = -2$, Power series becomes

$$\sum_{0}^{\infty} \frac{n(-2)^n}{(n+1)(n+2)2^n} = \sum_{0}^{\infty} \frac{(-1)^n\, n\, \cancel{2^n}}{(n+1)(n+2)\cancel{2^n}} = \sum_{0}^{\infty} \frac{(-1)^n\, n}{(n+1)(n+2)}$$

which is an alternating series

with $|a_n| = \frac{n}{(n+1)(n+2)}$ i.e

using alternating series-test

(i) $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n}{(n+1)(n+2)} = \lim_{n\to\infty} \frac{1}{n(1+\frac{1}{n})(1+\frac{2}{n})} = 0$

and $|a_n| - |a_{n+1}| = \frac{n}{(n+1)(n+2)} - \frac{n+1}{(n+2)(n+3)}$

$$= \frac{n(n+3) - (n+1)^2}{(n+1)(n+2)(n+3)} = \frac{n-1}{(n+1)(n+2)(n+2)}$$

$\because \dfrac{n-1}{(n+1)(n+2)(n+3)} \geq 0$ for all $n = 0, 1, 2, 3 \ldots$

$\Rightarrow |a_n| - |a_{n+1}| \geq 0$ or $|a_n| > |a_{n+1}|$

So by alternating Series test $\sum a_n$ Converges. $\rightarrow$ ①

also let $b_n = \frac{1}{n}$ $\Rightarrow \sum b_n = \sum \frac{1}{n}$ which is divergent

also $$\lim_{n\to\infty} \frac{|a_n|}{b_n} = \lim_{n\to\infty} \frac{n}{(n+1)(n+2)} \times \frac{n}{1} = \lim_{n\to\infty} \frac{n^2}{(n+1)(n+2)}$$

$$= \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})(1+\frac{2}{n})} = 1 \neq 0$$

$\Rightarrow \sum |a_n|$ and $\sum b_n$ behaves alike but $\sum b_n$ is divergent. So by L.C. test $\sum |a_n|$ diverges $\rightarrow$ ②

Combining ① & ②, implies the power series for $x = 2$, the series is conditionally convergent. and

for $x = 2$, Power series becomes

$$= \sum_{0}^{\infty} \frac{n \cdot 2^n}{(n+1)(n+2) \cdot 2^n} = \sum_{0}^{\infty} \frac{n}{(n+1)(n+2)} = \sum a_n$$

which is divergent ($\because \sum b_n = \sum \frac{1}{n}$ is divergent)

So interval of convergence is $]-2, 2[$

x——————x

Q.17 $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^n \sin^n x}{1}$ such that $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

Sol. $\Rightarrow |a_n| = 2^n \sin^n x$

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So using ratio test for absolute convergent i.e

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2^{n+1} (\sin x)^{n+1}}{2^n (\sin x)^n}\right| = \lim_{n\to\infty} 2|\sin x| = 2|\sin x|$$

$\Rightarrow$ The power series converges if $2|\sin x| < 1 \Rightarrow |\sin x| < \frac{1}{2}$ i.e. $|x| < \frac{\pi}{6}$

and " " " diverges if $2|\sin x| > 1 \Rightarrow |\sin x| > \frac{1}{2}$ i.e. $|x| > \frac{\pi}{6}$

So radius of convergence R is $= \frac{\pi}{6}$

Now when $|\sin x| = \frac{1}{2}$ $\Rightarrow |x| = \frac{\pi}{6}$ $\Rightarrow x = \pm\frac{\pi}{6}$

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For $x = \frac{\pi}{6}$, Power series becomes $\sum_0^\infty (-1)^n 2^n \left(\sin\frac{\pi}{6}\right)^n$

$= \sum_0^\infty (-1)^n 2^n \left(\frac{1}{2}\right)^n = \sum_0^\infty (-1)^n$

$\boxed{\sum_0^\infty (-1)^n = 1-1+1-1+\cdots \text{ is divergent. See Exm-3 Page 258}}$

which is an alternating series and is divergent.

