Exercise 7.1

Q1 Let $u=(u_1,u_2)$ and $v=(v_1,v_2)$ belong to R^2.
verify that $\langle u,v\rangle = u_1v_1-2u_1v_2-2u_2v_1+5u_2v_2$ is an I.P on R^2.

(i) $\langle u,u\rangle = u_1u_1-2u_1u_2-2u_2u_1+5u_2u_2$
$= u_1^2-4u_1u_2+5u_2^2$
$= u_1^2-4u_1u_2+(2u_2)^2-(2u_2)^2+5u_2^2$ completing square.
$= (u_1-2u_2)^2+u_2^2 \quad \geq 0$

If $\langle u,u\rangle = 0$
$\Leftrightarrow (u_1-2u_2)^2+u_2^2 = 0$
$\Leftrightarrow (u_1-2u_2)^2=0$ & $u_2^2=0$
$\Leftrightarrow u_1-2u_2=0$ & $\boxed{u_2=0}$
$\Leftrightarrow u_1-2(0)=0 \quad \because u_2=0$
$\Leftrightarrow \boxed{u_1=0}$
$\Leftrightarrow u=0 \quad \because u=(u_1,u_2)$

(ii) $\langle u,v\rangle = u_1v_1-2u_1v_2-2u_2v_1+5u_2v_2$
$= v_1u_1-2v_2u_1-2v_1u_2+5v_2u_2$
$= \langle v,u\rangle$

$au+bv=(au_1+bv_1, au_2+bv_2)$
$w=(w_1,w_2)$

(iii) $\langle au+bv, w\rangle = (au_1+bv_1)w_1-2(au_1+bv_1)w_2-2(au_2+bv_2)w_1+5(au_2+bv_2)w_2$
$= au_1w_1+bv_1w_1-2au_1w_2-2bv_1w_2-2au_2w_1-2bv_2w_1+5au_2w_2+5bv_2w_2$
$= a(u_1w_1-2u_1w_2-2u_2w_1+5u_2w_2)+b(v_1w_1-2v_1w_2-2v_2w_1+5v_2w_2)$
$= a\langle u,w\rangle + b\langle v,w\rangle$

All the three conditions for I.P are satisfied
Hence $\langle u,v\rangle$ as defined above is an I.P on R^2.

x ———— x

Q 1(ii) For what value of K $\langle u, v\rangle = u_1v_1 - 3u_1v_2 - 3u_2v_1 + Ku_2v_2$ is an I.P on R^2.

Sol

$$\langle u,u\rangle = u_1u_1 - 3u_1u_2 - 3u_2u_1 + Ku_2u_2 \qquad u=(u_1,u_2) \quad v=(v_1,v_2)$$
$$= u_1^2 - 6u_1u_2 + Ku_2^2$$

For an I.P condition (i) i.e $\langle u,u\rangle$ must be +ive

for this $\langle u,u\rangle$ must be in perfect square form

$$\therefore \langle u,u\rangle = u_1^2 - 6u_1u_2 + 9u_2^2 \quad \text{is perfect square form for } K\geq 9$$
$$= (u_1-3u_2)^2 \qquad \text{for } K=9$$
$$= (u_1-3u_2)^2 + u_2^2 \qquad \text{for } K=10$$

Cond (ii) & (iii) are also satisfied for $K\geq 9$.

Q2 Find the norm of $(2,3)\in R^2$ w.r.t Euclidean inner product on R^2

(i) $\langle u,v\rangle = u_1v_1 + u_2v_2 \qquad u=(u_1,u_2) \quad v=(v_1,v_2)$

$$\|u\| = \sqrt{\langle u,u\rangle} = \sqrt{u_1^2+u_2^2}$$
$$\|(2,3)\| = \sqrt{2^2+3^2} = \sqrt{13} \text{ Ans}$$

(ii) I.P defined as $\langle u,v\rangle = u_1v_1 - 2u_1v_2 - 2u_2v_1 + 5u_2v_2$

$$\|u\| = \sqrt{\langle u,u\rangle} = \sqrt{u_1u_1 - 2u_1u_2 - 2u_2u_1 + 5u_2u_2}$$
$$= \sqrt{u_1^2 - 4u_1u_2 + 5u_2^2}$$
$$\|(2,3)\| = \sqrt{2^2 - 4\cdot 2\cdot 3 + 5\cdot 3^2}$$
$$= \sqrt{4-24+45}$$
$$= \sqrt{25} = \boxed{5} \text{ Ans}$$

Q3 Find the norm of $\left(\frac{1}{2}, -\frac{1}{4}, \frac{1}{3}, \frac{1}{6}\right)\in R^4$ w.r.t Euclidean I.P on R^4

