(Exercise 6.3)

Q1 Check which of the following define linear transformations from R^3 to R^2?

(i) $T(x_1, x_2, x_3) = (x_1 - x_2, x_1 - x_3)$

Sol. Given transformation is

$$T(x_1, x_2, x_3) = (x_1 - x_2, x_1 - x_3)$$

Let $u_1 = (x_1, x_2, x_3)$

& $u_2 = (y_1, y_2, y_3) \in R^3$

(i) then we prove $T(u_1 + u_2) = T(u_1) + T(u_2)$

Now
$$\begin{aligned}
T(u_1 + u_2) &= T\big((x_1, x_2, x_3) + (y_1, y_2, y_3)\big) \\
&= T(x_1 + y_1, x_2 + y_2, x_3 + y_3) \\
&= \big((x_1 + y_1) - (x_2 + y_2), (x_1 + y_1) - (x_3 + y_3)\big) \\
&= (x_1 + y_1 - x_2 - y_2, x_1 + y_1 - x_3 - y_3) \\
&= (x_1 - x_2 + y_1 - y_2, x_1 - x_3 + y_1 - y_3) \\
&= (x_1 - x_2, x_1 - x_3) + (y_1 - y_2, y_1 - y_3) \\
&= T(x_1, x_2, x_3) + T(y_1, y_2, y_3) \\
&= T(u_1) + T(u_2)
\end{aligned}$$

(ii) Let $a \in R$ & $u_1 = (x_1, x_2, x_3) \in R^3$

then we prove $T(au_1) = aT(u_1)$

Now
$$\begin{aligned}
T(au_1) &= T\big(a(x_1, x_2, x_3)\big) \\
&= T(ax_1, ax_2, ax_3) \\
&= (ax_1 - ax_2, ax_1 - ax_3) \\
&= a(x_1 - x_2, x_1 - x_3) \\
&= aT(x_1, x_2, x_3) \\
&= aT(u_1)
\end{aligned}$$

Hence T is a linear transformation from R^3 to R^2

(ii) $T(x_1, x_2, x_3) = (|x_1|, x_2 - x_3)$

<u>Sol.</u> Given transformation is

$T(x_1, x_2, x_3) = (|x_1|, x_2 - x_3)$

let $u_1 = (x_1, x_2, x_3)$

& $u_2 = (y_1, y_2, y_3) \in R^3$ then we prove

(i) $T(u_1 + u_2) = T(u_1) + T(u_2)$

Now

$$T(u_1+u_2) = T((x_1, x_2, x_3) + (y_1, y_2, y_3))$$
$$= T(x_1+y_1, x_2+y_2, x_3+y_3)$$
$$= (|x_1+y_1|, (x_2+y_2) - (x_3+y_3))$$

$$\therefore T(u_1+u_2) = (|x_1+y_1|, x_2+y_2-x_3-y_3) \quad \text{———} \quad (1)$$

Now

$$T(u_1)+T(u_2) = T(x_1, x_2, x_3) + T(y_1, y_2, y_3)$$
$$= (|x_1|, x_2-x_3) + (|y_1|, y_2-y_3)$$
$$= (|x_1|+|y_1|, x_2-x_3+y_2-y_3)$$

$$\therefore T(u_1)+T(u_2) = (|x_1|+|y_1|, x_2+y_2-x_3-y_3) \quad \text{———} \quad (2)$$

From (1) & (2)

$$T(u_1+u_2) \neq T(u_1)+T(u_2)$$

Hence T is not a <u>linear</u> transformation from R^3 to R^2.

(iii) $T(x_1, x_2, x_3) = (x_1+1, x_2+x_3)$

<u>Sol.</u> Given transformation is

$T(x_1, x_2, x_3) = (x_1+1, x_2+x_3)$

let $u_1 = (x_1, x_2, x_3)$

& $u_2 = (y_1, y_2, y_3) \in R^3$ then we prove

(i) $T(u_1+u_2) = T(u_1) + T(u_2)$

Now

$$T(u_1+u_2) = T((x_1, x_2, x_3) + (y_1, y_2, y_3))$$

$$T(u_1+u_2) = T(x_1+y_1, x_2+y_2, x_3+y_3)$$
$$= (x_1+y_1+1, x_2+y_2+x_3+y_3)$$
$$T(u_1+u_2) = (x_1+y_1+1, x_2+x_3+y_2+y_3) \quad \text{——— (1)}$$

Now

$$T(u_1)+T(u_2) = T(x_1,x_2,x_3) + T(y_1,y_2,y_3)$$
$$= (x_1+1, x_2+x_3) + (y_1+1, y_2+y_3)$$
$$= (x_1+1+y_1+1, x_2+x_3+y_2+y_3)$$
$$T(u_1)+T(u_2) = (x_1+y_1+2, x_2+x_3+y_2+y_3) \quad \text{——— (2)}$$

From (1) & (2)

$$T(u_1+u_2) \neq T(u_1)+T(u_2)$$

Hence T is not a linear transformation from R^3 to R^2.

(iv) $T(x_1,x_2,x_3) = (0, x_3)$

Sol. Given transformation is

$$T(x_1,x_2,x_3) = (0, x_3)$$

Let $u_1 = (x_1,x_2,x_3)$

& $u_2 = (y_1,y_2,y_3) \in R^3$ then we prove.

(i) $T(u_1+u_2) = T(u_1)+T(u_2)$

Now

$$T(u_1+u_2) = T((x_1,x_2,x_3)+(y_1,y_2,y_3))$$
$$= T(x_1+y_1, x_2+y_2, x_3+y_3)$$
$$= (0, x_3+y_3)$$
$$= (0,x_3)+(0,y_3)$$
$$= T(x_1,x_2,x_3)+T(y_1,y_2,y_3)$$
$$\therefore T(u_1+u_2) = T(u_1)+T(u_2)$$

(ii) Let $a \in R$ & $u_1 = (x_1,x_2,x_3) \in R^3$ then we prove

$$T(au_1) = aT(u_1)$$

Now $T(au_1) = T(a(x_1,x_2,x_3))$

$$T(au_1) = T(ax_1, ax_2, ax_3)$$
$$= (0, ax_3)$$
$$= a(0, x_3)$$
$$= aT(x_1, x_2, x_3)$$
$$T(au_1) = aT(u_1)$$

Hence T is a linear transformation from R^3 to R^2.