For $x = -\frac{\pi}{6}$, Power series becomes $\sum_0^\infty (-1)^n 2^n \left[\sin\left(\frac{-\pi}{6}\right)\right]^n = \sum_0^\infty (-1)^n 2^n \left(\frac{-1}{2}\right)^n$

$= \sum_0^\infty (+1)^n$

$= 1+1+1+1\cdots\infty$

which is divergent. (See Exm-3. Page 258 in which $\Sigma a_n = a + a x + a x^2 + \cdots$ is divergent for $|x| = 1$)

Q.18 $\sum_{n=2}^{\infty} \frac{(x-e)^n \ln n}{e^n}$

Sol. Here $|a_n| = \frac{(x-e)^n \ln n}{e^n}$

appying ratio test for absolute convergent

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-e)^{n+1} \ln(n+1)}{e\, \cancel{e^n}} \cdot \frac{\cancel{e^n}}{\cancel{(x-e)^n} \ln n}\right|$$

$$= \lim_{n\to\infty} \frac{|x-e|}{e} \frac{\ln(n+1)}{\ln n}$$

$$= \frac{|x-e|}{e} \cdot 1 = \left|\frac{x-e}{e}\right|$$

$\because \lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} \left(\frac{\infty}{\infty}\right)$

$= \lim_{n\to\infty} \frac{\frac{1}{n+1}}{1/n}$ (L'Hospital Rule)

$= \lim_{n\to\infty} \frac{n}{1+n} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$

$\Rightarrow$ Power Series become convergent for $\left|\frac{x-e}{e}\right| < 1$ or $|x-e| < e$

" " " divergent for $\left|\frac{x-e}{e}\right| > 1$ or $|x-e| > e$

So radius of convergent 'R' is $= e$

For $\left|\frac{x-e}{e}\right| = 1$ i.e. $x - e = \pm e$ or $x = 0, 2e$

So for $x = 0$, Power series becomes $\sum_2^\infty \frac{(-e)^n \ln n}{e^n} = \sum_2^\infty (-1)^n \ln n$

which is an alternating Series and since

$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \ln n = \infty \neq 0 \Rightarrow$ A. Series diverges

and $|a_n| \neq |a_{n+1}|$ ($\because \ln n \neq \ln(n+1)$)

Now for $x = 2e$, Power Series becomes $\sum_2^\infty \frac{e^n \ln n}{e^n} = \sum_2^\infty \ln n$

which is a +ve term series and is divergent (Already proved)

Since $|x - e| < e$ is equivalent to $-e < x - e < e$

or $0 < x < 2e$

Thus interval of convergence is $]0, 2e[$

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Q-19 $\sum_{0}^{\infty} \frac{(ax+b)^n}{c^n}$, $a>0$, $c>0$

Sol. using root test for absolute convergent i.e

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \left|\frac{ax+b}{c}\right| = \left|\frac{ax+b}{c}\right|$$

$\Rightarrow$ Power series converges if $\left|\frac{ax+b}{c}\right| < 1$ i.e $|ax+b| < c$

or $|a(x+\frac{b}{a})| < c$ or $a|x+\frac{b}{a}| < c$ or $|x+\frac{b}{a}| < \frac{c}{a}$

and diverges if $|x+\frac{b}{a}| > \frac{c}{a}$

$\Rightarrow$ radius of convergence 'R' is $= \frac{c}{a}$

if $|x+\frac{b}{a}| = \frac{c}{a}$ $\Rightarrow$ $x+\frac{b}{a} = \pm\frac{c}{a}$ or $x = \pm\frac{c}{a} - \frac{b}{a}$

$\Rightarrow$ $x = \frac{c-b}{a}$, $-\frac{c+b}{a}$

For $x = \frac{c-b}{a}$ $\Rightarrow$ $ax+b = c$ or $\frac{ax+b}{c} = 1$

So Power Series becomes $\sum_{0}^{\infty} (1)^n = \sum_{0}^{\infty} 1$ which divergent

For $x = -\frac{c+b}{a}$ $\Rightarrow$ $ax = -b-c$ or $ax+b = -c$

or $\frac{ax+b}{c} = -1$ so power series becomes

$\sum_{2}^{\infty} (-1)^n$ i.e an alternating series and is divergent.