$\langle u,v\rangle = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4 \qquad$ Euclidean I.P on R^4

$$\|u\| = \sqrt{\langle u,u\rangle}$$
$$= \sqrt{u_1^2+u_2^2+u_3^2+u_4^2}$$
$$\left\|\left(\tfrac{1}{2},-\tfrac{1}{4},\tfrac{1}{3},\tfrac{1}{6}\right)\right\| = \sqrt{\left(\tfrac{1}{2}\right)^2 + \left(-\tfrac{1}{4}\right)^2 + \left(\tfrac{1}{3}\right)^2 + \left(\tfrac{1}{6}\right)^2}$$
$$= \sqrt{\frac{1}{4}+\frac{1}{16}+\frac{1}{9}+\frac{1}{36}}$$
$$= \sqrt{\frac{36+9+16+4}{144}} = \sqrt{\frac{65}{144}} = \boxed{\frac{\sqrt{65}}{12}} \text{ Ans}$$

Q4 Let V denote the vectorspace of 2×2 matrices over R. If $A, B \in V$ and $Tr(A)$ (called the Trace of A) denotes the sum of diagonal elements of A, show that $\langle A, B\rangle = Tr(B^t A)$ is an I.P on V. Also find norm of $A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$.

Sol Let $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$

$$B^t A = \begin{bmatrix} b_1 & b_3 \\ b_2 & b_4 \end{bmatrix}\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$$

$$B^t A = \begin{bmatrix} b_1a_1 + b_3a_3 & b_1a_2 + b_3a_4 \\ b_2a_1 + b_4a_3 & b_2a_2 + b_4a_4 \end{bmatrix}$$

$$\langle A, B\rangle = Tr(B^t A)$$
$$= b_1a_1 + b_2a_2 + b_3a_3 + b_4a_4 \quad \text{— (1)} \quad \text{(sum of diagonal element)}$$

(i) $\langle A, A\rangle = a_1a_1 + a_2a_2 + a_3a_3 + a_4a_4$
$= a_1^2 + a_2^2 + a_3^2 + a_4^2 \geq 0$

if $\langle A, A\rangle = 0 \iff a_1^2 + a_2^2 + a_3^2 + a_4^2 = 0$

$\iff a_1^2 = 0, a_2^2 = 0, a_3^2 = 0, a_4^2 = 0$

$\iff a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 0$

$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

(ii) $\langle A, B\rangle = b_1a_1 + b_2a_2 + b_3a_3 + b_4a_4$
$= a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4$
$= \langle B, A\rangle$

L, M are const. $C = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}$

(iii) $\langle LA + MB, C\rangle = Tr(C^t(LA + MB))$
$= c_1(La_1 + Mb_1) + c_2(La_2 + Mb_2) + c_3(La_3 + Mb_3) + c_4(La_4 + Mb_4)$ using (1)
$= L(c_1a_1 + c_2a_2 + c_3a_3 + c_4a_4) + M(c_1b_1 + c_2b_2 + c_3b_3 + c_4b_4)$
$= L\langle A, C\rangle + M\langle B, C\rangle$ Hence $\langle A, B\rangle$ is I.P on V.

$\|A\| = \sqrt{\langle A, A\rangle} = \sqrt{a_1^2 + a_2^2 + a_3^2 + a_4^2}$

$\left\| \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} \right\| = \sqrt{1^2 + 2^2 + 3^2 + (-4)^2} = \sqrt{1+4+9+16} = \boxed{\sqrt{30}}$

Q5 Let V be the vectorspace $F(x)$ of polynomials over R. Show that

$\langle f,g\rangle = \int_0^1 f(t)g(t)\,dt$ defines an inner product on V.

Sol
(i) $\langle f,f\rangle = \int_0^1 f(t)f(t)\,dt$

$= \int_0^1 f^2(t)\,dt \geq 0$ as definite integral represents area of plane region.

If $\langle f,f\rangle = 0$

$\Leftrightarrow \int_0^1 f^2(t)\,dt = 0$

$\Leftrightarrow f^2(t) = 0$

$\Leftrightarrow f(t) = 0$

(ii) $\langle f,g\rangle = \int_0^1 f(t)g(t)\,dt$

$= \int_0^1 g(t)f(t)\,dt$

$= \langle g,f\rangle$

(iii) $\langle af+bg,h\rangle = \int_0^1 (af(t)+bg(t))\,h(t)\,dt$

$= a\int_0^1 f(t)h(t)\,dt + b\int_0^1 g(t)h(t)\,dt$

$= a\langle f,h\rangle + b\langle g,h\rangle$

Hence $\langle f,g\rangle$ is I.P in V.

