DEFINITION Let $\{a_n\} = a_1, a_2, a_3, a_4 \ldots\ldots$ be an *Infinite Sequence*

$\sum_1^\infty a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ be *Infinite Series*

containing infinite number of terms.

Symbolically it is written as $\sum_{n=1}^\infty a_n$ or $\sum_1^\infty a_n$ or $\sum a_n$

where a_n is called n^{th} term of series

Sequence Of Partial Sum $\{S_n\}$

Let $\sum_{n=1}^\infty a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ be an Infinite series.

Consider $S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$S_4 = a_1 + a_2 + a_3 + a_4$

$S_n = a_1 + a_2 + a_3 + \cdots a_n$

S_n is called nth partial sum of the series $\sum_{n=1}^\infty a_n$

$\{S_n\} = S_1, S_2, S_3, \ldots S_n \ldots$ is called *Sequence of partial sums*.

DEFINITION

If the sequence $\{S_n\}$ converges to limit S, then series $\sum_1^\infty a_n$ is said to converge to S.

$\Rightarrow \lim_{n\to\infty} S_n = S$

$\Rightarrow \sum_1^\infty a_n = S$

(S_n is n^{th} partial sum and S is sum of series)

If S_n cgs to S then $\{S_n\}$ seq converges to S.

If $\{S_n\}$ cgs to S then Series $\sum a_n$ cgs to S.

Also if $\{S_n\}$ *diverges*, then the series $\sum_1^\infty a_n$ is divergent

SOME WELL-KNOWN INFINITE SERIES

(1) **INFINITE GEOMETRIC SERIES (IGS)**

Consider the infinite geometric series

$\sum_1^\infty a_n = \sum_1^\infty a r^{n-1} = a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$

We investigate the behaviour of this series for different values of r.

Here $S_n = a + ar + ar^2 + \cdots + ar^{n-1}$

or $S_n = \dfrac{a(1-r^n)}{1-r}$; $|r| < 1$

$= \dfrac{a(r^n - 1)}{r-1}$; $|r| > 1$

Case-1 if $|r| < 1$, then $r^n \longrightarrow 0$ as $n \longrightarrow \infty$

$\therefore \lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \dfrac{a(1-r^n)}{1-r} = \dfrac{a-0}{1-r} = \boxed{\dfrac{a}{1-r}}$ is a finite value

$\sum\limits_1^\infty a_n = \sum\limits_1^\infty ar^{n-1} = S_\infty = \lim\limits_{n\to\infty} S_n$ Converges to $\dfrac{a}{1-r}$

Case-2 If $|r| > 1$. Then $r^n \longrightarrow \infty$, as $n \longrightarrow \infty$

Then $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \dfrac{a(r^n-1)}{r-1} = \infty$

Thus $\{S_n\}$ diverge, so $\sum\limits_1^\infty a_n = \sum\limits_1^\infty ar^{n-1}$ diverges.

Case-3 If $r = 1$, Then

$S_\infty = \sum\limits_1^\infty ar^{n-1} = a + a + a + a + a + \cdots$

$\Rightarrow S_n = a + a + a + \cdots\cdots$ n times $= na$

So $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} na = \pm\infty$, according as

'a' is +ive or −ive $\Rightarrow \sum\limits_1^\infty ar^{n-1}$ diverges.

Case-4 if $r = -1$, Then

$\sum\limits_{n=1}^\infty ar^{n-1} = a - a + a - a + a - a + a - a + \cdots$

Then sequence of partial sums is

$\{S_n\} = S_1, S_2, S_3, S_4, S_5, \ldots = a, 0, a, 0, a, 0, a, \ldots$

$\because S_1 = a,\ S_2 = a - a = 0,\ S_3 = a - a + a = a, \ldots$

which diverges for $|r| = -1$ — limit not unique either '0' or 'a'

DEDUCTION An infinite G. series $\begin{cases} \text{(i) Converges if } |r| < 1 \\ \text{(ii) Diverges if } |r| \geq 1 \ \& \ r = -1 \end{cases}$

EXAMPLE Find the sum of the series

$$\sum_1^\infty a_n = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots$$

Sol Here $\sum_1^\infty a_n = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots$

Then nth partial sums is

$$S_n = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots + \frac{3}{10^n}$$

It is a G.S with $a = \frac{3}{10}$ and $r = \frac{1}{10} < 1$

using $S_n = \frac{a(1-r^n)}{1-r}$ put values

$$= \frac{\frac{3}{10}\left(1 - \frac{1}{10^n}\right)}{1 - \frac{1}{10}} = \frac{\frac{3}{10}\left(1 - \frac{1}{10^n}\right)}{9/10} = \frac{1}{3}\left[1 - \frac{1}{10^n}\right]$$

$$\Rightarrow \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{3}\left(1 - \frac{1}{10^n}\right) = \frac{1}{3}\left(1 - \frac{1}{\infty}\right) = \frac{1}{3}$$

as $\lim_{n\to\infty} S_n = \frac{1}{3}$, So $\{S_n\}$ is convergent and so $\sum_1^\infty a_n$ converges and so $\frac{1}{3}$ is sum of the given series.

(Easy) **ALTERNATE** We are given that

$$\sum_1^\infty a_n = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots$$

which is an infinite G. Series i.e

$$S_\infty = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots$$

Here $a = \frac{3}{10}$ and $\frac{1}{10} = r < 1$

So the given infinite G. Series is convergent

its sum can be found by the following formula

$$S_\infty = \frac{a}{1-r}$$ putting values of $a = \frac{3}{10}$ & $r = \frac{1}{10}$

$$\Rightarrow S_\infty = \frac{\frac{3}{10}}{1 - \frac{1}{10}} = \frac{\frac{3}{10}}{9/10} = \frac{1}{3}$$

$$\Rightarrow \sum_1^\infty a_n = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots = \frac{1}{3}.$$

EXAMPLE Determine whether the following series converges or diverges

$$\sum_{1}^{\infty} \frac{1}{(n+2)(n+3)} = \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \frac{1}{5\cdot 6} + \cdots\cdots$$

If it converges, find its sum.

SOL. The given series is

$$\frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \frac{1}{5\cdot 6} + \cdots\cdots + \frac{1}{(n+2)(n+3)} + \cdots\cdots$$

then its nth partial sum of given series

$$S_n = \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \frac{1}{5\cdot 6} + \cdots\cdots + \frac{1}{(n+2)(n+3)}$$

$$\Rightarrow S_n = \sum_{k=1}^{n} \frac{1}{(k+2)(k+3)} \longrightarrow (1)$$

$$\boxed{\because \frac{1}{(k+2)(k+3)} = \frac{(k+3)-(k+2)}{(k+2)(k+3)} = \frac{k+3}{(k+2)(k+3)} - \frac{k+2}{(k+2)(k+3)} = \frac{1}{k+2} - \frac{1}{k+3}}$$

$$= \sum_{k=1}^{n} \left[\frac{1}{k+2} - \frac{1}{k+3}\right]$$

$$= \left(\frac{1}{3} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{6}}\right) + \cdots + \left(\cancel{\frac{1}{n+2}} - \frac{1}{n+3}\right)$$

$$S_n = \frac{1}{3} - \frac{1}{n+3}$$

Now $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \left[\frac{1}{3} - \frac{1}{n+3}\right] = \frac{1}{3} - 0 = \frac{1}{3}$

since $\lim\limits_{n\to\infty} S_n = \frac{1}{3}$, so $\{S_n\}$ is convergent. Hence given series is convergent and its sum is $\frac{1}{3}$.

HARMONIC SERIES

Test for Convergence of the Harmonic series i.e.

$$\sum_{1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\cdots + \frac{1}{n} + \cdots\cdots$$

SOL: From given series:

$S_1 = 1$, $\quad S_2 = 1 + \frac{1}{2}$, $\quad S_3 = 1 + \frac{1}{2} + \frac{1}{3}$

$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$, ...

$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots\cdots + \frac{1}{n}$ $\quad (\because S_1 < S_2 < S_3 < S_4 < \cdots)$

It is obvious that $\{S_n\}$ is monotonically increasing sequence.

Now sequence $\{S_n\}$ is convergent if it bounded above but Here we shall prove that $\{S_n\}$ is unbounded and so it is divergent and diverges to ∞.

Now $1 > \frac{1}{2}$

$$\frac{1}{2}+\frac{1}{3} > \frac{1}{4}+\frac{1}{4} = \frac{1}{2}$$

$$\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7} > \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8} = \frac{1}{2}$$

$$\frac{1}{8}+\frac{1}{9}+\cdots\cdots+\frac{1}{15} > \frac{1}{16}+\frac{1}{16}+\cdots\cdots+\frac{1}{16}\ (8 \text{ times}) = \frac{1}{2}$$

So $1+\left(\frac{1}{2}+\frac{1}{3}\right)+\left(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right)+\left(\frac{1}{8}+\frac{1}{9}+\cdots+\frac{1}{15}\right)+\frac{1}{16}+\cdots > \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots$

Now consider the series

$$\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots\cdots$$

$$\Rightarrow \quad S_n = \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots \ n \text{ times} = \frac{n}{2}.$$

$$\Rightarrow \quad S_\infty = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\frac{n}{2} = \frac{\infty}{2} = \infty$$

So the series $\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots\cdots$ is divergent [illegible]

Consequently the series $1+\left(\frac{1}{2}+\frac{1}{3}\right)+\left(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right)+\cdots$ whose sum is greater than the series

$$\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots\cdots$$

is also divergent. So

$$\sum_1^\infty a_n = \sum_1^\infty \frac{1}{n} \text{ is divergent.}$$

THE EULER'S SERIES

Investiagate the behaviour series i.e.

$$\sum_1^\infty \frac{1}{n^2} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\cdots+\frac{1}{n^2}+\cdots$$

SOL: We have $S_n = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}$.

The Sequence is monotonically increasing, since

$S_1 = 1, \quad S_2 = 1+\frac{1}{4}, \quad S_3 = 1+\frac{1}{4}+\frac{1}{9}, \ldots$

and $S_1 < S_2 < S_3 < \cdots$

Now we check whether $\{S_n\}$ is bounded

$$\because S_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{n^2}$$

$$\text{or } S_n = 1 + \frac{1}{2\cdot 2} + \frac{1}{3\cdot 3} \cdots \frac{1}{4\cdot 4} + \cdots + \frac{1}{n\cdot n}$$

$$\text{or } S_n < 1 + \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(n-1)n}$$

$$= 1 + \left(1 - \tfrac{1}{2}\right) + \left(\tfrac{1}{2} - \tfrac{1}{3}\right) + \left(\tfrac{1}{3} - \tfrac{1}{4}\right) + \left(\tfrac{1}{4} - \tfrac{1}{5}\right) + \cdots + \left(\tfrac{1}{n-1} - \tfrac{1}{n}\right)$$

$$= 2 - \frac{1}{n}$$

$$\Rightarrow S_n < 2 - \frac{1}{n} \quad \forall\, n$$

$$\Rightarrow \lim_{n\to\infty} S_n < \lim_{n\to\infty}\left(2 - \frac{1}{n}\right) = 2 \qquad \text{i.e. } \lim_{n\to\infty} S_n < 2$$

$\lim_{n\to\infty} S_n < 2 \quad \therefore \{S_n\}$ is bounded above by 2

$\because \{S_n\}$ is monotonically increasing and bounded above so $\{S_n\}$ is convergent sequence. Since the sequence of partial sums of the series is convergent, so the series $\sum_1^\infty \frac{1}{n^2}$ is convergent.