Now Since $|x+\frac{b}{a}| < \frac{c}{a}$ is equivalent a

$-\frac{c}{a} < x+\frac{b}{a} < \frac{c}{a}$ or $-\frac{c}{a} - \frac{b}{a} < x < \frac{c}{a} - \frac{b}{a}$

or $-\left(\frac{c+b}{a}\right) < x < \frac{c-b}{a}$

$\Rightarrow$ Interval of convergence is $\left]-\left(\frac{b+c}{a}\right), \frac{c-b}{a}\right[$

Q.20 $\sum_{1}^{\infty} \frac{n(x-1)^n}{2^n(3n-1)}$

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Sol. $\Rightarrow$ $|a_n| = \left|\frac{n(x-1)^n}{2^n(3n-1)}\right|$

using ratio test for absolute convergent i.e

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)(x-1)^{n+1}}{2^{n+1}(3n+2)} \cdot \frac{2^n(3n-1)}{n(x-1)^n}\right|$$

$$= \lim_{n\to\infty} \left(\frac{(n+1)(3n-1)}{2n(3n+2)}\right)|x-1| = \lim_{n\to\infty} \frac{(1+\frac{1}{n})(3-\frac{1}{n})}{2(3+\frac{2}{n})}|x-1| = \frac{|x-1|}{2}$$

$\Rightarrow$ Power series converges if $\frac{|x-1|}{2} < 1$ or $|x-1| < 2$

and " " diverges if $|x-1| > 2$

So radius of convergence R is $= 2$

When $|x-1| = 2$ $\Rightarrow$ $x - 1 = \pm 2$, $x = 3, -1$

For $x = -1$, Power Series becomes

$$\sum_1^{\infty} \frac{n(-2)^n}{2^n(3n-1)} = \sum_1^{\infty} \frac{(-1)^n \cancel{2^n}\, n}{\cancel{2^n}(3n-1)} = \sum_1^{\infty} (-1)^n \frac{n}{3n-1} .$$

~~Alternating~~ an alternating Series.

So by divergent Series test

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n}{3n-1} = \frac{1}{3} \neq 0 ,$$

$\Rightarrow \sum |a_n|$ is divergent So $\sum a_n$ is divergent

because $|a_n| = \frac{n}{3n-1}$ and $|a_{n+1}| = \frac{n+1}{3n+2}$ $\quad \frac{3n-1}{n+1}$, $3n^2+2n-1$

and $|a_n| - |a_{n+1}| = \frac{n}{3n-1} - \frac{n+1}{3n+2} = \frac{3n^2+2n-3n^2-2n+1}{(3n-1)(3n+2)}$

$$= \frac{1}{(3n-1)(3n+2)} > 0$$

$\Rightarrow$ $|a_n| - |a_{n+1}| > 0$ or $|a_n| > |a_{n+1}|$

but first condition of alternating is not satisfied

(i.e $\lim_{n\to\infty} |a_n| = 0$)

So alternating Series $\sum_1^{\infty} (-1)^n \frac{n}{3n-1}$ is not convergent

Hence is divergent for $x = -1$

For $x = 3$, Power series becomes

$$\sum_1^{\infty} \frac{n\, \cancel{2^n}}{\cancel{2^n}(3n-1)} = \sum_1^{\infty} \frac{n}{3n-1} = \sum_1^{\infty} a_n$$

$\because \lim_{n\to\infty} a_n \neq 0$ So Power Series is divergent for

$x = 3$

$|x-1| < 2$ is equivalent to $-2 < x-1 < 2$

$-1 < x < 3$

$\Rightarrow$ Interval of convergence is $]-1, 3[$

Q.21 $\sum_{n=1}^{\infty} \frac{n^n x^n}{n!}$

SOL. Here $|a_n| = \frac{n^n x^n}{n!}$

and $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^{n+1} x^{n+1}}{(n+1)!} \times \frac{n!}{n^n x^n}\right|$

$= \left|\frac{\cancel{(n+1)}(n+1)^n \cancel{x^n} x \cdot \cancel{n!}}{\cancel{(n+1)} \cancel{n!}\; n^n \cancel{x^n}}\right| = \left(\frac{n+1}{n}\right)^n |x|$

