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(IPS) INNER PRODUCT SPACES

Inner product spaces are simply vector spaces over the field 'F' of real or complex numbers and with an Inner Product defined on them.

Def:

Let V be the vectorspace over the field F of real or complex numbers.

A mapping (function) $\langle\, ,\, \rangle : V \times V \to F$ is said to be an <u>INNER PRODUCT</u> (IP) on V if the following conditions are satisfied:

i) $\langle v, v\rangle \geq 0$

& $\langle v, v\rangle = 0$ iff $v = 0 \;\forall\, v \in V$

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ii) $\langle v_1, v_2\rangle = \overline{\langle v_2, v_1\rangle} \quad \forall\, v_1, v_2 \in V$

where $\overline{\langle v_2, v_1\rangle}$ is complex conjugate of $\langle v_2, v_1\rangle$

iii) $\langle av_1 + bv_2, v_3\rangle = a\langle v_1, v_3\rangle + b\langle v_2, v_3\rangle \quad \forall\, v_1, v_2, v_3 \in V \;\; a, b \in F$

The pair $(V, \langle\, ,\, \rangle)$ is called <u>Inner Product Space</u> (IPS) where V is a vectorspace over the field 'F' of real or complex numbers and $\langle\, ,\, \rangle$ is an inner product on V.

<u>Note</u> If F is taken as the field of real numbers then condition (ii) becomes $\langle v_1, v_2\rangle = \langle v_2, v_1\rangle$

$\because$ if z is real then $\bar{z} = z$
$\because \overline{\langle v_2, v_1\rangle} = \langle v_2, v_1\rangle$ only in real numbers.

<u>We shall consider innerproduct space over R only</u>

using cond II
So condition (iii) becomes

$$\langle v_3, av_1 + bv_2\rangle = \langle av_1 + bv_2, v_3\rangle \quad \text{using cond(ii)}$$

$$= a\langle v_1, v_3\rangle + b\langle v_2, v_3\rangle \quad \text{by (iii)}$$

$$\langle v_3, av_1 + bv_2\rangle = a\langle v_3, v_1\rangle + b\langle v_3, v_2\rangle \quad \text{by (iii)}$$

Example 1 Let $u, v \in R^n$ where $u = (u_1, u_2, \cdots u_n)$, $v = (v_1, v_2 - - - v_n)$

then the dot product $\langle u, v\rangle = u_1v_1 + u_2v_2 + \cdots \cdot u_nv_n$ is an inner product or R^n, verify.

Sol We show that $\langle u, v\rangle$ satisfies the three conditions.

C_1: $\langle u, u\rangle = u_1u_1 + u_2u_2 + - - - - u_nu_n$

$= u_1^2 + u_2^2 + \cdot - - \cdot u_n^2 > 0$ $\quad \because$ sum of squares.

Let $\langle u, u\rangle = 0$

$\Leftrightarrow u_1^2 + u_2^2 - - \cdot - - \cdot + u_n^2 = 0$

$\Leftrightarrow$ each $u_i^2 = 0$, $i = 1, 2 \cdots n$

$\Leftrightarrow$ each $u_i = 0$

So $u = (0, 0, - - - 0) = 0$

Hence $\langle u, u\rangle \geq 0$

C_2: $\langle u, v\rangle = u_1v_1 + u_2v_2 + - - - u_nv_n$

$= v_1u_1 + v_2u_2 + - \cdots v_nu_n$ $\quad \because u_i v_i \in R$

$\langle u, v\rangle = \langle v, u\rangle$

C_3: $\langle au + bv, w\rangle$ $\quad w = (w_1, w_2 \cdots - - w_n) \in R^n$.

$(au + bv) = a(u_1, u_2, \cdots u_n) + b(v_1, v_2 - - v_n) = au_1 + bv_1, au_2 + bv_2, \cdots, au_n + bv_n$

$= (au_1 + bv_1)w_1 + (au_2 + bv_2)w_2 + \cdots (au_n + bv_n)w_n$ (as defined)

$= au_1w_1 + bv_1w_1 + au_2w_2 + bv_2w_2 + \cdots au_nw_n + bv_nw_n$

$= a(u_1w_1 + u_2w_2 + \cdots u_nw_n) + b(v_1w_1 + v_2w_2 \cdots v_nw_n)$

$= a\langle u, w\rangle + b\langle v, w\rangle$

Hence $\langle u, v\rangle$ is an Inner Product on R^n.

<u>Note</u> $(R^n, \langle u, v\rangle)$ is called <u>Euclidean Inner Product Space on R^n</u>

where $\langle u, v\rangle$ = dot product

$= u_1v_1 + u_2v_2 - - - - u_nv_n$

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Example 2 Let V be the vector space of all $n\times 1$ matrices over R (Such matrices are called Column Vectors)

$c_1, c_2 \in V$ where $c_1 = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $c_2 = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$

$c_1^t = [x_1, x_2 \cdots x_n]$ $\quad c_2^t = [y_1, y_2 \cdots y_n]$

Show that $\langle c_1, c_2 \rangle$ is IP, where $\langle c_1, c_2\rangle = \det(c_1^t c_2) = |c_1^t c_2|$