(v) $T(x_1, x_2, x_3) = \left(\frac{x_1+x_2}{x_3}, x_3\right)$

<u>Sol.</u> Given transformation is

$$T(x_1, x_2, x_3) = \left(\frac{x_1+x_2}{x_3}, x_3\right)$$

Let $u_1 = (x_1, x_2, x_3)$

& $u_2 = (y_1, y_2, y_3) \in R^3$ then we prove

(i) $T(u_1+u_2) = T(u_1) + T(u_2)$

Now

$$T(u_1+u_2) = T\big((x_1, x_2, x_3) + (y_1, y_2, y_3)\big)$$
$$= T(x_1+y_1, x_2+y_2, x_3+y_3)$$
$$= \left(\frac{x_1+y_1+x_2+y_2}{x_3+y_3}, x_3+y_3\right) \text{ ——— } (1)$$

Now

$$T(u_1) + T(u_2) = T(x_1, x_2, x_3) + T(y_1, y_2, y_3)$$
$$= \left(\frac{x_1+x_2}{x_3}, x_3\right) + \left(\frac{y_1+y_2}{y_3}, y_3\right)$$
$$= \left(\frac{x_1+x_2}{x_3} + \frac{y_1+y_2}{y_3}, x_3+y_3\right)$$
$$= \left(\frac{y_3(x_1+x_2)+x_3(y_1+y_2)}{x_3 y_3}, x_3+y_3\right) \text{ ——— } (2)$$

From (1) & (2)

$$T(u_1+u_2) \neq T(u_1) + T(u_2)$$

Hence T is not a linear transformation from R^3 to R^2.

(vi) $T(x_1,x_2,x_3) = (3x_1-2x_2+x_3,\ x_3-3x_2-2x_1)$

Sol. Given transformation is

$T(x_1,x_2,x_3) = (3x_1-2x_2+x_3,\ x_3-3x_2-2x_1)$

Let $u_1 = (x_1,x_2,x_3)$

& $u_2 = (y_1,y_2,y_3) \in R^3$ then we prove

(i) $T(u_1+u_2) = T(u_1)+T(u_2)$

Now

$$\begin{aligned}
T(u_1+u_2) &= T((x_1,x_2,x_3)+(y_1,y_2,y_3)) \\
&= T(x_1+y_1,\ x_2+y_2,\ x_3+y_3) \\
&= (3(x_1+y_1)-2(x_2+y_2)+(x_3+y_3),\ (x_3+y_3)-3(x_2+y_2)-2(x_1+y_1)) \\
&= (3x_1-2x_2+x_3+3y_1-2y_2+y_3,\ x_3-3x_2-2x_1+y_3-3y_2-2y_1) \\
&= (3x_1-2x_2+x_3,\ x_3-3x_2-2x_1)+(3y_1-2y_2+y_3,\ y_3-3y_2-2y_1) \\
&= T(x_1,x_2,x_3)+T(y_1,y_2,y_3) \\
&= T(u_1)+T(u_2)
\end{aligned}$$

(ii) Let $a \in R$ & $u_1 = (x_1,x_2,x_3) \in R^3$ then we prove

$T(au_1) = aT(u_1)$

$$\begin{aligned}
\text{Now } T(au_1) &= T(a(x_1,x_2,x_3)) \\
&= T(ax_1,\ ax_2,\ ax_3) \\
&= (3ax_1-2ax_2+ax_3,\ ax_3-3ax_2-2ax_1) \\
&= a(3x_1-2x_2+x_3,\ x_3-3x_2-2x_1) \\
&= aT(x_1,x_2,x_3) \\
&= aT(u_1)
\end{aligned}$$

Hence T is a linear transformation from R^3 to R^2.

Q2 Show that each of the following defines linear transformation from R^3 to R^3.

(i) $T(x_1,x_2,x_3) = (x_1-x_2,\ x_2-x_3,\ x_1)$

Sol. Given transformation is

$T(x_1, x_2, x_3) = (x_1 - x_2, x_2 - x_3, x_1)$

Let $u_1 = (x_1, x_2, x_3)$

& $u_2 = (y_1, y_2, y_3) \in R^3$ then we prove

(i) $T(u_1 + u_2) = T(u_1) + T(u_2)$

Now

$$
\begin{aligned}
T(u_1+u_2) &= T\big((x_1, x_2, x_3) + (y_1, y_2, y_3)\big) \\
&= T(x_1+y_1, x_2+y_2, x_3+y_3) \\
&= \big((x_1+y_1) - (x_2+y_2), (x_2+y_2) - (x_3+y_3), x_1+y_1\big) \\
&= (x_1+y_1-x_2-y_2, x_2+y_2-x_3-y_3, x_1+y_1) \\
&= (x_1-x_2+y_1-y_2, x_2-x_3+y_2-y_3, x_1+y_1) \\
&= (x_1-x_2, x_2-x_3, x_1) + (y_1-y_2, y_2-y_3, y_1) \\
&= T(x_1, x_2, x_3) + T(y_1, y_2, y_3) \\
&= T(u_1) + T(u_2)
\end{aligned}
$$

Now

$$
\begin{aligned}
T(au_1) &= T\big(a(x_1, x_2, x_3)\big) \\
&= T(ax_1, ax_2, ax_3) \\
&= (ax_1 - ax_2, ax_2 - ax_3, ax_1) \\
&= a(x_1 - x_2, x_2 - x_3, x_1) \\
&= aT(x_1, x_2, x_3) \\
&= aT(u_1)
\end{aligned}
$$