⑥ Let $u_1=\begin{bmatrix}1\\2\\1\end{bmatrix}$ $u_2=\begin{bmatrix}2\\1\\2\end{bmatrix}$ $u_3=\begin{bmatrix}2\\1\\-4\end{bmatrix}$

Using Example 2, find i.e $\langle u_1,u_2\rangle=\det(u_1^t u_2)$

(i) the inner product of each pair of above column vectors

(ii) the norm of each vector

(iii) a vector orthogonal to u_1 & u_2

(iv) a vector orthogonal to u_1 & u_3

Sol (i) $\langle u_1,u_2\rangle=\det(u_1^t u_2)=\det\left(\begin{bmatrix}1&2&1\end{bmatrix}\begin{bmatrix}2\\1\\2\end{bmatrix}\right)=\det\left[1(2)+2(1)+1(2)\right]=\det[6]=6$

$\langle u_1,u_3\rangle=\det(u_1^t u_3)=\det\left(\begin{bmatrix}1&2&1\end{bmatrix}\begin{bmatrix}2\\1\\-4\end{bmatrix}\right)=\det\left[1(2)+2(1)+1(-4)\right]=0$

$\langle u_2,u_3\rangle=\det(u_2^t u_3)=\det\left[\begin{bmatrix}2&1&2\end{bmatrix}\begin{bmatrix}2\\1\\-4\end{bmatrix}\right]=\det\left[2(2)+1(1)+2(-4)\right]=-3$

(ii) $\|u_1\|=\sqrt{\langle u_1,u_1\rangle}=\sqrt{\det(u_1^t u_1)}=\sqrt{\det\left[\begin{bmatrix}1&2&1\end{bmatrix}\begin{bmatrix}1\\2\\1\end{bmatrix}\right]}=\sqrt{\det\left[1^2+2^2+1^2\right]}=\sqrt{\det[6]}=$

$\|u_2\|=\sqrt{\langle u_2,u_2\rangle}=\sqrt{\det(u_2^t u_2)}=\sqrt{\det\left[\begin{bmatrix}2&1&2\end{bmatrix}\begin{bmatrix}2\\1\\2\end{bmatrix}\right]}=\sqrt{\det\left[2^2+1^2+2^2\right]}=\sqrt{\det[9]}=$

$\|u_3\|=\sqrt{\langle u_3,u_3\rangle}=\sqrt{\det(u_3^t u_3)}=\sqrt{\det\left[\begin{bmatrix}2&1&-4\end{bmatrix}\begin{bmatrix}2\\1\\-4\end{bmatrix}\right]}=\sqrt{\det\left[2^2+1^2+4^2\right]}=\sqrt{\det[21]}=$

(iii) Suppose $u=\begin{bmatrix}x\\y\\z\end{bmatrix}$ be a vector orthogonal to u_1 & u_2

$\therefore \langle u,u_1\rangle=0 \Rightarrow \det(u^t u_1)=0 \Rightarrow \det\begin{bmatrix}x&y&z\end{bmatrix}\begin{bmatrix}1\\2\\1\end{bmatrix}=0$

$\Rightarrow x+2y+z=0$ —(I)

& $\langle u,u_2\rangle=0 \Rightarrow \det(u^t u_2)=0 \Rightarrow \det\begin{bmatrix}x&y&z\end{bmatrix}\begin{bmatrix}2\\1\\2\end{bmatrix}=0$

$\Rightarrow 2x+y+2z=0$ —(II)

from (I)&(II) $\begin{matrix}x+2y+z=0\\2x+y+2z=0\end{matrix}$ $\quad \dfrac{x}{4-1}=\dfrac{-y}{2-2}=\dfrac{z}{1-4}=k$

$x=3k$
$y=0$
$z=-3k$

$u=\begin{bmatrix}x\\y\\z\end{bmatrix}=3k\begin{bmatrix}1\\0\\-1\end{bmatrix}\Rightarrow \begin{bmatrix}1\\0\\-1\end{bmatrix}$ is orthogonal to u_1 & u_2

2nd Method

$\begin{vmatrix}i&j&k\\1&2&1\\2&1&2\end{vmatrix}$

$=3i-0j+-3k$

$=3(i-0j-k)$

$\therefore \begin{bmatrix}1\\0\\-1\end{bmatrix}$ is required vec

(iv) Let $u=\begin{bmatrix}x\\y\\z\end{bmatrix}$ be a vector orthogonal to u_1 & u_3

$\therefore \langle u,u_1\rangle = x+2y+z=0$ —(I)

$\langle u,u_3\rangle = 2x+y-4z=0$ —(II)

$\dfrac{x}{-8-1}=\dfrac{-y}{-4-2}=\dfrac{z}{1-4}=k$

$x=-9k$
$y=+6k$
$z=-3k$

$u=\begin{bmatrix}x\\y\\z\end{bmatrix}=-3k\begin{bmatrix}3\\-2\\1\end{bmatrix}\Rightarrow \begin{bmatrix}3\\-2\\1\end{bmatrix}$ is orthogonal to u_1 & u_3.

2nd Method

$\begin{vmatrix}i&j&k\\1&2&1\\2&1&-4\end{vmatrix}$

$=-9i-j(-6)+k(1-$

$=-3(3i-2j+k)$

$\therefore \begin{bmatrix}3\\-2\\1\end{bmatrix}$ is orthogonal u_1 & u_3

(i) Show that (1,1) (0,1) is a basis of R^2. Using the Gram-Schmidt process, find an orthonormal basis of R^2.