Sol

$$c_1^t c_2 = [x_1, x_2 \cdots x_n]\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$1\times n$, $n\times 1$ $= (x)$ = element

$$c_1^t c_2 = [(x_1y_1 + x_2y_2 + \cdots x_ny_n)]$$

$$|c_1^t c_2| = |(x_1y_1 + x_2y_2 + \cdots x_ny_n)| = x_1y_1 + x_2y_2 \cdots x_ny_n \quad \text{—— (1)}$$

c_1: $\langle c_1, c_1\rangle = |c_1^t c_1|$

$$= x_1x_1 + x_2x_2 + \cdots x_nx_n = x_1^2 + x_2^2 + \cdots x_n^2 > 0$$

Let $\langle c_1, c_1\rangle = 0$

$\Leftrightarrow x_1^2 + x_2^2 + \cdots x_n^2 = 0$

$\Leftrightarrow$ each $x_i = 0 \quad i = 1, 2, \cdots n.$

$\Leftrightarrow c_1 = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 0 \qquad \therefore \langle c_1, c_1\rangle = 0 \Rightarrow c_1 = 0$

c_2: $\langle c_1, c_2\rangle = |c_1^t c_2| = x_1y_1 + x_2y_2 + \cdots x_ny_n$

$$= y_1x_1 + y_2x_2 + \cdots y_nx_n$$
$$= |c_2^t c_1|$$
$$= \langle c_2, c_1\rangle$$

Let $a, b \in R$ & $c_3 = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_3 \end{bmatrix}$

$a c_1 + b c_2 = \begin{bmatrix} ax_1 + by_1 \\ ax_2 + by_2 \\ \vdots \\ ax_n + by_n \end{bmatrix}$

c_3: $\langle ac_1 + bc_2, c_3\rangle = |(ac_1 + bc_2)^t c_3|$

$$= |(ax_1 + by_1)z_1 + (ax_2 + by_2)z_2 + \cdots (ax_n + by_n)z_n|$$
$$= (ax_1 + by_1)z_1 + (ax_2 + by_2)z_2 + \cdots (ax_n + by_n)z_n$$
$$= a(x_1z_1 + x_2z_2 + \cdots x_nz_n) + b(y_1z_1 + y_2z_2 + \cdots y_nz_n)$$
$$= a\,|c_1^t c_3| + b\,|c_2^t c_3|$$
$$= a\langle c_1, c_3\rangle + b\langle c_2, c_3\rangle$$

Hence $\langle c_1, c_2\rangle$ is an I.P on V

<u>Example 3</u> Let $u, v \in R^2$, $u = (x_1, x_2)$ $v = (y_1, y_2)$

Then show that $\langle u, v \rangle = x_1y_1 - x_1y_2 - x_2y_1 + 3x_2y_2$ is an IP on R^2.

<u>Sol</u> (i) $\langle u, u \rangle = x_1x_1 - x_1x_2 - x_2x_1 + 3x_2x_2$

$$= x_1^2 - 2x_1x_2 + 3x_2^2$$
$$= x_1^2 - 2x_1x_2 + x_2^2 + 2x_2^2$$
$$= (x_1 - x_2)^2 + 2x_2^2 \geq 0$$

Let $\langle u, u \rangle = 0$

$\Leftrightarrow (x_1 - x_2)^2 + 2x_2^2 = 0$

$\Leftrightarrow (x_1 - x_2)^2 \,\&\, 2x_2^2 = 0$

$\Leftrightarrow (x_1 - x_2) = 0 \,\&\, \sqrt{2}x_2 = 0$

$\Leftrightarrow x_1 = x_2 \,\&\, x_2 = 0$

$\Leftrightarrow x_1 = 0, \,\&\, x_2 = 0$

$\Leftrightarrow u = 0$

(ii) $\langle u, v \rangle = x_1y_1 - x_1y_2 - x_2y_1 + 3x_2y_2$

$$= y_1x_1 - y_2x_1 - y_1x_2 + 3y_2x_2$$
$$= y_1x_1 - y_1x_2 - y_2x_1 + 3y_2x_2$$
$$= \langle v, u \rangle \quad \forall\, u, v \in R^2$$

(iii) $\langle au + bv, w \rangle = \langle (ax_1 + by_1, ax_2 + by_2), (z_1, z_2) \rangle$

Let $w = (z_1, z_2)$

$\because au + bv$
$= a(x_1, x_2) + b(y_1, y_2)$
$= (ax_1 + by_1, ax_2 + by_2)$

$$= (ax_1 + by_1)z_1 - (ax_1 + by_1)z_2 - (ax_2 + by_2)z_1 + 3(ax_2 + by_2)z_2$$
$$= ax_1z_1 + by_1z_1 - ax_1z_2 - by_1z_2 - ax_2z_1 - by_2z_1 + 3ax_2z_2 + 3by_2z_2$$
$$= a(x_1z_1 - x_1z_2 - x_2z_1 + 3x_2z_2) + b(y_1z_1 - y_1z_2 - y_2z_1 + 3y_2z_2)$$
$$= a\langle u, w \rangle + b\langle v, w \rangle$$

Thus all the three conditions are satisfied

So $(R^2, \langle \cdot, \cdot \rangle)$ is an IPS.

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Norm (or Length) of a vector.

Let V be an IPS and $v \in V$, then the real number $\sqrt{\langle v, v\rangle}$ is called the norm of v, it is denoted by $\|v\|$.

$$\therefore \|v\| = \sqrt{\langle v, v\rangle} \quad \& \quad \|v\|^2 = \langle v, v\rangle$$

If $\|v\| = 1$

i.e $\sqrt{\langle v, v\rangle} = 1$

i.e $\langle v, v\rangle = 1$ then v is called Unit vector or Normalized Vector.

Any non-zero vector $u \in V$ can be normalized by multiplying it with $\frac{1}{\|u\|}$

$\therefore \frac{u}{\|u\|}$ is a unit vector i.e vector 'u' has been Normalized.

Example 5 Find the norm of $v = (3,4) \in R^2$ with respect to the Euclidean inner product and the inner product defined in Example 3.