Theorem $\boxed{\text{If } \sum_1^\infty a_n \text{ converges then } \lim_{n\to\infty} a_n = 0}$

Proof Given series is (اگر کوئی series $\sum_1^\infty a_n$ cgs ہو تو اس کی nth term کی $\lim_{n\to\infty}$ zero کے برابر ہوگی)

لیکن اگر nth term کی $\lim_{n\to\infty}$ zero کے برابر ہو تو لازمی نہیں ہے کہ series $\sum_1^\infty a_n$ cgs

see Note on Next Page

$$\sum_1^\infty a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

then its nth partial sum is

(a_n is nth element of S_n)

$$S_n = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$$

$$\Rightarrow S_{n-1} = a_1 + a_2 + a_3 + \cdots + a_{n-1}$$

subtracting $\Rightarrow$ $S_n - S_{n-1} = a_n$

$$\lim_{n\to\infty}(S_n - S_{n-1}) = \lim_{n\to\infty} \sum_1^\infty a_n$$

Now since $\sum_1^\infty a_n$ converges (given)

So $\lim_{n\to\infty}(S_n - S_{n-1})$ converges

i.e. $\lim_{n\to\infty} S_n = S = \lim_{n\to\infty} S_{n-1}$ (Say) as $n\to\infty$ so $n-1\to\infty$

Now since $n \to \infty$ so $n-1 \to \infty$

So $\lim_{n\to\infty} S_{n-1} = S$

Now $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(S_n - S_{n-1})$

$$= \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$$

$$= S - S = 0$$

$$\Rightarrow \boxed{\lim_{n\to\infty} a_n = 0}$$

NOTE: If $\lim_{n\to\infty} a_n = 0$, then $\sum_1^\infty a_n$ is not necessarily convergent

e.g. if $a_n = \frac{1}{n}$, Then $\lim_{n\to\infty} \frac{1}{n} = 0$, but $\sum_1^\infty a_n = \sum_1^\infty \frac{1}{n}$ is divergent. (see Harmonic Series)

DIVERGENT TEST (COROLLARY)

If $\lim_{n\to\infty} a_n \neq 0$ then the series $\sum_1^\infty a_n$ diverges.

EXAMPLE Examine the series $\sum_1^\infty \frac{5n+2}{3n-1}$ for convergence

Sol.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{5n+2}{3n-1} = \lim_{n\to\infty} \frac{5+2/n}{3-\frac{1}{n}} = \frac{5}{3} \neq 0$$

So by divergent test the given series diverges.

THEOREMS: ① If $\sum_1^\infty a_n$ and $\sum_1^\infty b_n$ are convergent series with sums S and T, Then Series $\sum_1^\infty (a_n+b_n)$ and $\sum_1^\infty (a_n-b_n)$ are convergent and sums of these series are

$$\sum_{1}^{\infty}(a_n + b_n) = \sum_{1}^{\infty} a_n + \sum_{1}^{\infty} b_n = S+T$$ (sum of cgt series is cgt. (cgt + cgt = cgt))

and $$\sum_{1}^{\infty}(a_n - b_n) = \sum_{1}^{\infty} a_n - \sum_{1}^{\infty} b_n = S-T$$ (diff of cgt series is cgt. (cgt − cgt = cgt))

(ii) If $\sum_{1}^{\infty} a_n$ converges and $\sum_{1}^{\infty} b_n$ diverges, then $\sum_{1}^{\infty}(a_n + b_n)$ diverges. (sum of cgt + dgt series is dgt) (cgt + dgt = dgt)

(iii) If c is non-zero real number, then $\sum_{1}^{\infty} c a_n$ converges if $\sum_{1}^{\infty} a_n$ converges and $\sum_{1}^{\infty} a_n$ diverges if $\sum_{1}^{\infty} c a_n$ div.

(iv) '+' or '−' of a finite number of terms does not affect the convergence or divergence of an infinite series.

THE BASIC COMPARISON TEST (BCT)

let $\sum_{1}^{\infty} a_n$ and $\sum_{1}^{\infty} b_n$ be two positive terms series

then

(بڑی cvg تو چھوٹی cvg) (i) $\{$if $a_n \leq b_n \; ; \; \forall \; n$, & if $\sum_{1}^{\infty} b_n$ converges$\}$ then $\sum_{1}^{\infty} a_n$ converges.

(چھوٹی dvg تو بڑی dvg) (ii) $\sum_{1}^{\infty} a_n$ diverges if $\{a_n \geq b_n$ and $\sum_{1}^{\infty} b_n$ diverges$\}$

EXAMPLE

Determine whether series $\sum_{1}^{\infty} \frac{1}{1+n^2}$ converges or diverges.

Sol:- Given series is $\sum_{1}^{\infty} \frac{1}{n^2+1} \Rightarrow a_n = \frac{1}{n^2+1} \rightarrow$ (i)

let $b_n = \frac{1}{n^2} \longrightarrow$ (ii)

from (i) and (ii)

$$\frac{1}{n^2+1} < \frac{1}{n^2} \Rightarrow a_n < b_n$$

also from (ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n^2}$ is convergent

So by BCT Σa_n is convergent.

HOW TO FIND b_n from a_n

Power of n in b_n

= Power of n in numerator of a_n

− power of n in denominator of a_n

i.e $0-2 = -2 \therefore b_n = n^{-2} = \frac{1}{n^2}$

Example Show that $\sum_1^\infty \frac{n+5}{n^2+4}$ diverges

$a_n = \frac{n+5}{n^2+4}$

$b_n = \frac{1}{n}$ $\left(\because \frac{n}{n^2} = \frac{1}{n}\right)$

Now $\frac{n+5}{n^2+4} > \frac{1}{n}$

$a_n > b_n$

& $\because \sum_1^\infty b_n = \sum_1^\infty \frac{1}{n}$ is divergent

So $\sum_1^\infty a_n = \sum_1^\infty \frac{n+5}{n^2+4}$ is divergent by BCT

dgt چھوٹی تو بڑی dgt

2nd Method

$a_n = \frac{n+5}{n^2+4}$, $b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n+5}{n^2+4}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \left(\frac{n+5}{n^2+4}\right)\cdot \frac{n}{1}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}\left(1+\frac{5}{n}\right)}{\cancel{n^2}\left(1+\frac{4}{n^2}\right)}\cdot\frac{\cancel{n}}{1}$$

$$= \frac{1+0}{1+0} = \boxed{1} \text{ definite}$$

So both series behave alike

$\sum_1^\infty b_n = \sum_1^\infty \frac{1}{n}$ is dgt

So $\sum_1^\infty a_n = \sum_1^\infty \frac{n+5}{n^2+4}$ is also dgt.

Note

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+5}{n^2+4}$$

$$= \lim_{n\to\infty} \frac{n\left(1+\frac{5}{n}\right)}{n^2\left(1+\frac{4}{n^2}\right)}$$

$$= \lim_{n\to\infty} \frac{\left(1+\frac{5}{n}\right)}{n\left(1+\frac{4}{n^2}\right)}$$

$$= \frac{1}{\infty}$$

$$\boxed{\lim_{n\to\infty} a_n = 0}$$

By Theorem it does not imply that series cgs or dgs. So use some other test.

THE LIMIT COMPARISON TEST.

Let $\sum_1^\infty a_n$ and $\sum_1^\infty b_n$ be two +ive terms series

(i) If $\lim_{n\to\infty} \frac{a_n}{b_n} = l \neq 0$ then $\Rightarrow$ both series converges or both series diverges (depending on b_n.)

(ii) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and if $\sum_1^\infty b_n$ Converges then $\Longrightarrow \sum_1^\infty a_n$ also converges

(iii) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and if $\sum_1^\infty b_n$ diverges, then $\Longrightarrow \sum_1^\infty a_n$ also diverges.

EXAMPLE

use limit comparison test (L.C.T) determine each of following series converges or diverges

(i) $\sum_1^\infty \frac{n+1}{2n^2+1}$ (ii) $\frac{n-4}{n^3+n+3}$

SOL (i) $\because a_n = \frac{n+1}{2n^2+1}$ $\Rightarrow$ $b_n = \frac{1}{n}$

Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n+1}{2n^2+1} \Big/ \frac{1}{n} = \lim_{n\to\infty} \frac{n(n+1)}{2n^2+1}$

$$= \lim_{n\to\infty} \frac{\cancel{n^2}(1+\frac{1}{n})}{\cancel{n^2}(2+\frac{1}{n})} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2+\frac{1}{n}} = \frac{1}{2} = l \neq 0$$

$\Rightarrow$ both series behaves alike.

but $\sum_1^\infty b_n = \sum_1^\infty \frac{1}{n}$ diverges ($\because$ infinite H. Series)

So by Limit comparison test $\sum_1^\infty a_n = \sum_1^\infty \frac{n+1}{2n^2+1}$ is divergent.

(ii) $\sum_1^n \frac{n-4}{n^3+n+3}$

Here $a_n = \frac{n-4}{n^3+n+3}$ $\Rightarrow$ $b_n = \frac{1}{n^2}$

So $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n-4}{n^3+n+3} \times \frac{n^2}{1} = \lim_{n\to\infty} \frac{\cancel{n^3}(1-4/n)}{\cancel{n^3}(1+\frac{n}{n^3}+\frac{3}{n^3})}$

$$= \lim_{n\to\infty} \frac{1-\frac{4}{n}}{1+\frac{1}{n^2}+\frac{3}{n^3}} = 1 = l \neq 0$$

$\Rightarrow$ both series behaves alike.

but $\sum_1^\infty b_n = \sum_1^\infty \frac{1}{n^2}$ is ~~di~~ Convergent ($\because$ Euler Series)

So by Limit Comparison test $\sum_1^\infty a_n = \sum_1^\infty \frac{n-4}{n^3+n+3}$ Converges.

THE INTEGRAL TEST (or Cauchy's Integral Test)

Let $\sum_1^\infty a_n$ be +ive term series. If f is continuous and decreasing function on $[1,\infty[$, such that $f(n) = a_n$ for all +ive integers, then

$$\boxed{\begin{array}{l} \text{(i) } \sum_1^\infty a_n \text{ Converges if } \int_1^\infty f(x)\,dx \text{ Converges.} \\ \hline \text{(ii) } \sum_1^\infty a_n \text{ diverges if } \int_1^\infty f(x)\,dx \text{ diverges} \end{array}}$$

EXAMPLE Investigate the behaviour of harmonic series $\sum_1^\infty \frac{1}{n}$ (by using integral test)

SOL we have $\sum_1^\infty \frac{1}{n}$

Here $a_n = \frac{1}{n} \Rightarrow f(x) = \frac{1}{x}$

then $\int_1^\infty f(x)dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \left|\ln x\right|_1^t$

$= \lim_{t\to\infty} (\ln t - \ln 1) = \infty$ $\quad \boxed{\because \ln 1 = 0 \ \& \lim_{n\to\infty} \ln t = \ln\infty = \infty}$

Since $\int_1^\infty f(x)\,dx$ diverges so $\sum_1^\infty \frac{1}{n}$ diverges.

P_SERIES (OR HYPERHORMONIC SERIES)

Test for convergence the P_series or Hyperharmonic series i.e

$$\sum_{n=1}^\infty \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$$

SOL Here $a_n = \frac{1}{n^p} \Rightarrow f(x) = \frac{1}{x^p}$

Then $I = \int_1^\infty \frac{1}{x^p}\,dx = \int_1^\infty x^{-p}\,dx = \left|\frac{x^{-p+1}}{-p+1}\right|_1^\infty$

Case-1 when $p>1$ say $p = 1+q$, where q is +ive

then $I = \left|\frac{x^{-1-q+1}}{-1-q+1}\right| = \left|\frac{x^{-q}}{-q}\right|_1^\infty = \left|\frac{1}{-q x^q}\right|_1^\infty = 0 - \frac{1}{-q} = \frac{1}{q}$

$\Rightarrow \int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{x^p}\,dx$ Convergent. So $\sum_1^\infty \frac{1}{n^p}$ converges.