Sol: Let $(x,y) \in R^2$ $\qquad S = \{(1,1), (0,1)\}$

Consider $(x,y) = a(1,1) + b(0,1)$ $\qquad$ linear Combination of a, b & given vectors

$(x,y) = (a, a+b)$

$\Rightarrow \boxed{a = x}$

& $a+b = y \Rightarrow x+b = y \Rightarrow \boxed{b = y-x}$

$\therefore$ the linear combination exists, as (1,1) & (0,1) can be written as a linear combination of a & b. So the set 'S' is spanning set for R^2.

Now for Linear Independence.

$a(1,1) + b(0,1) = 0$

$\boxed{a = 0}$

$a+b = 0 \Rightarrow 0+b = 0 \Rightarrow \boxed{b = 0}$

Hence given vectors are Lin. independent.

2nd Method

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

It is in echelon form

Rank of A = 2 = No. of vectors in S

$\therefore$ S is Lin. independent.

S is spanning set & Linearly Independent So S is basis for R^2

Orthonormal Basis of R^2 by Gram Schmidt Process.

Let $v_1 = (1,1) \qquad v_2 = (0,1)$

Now $u_1 = \dfrac{v_1}{\|v_1\|} = \dfrac{(1,1)}{\sqrt{1^2+1^2}} = \left(\dfrac{1}{\sqrt2}, \dfrac{1}{\sqrt2}\right)$

$u_2 = \dfrac{\omega_2}{\|\omega_2\|}$ where $\omega_2 = v_2 - \langle v_2, u_1\rangle u_1$

$= (0,1) - \left\langle (0,1), \left(\frac{1}{\sqrt2}, \frac{1}{\sqrt2}\right)\right\rangle \left(\frac{1}{\sqrt2}, \frac{1}{\sqrt2}\right)$

$= (0,1) - \left(0\cdot\frac{1}{\sqrt2} + 1\cdot\frac{1}{\sqrt2}\right)\left(\frac{1}{\sqrt2}, \frac{1}{\sqrt2}\right)$

$= (0,1) - \frac{1}{\sqrt2}\left(\frac{1}{\sqrt2}, \frac{1}{\sqrt2}\right)$

$= (0,1) - \left(\frac{1}{2}, \frac{1}{2}\right)$

$= \left(0 - \frac{1}{2}, 1 - \frac{1}{2}\right)$

$\omega_2 = \left(-\frac{1}{2}, \frac{1}{2}\right)$

$\|\omega_2\| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{4}+\frac{1}{4}} = \frac{1}{\sqrt2}$

$\therefore u_2 = \dfrac{\left(-\frac{1}{2}, \frac{1}{2}\right)}{\frac{1}{\sqrt2}}$

$u_2 = \sqrt2\left(-\frac{1}{2}, \frac{1}{2}\right) = \left(-\frac{1}{\sqrt2}, \frac{1}{\sqrt2}\right)$

Thus the orthonormal basis is $\{u_1, u_2\} = \left\{\left(\frac{1}{\sqrt2}, \frac{1}{\sqrt2}\right), \left(-\frac{1}{\sqrt2}, \frac{1}{\sqrt2}\right)\right\}$

$$\|\omega_3\| = \sqrt{1^2+0^2+0^2} = 1$$

$$u_3 = \frac{\omega_3}{\|\omega_3\|} = \frac{(1,0,0)}{1} = (1,0,0)$$

Therefore orthonormal basis is $\{u_1, u_2, u_3\} = \{(0,0,1), (0,1,0), (1,0,0)\}$

(ii) Taking $v_1 = (1,0,1)$, $v_2 = (0,1,1)$, $v_3 = (0,0,1)$

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{(1,0,1)}{\sqrt{1^2+0^2+1^2}} = \boxed{\left(\frac{1}{\sqrt2}, 0, \frac{1}{\sqrt2}\right)}$$

$u_2 = \frac{\omega_2}{\|\omega_2\|}$ where $\omega_2 = v_2 - \langle v_2, u_1\rangle u_1$

$$= (0,1,1) - \left\langle (0,1,1), \left(\frac{1}{\sqrt2}, 0, \frac{1}{\sqrt2}\right)\right\rangle \left(\frac{1}{\sqrt2}, 0, \frac{1}{\sqrt2}\right)$$

$$= (0,1,1) - \frac{1}{\sqrt2}\left(\frac{1}{\sqrt2}, 0, \frac{1}{\sqrt2}\right)$$

$$= (0,1,1) - \left(\frac{1}{2}, 0, \frac{1}{2}\right)$$

$$\omega_2 = \left(-\frac{1}{2}, 1, \frac{1}{2}\right)$$

$$\|\omega_2\| = \sqrt{\frac{1}{4}+1+\frac{1}{4}} = \sqrt{\frac{6}{4}} = \sqrt{\frac{3}{2}}$$