Sol Euclidean inner product on R^n is defined as

$$\langle u, v\rangle = u_1v_1 + u_2v_2 + \cdots u_nv_n$$

where $u = (u_1, u_2, \cdots u_n) \in R^n$
$v = (v_1, v_2, \cdots v_n) \in R^n$

$$\|v\| = \sqrt{\langle v, v\rangle}$$
$$\|v\| = \sqrt{v_1^2 + v_2^2 + \cdots v_n^2} \quad \text{on } R^n$$
$$\|v\| = \sqrt{v_1^2 + v_2^2} \quad \text{on } R^2$$
$$= \sqrt{9+16}$$
$$= 5$$

$v = (3,4) \in R^2$
$\therefore v_1 = 3,\ v_2 = 4$

ii) $$\langle u, v\rangle = \langle (u_1, u_2), (v_1, v_2)\rangle$$
$$= u_1v_1 - u_1v_2 - u_2v_1 + 3u_2v_2$$

as defined in Example 3

$$\|v\| = \sqrt{\langle v, v\rangle}$$
$$= \sqrt{v_1v_1 - v_1v_2 - v_2v_1 + 3v_2v_2}$$
$$= \sqrt{3\cdot3 - 3\cdot4 - 4\cdot3 + 3\cdot4\cdot4} = \sqrt{9-12-12+48} = \sqrt{33} \text{ Ans.}$$

Example 4 Let V be the vectorspace of all real-valued continuous functions on the interval $a \le t \le b$, then for $f, g \in V$

$$\langle f, g\rangle = \int_a^b f(t)\, g(t)\, dt$$ is an inner product on V.

Sol (i) $\langle f, f\rangle = \int_a^b f(t) f(t)\, dt$

$= \int_a^b f^2(t)\, dt \geq 0$ ∵ the definite integral gives the area bounded by the curve, which is always +ive.

Let $\langle f, f\rangle = 0$

$\Leftrightarrow \int_a^b f^2(t)\, dt = 0$

$\Leftrightarrow \quad f^2(t) = 0$

$\Leftrightarrow \quad f(t) = 0$

(ii) $\langle f, g\rangle = \int_a^b f(t)\, g(t)\, dt$

$= \int_a^b g(t)\, f(t)\, dt$

$= \langle g, f\rangle$

(iii) $\langle a_1 f + b_1 g, h\rangle = \int_a^b (a_1 f(t) + b_1 g(t))\, h(t)\, dt$

$= \int_a^b (a_1 f(t) h(t) + b_1 g(t) h(t))\, dt$

$= a_1 \int_a^b f(t)\, h(t)\, dt + b_1 \int_a^b g(t)\, h(t)\, dt$

$= a_1 \langle f, h\rangle + b_1 \langle g, h\rangle \quad \forall a_1, b_1 \in R$
$\forall f, g, h \in V$

Thus all the three conditions of an I.P are satisfied

So $(V, \langle \cdot, \cdot \rangle)$ is an IPS

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Example 6

Normalize $v = (1,2,1) \in R^3$ w.r.t Euclidean I.P

Sol $\|v\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$ $\quad \because \|v\| = \sqrt{\langle v, v\rangle}$

$= \sqrt{1^2 + 2^2 + 1^2}$

$= \sqrt{6}$

Normalized Vector $= \dfrac{v}{\|v\|} = \dfrac{1}{\sqrt{6}}, \dfrac{2}{\sqrt{6}}, \dfrac{1}{\sqrt{6}}$

Th v.imp The Cauchy-Schwarz Inequality

Let u, v be the elements of an inner product space V or R

then $|\langle u, v\rangle| \leq \|u\| \cdot \|v\|$

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Proof If $v = 0$ then

LHS $\langle u, v\rangle = \langle u, 0\rangle$ $\quad \because v = 0$

$= \langle u, 0\cdot w\rangle$ $\quad \forall w \in V$

$= 0\langle u, w\rangle$ by cond(iii)

$= 0$

RHS $\|u\|\|v\| = \|u\|\|0\|$

$= 0$ $\quad \therefore$ LHS = RHS.

If $v \neq 0$ then for all real 't' $\in R$

$0 \leq \|u - tv\|^2$ $\quad a = 1, b = -t$

$= \langle u - tv, u - tv\rangle$

$= \langle u, u - tv\rangle - t\langle v, u - tv\rangle$ by cond iii

$= \langle u, u\rangle - t\langle u, v\rangle - t\langle v, u\rangle + t^2\langle v, v\rangle$ by cond iii

$= \|u\|^2 - 2t\langle u, v\rangle + t^2\|v\|^2$ $\quad$ For real t: $t \cdot \bar{t} = |t|^2$, $t^2 = |t|^2$

Let $t = \dfrac{\langle u, v\rangle}{\langle v, v\rangle}$, $\Rightarrow t^2 = \dfrac{|\langle u, v\rangle|^2}{\|v\|^4}$ $\quad \because (\langle u, v\rangle)(\langle u, v\rangle) = |\langle u, v\rangle|^2$

$0 \leq \|u\|^2 - 2\dfrac{\langle u, v\rangle}{\langle v, v\rangle}\langle u, v\rangle + \dfrac{|\langle u, v\rangle|^2}{\|v\|^4}\|v\|^2$

$0 \leq \|u\|^2 - 2\dfrac{|\langle u, v\rangle|^2}{\|v\|^2} + \dfrac{|\langle u, v\rangle|^2}{\|v\|^2}$

by LCM $\quad 0 \leq \|u\|^2\|v\|^2 - |\langle u, v\rangle|^2$ $\xrightarrow{\text{Taking Square Root}}$ $|\langle u, v\rangle| \leq \|u\|\|v\|$

proved.