$= \left(1+\frac{1}{n}\right)^n |x|$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n |x| = e|x|$ $\quad \boxed{\because \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e}$

so by ratio test for absolute convergent if

$e|x| < 1, \implies |x| < \frac{1}{e}$

and diverges if $|x| > \frac{1}{e}$

Thus radius of convergence is $R' = e$

if $e|x| = 1, \quad |x| = \frac{1}{e}, \quad x = \pm\frac{1}{e}$

for $x = -\frac{1}{e}$, Power series becomes $= \sum_{1}^{\infty} \frac{n^n}{n!}\left(-\frac{1}{e}\right)^n$

We know that for 'n' is very large $= \sum_{1}^{\infty} (-1)^n \frac{n^n}{n!} \cdot \frac{1}{e^n}$ (An alternating Series)

$n! = \sqrt{2\pi n}\, \frac{n^n}{e^n}$

$\Rightarrow \sum_{1}^{\infty} \frac{(-1)^n n^n}{n!\, e^n} = \sum_{1}^{\infty} \frac{(-1)^n \cancel{n^n}}{\cancel{e^n}} \frac{\cancel{e^n}}{\sqrt{2\pi n}\, \cancel{n^n}} = \sum_{1} \frac{(-1)^n}{\sqrt{2\pi n}}$

$\sum |a_n| = \sum \frac{1}{\sqrt{2\pi n}} = \frac{1}{\sqrt{2\pi}} \sum_{1}^{\infty} \frac{1}{\sqrt{n}}$ which is divergent $\boxed{\because p = \frac{1}{2} < 1 \text{ by } p\text{. Series}}$

Now using alternating Series test:

$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{\sqrt{2\pi n}} = 0$

$|a_n| = \frac{1}{\sqrt{2\pi n}} > \frac{1}{\sqrt{2(n+1)\pi}} = |a_{n+1}|$

$\Rightarrow$ A. Series converges but $\sum |a_n|$ diverges.

$\Rightarrow$ Given Series is conditionally convergent.

for $x = \frac{1}{e}$, P. Series becomes $\sum_{1}^{\infty} \frac{n^n}{n!\, e^n} = \sum_{1}^{\infty} \frac{1}{\sqrt{2\pi n}}$ +ive term Series and is divergent

So interval of convergence is $\left[-\frac{1}{e}, \frac{1}{e}\right[$ Ans.

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Q.22 $\sum_{1}^{\infty} (-1)^n \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4\cdot 6\cdot 8 \cdots 2n} x^{2n+1}$

Sol. Here $|a_n| = \left| \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4\cdot 6 \cdots 2n} x^{2n+1} \right|$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left| \frac{1\cdot 3\cdot 5 \cdots (2n-1)(2n+1)}{2\cdot 4\cdot 6 \cdots 2n\,(2n+2)} \cdot \frac{x^{2n+3}}{1} \times \frac{2\cdot 4\cdot 6 \cdots 2n}{1\cdot 3\cdot 5 \cdots (2n-1)} \cdot \frac{1}{x^{2n+1}} \right|$$

$$= \left| \frac{2n+1}{2n+2} \cdot \frac{x^{\cancel{2n+1}+2}}{x^{\cancel{2n+1}}} \right| = \frac{2n+1}{2n+2} |x^2|$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left(\frac{2+1/n}{2+\frac{2}{n}}\right) |x^2| = |x^2|$$

$\Rightarrow$ The Power series will converge if $|x^2| < 1$ or $|x|^2 < 1$ or $|x| < 1$

and " " diverge if $|x| > 1$

So Radius of convergence 'R' is $= 1$.