$$\therefore u_2 = \frac{\omega_2}{\|\omega_2\|} = \frac{\left(-\frac{1}{2}, 1, \frac{1}{2}\right)}{\sqrt{\frac{3}{2}}} = \frac{\sqrt2}{\sqrt3}\left(-\frac{1}{2}, 1, \frac{1}{2}\right) = \left(-\frac{1}{\sqrt6}, \frac{\sqrt2}{\sqrt3}, \frac{1}{\sqrt6}\right)$$

$$= \boxed{\left(-\frac{1}{\sqrt6}, \frac{2}{\sqrt6}, \frac{1}{\sqrt6}\right)}$$

$u_3 = \frac{\omega_3}{\|\omega_3\|}$ where $\omega_3 = v_3 - \langle v_3, u_1\rangle u_1 - \langle v_3, u_2\rangle u_2$

$$= (0,0,1) - \left\langle (0,0,1), \left(\frac{1}{\sqrt2}, 0, \frac{1}{\sqrt2}\right)\right\rangle \left(\frac{1}{\sqrt2}, 0, \frac{1}{\sqrt2}\right)$$

$$- \left\langle (0,0,1), \left(-\frac{1}{\sqrt6}, \frac{2}{\sqrt6}, \frac{1}{\sqrt6}\right)\right\rangle \left(-\frac{1}{\sqrt6}, \frac{2}{\sqrt6}, \frac{1}{\sqrt6}\right)$$

$$= (0,0,1) - \frac{1}{\sqrt2}\left(\frac{1}{\sqrt2}, 0, \frac{1}{\sqrt2}\right) - \frac{1}{\sqrt6}\left(-\frac{1}{\sqrt6}, \frac{2}{\sqrt6}, \frac{1}{\sqrt6}\right)$$

$$= (0,0,1) - \left(\frac{1}{2}, 0, \frac{1}{2}\right) - \left(-\frac{1}{6}, \frac{2}{6}, \frac{1}{6}\right)$$

$$= \left(0-\frac{1}{2}, 0, \frac{1}{2}\right) - \left(-\frac{1}{6}, \frac{2}{6}, \frac{1}{6}\right)$$

$$\omega_3 = \left(-\frac{1}{2}+\frac{1}{6}, -\frac{2}{6}, \frac{1}{2}-\frac{1}{6}\right) = \left(-\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}\right)$$

$$\|\omega_3\| = \sqrt{\frac{1}{9}+\frac{1}{9}+\frac{1}{9}} = \sqrt{\frac{3}{9}} = \sqrt{\frac{1}{3}}$$

$$u_3 = \frac{\omega_3}{\|\omega_3\|} = \frac{\left(-\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}\right)}{\frac{1}{\sqrt3}} = \sqrt3\left(-\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}\right) = \boxed{\left(-\frac{1}{\sqrt3}, -\frac{1}{\sqrt3}, \frac{1}{\sqrt3}\right)}$$

$\therefore \{u_1, u_2, u_3\}$ is orthonormal basis

Show that (1,0,1), (0,1,1), (0,0,1) is a basis of R^3. Using the Gram Schmidt process, find an orthonormal basis of R^3 by taking $U_1=(0,0,1)$

Let $S=\{(1,0,1),(0,1,1),(0,0,1)\} \subset R^3$

Let $(x,y,z) \in R^3$

Consider $(x,y,z) = a(1,0,1)+b(0,1,1)+c(0,0,1)$

$= (a, b, a+b+c)$.

$\boxed{a=x}$

$\boxed{b=y}$

$a+b+c=z \Rightarrow \boxed{c=z-x-y}$ So Linear combination of vectors of 'S' and a, b, c exists.
So S is spanning set for R^3.

Now $a(1,0,1)+b(0,1,1)+c(0,0,1)=0$

$a+0+0=0 \Rightarrow \boxed{a=0}$

$0+b+0=0 \Rightarrow \boxed{b=0}$ Hence 'S' is Lin Independent.

$a+b+c=0 \Rightarrow \boxed{c=0}$

So S is a basis of R^3 as S is spanning & Lin Independent.

To check Linear Independence.
2nd Method

$A=\begin{bmatrix}1&0&1\\0&1&1\\0&0&1\end{bmatrix}$

Rank of A = 3 = No. of vectors in S.

$\therefore$ S is Independent.

Gram Schmidt Orthonormalization Process.