Theorem The norm in an inner product space V satisfies the following axioms

(i) $\|v\| \geq 0$ and $\|v\| = 0$ iff $v = 0$, $v \in V$

(ii) $\|Kv\| = |K|\|v\|$ for all $v \in V$ and $K \in R$

(iii) $\|u+v\| \leq \|u\| + \|v\|$ for all $u, v \in V$

Proof (i) $\|v\| = \sqrt{\langle v, v\rangle}$

$\geq 0 \quad \because \langle v, v\rangle \geq 0$ by cond(i)

$\therefore \|v\| \geq 0$

Further $\|v\| = 0 \iff \sqrt{\langle v, v\rangle} = 0$

$\iff \langle v, v\rangle = 0$

$\iff v = 0$

(ii) $\|Kv\|^2 = \langle Kv, Kv\rangle \quad K \in R$

$= K\langle v, Kv\rangle$

$= K^2\langle v, v\rangle$

$= K^2\|v\|^2 \quad \because \sqrt{\langle v, v\rangle} = \|v\|$

$= |K|^2\|v\|^2 \quad \because K \in R$

(iii) $\|u+v\|^2 = \langle u+v, u+v\rangle$

$= \langle u, u\rangle + \langle u, v\rangle + \langle v, u\rangle + \langle v, v\rangle \quad z + \bar{z} = 2\mathrm{Re}\, z$

$= \|u\|^2 + 2\langle u, v\rangle + \|v\|^2 \quad \because$ cond ii $\because \langle u, v\rangle = \langle v, u\rangle$

$\leq \|u\|^2 + 2|\langle u, v\rangle| + \|v\|^2 \quad \mathrm{Re}\, z \leq |z|$

$\leq \|u\|^2 + 2\|u\|\cdot\|v\| + \|v\|^2$ By Cauchy Schwarz $\quad \langle u, v\rangle \leq |\langle u, v\rangle|$

$= (\|u\| + \|v\|)^2$

$\|u+v\|^2 \leq (\|u\| + \|v\|)^2$

$\|u+v\| \leq \|u\| + \|v\|$

x ———

ORTHOGONALITY

Let 'θ' be the Angle between two vectors $u, v \in V$ (IPS), then

$$\cos\theta = \frac{\langle u,v\rangle}{\|u\|\|v\|} \qquad 0 \leq \theta \leq \pi$$

$\because$ By Cauchy Schwarz

$$|\langle u,v\rangle| \leq \|u\|\|v\|$$

$$\frac{|\langle u,v\rangle|}{\|u\|\|v\|} \leq \frac{\|u\|\|v\|}{\|u\|\|v\|}$$

$$\frac{|\langle u,v\rangle|}{\|u\|\|v\|} \leq 1$$

If $\theta = 90$

$$\cos(90) = \frac{\langle u,v\rangle}{\|u\|\|v\|}$$

$$0 = \frac{\langle u,v\rangle}{\|u\|\|v\|}$$

$$0 = \langle u,v\rangle \quad \therefore$$

$\therefore$ u & v are orthogonal iff $\langle u,v\rangle = 0$
($u \perp v$, read u is orthogonal to v)

Note

1) If $u \perp v \Rightarrow \langle u,v\rangle = 0$
If $v \perp u \Rightarrow \langle v,u\rangle = 0$
$\therefore \langle u,v\rangle = \langle v,u\rangle$
Hence relation of orthogonality is symmetric.

2) The vector '0' is orthogonal to every $v \in V$

$\because \langle 0,v\rangle = \langle 0v, v\rangle = 0\cdot\langle v,v\rangle = 0$ by cond iii

3) If u is orthogonal to itself then $u = 0$

$\because \langle u,u\rangle = 0$ by def of orthogonality

$\|u\|^2 = 0 \Rightarrow \|u\| = 0 \Rightarrow u = 0$

Example 7 Show that $x \perp y$ where $x, y \in R^3$

$x = (1,-1,2) \qquad y = (-1,1,1)$

Sol $\langle x,y\rangle = (1)(-1) + (-1)(1) + (2)(1)$

$= -1 - 1 + 2$

$\langle x,y\rangle = 0$ So $x \perp y$

Show that $x \perp y$ where $x, y \in R^4$

$x = (1,-1,1,-1) \cdot y = (-1,2,2,-1)$

$\langle x,y\rangle = (1)(-1) + (-1)(2) + (1)(2) + (-1)(-1)$

$= -1 - 2 + 2 + 1$

$= 0$ So $x \perp y$.

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Example 8 If u is orthogonal to v then every scalar multiple of u is also orthogonal to v. (Ku is multiple of u)

Sol If $u \perp v \Rightarrow \langle u, v\rangle = 0$

then $\langle ku, v\rangle = k\langle u, v\rangle$ by cond iii

$= K(0)$

$\langle Ku, v\rangle = 0$

Hence $Ku \perp v$

Example 9 Find a unit vector orthogonal to both $(1,1,2)$ and $(0,1,3)$ in R^3.

Sol Let $(x,y,z) \in R^3$ be a vector orthogonal to given vectors

$\therefore \langle (x,y,z), (1,1,2)\rangle = 0$ $\quad \because (x,y,z) \perp (1,1,2)$

$x(1) + y(1) + z(2) = 0$ ——— (i)

Also $\langle (x,y,z), (0,1,3)\rangle = 0$ $\quad \because (x,y,z) \perp (0,1,3)$

$x(0) + y(1) + z(3) = 0$ ——— (ii)

using (ii) in (i) $\quad y = -3z$

$x - 3z + 2z = 0$

$x - z = 0$

$x = z$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ -3z \\ z \end{bmatrix} = z\begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$$

$\Rightarrow (1,-3,1)$ is $\perp$ to both u & v

$\|(1,-3,1)\| = \sqrt{1^2 + 3^2 + 1^2} = \sqrt{11}$

Unit Vector $= \left(\frac{1}{\sqrt{11}}, \frac{-3}{\sqrt{11}}, \frac{1}{\sqrt{11}}\right)$

2nd Method

$$w = \text{vector} \perp u \,\&\, v = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 1 & 2 \\ 0 & 1 & 3 \end{vmatrix}$$