Case-2 when $p<1$, say $p = 1-q$, where q is +ive

then $I = \int_1^\infty \frac{1}{x^p}\,dx = \left|\frac{x^{-p+1}}{-p+1}\right|_1^\infty = \left|\frac{x^{-1+q+1}}{-1+q+1}\right|^\infty$

$$= \left| \frac{x^q}{q} \right|_1^\infty = \infty$$

$$\Rightarrow \int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{x^p}\,dx \text{ is divergent}$$

So $\sum_1^n \frac{1}{n^p}$ is divergent for $p<1$

If $p=1$, Then p– Series becomes Harmonic which divergent.

DEDUCTION $\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$

(i) $\sum \frac{1}{n^p}$ is Convergent for $p>1$
(ii) $\sum \frac{1}{n^p}$ is divergent for $p \leq 1$

EXAMPLE Determine whether the series $\sum_1^\infty \frac{\tan^{-1} n}{1+n^2}$ converges or diverges.

SOL:– Given series is $\sum_1^\infty \frac{\tan^{-1} n}{1+n^2}$

$$\Rightarrow a_n = \frac{\tan^{-1} n}{1+n^2} \quad \Rightarrow \quad f(x) = \frac{\tan^{-1} x}{1+x^2}$$

$$\Rightarrow f(x) > 0 \quad \text{for } x \geq 1$$

and $f'(x) = \dfrac{\cancel{(1+x^2)} \cdot \frac{1}{\cancel{1+x^2}} - 2x\tan^{-1}x}{(1+x^2)^2} = \dfrac{1-2x\tan^{-1}x}{(1+x^2)^2}$

Now $f'(x) < 0$ for $x \geq 1$

Hence $f(x)$ is decreasing function for $x \geq 1$ $\left(\because \begin{array}{c} f'(x)<0 \\ \text{for } x\geq 1 \end{array}\right)$

Now $\int_1^\infty f(x)dx = \lim_{t\to\infty} \int_1^t \frac{\tan^{-1}x}{1+x^2}dx = \lim_{t\to\infty} \left| \frac{(\tan^{-1}x)^2}{2} \right|_1^t$

$$= \frac{1}{2} \lim_{t\to\infty} \left\{ \tan^{-1}t - \tan^{-1}1 \right\} = \frac{1}{2}\left\{ \left(\frac{\pi}{2}\right)^2 - \left(\frac{\pi}{4}\right)^2 \right\}$$

$$= \frac{1}{2}\left[\frac{\pi^2}{4} - \frac{\pi^2}{16} \right] = \frac{3\pi^2}{32}$$

So $\int_1^\infty f(x)dx$ Converges. So $\sum_1^\infty \frac{\tan^{-1} n}{1+n^2}$ is Convergent.

EXERCISE 8.2

Determine whether the given series converges or diverges if it converges find its sum. (Problems 1–5):

Q.1 $\sum_{1}^{\infty} \cos \pi n$

Sol. Here $a_n = \cos \pi n$

Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \cos \pi n = \begin{cases} 1, & \text{if } n \text{ is even} \\ -1, & \text{if } n \text{ is odd} \end{cases}$

Since $\lim_{n\to\infty} a_n \neq$ definite Real Value (limit is not unique so diverges)
So by divergence test

So $\sum_{1}^{\infty} \cos \pi n$ is divergent.

Q.2 $\sum_{0}^{\infty} \frac{1}{(2+x)^n}$; $|x| < 1$

IGS is convergent if $r < 1$
IGS is divergent if $r \geq 1$ or $r = -1$

Here $a_n = \frac{1}{(2+x)^n}$

$$\sum_{0}^{\infty} a_n = \sum_{0}^{\infty} \frac{1}{(2+x)^n} = 1 + \frac{1}{2+x} + \frac{1}{(2+x)^2} + \frac{1}{(2+x)^3} + \cdots$$

Infinite G. Series

$a = 1$ & $r = \left|\frac{1}{2+x}\right| < 1$ $\quad \because |x| < 1$

since $|r| < 1$ $\therefore$ Series is Convergent.

$a_n = \frac{1}{(2+x)^n}$

$\lim_{n\to\infty} a_n = \frac{1}{\infty} = 0$ which does not imply that Series is definitely convergent, so check Infinite Geometric Series.

Its sums $S_\infty = \frac{a}{1-r}$

$$S_\infty = \frac{1}{1 - \frac{1}{2+x}}$$

$$\Rightarrow \quad S_\infty = \frac{1}{\frac{2+x-1}{2+x}} = \frac{2+x}{1+x}$$

Q.3 $\sum_{0}^{\infty} \frac{2^{n/2}}{3^n}$

Sol we have $a_n = \frac{2^{\frac{n}{2}}}{3^n} = \left(\frac{\sqrt{2}}{3}\right)^n$

The Series is $1 + \frac{\sqrt{2}}{3} + \left(\frac{\sqrt{2}}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^3 + \cdots \infty$ I.G.S

$a = 1 \quad r = \frac{\sqrt{2}}{3}$

$r < 1$

So Series is cgt.

$$\therefore S_\infty = \frac{a}{1-r}$$

$$= \frac{1}{1-\frac{\sqrt{2}}{3}}$$

$$= \frac{1}{\frac{3-\sqrt{2}}{3}} = \frac{3}{3-\sqrt{2}}$$

2nd Method.

Apply Root test ....

$(a_n)^{\frac{1}{n}} = \left(\frac{2^{\frac{n}{2}}}{3^n}\right)^{\frac{1}{n}}$

$= \frac{2^{\frac{n}{2}\cdot\frac{1}{n}}}{3^{\frac{n}{n}}}$

$= \frac{\sqrt{2}}{3} = .47 < 1$

$(a_n)^{\frac{1}{n}} < 1$

So converges.

Hence given Series is convergent and its sum is $\frac{3}{3-\sqrt{2}}$

Q.4 $\sum_1^\infty \left(\frac{1}{2^n} - \frac{1}{2^{n+1}}\right)$

Sol. Given series is $\sum_1^\infty \left(\frac{1}{2^n} - \frac{1}{2^{n+1}}\right)$

Here $a_1 = \frac{1}{2^1} - \cancel{\frac{1}{2^2}}$

$a_2 = \cancel{\frac{1}{2^2}} - \cancel{\frac{1}{2^3}}$

$a_3 = \cancel{\frac{1}{2^3}} - \cancel{\frac{1}{2^4}}$

- - - - - - - - -

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$a_{n-1} = \cancel{\frac{1}{2^{n-1}}} - \cancel{\frac{1}{2^n}}$

$a_n = \cancel{\frac{1}{2^n}} - \frac{1}{2^{n+1}}$

$S_n = a_1 + a_2 + \cdots + a_n = \frac{1}{2} - \frac{1}{2^{n+1}}$

so $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{1}{2} - \frac{1}{2^{n+1}}\right) = \frac{1}{2} - 0 = \boxed{\frac{1}{2}}$ definite value

So the Series $\sum_1^\infty a_n = \sum_1^\infty \left(\frac{1}{2^n} - \frac{1}{2^{n+1}}\right)$ is convergent

2nd Method Easy.

$a_n = \frac{1}{2^n} - \frac{1}{2^{n+1}}$

$= \frac{1}{2^n}\left(1 - \frac{1}{2}\right)$

$= \frac{1}{2^n}\left(\frac{1}{2}\right) = \boxed{\frac{1}{2^{n+1}}}$

$\sum_1^\infty a_n = \sum_1^\infty \frac{1}{2^{n+1}} = \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} - - -$ IGS

Infinite G. Series with

$a = \frac{1}{4} \quad r = \frac{1}{2^3}\cdot\frac{2^2}{1} = \frac{1}{2} < 1$

So Series is Convergent

$S_\infty = \frac{a}{1-r}$

$= \frac{\frac{1}{4}}{1-\frac{1}{2}} = \frac{\frac{1}{4}}{\frac{1}{2}}$

$= \frac{1}{4}\cdot\frac{2}{1} = \boxed{\frac{1}{2}}$

$\because$ If $\lim_{n\to\infty} S_n = S$ definite value then $\sum_1^\infty a_n$ converge

detail see page 7

Q.5 $\sum_{1}^{\infty} \frac{1}{9n^2+3n-2}$

Sol: we have $\sum_{1}^{\infty} \frac{1}{9n^2+3n-2} = \frac{1}{(3n+2)(3n-1)}$

$9n^2+6n-3n-2$
$= 3n(3n+2)-1(3n+2)$
$= (3n-1)(3n+2)$

By P. Fraction

$$= \sum_{1}^{\infty} \frac{1}{(3n+2)(3n-1)} = \frac{A}{(3n-1)} + \frac{B}{(3n+2)}$$

$$\Rightarrow a_n = \frac{1}{(3n+2)(3n-1)} = \frac{1}{3}\left[\frac{1}{3n-1}\right] - \frac{1}{3}\left[\frac{1}{3n+2}\right]$$

$$\frac{1}{(3n+2)(3n-1)} = \frac{1}{3}\left(\frac{(3n+2)-(3n-1)}{(3n+2)(3n-1)}\right)$$

$$= \frac{1}{3}\left(\frac{3n+2}{(3n+2)(3n-1)} - \frac{3n-1}{(3n+2)(3n-1)}\right)$$

$$= \frac{1}{3}\left(\frac{1}{3n-1} - \frac{1}{3n+2}\right)$$

$$\Rightarrow a_1 = \frac{1}{3}\cdot\frac{1}{2} - \cancel{\frac{1}{3}\cdot\frac{1}{5}}$$

$$a_2 = \cancel{\frac{1}{3}\cdot\frac{1}{5}} - \cancel{\frac{1}{3}\cdot\frac{1}{8}}$$

$$a_3 = \cancel{\frac{1}{3}\cdot\frac{1}{8}} - \cancel{\frac{1}{3}\cdot\frac{1}{11}}$$

$$\cdots\cdots\cdots$$

$$a_{n-1} = \frac{1}{3}\left(\cancel{\frac{1}{3n-4}}\right) - \frac{1}{3}\left(\cancel{\frac{1}{3n-1}}\right)$$

$$a_n = \frac{1}{3}\left(\cancel{\frac{1}{3n-1}}\right) - \frac{1}{3}\left(\frac{1}{3n+2}\right)$$

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = \frac{1}{3}\cdot\frac{1}{2} - \frac{1}{3}\cdot\frac{1}{3n+2} = \frac{1}{3}\left[\frac{1}{2} - \frac{1}{3n+2}\right]$$

$$\Rightarrow \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{3}\left(\frac{1}{2} - \frac{1}{3n+2}\right) = \frac{1}{6}$$

$\Rightarrow$ The Sequence $\{S_n\}$ is convergent. Hence the series $\sum_{1}^{\infty} a_n = \sum_{1}^{\infty} \frac{1}{9n^2+3n-2}$ is Convergent.

x ———— x

Each of the following is the nth partial sums of an infinite series. Determine the series and check whether it converges (Problems 6 – 8).