Taking $v_1=(0,0,1)$ $v_2=(0,1,1)$, $v_3=(1,0,1)$

$U_1 = \frac{v_1}{\|v_1\|} = \frac{(0,0,1)}{\sqrt{0^2+0^2+1^2}} = (0,0,1)$

$U_2 = \frac{\omega_2}{\|\omega_2\|}$ where $\omega_2 = v_2 - \langle v_2, U_1\rangle U_1$

$= (0,1,1) - \langle (0,1,1),(0,0,1)\rangle (0,0,1)$

$= (0,1,1) - (0+0+1)(0,0,1)$

$= (0,1,1) - (0,0,1)$

$\omega_2 = (0,1,0)$

$\|\omega_2\| = \sqrt{0^2+1^2+0^2} = 1$

$\therefore U_2 = \frac{(0,1,0)}{1} = (0,1,0)$

$U_3 = \frac{\omega_3}{\|\omega_3\|}$ where $\omega_3 = v_3 - \langle v_3, U_1\rangle U_1 - \langle v_3, U_2\rangle U_2$

$= (1,0,1) - \langle (1,0,1),(0,0,1)\rangle (0,0,1) - \langle (1,0,1),(0,1,0)\rangle (0,1,0)$

$= (1,0,1) - 1(0,0,1) - 0(0,1,0)$

$= (1,0,1) - (0,0,1)$

$\omega_3 = (1,0,0)$

(9) Show that $\{(1,-1,0),(2,-1,-2),(1,-1,-2)\}$ is a basis of R^3.
Find an orthonormal basis of R^3 using Graham Schmidt Process.

Sol $S=\{(1,-1,0),\ (2,-1,-2),\ (1,-1,-2)\}$

Let $(x,y,z)\in R^3$

Consider $(x,y,z)=a(1,-1,0)+b(2,-1,-2)+c(1,-1,-2)$ — (I)

from

$x=a+2b+c$ — (II)

$y=-a-b-c$ — (III)

$z=-2b-2c$ — (IV)

from (II) & (III) $\boxed{x+y=b}$ Put in (IV)

$z=-2x-2y-2c \Rightarrow 2c=-2x-2y-z$

$\boxed{c=-x-y-\frac{z}{2}}$ Put in

from (III) $y=-a-(x+y)-(-x-y-\frac{z}{2})$

$a=-y-\cancel{x}-\cancel{y}+\cancel{x}+\cancel{y}+\frac{z}{2} \Rightarrow \boxed{a=\frac{z}{2}-y}$

Linear combination of a,b,c & vectors of S exist. So S is spaning set.

Now

$a(1,-1,0)+b(2,-1,-2)+c(1,-1,-2)=0$

$a+2b+c=0$ — (V)

$-a-b-c=0$ — (VI)

$-2b+(-2c)=0 \Rightarrow b=\frac{\cancel{2}c}{-\cancel{2}}=\boxed{-c}$ — (VII)

$-a-b-c=0 \Rightarrow -a-(-c)-c=0$

$\Rightarrow -a+c-c=0$

$\Rightarrow \boxed{a=0}$ Put in (V)

$a+2b+c=0 \Rightarrow 0+2(-c)+c=0$

$\Rightarrow \boxed{c=0}$

by Putting c = 0 in (VII) $\boxed{b=0}$

Hence S is linearly Independent.

$\because$ S is spaning & linearly Independent
so S is basis of R^3.

2nd Method

$$A=\begin{bmatrix}1&-1&0\\2&-1&-2\\1&-1&-2\end{bmatrix}$$

$$\sim\begin{bmatrix}1&-1&0\\0&1&-2\\0&0&-2\end{bmatrix} \text{ Echelon form}$$

Rank of A = 3 = No. of vectors in S.

So S is linearly Independent

Gram Schmidt Process.

$S = \{(1,-1,0),\ (2,-1,-2),\ (1,-1,-2)\}$

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{(1,-1,0)}{\sqrt{1^2+1^2+0}} = \left(\frac{1}{\sqrt2}, \frac{-1}{\sqrt2}, 0\right)$$

Let $u_2 = \dfrac{\omega_2}{\|\omega_2\|}$ where $\omega_2 = v_2 - \langle v_2, u_1\rangle u_1$

$$= (2,-1,-2) - \left\langle (2,-1,-2), \left(\tfrac{1}{\sqrt2}, -\tfrac{1}{\sqrt2}, 0\right)\right\rangle \left(\tfrac{1}{\sqrt2}, -\tfrac{1}{\sqrt2}, 0\right)$$

$$= (2,-1,-2) - \left(\tfrac{2}{\sqrt2} + \tfrac{1}{\sqrt2} + 0\right)\left(\tfrac{1}{\sqrt2}, -\tfrac{1}{\sqrt2}, 0\right)$$

$$= (2,-1,-2) - \tfrac{3}{\sqrt2}\left(\tfrac{1}{\sqrt2}, -\tfrac{1}{\sqrt2}, 0\right)$$

$$= (2,-1,-2) - \left(\tfrac{3}{2}, -\tfrac{3}{2}, 0\right)$$

$$\omega_2 = \left(\tfrac{1}{2}, +\tfrac{1}{2}, -2\right)$$

$$\|\omega_2\| = \sqrt{\left(\tfrac12\right)^2 + \left(\tfrac12\right)^2 + (-2)^2} = \sqrt{\tfrac14 + \tfrac14 + 4} = \sqrt{\tfrac{18}{4}}$$

$$= \sqrt{\tfrac{9}{2}} = \tfrac{3}{\sqrt2}$$

$$\therefore u_2 = \frac{\omega_2}{\|\omega_2\|} = \frac{\left(\frac12, \frac12, -2\right)}{\frac{3}{\sqrt2}} = \frac{\sqrt2}{3}\left(\tfrac12, \tfrac12, -2\right) = \left(\tfrac{1}{3\sqrt2}, \tfrac{1}{3\sqrt2}, -\tfrac{2\sqrt2}{3}\right)$$