$e_1 = (1,0,0) = i$

$e_2 = (0,1,0) = j$

$e_3 = (0,0,1) = k$

$w = e_1(3-2) - e_2(3-0) + e_3(1-0)$

$= e_1 - 3e_2 + e_3 = 1, -3, 1$

Unit vector $= \frac{w}{\|w\|} = \frac{1}{\sqrt{11}}(e_1 - 3e_2 + e_3) = \frac{1}{\sqrt{11}}e_1, \frac{-3}{\sqrt{11}}e_2, \frac{1}{\sqrt{11}}e_3$

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<u>Orthogonal Complement</u>:

Let 'W' be a subset of an Inner Product Space 'V' over R. The orthogonal complement of 'W' denoted by $W^{\perp}$ and read as 'W perp', consists of those vectors in V which are orthogonal to every $w \in W$. Thus

$$W^{\perp} = \{v \in V : \langle v, w \rangle = 0 \quad \forall\, w \in W\}$$

<u>Prove that $W^{\perp}$ is subspace of V.</u>

$\left.\begin{array}{l} \text{Let } u, v \in W^{\perp} \\ \text{and } a, b \in R \end{array}\right\}$ then for $w \in W$ $\quad \langle u, w \rangle = 0 \quad \because u, v \in W^{\perp}$

$\langle v, w \rangle = 0$

Since $\langle au + bv , w \rangle = a\langle u, w \rangle + b\langle v, w \rangle$

$= a \cdot 0 + b \cdot 0$

$= 0$

So $au + bv \in W^{\perp}$. Thus $W^{\perp}$ is a subspace of V.

<u>Orthogonal Systems</u>

A set 'S' of vectors in an I.P. Space 'V' over R is said to be an orthogonal system if its <u>distinct</u> vectors are orthogonal i.e if $\langle u_i, u_j \rangle = 0 \quad \forall\, u_i, u_j \in S \quad i \neq j$

<u>Orthonormal System</u>

A set S of vectors in an I.P. Space 'V' over R is said to be an orthonormal system if

$$\langle u_i, u_j \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

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Example 10 Show that the system $\{ u_1 = (1,-1,1,-1)$
$u_2 = (3,1,-1,1)$
$u_3 = (0,2,1,-1)$
$u_4 = (0,0,1,1) \}$ is an orthonormal system in R^4!

Sol
$$\langle u_1, u_2 \rangle = 1\cdot 3 + (1)(-1) + 1(-1) + (-1)(1)$$
$$= 3-1-1-1$$
$$= \boxed{0}$$
$$\langle u_1, u_3 \rangle = 1\cdot 0 + (-1)(2) + 1(1) + (-1)(-1)$$
$$= 0-2+1-1$$
$$= \boxed{0}$$
$$\langle u_1, u_4 \rangle = 1\cdot 0 + (-1)(0) + 1(1) + (-1)(1)$$
$$= 0+0+1-1$$
$$= \boxed{0}$$
$$\langle u_2, u_3 \rangle = (3)0 + (1)(2) + (-1)(1) + 1(-1)$$
$$= 0+2-1-1$$
$$= \boxed{0}$$
$$\langle u_2, u_4 \rangle = (3)(0) + 1(0) + (-1)(1) + 1(1)$$
$$= 0+0-1+1$$
$$= \boxed{0}$$
$$\langle u_3, u_4 \rangle = (0)(0) + (-1)(0) + (1)(1) + (-1)(1)$$
$$= 0+0+1-1$$
$$= \boxed{0}$$
$$\langle u_1, u_1 \rangle = (1)(1) + (-1)(-1) + (1)(1) + (-1)(-1)$$
$$= 1+1+1+1$$
$$= 4 \neq 1$$
$$\langle u_2, u_2 \rangle = (3)(3) + (1)(1) + (-1)(-1) + (1)(1)$$
$$= 9+1+1+1$$
$$= 12 \neq 1$$

So this is orthogonal system not orthonormal system.

Def.

Example 11 Let V be the vectorspace of real-valued continuous fns on the interval $-\pi \le t \le \pi$ with inner product defined by

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(t)\cdot g(t)\, dt \quad \forall f, g \in V$$

then $\{1, \cos t, \cos 2t, \ldots \sin t, \sin 2t \ldots\}$ is an orthogonal system in V.

Sol

$$\langle 1, \cos mt\rangle = \int_{-\pi}^{\pi} 1\cdot \cos mt\, dt = \left|\frac{\sin mt}{m}\right|_{-\pi}^{\pi} = \frac{1}{m}\left(\sin(m\pi) - \sin(-m\pi)\right) = \frac{1}{m}(0-0) = \boxed{0}$$

$$\langle 1, \sin mt\rangle = \int_{-\pi}^{\pi} 1\cdot \sin mt\, dt = \left|\frac{-\cos mt}{m}\right|_{-\pi}^{\pi} = \frac{1}{m}\left(\cos m\pi + \cos(-m\pi)\right) = \frac{1}{m}\left(\cancel{\cos m\pi} - \cancel{\cos m\pi}\right) = \boxed{0}$$

when $m \ne n$

$$\langle \cos mt, \sin nt\rangle = \int_{-\pi}^{\pi} \cos(mt)\sin(nt)\, dt = \frac{1}{2}\int_{-\pi}^{\pi} 2\cos(mt)\sin(nt)\, dt$$

$$= \frac{1}{2}\int_{-\pi}^{\pi}\left\{\sin(m+n)t - \sin(m-n)t\right\} dt$$

$\because 2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$ $\boxed{\text{where } m > n}$

$$= \frac{1}{2}\left| -\frac{\cos(m+n)t}{(m+n)} - \left(-\frac{\cos(m-n)t}{m-n}\right)\right|_{-\pi}^{\pi}$$