Hence T is a linear transformation from R^3 to R^3

(ii) $T(x_1, x_2, x_3) = (x_1 + x_2, -x_1 - x_2, x_3)$

Sol. Given transformation is

$T(x_1, x_2, x_3) = (x_1 + x_2, -x_1 - x_2, x_3)$

let $u_1 = (x_1, x_2, x_3)$

& $u_2 = (y_1, y_2, y_3) \in R^3$ then we prove

(i) $T(u_1 + u_2) = T(u_1) + T(u_2)$

Now

$$
\begin{aligned}
T(u_1+u_2) &= T\big((x_1,x_2,x_3)+(y_1,y_2,y_3)\big)\\
&= T(x_1+y_1, x_2+y_2, x_3+y_3)\\
&= \big(x_1+y_1+x_2+y_2,\ -(x_1+y_1)-(x_2+y_2),\ x_3+y_3\big)\\
&= (x_1+y_1+x_2+y_2,\ -x_1-y_1-x_2-y_2,\ x_3+y_3)\\
&= (x_1+x_2+y_1+y_2,\ -x_1-x_2-y_1-y_2,\ x_3+y_3)\\
&= (x_1+x_2, -x_1-x_2, x_3)+(y_1+y_2, -y_1-y_2, y_3)\\
&= T(x_1,x_2,x_3)+T(y_1,y_2,y_3)\\
&= T(u_1)+T(u_2)
\end{aligned}
$$

(ii) Let $a \in R$ & $u_1 = (x_1,x_2,x_3) \in R^3$ then we prove

$T(au_1) = aT(u_1)$

Now

$$
\begin{aligned}
T(au_1) &= T\big(a(x_1,x_2,x_3)\big)\\
&= T(ax_1, ax_2, ax_3)\\
&= (ax_1+ax_2,\ -ax_1-ax_2,\ ax_3)\\
&= a(x_1+x_2, -x_1-x_2, x_3)\\
&= aT(x_1,x_2,x_3)\\
&= aT(u_1)
\end{aligned}
$$

Hence T is a linear transformation from R^3 to R^3

(iii) $T(x_1,x_2,x_3) = (x_2, -x_1, -x_3)$

Sol. Given transformation is

$T(x_1,x_2,x_3) = (x_2, -x_1, -x_3)$

Let $u_1 = (x_1,x_2,x_3)$

& $u_2 = (y_1,y_2,y_3) \in R^3$ then we prove

(i) $T(u_1+u_2) = T(u_1)+T(u_2)$

Now

$T(u_1+u_2) = T\big((x_1,x_2,x_3)+(y_1,y_2,y_3)\big)$

$$T(u_1+u_2) = T(x_1+y_1, x_2+y_2, x_3+y_3)$$
$$= (x_2+y_2, -(x_1+y_1), -(x_3+y_3))$$
$$= (x_2+y_2, -x_1-y_1, -x_3-y_3).$$
$$= (x_2, -x_1, -x_3) + (y_2, -y_1, -y_3)$$
$$= T(x_1,x_2,x_3) + T(y_1,y_2,y_3)$$
$$= T(u_1) + T(u_2)$$

(ii) Let $a \in R$ & $u_1 = (x_1,x_2,x_3) \in R^3$ then we prove

$$T(au_1) = aT(u_1)$$

Now

$$T(au_1) = T(a(x_1,x_2,x_3))$$
$$= T(ax_1, ax_2, ax_3)$$
$$= (ax_2, -ax_1, -ax_3)$$
$$= a(x_2, -x_1, -x_3)$$
$$= aT(x_1,x_2,x_3)$$
$$= aT(u_1)$$

Hence T is a linear transformation from R^3 to R^3.

(iv) $T(x_1,x_2,x_3) = (x_1-3x_2-2x_3, x_2-4x_3, x_3)$

Sol: Given transformation is

$$T(x_1,x_2,x_3) = (x_1-3x_2-2x_3, x_2-4x_3, x_3)$$

Let $u_1 = (x_1,x_2,x_3)$

& $u_2 = (y_1,y_2,y_3) \in R^3$ then we prove

(i) $T(u_1+u_2) = T(u_1) + T(u_2)$

Now

$$T(u_1+u_2) = T((x_1,x_2,x_3)+(y_1,y_2,y_3))$$
$$= T(x_1+y_1, x_2+y_2, x_3+y_3)$$
$$= ((x_1+y_1)-3(x_2+y_2)-2(x_3+y_3), (x_2+y_2)-4(x_3+y_3), x_3+y_3)$$
$$= (x_1-3x_2-2x_3+y_1-3y_2-2y_3, x_2-4x_3+y_2-4y_3, x_3+y_3)$$

$$T(u_1+u_2) = (x_1-3x_2-2x_3\,,\ x_2-4x_3\,,\ x_3) + (y_1-3y_2-2y_3\,,\ y_2-4y_3\,,\ y_3)$$
$$= T(x_1,x_2,x_3) + T(y_1,y_2,y_3)$$
$$= T(u_1) + T(u_2)$$

(ii) let $\alpha \in R$ & $u_1 = (x_1,x_2,x_3) \in R^3$ then we prove
$T(\alpha u_1) = \alpha T(u_1)$

Now

$$T(\alpha u_1) = T\big(\alpha(x_1,x_2,x_3)\big)$$
$$= T(\alpha x_1, \alpha x_2, \alpha x_3)$$
$$= (3\alpha x_1 - 3\alpha x_2 - 2\alpha x_3\,,\ \alpha x_2 - 4\alpha x_3\,,\ \alpha x_3)$$
$$= \alpha(3x_1-3x_2-2x_3\,,\ x_2-4x_3\,,\ x_3)$$
$$= \alpha T(x_1,x_2,x_3)$$
$$= \alpha T(u_1)$$

Hence T is a linear transformation from R^3 to R^3.