$S_n = \cancel{a_1} + \cancel{a_2} + \cdots + \cancel{a_{n-1}} + a_n$
$S_{n-1} = \cancel{a_1} + \cancel{a_2} + \cdots + \cancel{a_{n-1}}$

$S_n - S_{n-1} = a_n$

⑥ $S_n = \frac{3n}{4n+1}$

Sol $\because a_n = S_n - S_{n-1}$

$$= \frac{3n}{4n+1} - \frac{3(n-1)}{4(n-1)+1} = \frac{3n}{4n+1} - \frac{3n-3}{4n-3}$$

$$= \frac{12n^2 - 9n - 12n^2 + 9n + 3}{(4n+1)(4n-3)} = \frac{3}{(4n+1)(4n-3)}$$

$$\Rightarrow a_n = \frac{3}{(4n+1)(4n-3)}$$

$\Rightarrow \therefore$ Required Infinite Series is

So $\boxed{\sum_1^{\infty} a_n = \sum_1^{\infty} \frac{3}{(4n+1)(4n-3)}}$

$\therefore$ sum of I. Series is

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{3n}{4n+1}$$

$$= \lim_{n\to\infty} \frac{3\cancel{n}}{\cancel{n}(4+\frac{1}{n})}$$

$$= \lim_{n\to\infty} \frac{3}{4+\frac{1}{n}}$$

$\boxed{\lim_{n\to\infty} S_n = \frac{3}{4}}$ (definite value)

$\Rightarrow$ The Sequence of partial sum i.e $\{S_n\}$ converges. Hence the series $\sum_1^{\infty} a_n = \sum_1^{\infty} \frac{3}{(4n+1)(4n-3)}$ Converges and its sum is $= \frac{3}{4}$

Difficult ALTERNATE (For sum of infinite series)

$\because a_n = \frac{3}{(4n+1)(4n-3)} = 3\left[\frac{(4n+1)-(4n-3)}{(4n+1)(4n-3)}\right]$

$a_n = \frac{3}{4}\left[\frac{1}{4n-3} - \frac{1}{4n+1}\right]$ by partial fraction

$\Rightarrow a_1 = \frac{3}{4}\left[\frac{1}{1} - \cancel{\frac{1}{5}}\right]$

$a_2 = \frac{3}{4}\left[\cancel{\frac{1}{5}} - \cancel{\frac{1}{9}}\right]$

$a_3 = \frac{3}{4}\left[\cancel{\frac{1}{9}} - \cancel{\frac{1}{13}}\right]$

$\cdots\cdots\cdots$

$a_{n-1} = \frac{3}{4}\left[\cancel{\frac{1}{4n-7}} - \cancel{\frac{1}{4n-3}}\right]$

$a_n = \frac{3}{4}\left[\cancel{\frac{1}{4n-3}} - \frac{1}{4n+1}\right]$

$+$

$S_n = a_1 + a_2 + \cdots + a_n = \frac{3}{4}\left(1 - \frac{1}{4n+1}\right)$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{3}{4}\left(1 - \frac{1}{4n+1}\right) = \frac{3}{4}$

Q.7 $S_n = \frac{n^2}{n+1}$

Sol: $\because a_n = S_n - S_{n-1}$

$$\Rightarrow a_n = \frac{n^2}{n+1} - \frac{(n-1)^2}{n}$$

$$= \frac{n^3 - (n+1)(n^2-2n+1)}{(n+1)(n)}$$

$$= \frac{\cancel{n^3} - \cancel{n^3} + n^2 + n - 1}{n(n+1)}$$

$$a_n = \frac{n^2+n-1}{n^2+n}$$

So required infinite series is

$\boxed{\sum_1^{\infty} a_n = \sum_1^{\infty} \frac{n^2+n-1}{n^2+n}}$ ✱

So $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2+n-1}{n^2+n}$

$= \lim_{n\to\infty} \frac{1 + 1/n - \frac{1}{n^2}}{1+\frac{1}{n}} = 1$

$\boxed{\lim_{n\to\infty} a_n \neq 0}$

so by divergent test $\sum a_n$ is divergent

✱ First find lim, then ALTERNATE Easy.

$$S_n = \frac{n^2}{n+1}$$

So $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n^2}{n+1}$

$= \lim_{n\to\infty} \frac{n^2}{n(1+\frac{1}{n})}$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n}{1+\frac{1}{n}} = \infty$ not definite value

$\boxed{\lim_{n\to\infty} S_n = \infty}$ So $\{S_n\}$ seq is Divergent

$\Rightarrow \therefore$ The series is divergent

Q.8 $S_n = \dfrac{1}{2^n}$

SOL $a_n = S_n - S_{n-1}$

$\therefore a_n = \dfrac{1}{2^n} - \dfrac{1}{2^{n-1}} = \dfrac{1}{2^n} - \dfrac{1}{\frac{2^{n+1}}{2}} = \dfrac{1}{2^n} - \dfrac{2}{2^n} = \dfrac{-1}{2^n}$

so $\boxed{\sum_1^\infty a_n = \sum_1^\infty \left(\frac{-1}{2^n}\right)}$ is required Infinite Series

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \dfrac{1}{2^n} = 0$ is the sum of required Infinite Series

$\boxed{\lim_{n\to\infty} S_n = 0}$ definite value

So Seq. $\{S_n\}$ is convergent and its sum $= 0$

$\therefore \sum_1^\infty a_n = \sum_1^\infty \left(\frac{-1}{2^n}\right)$ is convergent and its sum $= 0$

Q.9 Prove that if a positive term series $\sum_1^\infty a_n$ Converges, then the series $\sum_1^\infty \sqrt{a_n \cdot a_{n+1}}$ converges.

SOL Since a_n and a_{n+1} are both +ive terms $\forall$ all +ive integral values of n

$G.M = \sqrt{ab}$
$A.M = \frac{a+b}{2}$

G.Mean, $G = \sqrt{a_n \cdot a_{n+1}}$ and $A.M = \dfrac{a_n + a_{n+1}}{2}$

but $A.M. > G.M.$ (always)

$\boxed{\because A > G > H \text{ for } a \text{ and } b}$

$\Rightarrow \dfrac{a_n + a_{n+1}}{2} > \sqrt{a_n \cdot a_{n+1}}$

$\Rightarrow \sum_1^\infty \left(\dfrac{a_n + a_{n+1}}{2}\right) > \sum_1^\infty \sqrt{a_n \cdot a_{n+1}}$

$\Rightarrow \dfrac{1}{2} \sum_1^\infty (a_n + a_{n+1}) > \sum_1^\infty \sqrt{a_n \cdot a_{n+1}}$

Since $\sum a_n$ is convergent (given)

So $\sum a_{n+1}$ " " also

$\Rightarrow \sum a_n + \sum a_{n+1}$ is Convergent

$\Rightarrow \frac{1}{2}\left(\sum a_n + \sum a_{n+1}\right)$ " "

Then by comparison test

$\sum_1^\infty \sqrt{a_n \cdot a_{n+1}}$ is Convergent

(بڑی تو تو چھوٹی تو)

Q.10 Give an example in which both $\sum_1^\infty a_n$ and $\sum_1^\infty b_n$ diverges but $\sum_1^\infty (a_n + b_n)$ converges

SOL: Let $\sum_1^\infty a_n = \sum_1^\infty \left(\frac{1}{n^2} - \frac{1}{n}\right)$ and $\sum_1^\infty b_n = \sum_1^\infty \left(\frac{1}{n^2} + \frac{1}{n}\right)$

both are divergent series

$\left(\because \text{Convergent} \pm \text{divergent} = \text{Divergent} \right)$

but
$$\sum_1^\infty (a_n + b_n) = \sum_1^\infty a_n + \sum_1^\infty b_n$$
$$= \sum_1^\infty \left(\frac{1}{n^2} - \frac{1}{n}\right) + \sum_1^\infty \left(\frac{1}{n^2} + \frac{1}{n}\right)$$
$$= \sum_1^\infty \left(\frac{1}{n^2} - \cancel{\frac{1}{n}} + \frac{1}{n^2} + \cancel{\frac{1}{n}}\right) = \sum_1^\infty \frac{2}{n^2}$$
$$= 2 \sum_1^\infty \frac{1}{n^2} \quad \text{(Euler Series)}$$
$$= \text{Convergent Series}$$

Using Comparison Tests, investigate convergence or divergence of the series in Problems (11 —— 20)

BCT **Q.11** $\sum_1^\infty \frac{1}{n^2 + n^{3/2}}$

SOL: Here $a_n = \frac{1}{n^{\frac{1}{2}} + n^{\frac{3}{2}}}$

$\because 0 - \frac{3}{2} = -\frac{3}{2}$

$b_n = n^{-3/2}$

$b_n = \frac{1}{n^{3/2}}$

Now

$\because n^{\frac{1}{2}} + n^{\frac{3}{2}} > n^{\frac{3}{2}}$

$\Rightarrow \frac{1}{n^{\frac{1}{2}} + n^{\frac{3}{2}}} < \frac{1}{n^{\frac{3}{2}}}$

$\Rightarrow a_n < b_n$

but $\sum_1^\infty b_n = \sum \frac{1}{n^{\frac{3}{2}}}$ is Convergent $\left(\because P = \frac{3}{2} > 1\right)$

So by comparison test (gt.3, gt 6-?)

$\sum_1^\infty a_n = \sum_1^\infty \frac{1}{n^{\frac{1}{2}} + n^{\frac{3}{2}}}$ is Convergent.

B COMPARISON TEST

If $\sum a_n$ & $\sum b_n$ two +ive terms series, Then

(i) $\sum a_n$ Converges iff $a_n < b_n$ and $\sum b_n$ converges.

(ii) $\sum a_n$ diverges iff $a_n > b_n$ and $\sum b_n$ is divergent.

P_ Series

$\sum_1^\infty \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \le 1$

By L.C.T
ALTERNATE $a_n = \dfrac{1}{n^{\frac{1}{2}} \cdot n^{3/2}}$; take $b_n = \dfrac{1}{n^{3/2}}$

then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{n^{3/2}}{n^{3/2} + n^{1/2}} = \lim\limits_{n\to\infty} \dfrac{n^{3/2}}{n^{3/2}(1+\frac{1}{n})}$

$= \lim\limits_{n\to\infty} \dfrac{1}{1+\frac{1}{n}} = 1 \neq 0$. (definite number but not equal to zero)

$\Rightarrow$ both series behave alike.

but $\Sigma b_n = \Sigma \dfrac{1}{n^{3/2}}$ $\left(\because P = \frac{3}{2} > 1\right)$ is convergent

so by limit comparison test Σa_n is convergent

Wrong **Q.12** $\displaystyle\sum_{1}^{\infty} \frac{\sqrt{n}}{n+1}$

SOL:- Here $a_n = \dfrac{\sqrt{n}}{n+1} = \dfrac{1}{\frac{n+1}{\sqrt{n}}} = \dfrac{1}{\sqrt{n}+\frac{1}{\sqrt{n}}}$

take $b_n = \dfrac{1}{n^{\frac{1}{2}}}$

$\because \sqrt{n} + \dfrac{1}{\sqrt{n}} > \sqrt{n}$

$\Rightarrow \dfrac{1}{\sqrt{n}+\frac{1}{\sqrt{n}}} < \dfrac{1}{\sqrt{n}}$

بڑی cgt تو چھوٹی cgt
چھوٹی dgt تو بڑی dgt

$\Rightarrow a_n < b_n$

Also $\Sigma b_n = \Sigma \dfrac{1}{\sqrt{n}} = \Sigma \dfrac{1}{n^{1/2}}$ $\left(\because P = \frac{1}{2} < 1\right)$ is Divergent

$\because$ Bigger term is dgt so we can not say anything about smaller term

Q12 DO By L.C.T
ALTERNATE

$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{\sqrt{n}}{n+1} \times \dfrac{\sqrt{n}}{1}$

$= \lim\limits_{n\to\infty} \dfrac{n}{n+1}$

$= \lim\limits_{n\to\infty} \dfrac{n}{n(1+\frac{1}{n})}$

$= \lim\limits_{n\to\infty} \dfrac{1}{1+\frac{1}{n}} = 1 = l \neq 0$

$\Rightarrow$ both series Σa_n & Σb_n behave alike.

but $\Sigma b_n = \Sigma \dfrac{1}{\sqrt{n}}$ is div.