$$= \frac{1}{2}\left\{\left(-\frac{\cos(m+n)\pi}{m+n} + \frac{\cos(m-n)\pi}{m-n}\right) - \left(-\frac{\cos(m+n)(-\pi)}{m+n} + \frac{\cos(m-n)(-\pi)}{m-n}\right)\right.$$

$$= \frac{1}{2}\left\{-\cancel{\frac{\cos(m+n)\pi}{m+n}} + \cancel{\frac{\cos(m-n)\pi}{m-n}} + \cancel{\frac{\cos(m+n)(-\pi)}{m+n}} - \cancel{\frac{\cos(m-n)(-\pi)}{m-n}}\right.$$

$$= \boxed{0}$$

$\because \cos(-\pi) = \cos\pi$

when $m = n$

$$\langle \cos mt, \sin mt\rangle = \int_{-\pi}^{\pi} \cos mt \sin mt\, dt$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} 2\cos mt \sin mt\, dt = \frac{1}{2}\int_{-\pi}^{\pi} \sin 2mt\, dt \quad \because 2\cos\theta\sin\theta = \sin 2\theta$$

$$= \frac{1}{2}\left|\frac{\cos 2mt}{2m}\right|_{-\pi}^{\pi} = \frac{1}{4m}\left(\cancel{\cos 2m\pi} + \cancel{\cos 2m(-\pi)}\right)$$

$$= \boxed{0}$$

PTO

when $m \neq n$

$$\langle \cos mt, \cos nt \rangle = \int_{-\pi}^{\pi} \cos mt \cos nt \, dt$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} 2\cos mt \cos nt \, dt$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \cos(m+n)t + \cos(m-n)t \, dt$$

$$= \frac{1}{2}\left| \frac{\sin(m+n)t}{m+n} + \frac{\sin(m-n)t}{m-n} \right|_{-\pi}^{\pi}$$

$$= \frac{1}{2}\left[\frac{\sin(m+n)\pi}{m+n} + \frac{\sin(m-n)\pi}{m-n} - \frac{\sin(m+n)(-\pi)}{m+n} - \frac{\sin(m-n)(-\pi)}{m-n} \right]$$

$$= \boxed{0}$$

$\because \sin\pi = 0 = \sin(-\pi)$

(In orthogonal system distinct vectors are orthogonal So we will not take $\langle \cos mt, \cos mt \rangle$ $\langle \sin mt, \sin mt \rangle$)

when $m \neq n$

$$\langle \sin mt, \sin nt \rangle = \int_{-\pi}^{\pi} \sin mt, \sin nt \, dt$$

$$= -\frac{1}{2}\int_{-\pi}^{\pi} -2\sin mt \sin nt \, dt$$

$$= -\frac{1}{2}\int_{-\pi}^{\pi} \cos(m+n)t - \cos(m-n)t \, dt$$

$$= -\frac{1}{2}\left| \frac{\sin(m+n)t}{m+n} - \frac{\sin(m-n)t}{m-n} \right|_{-\pi}^{\pi}$$

$$= -\frac{1}{2}\left(\sin(m+n)\pi - \frac{\sin(m-n)\pi}{m-n} - \frac{\sin(m+n)(-\pi)}{m+n} + \frac{\sin(m-n)(-\pi)}{m-n} \right)$$

$$= \boxed{0}$$

Thus the set $\{1, \cos t, \cos 2t, \ldots \sin t, \sin 2t, \ldots\}$ is an orthogonal system in V.

Extra when $m = n$

$$\langle \cos mt, \cos nt \rangle = \int_{-\pi}^{\pi} \cos^2 mt \, dt = \int_{-\pi}^{\pi} \left(\frac{1+\cos 2mt}{2}\right) dt = \frac{1}{2}\left| t + \frac{\sin 2mt}{2m} \right|_{-\pi}^{\pi} =$$

$$= \frac{1}{2}\left[\pi + \frac{\sin 2m\pi}{2m} - (-\pi) - \frac{\sin 2m(-\pi)}{2m} \right] = \frac{1}{2}(\pi+\pi) = \boxed{\pi}$$

when $m = n$

$$\langle \sin mt, \sin nt \rangle = \int_{-\pi}^{\pi} \sin^2 mt \, dt = \int_{-\pi}^{\pi} \left(\frac{1-\cos 2mt}{2}\right) dt = \frac{1}{2}\left| t - \frac{\sin 2mt}{2m} \right|_{-\pi}^{\pi}$$

$$= \frac{1}{2}\left(\pi - \frac{\sin 2m\pi}{2m} - (-\pi) + \frac{\sin 2m(-\pi)}{2m} \right) = \frac{1}{2}(\pi+\pi) = \boxed{\pi}$$

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The Gram-Schmidt Process

V is an I.P. Space over 'R'.

$\{v_1, v_2 \ldots v_n\}$ is a basis of $V(R)$.

An orthonormal basis $\{u_1, u_2 \ldots u_n\}$ of V can be constructed as,

Step 1 Let $u_1 = v_1 \cdot \frac{1}{\|v_1\|}$

$$\|u_1\| = \|v_1\| \cdot \frac{1}{\|v_1\|} = 1$$

u_1 is written in the linear combination of v_1 & $\|v_1\|$ where $\|v_1\|$ is scalar. Similarly $v_1 = u_1 \|v_1\|$ so $[u_1] = [v_1]$

Step 2

Let $u_2 = \frac{w_2}{\|w_2\|}$ where $w_2 = v_2 - \langle v_2, u_1\rangle u_1$

$$\langle u_1, u_2\rangle = \left\langle u_1, \frac{w_2}{\|w_2\|}\right\rangle$$

$$= \frac{1}{\|w_2\|}\langle u_1, w_2\rangle$$

$$= \frac{1}{\|w_2\|}\langle u_1, v_2 - \langle v_2, u_1\rangle u_1\rangle$$

$$= \frac{1}{\|w_2\|}\Big(\langle u_1, v_2\rangle - \langle v_2, u_1\rangle\langle u_1, u_1\rangle\Big)$$ by cond iii

$$= \frac{1}{\|w_2\|}\Big[\langle u_1, v_2\rangle - \langle u_1, v_2\rangle \cdot 1\Big]$$