(v) $T(x_1,x_2,x_3) = (x_1+x_3\,,\ x_1-x_3\,,\ x_2)$

Sol. Given Transformation is
$T(x_1,x_2,x_3) = (x_1+x_3\,,\ x_1-x_3\,,\ x_2)$

Let $u_1 = (x_1,x_2,x_3)$
& $u_2 = (y_1,y_2,y_3) \in R^3$ then we prove
$T(u_1+u_2) = T(u_1) + T(u_2)$

Now

$$T(u_1+u_2) = T\big((x_1,x_2,x_3)+(y_1,y_2,y_3)\big)$$
$$= T(x_1+y_1,\ x_2+y_2\,,\ x_3+y_3)$$
$$= \big((x_1+y_1)+(x_3+y_3)\,,\ (x_1+y_1)-(x_3+y_3)\,,\ x_2+y_2\big)$$
$$= (x_1+x_3+y_1+y_3\,,\ x_1-x_3+y_1-y_3\,,\ x_2+y_2)$$
$$= (x_1+x_3\,,\ x_1-x_3\,,\ x_2) + (y_1+y_3\,,\ y_1-y_3\,,\ y_2)$$
$$= T(x_1,x_2,x_3) + T(y_1,y_2,y_3)$$
$$= T(u_1) + T(u_2)$$

(ii) Let $a \in R$ & $u_1 = (x_1, x_2, x_3) \in R^3$ then we prove
$T(au_1) = aT(u_1)$

Now

$$T(au_1) = T(a(x_1, x_2, x_3))$$
$$= T(ax_1, ax_2, ax_3)$$
$$= (ax_1 + ax_3, ax_1 - ax_3, ax_2)$$
$$= a(x_1 + x_3, x_1 - x_3, x_2)$$
$$= aT(x_1, x_2, x_3)$$
$$= aT(u_1)$$

Hence T is a linear transformation from R^3 to R^3.

Q3 Show that each of the following transformations is not linear.

(i) $T : R^2 \longrightarrow R$ defined by $T(x_1, x_2) = x_1 x_2$

Sol. Given transformation is

$$T(x_1, x_2) = x_1 x_2$$

let $u_1 = (x_1, x_2)$

& $u_2 = (y_1, y_2) \in R^2$ then we prove

(i) $T(u_1 + u_2) = T(u_1) + T(u_2)$

Now

$$T(u_1 + u_2) = T((x_1, x_2) + (y_1, y_2))$$
$$= T(x_1 + y_1, x_2 + y_2)$$
$$= (x_1 + y_1)(x_2 + y_2) \quad \text{———} (1)$$

&

$$T(u_1) + T(u_2) = T(x_1, x_2) + T(y_1, y_2)$$
$$= x_1 x_2 + y_1 y_2 \quad \text{———} (2)$$

from (1) & (2) $T(u_1 + u_2) \neq T(u_1) + T(u_2)$.

Hence T is not a linear transformation from R^2 to R

(ii) $T: R^2 \longrightarrow R^3$ defined by $T(x_1, x_2) = (x_1+1, 2x_2, x_1+x_2)$

Sol: Given transformation is

$T(x_1, x_2) = (x_1+1, 2x_2, x_1+x_2)$

Let $u_1 = (x_1, x_2)$

& $u_2 = (y_1, y_2) \in R^2$ then we prove

(i) $T(u_1+u_2) = T(u_1) + T(u_2)$

Now

$$T(u_1+u_2) = T((x_1, x_2) + (y_1, y_2))$$
$$= T(x_1+y_1, x_2+y_2)$$
$$= (x_1+y_1+1, 2(x_2+y_2), (x_1+y_1)+(x_2+y_2))$$
$$= (x_1+y_1+1, 2x_2+2y_2, x_1+x_2+y_1+y_2) \text{ ——— } ①$$

$$T(u_1)+T(u_2) = T(x_1, x_2) + T(y_1, y_2)$$
$$= (x_1+1, 2x_2, x_1+x_2) + (y_1+1, 2y_2, y_1+y_2)$$
$$= (x_1+y_1+2, 2x_2+2y_2, x_1+x_2+y_1+y_2) \text{ ——— } ②$$

from ① & ②

$$T(u_1+u_2) \neq T(u_1) + T(u_2)$$

Hence T is not a linear transformation from R^2 to R^3.

(iii) $T: R^3 \longrightarrow R^2$ defined by $T(x_1, x_2, x_3) = (|x_1|, 0)$

Sol: Given transformation is

$T(x_1, x_2, x_3) = (|x_1|, 0)$

Let $u_1 = (x_1, x_2, x_3)$

& $u_2 = (y_1, y_2, y_3) \in R^3$ then we prove

(i) $T(u_1+u_2) = T(u_1) + T(u_2)$

Now

$$T(u_1+u_2) = T((x_1, x_2, x_3) + (y_1, y_2, y_3))$$
$$= T(x_1+y_1, x_2+y_2, x_3+y_3)$$

$T(u_1+u_2) = (|x_1+y_1|, 0)$ ——— ①

&

$$\begin{aligned} T(u_1)+T(u_2) &= T(x_1,x_2,x_3) + T(y_1,y_2,y_3) \\ &= (|x_1|, 0) + (|y_1|, 0) \\ &= (|x_1|+|y_1|, 0) \quad \text{———} ② \end{aligned}$$

From ① & ②

$T(u_1+u_2) \neq T(u_1)+T(u_2)$

Hence T is not a linear transformation from R^3 to R^2.

(iv) $T: R^2 \longrightarrow R^2$ defined by $T(x_1,x_2) = (x_1^2, x_2^2)$.