$\because P = \frac{1}{2} < 1$

so by Limit-C-Test Σa_n is also divergent

By LCT **Q.13** $\displaystyle\sum_{1}^{\infty} \frac{2}{\sqrt{n}+1}$

SOL. Here $a_n = \dfrac{2}{\sqrt{n}+1}$, take $b_n = \dfrac{1}{\sqrt{n}}$ $\left(\because 0 - \frac{1}{2} = -\frac{1}{2} \quad \therefore b_n = n^{-\frac{1}{2}} = \frac{1}{\sqrt{n}}\right)$

$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{2}{\sqrt{n}+1} \times \dfrac{\sqrt{n}}{1} = \lim\limits_{n\to\infty} \dfrac{2\sqrt{n}}{\sqrt{n}(1+\frac{1}{\sqrt{n}})} = \lim\limits_{n\to\infty} \dfrac{2}{1+\frac{1}{\sqrt{n}}} = 2 \neq 0$

$\Rightarrow$ both series behave alike

Now since $\Sigma b_n = \Sigma \dfrac{1}{n^{1/2}}$ $\left(\because P = \frac{1}{2} < 1\right)$ is divergent

So $\Sigma a_n = \Sigma \dfrac{2}{\sqrt{n}+1}$ is divergent

Q13 By B.C.T

ALTERNATE $a_n = \frac{2}{\sqrt{n}+1}$, take $b_n = \frac{1}{\sqrt{n}}$

Then $a_n - b_n = \frac{2}{\sqrt{n}+1} - \frac{1}{\sqrt{n}} \geq 0$ for $n = 1, 2, 3, \ldots$

$\Rightarrow \quad a_n \geq b_n$

and $\sum_1^\infty b_n = \sum_1^\infty \frac{1}{\sqrt{n}} = \sum_1^\infty \frac{1}{n^{1/2}}$ is divergent $(\because p = \frac{1}{2} < 1)$

So by B. comparison test $\sum_1^\infty a_n$ is divergent.

چھوٹی dgt تو بڑی dgt.

Q.14 $\sum_1^\infty \frac{1}{n^n}$

SOL here $a_n = \frac{1}{n^n}$ $\boxed{\text{WRONG}}$

$\Rightarrow \quad a_1 = \frac{1}{1} = 1$

$a_2 = \frac{1}{2^2} \not< \frac{1}{2^2} =$, $a_3 = \frac{1}{3^3} < \frac{1}{2^2} = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$a_4 = \frac{1}{4^4} < \frac{1}{2^3} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$

$a_5 = \frac{1}{5^5} < \frac{1}{2^4} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{16}$

$\ldots\ldots\ldots\ldots$

$\Rightarrow \sum_1^\infty \frac{1}{n^n} = 1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \frac{1}{5^5} + \ldots\ldots$

$\Rightarrow \sum_1^\infty \frac{1}{n^n} < 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \ldots$

= an infinite Geometric series with $r = \frac{1}{2} < 1$

$\therefore \Rightarrow \sum_1^\infty \frac{1}{2^n} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots\ldots$

is convergent series due to $r = \frac{1}{2} < 1$

also $\sum_1^\infty \frac{1}{n^n} < \sum_1^\infty \frac{1}{2^n}$

By B.C.T **Q.14** $\sum_1^\infty \frac{1}{n^n}$

SOL Given series is $\sum_1^\infty a_n = \sum_1^\infty \frac{1}{n^n}$

Now $a_1 = \frac{1}{1^1} \geq \frac{1}{2^1}$

$a_2 = \frac{1}{2^2} = \frac{1}{2^2}$

$a_3 = \frac{1}{3^3} < \frac{1}{2^3}$

$a_4 = \frac{1}{4^4} < \frac{1}{2^4}$

$a_5 = \frac{1}{5^5} < \frac{1}{2^5}$

- - - - - - - -

$\Rightarrow a_n = \frac{1}{n^n} < \frac{1}{2^n}$ for $n \geq 3$

2nd Method
Ratio Test

(14) $\Sigma \frac{1}{n^n}$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{n^{n+1}} \cdot \frac{n^n}{1}$

$= \lim_{n\to\infty} \frac{1}{\cancel{n^n}\, n} \cdot \frac{\cancel{n^n}}{1}$

$= \lim_{n\to\infty} \frac{1}{n}$

$= \frac{1}{\infty} = \boxed{0} < 1$

Hence series $\Sigma \frac{1}{n^n}$ Cgs

3rd Method

Q14 $\Sigma \frac{1}{n^n}$

We know $n^n > 2^n$ for $n \geq 3$

$\frac{1}{n^n} < \frac{1}{2^n}$

$\frac{1}{n^n} < (\frac{1}{2})^n$

But $\sum_3^\infty (\frac{1}{2})^n$ is IGS with $r = \frac{1}{2} < 1$

$\therefore$ It is Cgt.

So By B.CT $\Sigma \frac{1}{n^n}$ is Cgt.

but $\sum_1^\infty \frac{1}{2^n}$ is an infinite Geometric series with ratio $r = \frac{1}{2} < 1$. Hence $\sum_1^\infty \frac{1}{2^n} = \sum_1^\infty (\frac{1}{2})^n$ is convergent.

So by B. Comparison test $\sum_1^\infty \frac{1}{n^n}$ is also convergent.

BCT **Q.15** $\sum_0^\infty \frac{1}{n!}$

SOL $\Sigma a_n = \Sigma \frac{1}{n!}$

Now $a_1 = \frac{1}{1!} = \frac{1}{2^0}$

$a_2 = \frac{1}{2!} = \frac{1}{2^1}$

$a_3 = \frac{1}{3!} < \frac{1}{2^2}$

$a_4 = \frac{1}{4!} < \frac{1}{2^3}$

$a_5 = \frac{1}{5!} < \frac{1}{2^4}$

- - - - - - - -

$\Rightarrow a_n = \frac{1}{n!} < \frac{1}{2^{n-1}} = b_n$ (say)

Q15 2nd Method Ratio Test

$\Sigma \frac{1}{n!}$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} =$

$= \lim_{n\to\infty} \frac{1}{(n+1)!} \cdot \frac{n!}{1}$

$= \lim_{n\to\infty} \frac{1}{(n+1)\, n!} \cdot \frac{n!}{1}$

$= \lim_{n\to\infty} \frac{1}{n+1}$

$= \frac{1}{\infty+1} = \boxed{0} < 1$

Hence Series Cgs

Easy by Q. Shah

Q16 $\Sigma \frac{e^{2n}+e^{-2n}}{e^n+e^{-n}}$

$a_n = \frac{e^{2n}+e^{-2n}}{e^n+e^{-n}}$

$= \frac{e^{2n}(1+\frac{1}{e^{4n}})}{e^n(1+\frac{1}{e^{2n}})}$

$= e^n \frac{(1+\frac{1}{e^{4n}})}{(1+\frac{1}{e^{2n}})}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^n \left(\frac{1+\frac{1}{e^{4n}}}{1+\frac{1}{e^{2n}}}\right)$

$= \infty$ $\therefore$ Dgt.

but $\sum_1^\infty b_n = \sum_1^\infty \frac{1}{2^{n-1}}$ is an infinite Geometric series with $r = \frac{1}{2} < 1$. Hence it is convergent. So by B.C. Test.

So $\sum_1^\infty a_n$ and hence $\sum_0^\infty a_n$ is convergent. (cgt ~~~~ cgt ~~~)

By CT **Q.16** $\sum_1^\infty \frac{e^{2n}+e^{-2n}}{e^n+e^{-n}}$

SOL. Here $a_n = \frac{e^{2n}+e^{-2n}}{e^n+e^{-n}} = \frac{e^{2n}+\frac{1}{e^{2n}}}{e^n+\frac{1}{e^n}} = \frac{(e^{4n}+1)/e^{2n}}{(e^{2n}+1)/e^n}$

or $a_n = \frac{e^{4n}+1}{e^n(e^{2n}+1)} = \frac{e^{4n}+1}{e^{3n}+e^n}$

$e^{4n} - e^{3n} > e^n = b_n$

$e^{4n-3n} = e^n$

take $b_n = e^n$

$\Rightarrow \sum_1^\infty b_n = \sum_1^\infty e^n = e + e^2 + e^3 + e^4 + \cdots$ an infinite G. Series

with $r = e = 2.712818 > 1$

So $\sum_1^\infty b_n$ is divergent series

Also $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{e^{4n}+1}{e^{3n}+e^n} / e^n$

$$= \lim_{n\to\infty} \frac{e^{4n}+1}{e^{4n}+e^{2n}} = \lim_{n\to\infty} \frac{1+\frac{1}{e^{4n}}}{1+\frac{1}{e^{2n}}} = 1 \neq 0$$

$\Rightarrow$ both series Σa_n & Σb_n behave alike. Now Since Σb_n is divergent. So by Limit C. Test Σa_n also diverges.

LCT Q.17 $\sum_1^\infty \frac{\ln n}{n}$

(M.T 2014)

SOL Here $a_n = \frac{\ln n}{n}$ $\therefore b_n = \frac{1}{n}$

$\therefore b_n = \frac{0-1}{n} = \frac{-1}{n}$

Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln n}{n} \times \frac{n}{1} = \lim_{n\to\infty} \ln n$

$\Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = \infty$

$\because \sum_1^\infty b_n = \sum_1^\infty \frac{1}{n}$ is divergent.

$(\because P = 1)$ so by L.C.T.

$\sum_1^\infty a_n = \sum_1^\infty \frac{\ln n}{n}$ is divergent

ALTERNATE

$a_n = \frac{\ln n}{n} \Rightarrow f(x) = \frac{\ln x}{x}$

So $\int_1^\infty f(x)\,dx = \lim_{t\to\infty} \int_1^t \ln x \cdot \frac{1}{x}\,dx$

$= \lim_{t\to\infty} \left| \frac{(\ln x)^2}{2} \right|_1^t$

$= \lim_{t\to\infty} \left(\frac{(\ln t)^2}{2} - \frac{(\ln 1)^2}{2} \right)$

$= \infty$

$\Rightarrow$ According to integral test Σa_n is divergent

LCT Q.18 $\sum_1^\infty \left(1+\frac{1}{n}\right)^n$

SOL $a_n = \left(1+\frac{1}{n}\right)^n = \left(\frac{n+1}{n}\right)^n = \frac{(n+1)^n}{n^n}$

take $b_n = n^0 = 1$

and $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^n}{1} = e \neq 0 \Rightarrow$ both series behave alike

Now since $\sum_1^\infty b_n = \sum_1^\infty 1 = 1+1+1+1+\cdots = \infty$, $\Rightarrow \sum_1^\infty b_n$ is divergent

So by Limit comparison test $\sum_1^\infty a_n = \sum_1^\infty \left(1+\frac{1}{n}\right)^n$ is divergent

Q18 $\sum_1^\infty \left(1+\frac{1}{n}\right)^n$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e \neq 0$

$\Rightarrow \Sigma a_n = \Sigma\left(1+\frac{1}{n}\right)^n$ is dgt.