$u_3 = \dfrac{\omega_3}{\|\omega_3\|}$ where $\omega_3 = v_3 - \langle v_3, u_1\rangle u_1 - \langle v_3, u_2\rangle u_2$

$$= (1,-1,-2) - \left\langle (1,-1,-2), \left(\tfrac{1}{\sqrt2}, -\tfrac{1}{\sqrt2}, 0\right)\right\rangle \left(\tfrac{1}{\sqrt2}, -\tfrac{1}{\sqrt2}, 0\right) -$$
$$- \left\langle (1,-1,-2), \left(\tfrac{1}{3\sqrt2}, \tfrac{1}{3\sqrt2}, -\tfrac{2\sqrt2}{3}\right)\right\rangle \left(\tfrac{1}{3\sqrt2}, \tfrac{1}{3\sqrt2}, -\tfrac{2\sqrt2}{3}\right)$$

$$= (1,-1,-2) - \left(\tfrac{1}{\sqrt2} + \tfrac{1}{\sqrt2} + 0\right)\left(\tfrac{1}{\sqrt2}, -\tfrac{1}{\sqrt2}, 0\right) - \left(\cancel{\tfrac{1}{3\sqrt2}} - \cancel{\tfrac{1}{3\sqrt2}} + \tfrac{4\sqrt2}{3}\right)\left(\tfrac{1}{3\sqrt2}, \tfrac{1}{3\sqrt2}, -\tfrac{2\sqrt2}{3}\right)$$

$$= (1,-1,-2) - \left(\tfrac{2}{\sqrt2}\right)\left(\tfrac{1}{\sqrt2}, -\tfrac{1}{\sqrt2}, 0\right) - \left(\tfrac{4\sqrt2}{3}\right)\left(\tfrac{1}{3\sqrt2}, \tfrac{1}{3\sqrt2}, -\tfrac{2\sqrt2}{3}\right)$$

$$= (1,-1,-2) - \left(\tfrac{2}{2}, -\tfrac{2}{2}, 0\right) - \left(\tfrac49, \tfrac49, -\tfrac{16}{9}\right)$$

$$= (1-1, -1+1, -2-0) - \left(\tfrac49, \tfrac49, -\tfrac{16}{9}\right)$$

$$\omega_3 = (0, 0, -2) - \left(\tfrac49, \tfrac49, -\tfrac{16}{9}\right) = \left(-\tfrac49, -\tfrac49, -\tfrac29\right)$$

$$\|\omega_3\| = \sqrt{\tfrac{16}{81} + \tfrac{16}{81} + \tfrac{4}{81}} = \sqrt{\tfrac{36}{81}} = \tfrac69 = \tfrac23$$

$$\therefore u_3 = \frac{\omega_3}{\|\omega_3\|} = \frac{\left(-\frac49, -\frac49, -\frac29\right)}{2/3} = \frac32\left(-\tfrac49, -\tfrac49, -\tfrac29\right) = \left(-\tfrac23, -\tfrac23, -\tfrac13\right)$$

$\therefore$ Orthonormal basis is $\{u_1, u_2, u_3\} = \left\{\left(\tfrac{1}{\sqrt2}, \tfrac{-1}{\sqrt2}, 0\right), \left(\tfrac{1}{3\sqrt2}, \tfrac{1}{3\sqrt2}, -\tfrac{2\sqrt2}{3}\right), \left(\tfrac{-2}{3}, \tfrac{-2}{3}, \tfrac{-1}{3}\right)\right\}$

(10) Let $\{e_1, e_2, \cdots e_n\}$ be an orthonormal basis of an inner product space V over R. Show that for any $v \in V$

$$v = \langle v, e_1\rangle e_1 + \langle v, e_2\rangle e_2 + \cdots + \langle v, e_n\rangle e_n$$

Sol Since $\{e_1, e_2, \cdots e_n\}$ is orthonormal basis so $\langle e_i, e_j\rangle = 0, i \neq j$
and $\|e_i\| = \langle e_i, e_i\rangle = 1, i = j$

Also since $\{e_1, e_2, \cdots e_n\}$ is basis for V, so any vector $v \in V$ can be written as linear combination of basis vectors.