Sol. Given transformation is

$T(x_1,x_2) = (x_1^2, x_2^2)$

Let $u_1 = (x_1,x_2)$

& $u_2 = (y_1,y_2) \in R^2$ then we prove

(i) $T(u_1+u_2) = T(u_1)+T(u_2)$

Now

$$\begin{aligned} T(u_1+u_2) &= T((x_1,x_2)+(y_1,y_2)) \\ &= T(x_1+y_1, x_2+y_2) \\ &= ((x_1+y_1)^2, (x_2+y_2)^2) \quad \text{———} \end{aligned}$$

&

$$\begin{aligned} T(u_1)+T(u_2) &= T(x_1,x_2) + T(y_1,y_2) \\ &= (x_1^2, x_2^2) + (y_1^2, y_2^2) \\ &= (x_1^2+y_1^2, x_2^2+y_2^2) \quad \text{———} ② \end{aligned}$$

from ① & ②

$T(u_1+u_2) \neq T(u_1)+T(u_2)$

Hence T is not a linear transformation from R^2 to R^2

(v) $T : R^3 \longrightarrow R^3$ defined by $T(x_1, x_2, x_3) = (x_1, x_2, x_3) + (1,1,1)$

Sol. Given transformation is

$$T(x_1, x_2, x_3) = (x_1, x_2, x_3) + (1,1,1)$$

Let $u_1 = (x_1, x_2, x_3)$

& $u_2 = (y_1, y_2, y_3) \in R^3$ then we prove

(i) $T(u_1+u_2) = T(u_1) + T(u_2)$

Now

$$\begin{aligned} T(u_1+u_2) &= T\big((x_1,x_2,x_3)+(y_1,y_2,y_3)\big) \\ &= T(x_1+y_1,\ x_2+y_2,\ x_3+y_3) \\ &= (x_1+y_1,\ x_2+y_2,\ x_3+y_3) + (1,1,1) \\ &= (x_1+y_1+1,\ x_2+y_2+1,\ x_3+y_3+1) \quad \text{——— (1)} \end{aligned}$$

&

$$\begin{aligned} T(u_1)+T(u_2) &= T(x_1,x_2,x_3) + T(y_1,y_2,y_3) \\ &= (x_1,x_2,x_3)+(1,1,1)+(y_1,y_2,y_3)+(1,1,1) \\ &= (x_1+1,\ x_2+1,\ x_3+1) + (y_1+1,\ y_2+1,\ y_3+1) \\ &= (x_1+y_1+2,\ x_2+y_2+2,\ x_3+y_3+2) \quad \text{——— (2)} \end{aligned}$$

from (1) & (2)

$$T(u_1+u_2) \neq T(u_1) + T(u_2)$$

Hence T is not a linear transformation from R^3 to R^3.

Q3 Determine which of the following transformations are linear :

(a) $T : M_{22} \longrightarrow R$ defined by

(i) $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a+d$

Sol. Given transformation is

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a+d \quad \ldots$$

Let $A_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$

& $A_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in M_{22}$ then we prove

(i) $T(A_1 + A_2) = T(A_1) + T(A_2)$

Now

$$T(A_1 + A_2) = T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)$$

$$= T\left(\begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}\right)$$

$$= a_1 + a_2 + d_1 + d_2 \quad \text{———} \quad ①$$

&

$$T(A_1) + T(A_2) = T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)$$

$$= a_1 + d_1 + a_2 + d_2$$

$$= a_1 + a_2 + d_1 + d_2 \quad \text{———} \quad ②$$

from ① & ②

$$T(A_1 + A_2) = T(A_1) + T(A_2)$$

(ii) Let $a \in R$ & $A_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \in M_{22}$ then we prove

$$T(aA_1) = aT(A_1)$$

Now $T(aA_1) = T\left(a\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right)$

$$= T\left(\begin{bmatrix} aa_1 & ab_1 \\ ac_1 & ad_1 \end{bmatrix}\right)$$

$$= aa_1 + ad_1$$

$$= a(a_1 + d_1)$$

$$= aT\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right)$$

$$= aT(A_1)$$

Hence T is a linear transformation from M_{22} to R

(ii) $T: M_{22} \longrightarrow R$ defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

Sol. Given transformation is

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

Let $A_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$

& $A_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$

then we prove

$$T(A_1 + A_2) = T(A_1) + T(A_2)$$

Now

$$T(A_1 + A_2) = T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)$$

$$= T\left(\begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}\right)$$

$$= (a_1+a_2)(d_1+d_2) - (b_1+b_2)(c_1+c_2) \quad \text{———— } ①$$

Now

$$T(A_1) + T(A_2) = T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)$$

$$= a_1d_1 - b_1c_1 + a_2d_2 - b_2c_2 \quad \text{———— } ②$$

from ① & ②

$$\therefore T(A_1 + A_2) \neq T(A_1) + T(A_2)$$

Hence T is a linear transformation from M_{22} to R.

(b) $T: P_2(x) \longrightarrow P_2(x)$ defined by

(i) $T(a + bx + cx^2) = a + (b+c)x + (2a - 3b)x^2$

Sol. Given transformation is

$$T(a + bx + cx^2) = a + (b+c)x + (2a - 3b)x^2$$

Let $u = a+bx+cx^2$

& $v = p+qx+\lambda x^2 \in P_2(x)$ then we prove

(i) $T(u+v) = T(u)+T(v)$

Now

$$
\begin{aligned}
T(u+v) &= T\left((a+bx+cx^2)+(p+qx+\lambda x^2)\right)\\
&= T\left((a+p)+(b+q)x+(c+\lambda)x^2\right)\\
&= (a+p)+(b+q+c+\lambda)x+(2a+2p-3b-3q)x^2\\
&= (a+p)+(b+c+q+\lambda)x+(2a-3b+2p-3q)x^2\\
&= \left(a+(b+c)x+(2a-3b)x^2\right)+\left(p+(q+\lambda)x+(2p-3q)x^2\right)\\
&= T(a+bx+cx^2)+T(p+qx+\lambda x^2)\\
&= T(u)+T(v)
\end{aligned}
$$

(ii) Let $k \in R$ & $u = a+bx+cx^2$ then we prove

$T(ku) = kT(u)$

Now

$$
\begin{aligned}
T(ku) &= T\left(k(a+bx+cx^2)\right)\\
&= T\left(ka+kbx+kcx^2\right)\\
&= ka+(kb+kc)x+(2ka-3kb)x^2\\
&= k\left(a+(b+c)x+(2a-3b)x^2\right)\\
&= kT(a+bx+cx^2)\\
&= kT(u)
\end{aligned}
$$

Hence T is a linear transformation from $P_2(x)$ to $P_2(x)$.