$\because \|u_1\| = \sqrt{\langle u_1, u_1\rangle}$

$1 = \sqrt{\langle u_1, u_1\rangle}$

squaring $1 = \langle u_1, u_1\rangle$

$\langle u_1, u_2\rangle = 0$ & $\|u_2\| = \frac{\|w_2\|}{\|w_2\|} = 1$

Hence $[u_1, u_2]$ is orthonormal

Hence $[u_1, u_2]$ is O.N. System

Step 3 Let $u_3 = \frac{w_3}{\|w_3\|}$ where $w_3 = v_3 - \langle v_3, u_1\rangle u_1 - \langle v_3, u_2\rangle u_2$

Similarly $[u_1, u_2, u_3]$ is orthonormal.

and so on, then $u_n = \frac{w_n}{\|w_n\|}$ where $w_n = v_n - \langle v_n, u_1\rangle u_1 - \langle v_n, u_2\rangle u_2 \ldots$

$\ldots - \langle v_n, u_{n-1}\rangle u_{n-1}$

$\therefore$ The set $\{u_1, u_2, \ldots u_n\}$ is orthonormal

$\because$ We know orthonormal is linearly independent and

Span $\{u_1, u_2, \ldots u_n\}$ = Span $\{v_1, v_2, \ldots v_n\}$

Hence $\{u_1, u_2, \ldots u_n\}$ is orthonormal basis of V.

Theorem Every orthonormal system $\{u_1, u_2, \dots u_n\}$ is linearly independent. Moreover, for all $v \in V$, the vector $w = v - \sum_{K=1}^{n} \langle v, u_K \rangle u_K$ is orthogonal to each u_i, $1 \le i \le n$. i.e. $\{u_1, u_2 \dots u_n\}$

Proof Suppose $a_1 u_1 + a_2 u_2 + \dots a_n u_n = 0$ where a_i are scalars $a_i \in R$

Taking innerproduct of both sides with 'u_i'

$$\langle a_1 u_1 + a_2 u_2 \dots + a_n u_n , u_i \rangle = \langle 0, u_i \rangle$$

$$a_1 \langle u_1, u_i \rangle + a_2 \langle u_2, u_i \rangle + \dots a_{i-1} \langle u_{i-1}, u_i \rangle$$

$$+ a_i \langle u_i, u_i \rangle + a_{i+1} \langle u_{i+1}, u_i \rangle + \dots a_n \langle u_n, u_i \rangle = 0$$

$\because$ Cond iii $\because 0$ is orthogonal to every $v \in V$ $\therefore \langle 0, v \rangle = 0$

$$0 + 0 + \dots 0 + a_i \cdot 1 + 0 + \dots 0 = 0$$

$$a_i \cdot 1 = 0 \qquad \because 1 \ne 0 \quad a_i = 0$$

$$a_i = 0 \qquad \forall i = 1, 2, \dots n$$

$\because \{u_1, u_2 \dots u_n\}$ is orthonormal system $\therefore \langle u_i, u_j \rangle = 0$, $i \ne j$ Distinct; $\langle u_i, u_i \rangle = 1$ $i = j$

$\therefore \{u_1, u_2 \dots u_n\}$ is linearly independent.

Now to prove w is orthogonal to each u_i, $1 \le i \le n$

Consider $\langle w, u_i \rangle = \left\langle v - \sum_{K=1}^{n} \langle v, u_K \rangle u_K , u_i \right\rangle$

$$= \langle v, u_i \rangle - \left\langle \sum_{K=1}^{n} \langle v, u_K \rangle u_K , u_i \right\rangle \qquad \because \text{Cond iii}$$

$$= \langle v, u_i \rangle - \left\langle \langle v, u_1 \rangle u_1 + \langle v, u_2 \rangle u_2 , \dots \langle v, u_n \rangle u_n , u_i \right\rangle$$

$$= \langle v, u_i \rangle - (\langle v, u_1 \rangle) \langle u_1, u_i \rangle - (\langle v, u_2 \rangle) \langle u_2, u_i \rangle, \dots$$

$$- (\langle v, u_{i-1} \rangle) \langle u_{i-1}, u_i \rangle - \langle v, u_i \rangle \langle u_i, u_i \rangle \qquad \because \langle v, u \rangle \in F = R \therefore \text{by Cond iii}$$

$$- (\langle v, u_{i+1} \rangle) \langle u_{i+1}, u_i \rangle \dots - (\langle v, u_n \rangle) \langle u_n, u_i \rangle$$

$$= \langle v, u_i \rangle - 0 - 0 \dots 0 - \langle v, u_i \rangle \cdot 1 - 0 \dots 0 \qquad \left(\because \langle u_i, u_j \rangle = 0 \atop \langle u_i, u_i \rangle = 1 \right.$$

$$= \langle v, u_i \rangle - \langle v, u_i \rangle$$

$\langle w, u_i \rangle = \boxed{0}$ Hence w is orthogonal to each u_i, $1 \le i \le n$

x———x

Example 12. Show that $\{(1,1,1),(0,1,1),(0,0,1)\}$ is a basis of R^3. Using Gram-Schmidt orthogonalization process, transform this basis into an orthonormal basis.