For $|x| = 1 \Rightarrow x = \pm 1$.

when $x = 1$, then power series will become

$$\sum_{1}^{\infty} (-1)^n \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4\cdot 6 \cdots 2n}$$ which is an alternating Series

so applying divergent test i.e

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4\cdot 6 \cdots 2n} = \lim_{n\to\infty} \frac{1\cdot 3\cdot 5 \cdots \cancel{n}(2-\frac{1}{n})}{2\cdot 4\cdot 6 \cdots 2\cancel{n}}$$

$$= \lim_{n\to\infty} \frac{1\cdot 3\cdot 5 \cdots (2-\frac{1}{n})}{2\cdot 4\cdot 6 \cdots 2} \neq 0 \longrightarrow (1)$$

Similarly for $x = -1$, Power Series becomes

$$\sum_{1}^{\infty} (-1)^n \frac{1\cdot 3\cdot 5 \cdots (2n-1)\,(-1)^{2n+1}}{2\cdot 4\cdot 6 \cdots 2n} = \sum_{1}^{\infty} (-1)^{3n+1} \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4\cdot 6 \cdots 2n}$$

which is alternating series and also

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4\cdot 6 \cdots 2n} \neq 0 \text{ (already proved)} \longrightarrow (2)$$

So from (1) & (2) by divergent test. power Series diverges in both cases and by A. Series test Since 1st condition is not satisfied. Hence is not convergent. Thus in result P. Series diverges for $x = \pm 1$. So interval of Convergence is $]-1, 1[$

Q.23 $\sum_1^\infty \left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)x^n$

Sol. Here $|a_n| = \left|\left(1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{n}\right)x^n\right|$

and $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\left(1+\frac{1}{2}+\frac{1}{3}\cdots\cdots+\frac{1}{n+1}\right)x^{n+1}}{\left(1+\frac{1}{2}+\frac{1}{3}+\cdots\cdots\frac{1}{n}\right)\cdot x^n}\right|$

$\Rightarrow \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(\frac{1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{n+1}}{1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{n}}\right)|x|$

$\boxed{\because \left|\frac{x^{n+1}}{x^n}\right| = |x|}$

$= \lim_{n\to\infty} |x|$

$\boxed{\lim_{n\to\infty} 1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{n+1} = \lim_{n\to\infty} 1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\cdots}$

$\Rightarrow$ Power series converges for $|x|<1$

and " " diverges " $|x|>1$

so radius of convergence is $=1$

for $|x|=1 \Rightarrow x=\pm 1$.

if $x=1$, then power series becomes

$$\sum_1^\infty 1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{n}(+1)^n = \sum_1^\infty 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots = \sum_1^\infty \frac{1}{n}$$

which is divergent.

and if $x=-1$, then power series becomes

$\sum_1^\infty \left(1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{n}\right)(-1)^n$ an alternating series

$\Rightarrow \sum_1^\infty |a_n| = \sum_1^\infty 1+\frac{1}{2}+\frac{1}{n}+\cdots = \sum_1^\infty \frac{1}{n}$ which is divergent $\rightarrow$ (1)

~~but~~ also by alternating series test

$\lim_{n\to\infty}|a_n| = \lim_{n\to\infty} 1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{n} \neq 0$

and $|a_n| = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} < 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}+\frac{1}{n+1}$

$= 1+\frac{1}{2}+\frac{1}{3}\cdots+\frac{1}{n+1} = |a_{n+1}|$

So by alternating series test Power series is not convergent $\rightarrow$ (2)

So from (1) and (2), we concluded that power series diverges for $x=-1$. So

interval of convergence is $]-1, 1[$.

Q.24 $\sum_{1}^{\infty} \frac{n!\, x^n}{1\cdot4\cdot7\cdots(3n-2)}$

SOL So $\frac{a_{n+1}}{a_n} = \frac{(n+1)!\, x^{\cancel{n}+1}}{\cancel{1\cdot4\cdot7\cdots(3n-2)}(3n+1)} \times \frac{\cancel{1\cdot4\cdot7\cdots(3n-2)}}{n!\, \cancel{x^n}}$

$$= \frac{(n+1)!\, x}{(3n+1)\, n!} = \frac{(n+1)\, \cancel{n!}\, x}{(3n+1)\, \cancel{n!}} = \frac{(n+1)\, x}{(3n+1)}$$

$$\Rightarrow \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)\, x}{(3n+1)}\right| = \lim_{n\to\infty} \left(\frac{1+\frac{1}{n}}{3+\frac{1}{n}}\right)|x| = \frac{|x|}{3}$$