(iii). $T: P_2(x) \longrightarrow P_2(x)$ defined by

$T(a+bx+cx^2) = (a+1)+bx+cx^2$

Sol. Given transformation is

$T(a+bx+cx^2) = (a+1)+bx+cx^2$

Let $u = a+bx+cx^2$

& $v = p+qx+\lambda x^2 \in P_2(x)$ then we prove

(i) $T(u+v) = T(u) + T(v)$

Now

$$T(u+v) = T\big((a+bx+cx^2)+(p+qx+rx^2)\big)$$
$$= T\big((a+p)+(b+q)x+(c+r)x^2\big)$$
$$= (a+p+1)+(b+q)x+(c+r)x^2 \quad \text{——— (1)}$$

&

$$T(u)+T(v) = T(a+bx+cx^2)+T(p+qx+rx^2)$$
$$= (a+1)+bx+cx^2+(p+1)+qx+rx^2$$
$$= (a+p+2)+(b+q)x+(c+r)x^2 \quad \text{——— (2)}$$

From (1) & (2)

$$T(u+v) \neq T(u)+T(v)$$

Hence T is not a linear transformation from $P_2(x)$ to $P_2(x)$.

Q5. If A is an $m\times n$ matrix, show that $T(x) = Ax$ is a linear transformation from R^n to R^m

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in R^n$$

Sol:- Given transformation is

$$T(x) = Ax$$

Suppose

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \& \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \in R^n$$

Then we prove

$$T(x+y) = T(x)+T(y)$$

Now

$T(x+y) =$

Q5 Determine which of the following linear transformations are one-to-one:

(i) $T: R^2 \longrightarrow R^3$ defined by $T(x_1, x_2) = (x_1 + x_2, x_1 - x_2, x_1 + 2x_2)$

Sol: Given linear transformation is

$$T(x_1, x_2) = (x_1 + x_2, x_1 - x_2, x_1 + 2x_2)$$

Let $x = (x_1, x_2)$

& $y = (y_1, y_2) \in R^2$

Then $T(x) = (x_1 + x_2, x_1 - x_2, x_1 + 2x_2)$

& $T(y) = (y_1 + y_2, y_1 - y_2, y_1 + 2y_2)$

Suppose $T(x) = T(y)$

$\Rightarrow (x_1 + x_2, x_1 - x_2, x_1 + 2x_2) = (y_1 + y_2, y_1 - y_2, y_1 + 2y_2)$

$$\Rightarrow \quad x_1 + x_2 = y_1 + y_2 \quad \text{——— (1)}$$

$$x_1 - x_2 = y_1 - y_2 \quad \text{——— (2)}$$

$$x_1 + 2x_2 = y_1 + 2y_2 \quad \text{——— (3)}$$

Adding (1) & (2)

$$2x_1 = 2y_1$$

$$\boxed{x_1 = y_1}$$

Put in (1)

$$x_1 + x_2 = x_1 + y_2$$

$$\Rightarrow \boxed{x_2 = y_2}$$

Hence $(x_1, x_2) = (y_1, y_2)$

or $x = y$

Hence $T(x) = T(y) \Rightarrow x = y$

Hence T is one-to-one

(ii) $T: R^3 \longrightarrow R^2$ defined by $T(x_1, x_2, x_3) = (x_1 - x_2, x_3)$

Sol: Given linear transformation is

$$T(x_1, x_2, x_3) = (x_1 - x_2, x_3)$$

Let $x = (x_1, x_2, x_3)$

& $y = (y_1, y_2, y_3) \in R^3$

Then $T(x) = (x_1 - x_2, x_3)$

& $T(y) = (y_1 - y_2, y_3)$

Suppose $T(x) = T(y)$

$\Rightarrow (x_1 - x_2, x_3) = (y_1 - y_2, y_3)$

$\Rightarrow \quad x_1 - x_2 = y_1 - y_2$ ——— ①

$x_3 = y_3$ ——— ②

From ① we cannot conclude that

$x_1 = y_1$ & $x_2 = y_2$

Hence $T(x) = T(y) \not\Rightarrow x = y$

So T is not one-to-one.

(iii) $T: R^2 \longrightarrow R^3$ defined by $T(x_1, x_2) = (x_1, x_1 + x_2, x_1 - x_2)$

<u>Sol.</u> Given linear transformation is

$T(x_1, x_2) = (x_1, x_1 + x_2, x_1 - x_2)$

Let $x = (x_1, x_2)$

& $y = (y_1, y_2) \in R^2$

Then $T(x) = (x_1, x_1 + x_2, x_1 - x_2)$

& $T(y) = (y_1, y_1 + y_2, y_1 - y_2)$

Suppose $T(x) = T(y)$

$\Rightarrow (x_1, x_1 + x_2, x_1 - x_2) = (y_1, y_1 + y_2, y_1 - y_2)$

or $\quad x_1 = y_1$ ——— ①

$x_1 + x_2 = y_1 + y_2$ ——— ②

$x_1 - x_2 = y_1 - y_2$ ——— ③

① $\Rightarrow \boxed{x_1 = y_1}$

Subt. ② & ③

$2x_2 = 2y_2 \quad \Rightarrow \boxed{x_2 = y_2}$

Hence $(x_1, x_2) = (y_1, y_2)$

or $x = y$

So $T(x) = T(y) \Rightarrow x = y$

Hence T is one-to-one.