Sol. Let $(x,y,z) \in R^3$

Suppose $(x,y,z) = a(1,1,1) + b(0,1,1) + c(0,0,1)$ — linear combination of a, b, c & given set of vectors

$(x,y,z) = (a,\ a+b,\ a+b+c)$

$\therefore \boxed{a = x}$

$a+b = y \Rightarrow b = y-a \Rightarrow \boxed{b = y-x}$

$a+b+c = z \Rightarrow c = z-a-b \Rightarrow c = z-x-(y-x)$

$c = z-x-y+x$

$\boxed{c = z-y}$

$\because$ the given set of vectors $\{(1,1,1),(0,1,1),(0,0,1)\}$ can be written as a linear combination of a, b, c, as values of a, b, c exist.

$\therefore$ the set $\{(1,1,1),(0,1,1),(0,0,1)\}$ is a spanning set for R^3

Now check Linear Independence of $\{(1,1,1),(0,1,1),(0,0,1)\}$

$a(1,1,1) + b(0,1,1) + c(0,0,1) = 0$

$(a,\ a+b,\ a+b+c) = 0$

$\boxed{a = 0}$

$a+b = 0 \Rightarrow 0+b=0 \Rightarrow \boxed{b=0}$

$a+b+c = 0 \Rightarrow 0+0+c=0 \Rightarrow \boxed{c=0}$

Hence given vectors are Lin. Independent.

2nd Method

Since the Matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Rank of $A = 3$ = No. of vectors $= 3$

is in Echelon form

So the given three vectors $(1,1,1),(0,1,1),(0,0,1)$ are Linearly Independent.

Since the given set of vectors is a Spanning Set for R^3 and is Linearly Independent. Hence

Hence the given set of vectors $\{(1,1,1),(0,1,1),(0,0,1)\}$ is a basis of R^3.

P.T.O.

Available at
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by <u>Gram Schmidt Orthogonalisation</u>

Let $v_1=(1,1,1)$, $v_2=(0,1,1)$, $v_3=(0,0,1)$

Now $u_1 = \frac{v_1}{\|v_1\|} = \frac{(1,1,1)}{\sqrt{1^2+1^2+1^2}} = \left(\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, \frac{1}{\sqrt3}\right)$

$u_2 = \frac{\omega_2}{\|\omega_2\|}$ where $\omega_2 = v_2 - \langle v_2, u_1\rangle u_1$

$$= (0,1,1) - \left\langle (0,1,1), \left(\tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}\right)\right\rangle \left(\tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}\right)$$

$$= (0,1,1) - \left(0+\tfrac{1}{\sqrt3}+\tfrac{1}{\sqrt3}\right)\left(\tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}\right)$$

$$= (0,1,1) - \tfrac{2}{\sqrt3}\left(\tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}\right)$$

$$= (0,1,1) - \left(\tfrac{2}{3}, \tfrac{2}{3}, \tfrac{2}{3}\right)$$

$$\omega_2 = \left(-\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right)$$

$$\|\omega_2\| = \sqrt{\tfrac{4}{9}+\tfrac{1}{9}+\tfrac{1}{9}} = \sqrt{\tfrac{6}{9}} = \sqrt{\tfrac{2}{3}}$$

$$\therefore u_2 = \frac{\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)}{\sqrt{\frac{2}{3}}}$$

$$u_2 = \sqrt{\tfrac{3}{2}}\left(-\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right) = \left(-\tfrac{2}{\sqrt6}, \tfrac{1}{\sqrt6}, \tfrac{1}{\sqrt6}\right)$$

$u_3 = \frac{\omega_3}{\|\omega_3\|}$ where $\omega_3 = v_3 - \langle v_3, u_1\rangle u_1 - \langle v_3, u_2\rangle u_2$

$$= (0,0,1) - \left\langle (0,0,1), \left(\tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}\right)\right\rangle \left(\tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}\right) -$$

$$- \left\langle (0,0,1), \left(-\tfrac{2}{\sqrt6}, \tfrac{1}{\sqrt6}, \tfrac{1}{\sqrt6}\right)\right\rangle \left(-\tfrac{2}{\sqrt6}, \tfrac{1}{\sqrt6}, \tfrac{1}{\sqrt6}\right)$$

$$= (0,0,1) - \left(\tfrac{1}{\sqrt3}\right)\left(\tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}\right) - \left(\tfrac{1}{\sqrt6}\right)\left(-\tfrac{2}{\sqrt6}, \tfrac{1}{\sqrt6}, \tfrac{1}{\sqrt6}\right)$$

$$= (0,0,1) - \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right) - \left(-\tfrac{2}{6}, \tfrac{1}{6}, \tfrac{1}{6}\right)$$

$$= \left(0 - \tfrac{1}{3} + \tfrac{2}{6},\ 0 - \tfrac{1}{3} - \tfrac{1}{6},\ 1 - \tfrac{1}{3} - \tfrac{1}{6}\right)$$

$$= \left(0,\ \tfrac{-2-1}{6},\ \tfrac{6-2-1}{6}\right)$$

$$\omega_3 = \left(0,\ -\tfrac{1}{2},\ \tfrac{1}{2}\right)$$

$$\|\omega_3\| = \sqrt{0+\tfrac{1}{4}+\tfrac{1}{4}} = \tfrac{1}{\sqrt2}$$

$$\therefore u_3 = \frac{\left(0, -\frac{1}{2}, \frac{1}{2}\right)}{\frac{1}{\sqrt2}}$$

$$= \sqrt2\left(0, -\tfrac{1}{2}, \tfrac{1}{2}\right) = \left(0, -\tfrac{1}{\sqrt2}, \tfrac{1}{\sqrt2}\right)$$

Thus the orthonormal basis is $\{u_1, u_2, u_3\} = \left\{\left(\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, \frac{1}{\sqrt3}\right), \left(-\frac{2}{\sqrt6}, \frac{1}{\sqrt6}, \frac{1}{\sqrt6}\right), \left(0, -\frac{1}{\sqrt2}, \frac{1}{\sqrt2}\right)\right\}$