$\Rightarrow$ Power Series converges for $\frac{|x|}{3} < 1$ or $|x| < 3$

and " " diverges for $|x| > 3$

So radius of convergence 'R' = 3

for $|x| = 3$ i.e $x = \pm 3$

if $x = 3$, power Series becomes $\sum_{1}^{\infty} \frac{3^n\, n!}{1\cdot4\cdot7\cdots 3n-2}$

which +ve term Series

and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3^n\, n!}{1\cdot4\cdot7\cdots(3n-2)} = \lim_{n\to\infty} \frac{3^n\; 1\times2\times3\times4\cdots\cdots \cancel{n}}{1\times4\times7\cdots\cdots \cancel{n}(3-\frac{2}{n})}$

$$= \lim_{n\to\infty} 3^n \frac{1\cdot2\cdot3\times4\times\cdots}{1\cdot4\times7\times\cdots(3-\frac{2}{n})} = \infty \neq 0$$

So by divergent test, the series diverges for $x = 3$ $\to$ (1)

when $x = -3$, power series becomes $\sum_{1}^{\infty} \frac{(-1)^n\, 3^n\, n!}{1\cdot4\cdot7\cdots 3n-2}$

which is an alternating Series

$\because |a_n| = \frac{3^n\, n!}{1\cdot4\cdot7\cdots 3n-2}$ and $|a_{n+1}| = \frac{3^{n+1}(n+1)!}{1\cdot4\cdot7\cdots 3n+1}$

and $\frac{|a_{n+1}|}{|a_n|} = \frac{3^{\cancel{n}+1}(n+1)\cancel{n!}}{\cancel{1\cdot4\cdot7\cdots(3n-2)}(3n+1)} \times \frac{\cancel{1\cdot4\cdot7\cdots(3n-2)}}{\cancel{3^n}\,\cancel{n!}}$

$$= \frac{3(n+1)}{3n+1} = \frac{3n+3}{3n+1} > 1$$

$\Rightarrow \frac{|a_{n+1}|}{|a_n|} > 1$ or $|a_{n+1}| > |a_n|$ So by A. Series test

Series is not convergent $\longrightarrow$ (1)

and by divergent test

$\lim_{n\to\infty} |a_n| \neq 0$ (already proved)

The Series is divergent $\longrightarrow$ (2)

form (1) and (2) we concluded that

Series diverges for $x = -3$

Thus interval of convergence is $]-3, 3[$

Q.25 Obtain a power Series representation of $\frac{x}{(1+x^2)^2}$ if $|x|<1$

Sol: $\because \frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 + \cdots x^n + \cdots$

replacing x by x^2, we get

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 + \cdots x^{2n} + \cdots$$

$$\Rightarrow \frac{1}{(1+x^2)^2} = (1 - x^2 + x^4 - x^6 + x^8 + \cdots x^{2n} + \cdots)(1 - x^2 + x^4 - x^6 + x^8 + \cdots x^{2n} + \cdots)$$

$$= (1 - x^2 + x^4 - x^6 + x^8 + \cdots) + (-x^2 + x^4 - x^6 + x^8 - x^{10} + \cdots) + (x^4 - x^6 + x^8 - x^{10} + x^{12} + \cdots)$$
$$+ (-x^6 + x^8 - x^{10} + x^{12} - x^{14} + \cdots) + (x^8 - x^{10} + x^{12} - x^{14} + x^{16} + \cdots)$$

$$= 1 - 2x^2 + 3x^4 - 4x^6 + 5x^8 - 6x^{10} + \cdots$$

$$\Rightarrow \frac{x}{(1+x^2)^2} = x - 2x^3 + 3x^5 - 4x^7 + 5x^9 - 6x^{11} + \cdots$$

$$= \sum_{0}^{\infty} (-1)^n (n+1) x^{2n+1}$$

is required power series

$1, 3, 5, 7, 9, \ldots$ n terms

$a_n = 1 + (n-1)2 = 2n-1$

FREQUENTLY USED MACLAURIN SERIES

(1) $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \sum_{0}^{\infty} x^n$; $|x|<1$

(2) $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots + (-x)^n + \cdots = \sum_{0}^{\infty} (-1)^n x^n$; $|x|<1$