Q7 Let C be the vector space of Complex number over the field of reals & $T : C \longrightarrow C$ be defined by $T(z) = \bar{z}$ where $\bar{z}$ denotes the Complex Conjugate of z. Show that T is linear.

Sol. Given transformation is

$T(z) = \bar{z}$

Let $z_1, z_2 \in C$ then we prove

(i) $T(z_1 + z_2) = T(z_1) + T(z_2)$

Now

$$T(z_1 + z_2) = \overline{z_1 + z_2}$$
$$= \bar{z}_1 + \bar{z}_2$$
$$= T(z_1) + T(z_2)$$

(ii) Let $a \in R$ & $z_1 \in C$ then we prove

$T(az_1) = aT(z_1)$

Now

$$T(az_1) = \overline{az_1}$$
$$= a\bar{z}_1 \qquad \because a \in R$$
$$= aT(z_1)$$

Hence T is a linear transformation from C to C.

Q8 Let V be the vector space $P_n(x)$ of polynomials $p(x)$ with real cofficients & of degree not exceeding n together with the zero polynomial. Let $T : V \longrightarrow V$

be defined by $T(p(x)) = p(x+1)$

Show that T is linear.

Sol. Given transformation is

$$T(p(x)) = p(x+1)$$

Let $p_1(x) = a_0 + a_1x + a_2x^2 + \dots\dots + a_nx^n$

& $p_2(x) = b_0 + b_1x + b_2x^2 + \dots\dots + b_nx^n \in V$

then we prove $T(p_1(x) + p_2(x)) = T(p_1(x)) + T(p_2(x))$

Now

$$\begin{aligned}
T(p_1(x)+p_2(x)) &= T((a_0+a_1x+\dots\dots+a_nx^n)+(b_0+b_1x+\dots\dots+b_nx^n)) \\
&= T((a_0+b_0)+(a_1+b_1)x+\dots\dots+(a_n+b_n)x^n) \\
&= (a_0+b_0)+(a_1+b_1)(x+1)+\dots\dots+(a_n+b_n)(x+1)^n \\
&= [a_0+a_1(x+1)+\dots\dots+a_n(x+1)^n]+[b_0+b_1(x+1)+\dots\dots+b_n(x+1)^n] \\
&= T(a_0+a_1x+\dots\dots+a_nx^n)+T(b_0+b_1x+\dots\dots+b_nx^n) \\
&= T(p_1(x))+T(p_2(x))
\end{aligned}$$

(ii) Let $a \in R$ & $p_1(x) = a_0 + a_1x + \dots\dots + a_nx^n$

then we prove $T(ap_1(x)) = aT(p_1(x))$

Now

$$\begin{aligned}
T(ap_1(x)) &= T(a(a_0+a_1x+\dots\dots+a_nx^n)) \\
&= T(aa_0+aa_1x+\dots\dots+aa_nx^n) \\
&= aa_0+aa_1(x+1)+\dots\dots+aa_n(x+1)^n \\
&= a(a_0+a_1(x+1)+\dots\dots+a_n(x+1)^n) \\
&= aT(a_0+a_1x+\dots\dots+a_nx^n) \\
&= aT(p_1(x))
\end{aligned}$$

Hence T is a linear transformation from V to V.

Q9 Let $V_1 = (1,1,1)$, $V_2 = (1,1,0)$ & $V_3 = (1,0,0)$ be a basis for R^3. Find a linear transformation $T: R^3 \longrightarrow R^2$ s.t. $T(V_1) = (1,0)$, $T(V_2) = (2,-1)$ & $T(V_3) = (4,3)$

Sol. Let $x = (x_1, x_2, x_3)$ be any vector of R^3 then for scalars a_1, a_2, a_3

$$x = a_1V_1 + a_2V_2 + a_3V_3$$

$$= a_1(1,1,1) + a_2(1,1,0) + a_3(1,0,0)$$

$$(x_1, x_2, x_3) = (a_1 + a_2 + a_3,\ a_1 + a_2,\ a_1)$$

$$\Rightarrow \quad a_1 + a_2 + a_3 = x_1 \quad \text{———} (1)$$

$$a_1 + a_2 = x_2 \quad \text{———} (2)$$

$$a_1 = x_3 \quad \text{———} (3)$$

$(3) \Rightarrow \boxed{a_1 = x_3}$

Put in (2)

$$x_3 + a_2 = x_2 \Rightarrow \boxed{a_2 = x_2 - x_3}$$

Put in (1)

$$\cancel{x_3} + x_2 - \cancel{x_3} + a_3 = x_1$$

$$\boxed{a_3 = x_1 - x_2}$$

So

$$x = x_3V_1 + (x_2 - x_3)V_2 + (x_1 - x_2)V_3$$

Applying T on both sides

$$T(x) = T(x_3V_1 + (x_2 - x_3)V_2 + (x_1 - x_2)V_3$$

$$= x_3T(V_1) + (x_2 - x_3)T(V_2) + (x_1 - x_2)T(V_3) \qquad (\text{As } T \text{ is linear})$$

$$= x_3(1,0) + (x_2 - x_3)(2,-1) + (x_1 - x_2)(4,3)$$

$$= (x_3, 0) + (2x_2 - 2x_3, -x_2 + x_3) + (4x_1 - 4x_2, 3x_1 - 3x_2)$$

$$= (x_3 + 2x_2 - 2x_3 + 4x_1 - 4x_2,\ -x_2 + x_3 + 3x_1 - 3x_2)$$

$$T(x_1, x_2, x_3) = (4x_1 - 2x_2 - x_3,\ 3x_1 - 4x_2 + x_3)$$

Which is req. linear transformation from R^3 to R^2

Q10 Let $T : R^2 \to R$ be the linear transformation for which $T(1,1) = 3$ & $T(0,1) = -2$. Find $T(x_1, x_2)$

Sol. First we prove that the vectors $(1,1)$ & $(0,1)$ form a basis for R^2.