Therefore $v = a_1 e_1 + a_2 e_2 + \cdots a_n e_n$ —— (1)

Taking inner product of (1) with $e_1, e_2, \ldots e_n$

$$\langle v, e_1\rangle = \langle a_1 e_1 + a_2 e_2 + \cdots a_n e_n, e_1\rangle$$

$$\langle v, e_1\rangle = a_1\langle e_1, e_1\rangle + a_2\langle e_2, e_1\rangle + \cdots a_n\langle e_n, e_1\rangle$$

$$= a_1(1) + a_2(0) + \cdots a_n(0)$$

$$\boxed{\langle v, e_1\rangle = a_1}$$

$$\langle v, e_2\rangle = a_1\langle e_1, e_2\rangle + a_2\langle e_2, e_2\rangle + \cdots a_n\langle e_n, e_2\rangle$$

$$= a_1(0) + a_2(1) + \cdots a_n(0)$$

$$\boxed{\langle v, e_2\rangle = a_2}$$

Similarly $\boxed{\langle v, e_3\rangle = a_3}$

and so on $\boxed{\langle v, e_n\rangle = a_n}$

$\therefore$ (1) becomes $v = \langle v, e_1\rangle e_1 + \langle v, e_2\rangle e_2 + \cdots \langle v, e_n\rangle e_n$

Matrix of Linear Transformation:-

Let V & W be two finite dimensional vectorspaces over the same field F. $\dim V = n$ & $\dim W = m$

Let $B = \{v_1, v_2, \dots v_n\}$ and $E = \{w_1, w_2 \dots w_m\}$ be any bases for V & W resp.

Any vector v in V can be expressed in a unique way as a linear combination of $v_1, v_2 \dots v_n$ i.e basis B.

$\therefore \quad v = a_1 v_1 + a_2 v_2 + \dots a_n v_n$ $\qquad a_i \in F,\ i = 1, 2, \dots n$

a_i = coordinate vector of v relative to B

Let $T: V \to W$ be a linear transformation.

The images $T(v_1), T(v_2), \dots T(v_n)$ are elements of W

Each image can be expressed uniquely as a linear combination of the basis vectors $w_1, w_2, \dots w_m$.

where $a_{ij} \in F$

$$T(v_1) = a_{11} w_1 + a_{21} w_2 + \dots a_{i1} w_i \dots + a_{m1} w_m$$
$$T(v_2) = a_{12} w_1 + a_{22} w_2 + \dots a_{i2} w_i \dots + a_{m2} w_m$$
$$\vdots$$
$$T(v_j) = a_{1j} w_1 + a_{2j} w_2 + \dots + a_{ij} w_i \dots + a_{mj} w_m$$
$$\vdots$$
$$T(v_n) = a_{1n} w_1 + a_{2n} w_2 + \dots a_{in} w_i + \dots a_{mn} w_m$$

The $m \times n$ Matrix of T is called Matrix of linear Transformation

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & \boxed{a_{ij}} & \dots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mj} & & a_{mn} \end{bmatrix}$$

a_{ij} is coeff of w_i in the image of $T(v_j)$

Q10(ii) If $T: V \to V$ is linear transformation show that $\langle T(e_j), e_i \rangle$ is the ij^{th} entry of the matrix representing T in the given basis $\{e_1, e_2 \dots e_n\}$.

Sol $\quad T: V \to V \qquad$ Let $\{v_1, v_2 \dots v_n\} = \{w_1, w_2 \dots w_n\} = \{e_1, e_2 \dots e_n\}$

$$T(v_j) = a_{1j} w_1 + a_{2j} w_2 + \dots a_{ij} w_i + \dots a_{nj} w_n$$
$$T(e_j) = a_{1j} e_1 + a_{2j} e_2 + \dots a_{ij} e_i + \dots a_{nj} e_n$$

Take I.P with e_i

$$\langle T(e_j), e_i \rangle = a_{1j} \langle e_1, e_i \rangle + a_{2j} \langle e_2, e_i \rangle + \dots a_{ij} \langle e_i, e_i \rangle + \dots a_{nj} \langle e_n, e_i \rangle$$
$$= 0 + 0 + \dots a_{ij}(1) + 0 \dots 0$$

$\because \{e_1, e_2 \dots e_n\}$ is orthonormal basis

$\langle T(e_j), e_i \rangle = a_{ij} = ij^{th}$ entry in the matrix of T.

2(ii) Let W be a subspace of an inner product space V. Show that there is an orthonormal basis of W which is part of an orthonormal basis of V.

Sol W is subspace of inner product space $V(R)$

Let $\{w_1, w_2, \cdots w_t\}$ be a basis for W

$\dim(W) = t$

Let $\dim(V) = n$

As $\{w_1, w_2 \cdots w_t\}$ is a linearly independent set of vectors in V {being basis for W.}

So by theorem "Any linearly independent set of vectors in a finite dimensional vectorspace 'V' can be extended to a basis for V"

Another basis for $V = \{w, w_2, \cdots w_t, u_1, u_2, \cdots u_{n-t}\}$

$\Rightarrow$ Basis for $W \subseteq$ Basis for V ——— ①

Transforming by Gram Schmidt process into orthonormal basis for W & V

① becomes orthonormal basis for $W \subseteq$ orthonormal basis for V.

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