(3) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^{n-1} \frac{x^{2n-1}}{2n-1} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1}$; $|x| \leq 1$

(4) $\ln\left(\frac{1+x}{1-x}\right) = 2\tanh^{-1} x = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n+1}}{2n+1} + \cdots\right) = 2\sum_{0}^{\infty} \frac{x^{2n+1}}{2n+1}$; $|x|<1$

(5) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots = \sum_{1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$; $-1 < x \leq 1$

(6) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{0}^{\infty} \frac{x^n}{n!}$; $|x| < \infty$

Q 26 Find the sum of the series

$2+6x+12x^2+20x^3+\cdots$

Sol $\because$ Maclaurins Series for $\frac{1}{1-x}$ is. or I.G.S

$$\frac{1}{1-x} = 1+x+x^2+x^3+x^4+\cdots$$

$$\Rightarrow \frac{1}{(1-x)^2} = (1+x+x^2+x^3+\cdots)(1+x+x^2+x^3+\cdots)$$

$$= (1+x+x^2+x^3+x^4+\cdots)+(x+x^2+x^3+x^4+x^5+\cdots)$$

$$= (1+x+x^2+x^3+x^4+\cdots)+(x^3+x^4+x^5+x^6+\cdots)$$
$$+(x^2+x^3+x^4+x^5+x^6+\cdots)$$

$$\frac{1}{(1-x)^2} = 1+2x+3x^2+4x^3+\cdots$$

$$\Rightarrow \frac{1}{(1-x)^3} = (1+2x+3x^2+4x^3+\cdots)(1+x+x^2+x^3+\cdots)$$

$$= (1+x+x^2+x^3+\cdots)+(2x+2x^2+2x^3+2x^4+\cdots)$$
$$+(3x^2+3x^3+3x^4+3x^5+\cdots)+(4x^3+4x^4+4x^5+4x^6+\cdots)$$

$$= 1+3x+6x^2+10x^3+\cdots$$

$$\Rightarrow \frac{2}{(1-x)^3} = 2(1+3x+6x^2+10x^3+\cdots)$$

$$= 2+6x+12x^2+20x^3+\cdots$$

Thus sum of Series $2+6x+12x^2+20x^3+\cdots$ is $\frac{2}{(1-x)^3}$ ans.

Q.27 using power series representation of $\frac{e^x-1}{x}$,

show that $\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = 1$.

Sol: Since $e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$

$$\Rightarrow \frac{e^x-1}{x} = \frac{\cancel{1}+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots - \cancel{1}}{x}$$

$$= 1+\frac{x}{2!}+\frac{x^2}{3!}+\frac{x^3}{4!}+\cdots = \sum_{0}^{\infty}\frac{x^n}{(n+1)!}$$

then $\dfrac{d}{dx} \sum_{0}^{n} \dfrac{x^n}{(n+1)!} = \sum_{0}^{\infty} \dfrac{d}{dx}\left(\dfrac{x^n}{(n+1)!}\right)$

$$= \sum_{1}^{\infty} \frac{n\, x^{n-1}}{(n+1)!} \longrightarrow (1)$$

$$\left(\frac{d}{dx}\left(1+\frac{x}{2!}+\frac{x^2}{3!}+\cdots\right) = \frac{1}{2!}+\frac{2x}{3!}+\frac{3x^2}{4!}+\cdots = \sum_{1}^{\infty}\frac{n\,x^{n-1}}{(n+1)!}\right)$$

and $\dfrac{d}{dx}\left(1+\dfrac{x}{2!}+\dfrac{x^2}{3!}+\cdots\right) = \dfrac{d}{dx}\left(\dfrac{e^x-1}{x}\right)$

$$= \frac{x(e^x)-(e^x-1)\cdot 1}{x^2}$$

$$= \frac{xe^x - e^x + 1}{x^2} = \frac{e^x(x-1)+1}{x^2} \longrightarrow (2)$$

Combining (1) and (2), we get

$$\frac{e^x(x-1)+1}{x^2} = \sum_{1}^{\infty} \frac{n\,x^{n-1}}{(n+1)!}$$

putting $x = 1$; both sides we get

$$\frac{e^1(1-1)+1}{1^2} = \sum_{1}^{\infty}\frac{n\,1^{n-1}}{(n+1)!}$$

or $\quad 1 = \sum_{1}^{\infty} \dfrac{n}{(n+1)!}$. Proved.