LCT Easy **Q.19** $\sum_1^\infty \sin\frac{\pi}{n}$

Sol $a_n = \sin\frac{\pi}{n}$

$b_n = \frac{\pi}{n}$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\sin(\frac{\pi}{n})}{(\frac{\pi}{n})} = 1$$

$\left(\begin{array}{l}\frac{\pi}{n}=x \;\; \because \lim_{x\to 0}\frac{\sin x}{x} \\ \text{as } n\to\infty \\ x\to 0 \qquad\quad = 1\end{array}\right.$

so $\lim_{n\to\infty}\frac{a_n}{b_n} = 1 \neq 0$ ∴ both series behave alike

but $\sum_1^\infty b_n = \sum_1^\infty \frac{\pi}{n} = \pi\sum_1^\infty \frac{1}{n}$ is divergent. so by Limit C.T. $\sum_1^\infty a_n = \sum_1^\infty \sin\frac{\pi}{n}$ is divergent

ALTERNATE LCT

Here $a_n = \sin\frac{\pi}{n}$

take $b_n = \frac{1}{n}$

Then $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\sin\frac{\pi}{n}}{1/n}$

$= \lim_{n\to\infty} \pi\,\frac{\sin\frac{\pi}{n}}{\pi/n}$

let $\frac{\pi}{n} = x$, so as $n\to\infty$, $x\to 0$

$\Rightarrow \lim_{n\to\infty}\frac{a_n}{b_n} = \pi\lim_{x\to 0}\frac{\sin x}{x} = \pi\cdot 1 = \pi \neq 0$

$\Rightarrow$ both series behave alike

but $\sum_1^\infty b_n = \sum_1^\infty \frac{1}{n}$ is divergent

$\therefore$ by L.C.T. $\sum_1^\infty a_n$ is div.

BCT **Q.20** $\sum_1^\infty \frac{1}{e^{n^2}}$

Sol: $a_n = \frac{1}{e^{n^2}}$

take $b_n = \frac{1}{e^n}$ (say)

Now $\frac{1}{e^{n^2}} < \frac{1}{e^n}$ for $n \geq 1$ (Note $e^{n^2} \geq e^{n}$)

$\Rightarrow a_n < b_n$

but $\sum_1^\infty b_n = \sum_1^\infty \frac{1}{e^n} = \frac{1}{e} + \frac{1}{e^2} + \frac{1}{e^3} + \cdots$ IGS

$\therefore r = \frac{1}{e} < 1$

so $\sum_1^\infty b_n$ is Convergent

Hence by B. Comparison test

$\sum a_n = \sum_1^\infty \frac{1}{e^{n^2}}$ is Convergent

Q20 Ratio Test

$\sum \frac{1}{e^{n^2}}$

$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}$

$= \lim_{n\to\infty}\frac{1}{e^{(n+1)^2}}\cdot\frac{e^{n^2}}{1}$

$= \frac{1}{e^{n^2+2n+1}}\cdot\frac{e^{n^2}}{1}$

$= \lim_{n\to\infty}\frac{1}{e^{2n+1}}$

$= \frac{1}{e^\infty}$

$= \boxed{0} < 1$

Hence $\sum\frac{1}{e^{n^2}}$ Converges.

Q20 Root Test

$\sum \frac{1}{e^{n^2}}$

$\lim_{n\to\infty}(a_n)^{\frac{1}{n}}$

$= \lim_{n\to\infty}\left(\frac{1}{e^{n^2}}\right)^{\frac{1}{n}}$

$= \lim_{n\to\infty}\left(e^{-n^2}\right)^{\frac{1}{n}}$

$= \lim_{n\to\infty}\left(e^{-n}\right)$

$= \lim_{n\to\infty}\left(\frac{1}{e^n}\right)$

$= \frac{1}{\infty}$

$= \boxed{0} < 1$

Hence series converges

In Problems (21 — 40), test each of following Series Converges or diverges.

Q21 $\sum_{1}^{\infty} n^2 e^{-n^3}$

Sol $a_n = n^2 e^{-n^3}$

$f(x) = x^2 e^{-x^3}$

$f'(x) = 2x e^{-x^3} + x^2 e^{-x^3}(-3x^2)$

$= x e^{-x^3}(2 - 3x^3) \quad < 0$

$\therefore f(x)$ is Montonically decreasing

Now $\int_1^{\infty} f(x)dx$

$= \int_1^{\infty} x^2 e^{-x^3} dx$

$= \lim_{t\to\infty} \left(-\frac{1}{3}\right)\int_1^t e^{-x^3}(-3x^2)dx$

$= \lim_{t\to\infty} \left(-\frac{1}{3}\right)\left| e^{-x^3}\right|_1^t$

$= \lim_{t\to\infty} \left(-\frac{1}{3}\right)\left[e^{-t^3} - e^{-1}\right]$

$= \left(-\frac{1}{3}\right)\left(\frac{1}{\infty} - \frac{1}{e}\right) = \boxed{\frac{1}{3e}}$ definite value

So $\int_1^{\infty} f(x)dx$ is cgt $\Rightarrow$ series $\sum_1^{\infty} n^2 e^{-n^3}$ also cgs

Note $\int x^2 e^{-x^3} dx$

$-\frac{1}{3}\int e^{-x^3}(-3x^2)\,dx$

Put $-x^3 = t$

$-3x^2 dx = dt$

$= -\frac{1}{3}\int e^t dt$

$= -\frac{1}{3} e^t$

$= -\frac{1}{3} e^{-x^3}$

Q21 2nd Method by Rat

$\sum a_n = \sum n^2 e^{-n^3}$

$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{e^{(n+1)^3}} \cdot \frac{e^{n^3}}{n^2}$

$= \left(\frac{n+1}{n}\right)^2 \frac{1}{e^{(n+1)^3 - n^3}}$

$= \left(1 + \frac{1}{n}\right)^2 \frac{1}{e^{n^3 + 3n^2 + 3n + 1 - n^3}}$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^2 \frac{1}{e^{3n^2+3n+1}}$

$= \boxed{0 < 1}$

The series converges.

LCT **Q22** $\sum_1^{\infty} \frac{1}{(2n-1)^{\frac{1}{3}}}$

Sol

$a_n = \frac{1}{(2n-1)^{\frac{1}{3}}}, \quad b_n = \frac{1}{n^{\frac{1}{3}}}$

$0 - \frac{1}{3} = -\frac{1}{3}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{\frac{1}{3}}}{(2n-1)^{\frac{1}{3}}}$

$= \lim_{n\to\infty} \left(\frac{n}{2n-1}\right)^{\frac{1}{3}}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{1}{2 - \frac{1}{n}}\right)^{\frac{1}{3}} = \left(\frac{1}{2}\right)^{\frac{1}{3}} = l \neq 0$

both series behave alike

but $\sum_1^{\infty} b_n = \sum_1^{\infty} \frac{1}{n^{\frac{1}{3}}}$ is divergent $\because P = \frac{1}{3} < 1$

So by L.C.T. $\sum_1^{\infty} \frac{1}{(2n-1)^{1/3}}$ is divergent

Integral Test **ALTERNATE**

$a_n = \frac{1}{(2n-1)^{1/3}}$

$\Rightarrow f(x) = \frac{1}{(2x-1)^{1/3}}$

So $\int_1^{\infty} f(x)dx = \lim_{t\to\infty} \int_1^t \frac{1}{(2x-1)^{1/3}} dx$

$= \lim_{t\to\infty} \int_1^t \frac{(2x-1)^{-\frac{1}{3}}(2)\,dx}{2}$

$= \lim_{t\to\infty} \frac{1}{2}\left|\frac{(2x-1)^{\frac{2}{3}}}{\frac{2}{3}}\right|_1^t$

$= \lim_{t\to\infty} \frac{3}{4}\left((2t-1)^{\frac{2}{3}} - 1\right)$

$= \frac{3}{4}(\infty - 1) = \infty$

$\Rightarrow \sum_1^{\infty} a_n = \sum_1^{\infty} \frac{1}{(2n-1)^{\frac{1}{3}}}$ is divergent

Q23 Ex 8.3

DT **Q.23** $\sum_{1}^{\infty} \frac{2^n}{3^n+1}$

SOL. Here $a_n = \frac{2^n}{3^n+1}$

We know $\frac{2^n}{3^n+1} < \frac{2^n}{3^n}$

also $\frac{2^n}{3^n+1} < \left(\frac{2}{3}\right)^n$ $\forall$ values of n

$\Rightarrow a_n < b_n$

But $\sum_{1}^{\infty} b_n = \sum_{1}^{\infty}\left(\frac{2}{3}\right)^n$

$= \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \cdots$ IGS $r = \frac{2}{3} < 1$

so by Basic Comparison Test $\sum_{1}^{\infty} a_n = \sum_{1}^{\infty} \frac{2^n}{3^n+1}$ is convergent

NOTE 2nd Method

Q23 by Root test

$\lim_{n\to\infty} (b_n)^{\frac{1}{n}} = \lim_{n\to\infty}\left[\left(\frac{2}{3}\right)^n\right]^{\frac{1}{n}} = \frac{2}{3} = l < 1$

$\Rightarrow \sum b_n$ is convergent

Q23' $\sum_{1}^{\infty} \frac{2^n}{3^n+1}$ see Ex 8.3 Q23

DT **Q.24** $\sum_{1}^{\infty} \frac{\ln n}{1+\ln n}$

SOL. Here $a_n = \frac{\ln n}{1+\ln n}$

Applying divergent test i.e

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln n}{1+\ln n}$ $\left(\frac{\infty}{\infty} \text{ form}\right)$ apply L'Hospital Rule

$= \lim_{n\to\infty} \frac{1/n}{1/n}$ $\left(\frac{0}{0} \text{ form}\right)$

$= \lim_{n\to\infty} \frac{\cancel{n}^{\,1}}{\cancel{n}^{\,1}}$ $\left(\frac{\infty}{\infty} \text{ form}\right)$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(1) = 1 \neq 0$

$\therefore \sum \frac{\ln n}{1+\ln n}$ is divergent.

LCT **Q.25** $\sum_{1}^{\infty} \frac{\ln(n+1)}{n+2}$

SOL. Here $a_n = \frac{\ln(n+1)}{n+2} = \frac{n^0 \ln(n+1)}{n+2}$

so take $b_n = \frac{1}{n}$

$\Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln(n+1)}{n+2}\cdot\frac{n}{1}$

$= \lim_{n\to\infty} \frac{\cancel{n}\ln(n+1)}{\cancel{n}\left(1+\frac{2}{n}\right)} = \lim_{n\to\infty} \frac{\ln(n+1)}{1+\frac{2}{n}} = \infty$

NOTE: We can also take $b_n = \frac{1}{n+1}$ in Q25 but $\sum_1^\infty b_n = \sum_1^\infty \frac{1}{n}$ is divergent

so by limit comparison test $\sum_1^\infty a_n = \sum_1^\infty \frac{\ln(n+1)}{n+2}$ diverge.