Q.28 Find a Series of powers of x, that converges to $\tan x$; $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

Sol We know that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots = \sum_{0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

and $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

both series converges for all x.
So interval of convergence is $]-\infty, \infty[$

Now $\dfrac{1}{\cos x}$ exist if $-\frac{\pi}{2} < x < \frac{\pi}{2}$

$\Rightarrow$ Common interval of convergence is $]-\frac{\pi}{2}, \frac{\pi}{2}[$

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So

$$\sin x \cdot \frac{1}{\cos x} = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \right)\left(\frac{1}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots} \right)$$

$$= x + \frac{x^3}{3} + \frac{2x^5}{15} - \cdots \quad , \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

NOTE: The ans. given in the book is wrong.

$$\begin{array}{r|l} & x + x^3/3 + \frac{2x^5}{15} \\ \hline 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots & x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \\ & x - \frac{x^3}{2!} + \frac{x^5}{4!} - \cdots \\ \hline & \frac{x^3}{3} + \frac{x^5}{30} \\ & \frac{x^3}{3} - \frac{x^5}{6} \\ \hline & \frac{2x^5}{15} - \cdots \\ & \frac{2x^5}{15} - \frac{x^7}{30} \end{array}$$

$$\frac{x^5}{120} - \frac{x^5}{24} = \frac{x^5 - 5x^5}{120} = -\frac{4x^5}{120} = -\frac{x^5}{30}$$

$$\frac{x^5}{30} + \frac{x^5}{6} = \frac{x^5 + 5x^5}{30} = \frac{6x^5}{30} = \frac{2x^5}{15}$$

$$-\frac{x^3}{3!} + \frac{x^3}{2!} = -\frac{x^3}{6} + \frac{x^3}{2} = \frac{-x^3 + 3x^3}{6} = \frac{2x^3}{6} = \frac{x^3}{3}$$

Q-29 Use power series to find the value of $\int_0^{\frac{1}{2}} \frac{dx}{1+x^4}$, to three places of decimal.

Sol. We know that

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \quad \text{①}$$

replace x by x^4 in ① we get

$$\frac{1}{1+x^4} = 1 - x^4 + x^8 - x^{12} + \cdots$$

$$\Rightarrow \int_0^{\frac{1}{2}} \frac{1}{1+x^4}\,dx = \int_0^{\frac{1}{2}} \left(1 - x^4 + x^8 - x^{12} + \cdots\right) dx$$

$$= \left| x - \frac{x^5}{5} + \frac{x^9}{9} - \frac{x^{13}}{13} + \cdots \right|_0^{1/2} = \left(\frac{1}{2} - \frac{(1/2)^5}{5} + \frac{(\frac{1}{2})^9}{9} - \frac{(\frac{1}{2})^{13}}{13} + \cdots \right) - 0$$

$$= \frac{1}{2} - \frac{1}{32(5)} + \frac{1}{512(9)} - \cdots = 0.493967 = 0.494 \text{ ans.}$$

Q-30 Estimate $\int_0^1 x^2 e^{-x^2} dx$ to three places of decimals.

Sol. $\because e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \Rightarrow e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots$

$$\Rightarrow x^2 e^{-x^2} = x^2\left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots\right) = x^2 - x^4 + \frac{x^6}{2!} - \frac{x^8}{3!} + \cdots$$

$$\therefore \int_0^1 x^2 e^{-x^2} dx = \int_0^1 \left(x^2 - x^4 + \frac{x^6}{2!} - \frac{x^8}{3!} + \cdots \right) dx = \left| \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{14} - \frac{x^9}{54} + \cdots \right|_0^1 = \frac{1}{3} - \frac{1}{5} + \frac{1}{14} - \frac{1}{54} + \cdots = \frac{352}{1890}$$

$$\approx 0.187 \text{ ans.}$$

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