NOTE We cannot take $b_n = \frac{1}{n^3}$ ∵ by L.C.T

If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ & $\sum b_n$ dgs then $\sum a_n$ should diverge but here $\sum a_n$ is cgt so it is not satisfied

Q.26 $\sum_1^\infty \frac{\ln n}{n^3}$

Sol. Here $a_n = \frac{\ln n}{n^3} = \frac{n^0 \ln n}{n^3}$

so take $b_n = \frac{1}{n^3}$

See Note

then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln n}{n^3} \times \frac{n^3}{1} = \lim_{n\to\infty} \ln n = \infty \neq 0$

(WRONG) $\rightarrow$ But $\sum_1^\infty b_n = \sum_1^\infty \frac{1}{n^3}$ is Convergent ($\because p = 3 > 1$)

So by L. C. test $\sum_1^\infty a_n = \sum_1^\infty \frac{\ln n}{n^3}$ is Convergent

✓**ALTERNATE**

BCT **Q26** $a_n = \frac{\ln n}{n^3}$

We know $\ln n < n \quad \forall n \geq 1$

$\ln n < n$
$\ln 1 = 0 < 1$
$\ln 2 = .69 < 2$
$\ln 3 = 1.09 < 3$

$\Rightarrow \frac{\ln n}{n^3} < \frac{n}{n^3}$

$\Rightarrow \frac{\ln n}{n^3} < \frac{1}{n^2}$

$\because \sum \frac{1}{n^2}$ is cgt $\therefore \sum \frac{\ln n}{n^3}$ is also cgt by BCT

cgt [illegible] cgt ہے

BCT

Q.27 $\sum_1^\infty \frac{\sqrt{n}}{n^2 + \cos(2n-6)}$

Sol. Here $a_n = \frac{\sqrt{n}}{n^2 + \cos(2n-6)}$

Since $n^2 + \cos(2n-6) > n^2$

$\Rightarrow \frac{1}{n^2 + \cos(2n-6)} < \frac{1}{n^2}$

So $a_n = \frac{\sqrt{n}}{n^2 + \cos(2n-6)} < \frac{\sqrt{n}}{n^2} = \frac{1}{n^{\frac{3}{2}}} = b_n \quad \forall n \geq 1$

Now $\sum_1^\infty b_n = \sum_1^\infty \frac{1}{n^{3/2}}$ is convergent ($\because p = \frac{3}{2} > 1$)

So by Basic C.T. $\sum_1^\infty a_n = \sum_1^\infty \frac{\sqrt{n}}{n^2 + \cos(2n-6)}$ is convergent

(28) Integral Test $\sum_{1}^{\infty} \frac{1}{(n+1)\left[\ln(n+1)\right]^2}$

$$a_n = \frac{1}{(n+1)\left[\ln(n+1)\right]^2}$$

$$f(x) = \frac{1}{(x+1)\left[\ln(x+1)\right]^2} = \left[\ln(x+1)\right]^{-2}(x+1)^{-1}$$

$$f'(x) = -2\left[\ln(x+1)\right]^{-3} \cdot \frac{1}{(x+1)} \cdot (x+1)^{-1} + \left[\ln(x+1)\right]^{-2}\left[-(x+1)^{-2}\right]$$

$$= \frac{-2}{\left[\ln(x+1)\right]^3 (x+1)^2} - \frac{1}{\left[\ln(x+1)\right]^2 (x+1)^2} < 0 \quad \forall x \in [1,\infty[$$

So $f(x)$ is Monotonic Decreasing

Now $\int_1^{\infty} f(x)\,dx = \int \frac{1}{(x+1)\left[\ln(x+1)\right]^2}\,dx$

$$= \lim_{t\to\infty} \int_1^t \ln(x+1)^{-2} \cdot \frac{1}{(x+1)}\,dx$$

$$= \lim_{t\to\infty} \left| \frac{\left[\ln(x+1)\right]^{-1}}{-1} \right|_1^t$$

$$= \lim_{t\to\infty} \left| \frac{-1}{\ln(x+1)} \right|_1^t$$

$$= \lim_{t\to\infty} \left(\frac{-1}{\ln(t+1)} + \frac{1}{\ln(1+1)} \right)$$

$$= \frac{-1}{\ln(\infty+1)} + \frac{1}{\ln 2}$$

$$\int_1^{\infty} f(x)\,dx = 0 + \frac{1}{\ln 2} \neq \infty$$

Since $\int_1^{\infty} f(x)\,dx$ is convergent so

So $\sum_{1}^{\infty} \frac{1}{(n+1)\left[\ln(n+1)\right]^2}$ is Convergent.

Div. Test (29) $\sum_{1}^{\infty} n^2 \sin^2 \frac{1}{n}$

$$a_n = n^2 \sin^2 \frac{1}{n}$$

$$= \left(n \sin \frac{1}{n}\right)^2$$

$$a_n = \left(\frac{\sin \frac{1}{n}}{\frac{1}{n}}\right)^2$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{\sin \frac{1}{n}}{\frac{1}{n}}\right)^2$$

$$= \left(\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)}\right)^2$$

$$\lim_{n\to\infty} a_n = 1^2 = 1 \neq 0$$

So By Divergence Test

$\sum n^2 \sin^2 \frac{1}{n}$ is Divergent.

Integral

Q30 $\sum_{2}^{\infty} \frac{1}{n(\ln n)^p}$

$$a_n = \frac{1}{n(\ln n)^p}$$

$$f(x) = \frac{1}{x(\ln x)^p} = x^{-1}(\ln x)^{-p}$$

$$f'(x) = \frac{(-1)}{x^2}\cdot(\ln x)^{-p} + \frac{(-p)}{(\ln x)^{p+1}}\left(\frac{1}{x}\right)\cdot x^{-1}$$

$$f'(x) = \frac{-1}{x^2(\ln x)^p} - \frac{p}{(\ln x)^{p+1}(x^2)} < 0 \quad \forall\, x \in [2,\infty[$$

So $f(x)$ is Monotonically decreasing.

Now $\int_2^{\infty} f(x)dx = \int_2^{\infty} \frac{1}{x(\ln x)^p}\, dx$

$$= \lim_{t\to\infty}\int_2^{t}(\ln x)^{-p}\cdot\left(\frac{1}{x}\right)dx$$

$$= \lim_{t\to\infty}\left|\frac{(\ln x)^{-p+1}}{-p+1}\right|_2^{t}$$

$$= \lim_{t\to\infty}\left[\frac{(\ln t)^{1-p}}{1-p} - \frac{(\ln 2)^{1-p}}{1-p}\right]$$

$$= \frac{1}{(1-p)}\left[\frac{1}{(\ln \infty)^{p-1}} - \frac{1}{(\ln 2)^{p-1}}\right]$$

Now $\int_2^{\infty} f(x)dx = \begin{cases} \frac{1}{1-p}\left[0 - \frac{1}{(\ln 2)^{p-1}}\right] & \text{if } \underline{p>1} \\ \infty & \text{if } \underline{p\leq 1} \end{cases}$

So $\int_2^{\infty} f(x)dx$ converges if $p>1$ $\therefore \sum_2^{\infty}\frac{1}{n(\ln n)^p}$ cgs

& $\int_2^{\infty} f(x)dx$ diverges if $p\leq 1$ $\therefore \sum_2^{\infty}\frac{1}{n(\ln n)^p}$ dgs.

$\times$———————

LCT **Q.31** $\sum_{1}^{\infty} \dfrac{2n-1}{n(n+1)(n+2)}$

SOL. Here $a_n = \dfrac{2n+1}{n(n+1)(n+2)}$

take $b_n = \dfrac{1}{n^2}$

So $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{2n+1}{n(n+1)(n+2)} \times n^2 = \lim_{n\to\infty} \dfrac{\cancel{n^3}\left(2+\frac{1}{n}\right)}{\cancel{n^3}\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$

$= \lim_{n\to\infty} \dfrac{2+\frac{1}{n}}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} = 2 \neq 0$

$\Rightarrow$ both series behave alike.

But $\sum_{1}^{\infty} b_n = \sum_{1}^{\infty} \dfrac{1}{n^2}$ is convergent (Euler's Series or $p = 2 > 1$)

So by L.C.T, $\sum_{1}^{\infty} a_n$ is convergent.

BCT **Q.32** $\sum_{1}^{\infty} \dfrac{1}{n^{n-1}}$

SOL. Here $a_n = \dfrac{1}{n^{n-1}}$

$\Rightarrow a_1 = \dfrac{1}{1^0} = 1 = \dfrac{1}{2^0} = b_1$

$a_2 = \dfrac{1}{2^1} = \dfrac{1}{2} = b_2$

$a_3 = \dfrac{1}{3^2} < \dfrac{1}{2^2} = b_3$

$a_4 = \dfrac{1}{4^3} < \dfrac{1}{2^3} = b_4$

$a_5 = \dfrac{1}{5^4} < \dfrac{1}{2^4} = b_5$

$\vdots$

$a_n = \dfrac{1}{n^{n-1}} < \dfrac{1}{2^{n-1}} = b_n$ for $n \geq 3$

but $\sum_{1}^{\infty} b_n = \sum_{1}^{\infty} \dfrac{1}{2^{n-1}}$ is IGS $r = \frac{1}{2} < 1$

Hence it is convergent.

Root Test **Q32** $\sum_{1}^{\infty} \dfrac{1}{n^{n-1}}$

$\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \lim_{n\to\infty} \left(\dfrac{1}{n^{n-1}}\right)^{\frac{1}{n}}$

$= \lim_{n\to\infty} \left(n^{1-n}\right)^{\frac{1}{n}}$

$= \lim_{n\to\infty} (n)^{\frac{1-n}{n}}$

$= \lim_{n\to\infty} n^{\frac{1}{n}}\, n^{-1}$

$= \left(\lim_{n\to\infty} n^{\frac{1}{n}}\right) \lim_{n\to\infty} \dfrac{1}{n}$

$= 1 \cdot 0 = \boxed{0} < 1$

$\therefore$ Hence Cgt

Note first prove $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$ using L'Hospital as in Q15 8.3

Q33 $\sum \dfrac{e^{\tan^{-1} n}}{1+n^2}$

Integral Test (33) $\sum_{1}^{\infty} \frac{e^{\tan^{-1}n}}{1+n^2}$

$a_n = \frac{e^{\tan^{-1}n}}{1+n^2}$

$f(x) = \frac{e^{\tan^{-1}x}}{1+x^2}$

$f'(x) = \frac{(1+x^2)\, e^{\tan^{-1}x} \cdot \left(\frac{1}{1+x^2}\right) - e^{\tan^{-1}x} \cdot (2x)}{(1+x^2)^2}$

$f'(x) = e^{\tan^{-1}x} \frac{(1-2x)}{(1+x^2)^2} < 0 \quad \forall x \in [1, \infty]$

So $f(x)$ is Monotonically decreasing

Now $\int_1^{\infty} f(x)dx = \int_1^{\infty} \frac{e^{\tan^{-1}x}}{1+x^2} dx$

$= \lim_{t\to\infty} \int_1^t e^{\tan^{-1}x} \cdot \frac{dx}{1+x^2}$

$= \lim_{t\to\infty} \left| e^{\tan^{-1}x} \right|_1^t$

$= \lim_{t\to\infty} \left(e^{\tan^{-1}t} - e^{\tan^{-1}(1)} \right)$

$= \left(e^{\tan^{-1}\infty} - e^{\pi/4} \right)$

$\int_1^{\infty} f(x)dx = e^{\pi/2} - e^{\pi/4} \neq \infty$

So $\int_1^{\infty} f(x)dx$ converges

$\Rightarrow \sum_{1}^{\infty} \frac{e^{\tan^{-1}n}}{1+n^2}$ converges.

LCT (34) $a_n = \frac{1}{n\sqrt{n^2-1}}$,

$a_n = \frac{1}{n^2\sqrt{1-\frac{1}{n^2}}}$ $\quad b_n = \frac{1}{n^2}$ $\quad \left(n^{0-2} = \frac{1}{n^2}\right)$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1}{n^2\sqrt{1-\frac{1}{n^2}}} \times \frac{n^2}{1} = \lim_{n\to\infty} \frac{1}{\sqrt{1-\frac{1}{n^2}}} = \boxed{1}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = 1$ (definite). so both behaves alike.

$\sum b_n = \sum \frac{1}{n^2}$ is cgt. so $\sum a_n$ is cgt.

Integral Test (34) $\sum_{2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$

$a_n = \frac{1}{n\sqrt{n^2-1}}$

$f(x) = \frac{1}{x\sqrt{x^2-1}} = x^{-1}(x^2-1)^{-\frac{1}{2}}$

$f'(x) = \frac{-1}{x^2} \cdot (x^2-1)^{-\frac{1}{2}} + \left(-\frac{1}{2}\right)(x^2-1)^{-\frac{3}{2}} \cdot 2x \cdot x^{-1}$

$f'(x) = \frac{-1}{x^2\sqrt{x^2-1}} - \frac{1}{(x^2-1)^{3/2}} \cancel{\frac{2x}{2x}}$

$f'(x) < 0 \quad \forall x \in [2, \infty[$

So $f(x)$ is Monotonically Decreasing

Now $\int_2^{\infty} f(x)dx = \int_2^{\infty} \frac{1}{x\sqrt{x^2-1}} dx$

$= \lim_{t\to\infty} \int_2^t \frac{1}{x\sqrt{x^2-1}} dx$

$= \lim_{t\to\infty} \left| \sec^{-1} x \right|_2^t$

$= \lim_{t\to\infty} \left[\sec^{-1} t - \sec^{-1} 2 \right]$

$= \left[\sec^{-1} \infty - \sec^{-1} 2 \right]$

$= \cos^{-1}\left(\frac{1}{\infty}\right) - \cos^{-1}\frac{1}{2}$

$= \cos^{-1}(0) - \cos^{-1}\left(\frac{1}{2}\right)$

$\int_2^{\infty} f(x)dx = \frac{\pi}{2} - \frac{\pi}{3} \neq \infty$

So $\int_2^{\infty} f(x)dx$ cgs

$\Rightarrow \sum_{2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$ cgs

Note

$\sec^{-1} \infty = y$

$\infty = \sec y$

$\infty = \frac{1}{\cos y}$

$\cos y = \frac{1}{\infty}$

$y = \cos^{-1}\left(\frac{1}{\infty}\right)$

$\sec^{-1}\infty = \cos^{-1}\left(\frac{1}{\infty}\right)$

Integral test (35) $\sum_1^\infty \frac{n^2}{e^n}$ $\quad a_n = \frac{n^2}{e^n}$

$f(x) = \frac{x^2}{e^x} = x^2 e^{-x}$

$f'(x) = x^2(-e^{-x}) + 2x e^{-x} = x e^{-x}(2-x) < 0 \quad \forall\, x \in]2, \infty[$

So $f(x)$ is <u>Monotonic Decreasing</u>

Now $\int_1^\infty f(x)\,dx = \int_1^\infty \underset{I}{x^2}\,\underset{II}{e^{-x}}\,dx = -x^2 e^{-x} + 2\int x\, e^{-x}\,dx$

$= -x^2 e^{-x} + 2\left(-x e^{-x} + \int e^{-x}\,dx\right)$

$= -x^2 e^{-x} - 2x e^{-x} - 2e^{-x}$

$\Rightarrow \int_1^\infty f(x)\,dx = \lim_{t\to\infty} \int_1^t f(x)\,dx = \lim_{t\to\infty} \left| -x^2 e^{-x} - 2x e^{-x} - 2e^{-x} \right|_1^t$

$= \lim_{t\to\infty} \left(-\frac{t^2}{e^t} - \frac{2t}{e^t} - \frac{1}{e^t} + \frac{1}{e} + \frac{2}{e} + \frac{2}{e} \right)$

$= \lim_{t\to\infty} \left(-\frac{t^2+2t+1}{e^t} + \frac{5}{e} \right) = \lim_{t\to\infty} \left[\frac{5}{e} - \frac{(t+1)^2}{e^t} \right]$

$= \lim_{t\to\infty} \frac{5}{e} - \lim_{t\to\infty} \frac{(t+1)^2}{e^t} = \frac{5}{e} - \lim_{t\to\infty} \frac{(t+1)^2}{e^t}$

$\int_1^\infty f(x)\,dx = \frac{5}{e} - \lim_{t\to\infty} \frac{2(t+1)}{e^t} = \frac{5}{e} - \lim_{t\to\infty} \frac{2}{e^t} = \frac{5}{e} - 0 = \frac{5}{e} \neq \infty$

So by integral test $\sum_1^\infty \frac{n^2}{e^n}$ Cgs.

Q35 Root Test

$\sum_1^\infty \frac{n^2}{e^n}$

$\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{n^2}{e^n}\right)^{\frac{1}{n}}$

$= \lim_{n\to\infty} \frac{n^{2/n}}{e}$

$= \lim_{n\to\infty} \frac{(n^{\frac{1}{n}})^2}{e}$

$= \frac{(1)^2}{e} = \frac{1}{e} < 1$

Hence Series Cgs. $\because \lim_{n\to\infty} n^{\frac{1}{n}} = 1$ proved as Q15 8.3

Q.36 $\sum_1^\infty \frac{2\cdot 4\cdot 6 \cdots\cdots 2n}{1\cdot 3\cdot 5 \cdots\cdots 2n-1}$

SOL. $a_n = \frac{2\cdot 4\cdot 6 \cdots\cdots 2n}{1\cdot 3\cdot 5 \cdots\cdots (2n-1)}$

$\Rightarrow a_n > \frac{2}{2}\cdot\frac{4}{4}\cdot\frac{6}{6}\cdots\cdots\frac{2n}{2n}$

$\Rightarrow a_n > 1\cdot 1\cdot 1\cdot 1 \cdots\cdots 1$

$\Rightarrow a_n > 1 = b_n$ (say)

but $\sum_1^\infty b_n = \sum_1^\infty 1 = 1+1+1\cdots = \infty$

So $\sum b_n$ is divergent $\therefore \sum a_n$ is divergent by BCT

BCT **Q.37** $\sum_{1}^{\infty} \frac{1}{n-\sqrt{n}}$

SOL. Here $a_n = \frac{1}{n-\sqrt{n}}$, Let $b_n = \frac{1}{n}$

$$n-\sqrt{n} < n$$

$$\therefore \frac{1}{n-\sqrt{n}} > \frac{1}{n}$$

but $\sum_{1}^{\infty} b_n = \sum_{1}^{\infty} \frac{1}{n}$ is divergent

(چھوٹی dgt تو بڑی dgt)

So by BCT $\sum_{1}^{\infty} \frac{1}{n-\sqrt{n}}$ is divergent.

Imp LCT 2000 **Q.38** $\sum_{0}^{\infty} \frac{5^n+n}{6^n+n}$

SOL. $a_n = \frac{5^n+n}{6^n+n} = \frac{5^n\left(1+\frac{n}{5^n}\right)}{6^n\left(1+\frac{n}{6^n}\right)}$. take $b_n = \left(\frac{5}{6}\right)^n$

$$= \left(\frac{5}{6}\right)^n \left[\frac{1+\frac{n}{5^n}}{1+\frac{n}{6^n}}\right]$$

and $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\cancel{\left(\frac{5}{6}\right)^n}\left(\frac{1+\frac{n}{5^n}}{1+n/6^n}\right)}{\cancel{\left(5/6\right)^n}}$

$$= \lim_{n\to\infty} \left(\frac{1+\frac{n}{5^n}}{1+\frac{n}{6^n}}\right)$$

$\because \left(\frac{1+\frac{\infty}{\infty}}{1+\frac{\infty}{\infty}}\right)$ So first find $\frac{n}{5^n}$ & $\frac{n}{6^n}$

$\because$ By L'Hospital Rule

$\frac{n}{5^n} = \lim_{n\to\infty} \frac{1}{5^n \ln 5} = \frac{1}{\infty} = 0$

Similarly $\frac{n}{6^n} \to 0$ as $n \to \infty$

$$= \frac{1+0}{1+0}$$

$$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = 1 \neq 0$$

$\Rightarrow$ both series behaves alike. but

$\sum_{0}^{\infty} \left(\frac{5}{6}\right)^n = \left(\frac{5}{6}\right)^0 + \left(\frac{5}{6}\right)^1 + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^3 + \cdots$ is an infinite G. Series with $r<1$. Hence $\sum b_n$ is convergent $\Rightarrow \sum_{0}^{\infty} a_n$ is convergent.

LCT **Q.39** $\sum_{1}^{\infty} \frac{\ln(n+1)}{n^2}$

SOL Here $a_n = \frac{\ln(n+1)}{n^2}$

Let $b_n = \frac{1}{n^{3/2}}$

Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln(n+1)}{n^2} \times n^{\frac{3}{2}}$

$= \lim_{n\to\infty} \frac{\ln(n+1)}{\sqrt{n}} \quad \left(\frac{\infty}{\infty}\right)$

$= \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{2\sqrt{n}}} = \lim_{n\to\infty} \frac{2\sqrt{n}}{n+1} \quad \left(\frac{\infty}{\infty}\right)$

$= \lim_{n\to\infty} \frac{\frac{2}{2\sqrt{n}}}{1+0} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

But $\sum_{1}^{\infty} b_n = \sum_{1}^{\infty} \frac{1}{n^{3/2}}$ is Cgt ($\because P = \frac{3}{2} > 1$)

So by L.C.T $\sum_{1}^{\infty} \frac{\ln(n+1)}{n^2}$ is Cgt.

NOTE We cannot take $b_n = \frac{1}{n^2}$, because

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln(n+1)}{n^2} \cdot n^2 = \infty$

but $\sum b_n = \sum \frac{1}{n^2}$ is convergent. whereas according to L.C.T. when $\boxed{\lim_{n\to\infty} \frac{a_n}{b_n} = \infty}$ Then $\sum b_n$ should be div. (which is contradiction to the Th.)

Similarly we can not take $b_n = \frac{1}{n}$

$\because \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln(n+1)}{n^2} \cdot n$

$= \lim_{n\to\infty} \frac{\ln(n+1)}{n} \quad \frac{\infty}{\infty}$

$= \lim_{n\to\infty} \frac{1}{n+1}$ L.H. Rule

$\lim_{n\to\infty} \frac{a_n}{b_n} = 0$

Now $\sum b_n$ should converge but $\sum b_n = \sum \frac{1}{n}$ diverges. So again contradiction of Theorem.

So we take $b_n = \frac{1}{n^{3/2}}$

Integral Test **Q.40** $\sum_{10}^{\infty} \frac{1}{n(\ln n)\ln(\ln n)}$

SOL Here $a_n = \frac{1}{n(\ln n)(\ln \ln n)}$

So $f(x) = \frac{1}{x \ln x (\ln \ln x)}$

$\Rightarrow f'(x) = \frac{0 - (1)\cdot \ln x \ln(\ln x) - \cancel{x}\cdot\frac{1}{\cancel{x}}\ln(\ln x) - \cancel{x}\cancel{\ln x}\cdot\frac{1}{\cancel{\ln x}}\cdot\frac{1}{\cancel{x}}}{[x \ln x (\ln(\ln x))]^2}$

$f'(x) = \frac{-\ln x \ln(\ln x) - \ln(\ln x) - 1}{\{x \ln x \ln(\ln x)\}^2} < 0 \quad \forall x \in [10, \infty[$

So $f(x)$ is Monotonically decreasing on $[10, \infty[$

Now $\int_{10}^{\infty} f(x) = \lim_{t\to\infty} \int_{10}^{t} \frac{1}{x \ln x (\ln \ln x)} dx = \lim_{t\to\infty} \int_{10}^{\infty} \frac{1}{\ln(\ln x)} \cdot \left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right) dx \quad \because \int \frac{dz}{z} = \ln z$

$= \lim_{t\to\infty} \left| \ln(\ln(\ln x)) \right|_{10}^{t} = \lim_{t\to\infty} \left[\ln(\ln(\ln t)) - \ln(\ln(\ln 10))\right] = \infty$ so dgt

Since $\int_{10}^{\infty} f(x)\,dx$ is divergent

So $\sum_{10}^{\infty} a_n = \sum_{10}^{\infty} \frac{1}{n \ln n (\ln(\ln n))}$ is divergent.