Suppose for scalars $a, b \in R$

$$a(1,1) + b(0,1) = 0$$
$$(a,a) + (0,b) = 0$$
$$(a, a+b) = 0$$
$$\Rightarrow a = 0 \quad \text{---} \ (1)$$
$$a+b = 0 \quad \text{---} \ (2)$$

(1) $\Rightarrow$ $\boxed{a = 0}$

(2) $\Rightarrow$ $\boxed{b = 0}$

Hence vectors $(1,1)$ & $(0,1)$ are linearly independent.

As there are two linearly independent vectors in R^2

So $(1,1)$ & $(0,1)$ form a basis for R^2

Suppose $(x_1, x_2) \in R^2$ be an arbitrary vector

then $(x_1, x_2) = a(1,1) + b(0,1)$ where $a, b \in R$

or $(x_1, x_2) = (a, a+b)$

$$\Rightarrow a = x_1 \quad \text{---} \ (1)$$
$$a+b = x_2 \quad \text{---} \ (2)$$

(1) $\Rightarrow$ $\boxed{a = x_1}$

Put in (2)

$x_1 + b = x_2 \Rightarrow \boxed{b = x_2 - x_1}$

So

$$(x_1, x_2) = x_1(1,1) + (x_2 - x_1)(0,1)$$

Applying T on both sides

$$T(x_1,x_2) = T\left(x_1(1,1) + (x_2-x_1)(0,1)\right)$$
$$= x_1 T(1,1) + (x_2 - x_1) T(0,1) \qquad (\because T \text{ is linear})$$
$$= x_1(3) + (x_2 - x_1)(-2)$$
$$= 3x_1 - 2x_2 + 2x_1$$

$T(x_1,x_2) = 5x_1 - 2x_2$ which is T in terms of co-ords.

Q11 Let $D: P_2(x) \longrightarrow P_2(x)$ be the differentiation operator & $D(p(x)) = p'(x)$ for all $p(x) \in P_2(x)$. Find $N(D)$.

Sol. Given operator is

$$D(p(x)) = p'(x)$$

Here $N(D)$ will consist of those polynomials in $P_2(x)$ for which $D(p(x)) = 0$

Since we know that

$$D(p(x)) = 0 \quad \text{if} \quad p(x) = \text{Const polynomial}$$

So $N(D)$ will consist of all Const. polynomials.

Q12 Define $T: R^3 \longrightarrow R^3$ by $T(x_1, x_2, x_3) = (-x_3, x_1, x_1 + x_3)$. Find $N(T)$. Is T one-to-one?

Sol. Given transformation is

$$T(x_1, x_2, x_3) = (-x_3, x_1, x_1 + x_3)$$

Here $N(T) = \{(x_1, x_2, x_3) \in R^3 : T(x_1, x_2, x_3) = (0, 0, 0)\}$

Now $T(x_1, x_2, x_3) = (0, 0, 0)$

$\Rightarrow (-x_3, x_1, x_1 + x_3) = (0, 0, 0)$

$$\Rightarrow \quad -x_3 = 0 \quad \text{--- (1)}$$
$$x_1 = 0 \quad \text{--- (2)}$$
$$x_1 + x_3 = 0 \quad \text{--- (3)}$$

(1) $\Rightarrow x_3 = 0$

(2) $\Rightarrow x_1 = 0$

Which shows that $N(T)$ will consist of all vectors of the form $(0, x_2, 0)$. Which is x_2-axis.

i.e., $N(T) = \{(0, x_2, 0) \in R^3 : x_2 \in R\}$

Since $N(T) = (0, x_2, 0) \neq (0, 0, 0)$. So T is not one-to-one.

Q13 Suppose U, V & W are vector spaces over the same field F. Let $T : U \to V$ & $S : V \to W$ be linear transformations. The transformation $S \circ T : U \to W$ is defined by $(S\circ T)(u) = S(T(u))$ for all $u \in U$. Show that $S\circ T$ is a linear transformation.

Sol:

Here $S\circ T : U \to W$ be defined as

$$(S\circ T)(u) = S(T(u)) \quad \text{for all } u \in U$$

Let $u_1, u_2 \in U$ then we prove

$$(S\circ T)(u_1+u_2) = (S\circ T)(u_1) + (S\circ T)(u_2)$$

Now

$$(S\circ T)(u_1+u_2) = S(T(u_1+u_2)) \qquad \text{By def. of } S\circ T$$
$$= S(T(u_1)+T(u_2)) \qquad \because T \text{ is linear}$$
$$= S(T(u_1)) + S(T(u_2)) \qquad \because S \text{ is linear}$$
$$= (S\circ T)(u_1) + (S\circ T)(u_2)$$

(ii) Let $a \in F$ & $u \in U$ then we prove

$$(S\circ T)(au) = a(S\circ T)(u)$$

Now

$$(S\circ T)(au) = S(T(au)) \qquad \text{By def of } S\circ T$$
$$= S(aT(u)) \qquad \because T \text{ is linear}$$
$$= aS(T(u)) \qquad \because S \text{ is linear}$$
$$= a(S\circ T)(u)$$

Hence $S\circ T$ is a linear transformation from U to W.

Q14 Let U & V be two vector spaces over the same field F. Denote the set of all linear transformations from U into V by $L(U,V)$. Show that $L(U,V)$ is a vector space over F with vector space operations as defined in example 31

Sol: Consider the set $L(U,V)$. Let $S,T \in L(U,V)$ then $S: U \to V$ & $T: U \to V$ be two linear transformations. Define

$S+T : U \to V$ & $aS : U \to V$ by

$$(S+T)(u) = S(u) + T(u)$$

$$(aS)(u) = aS(u) \qquad \text{for all } u \in U \text{ \& } a \in F$$

First we show that $L(U,V)$ is an abelian gr. under +.

(i) Closure law

Let $S,T \in L(U,V)$, then we show $S+T \in L(U,V)$.

Now $(S+T)(u_1+u_2) = S(u_1+u_2) + T(u_1+u_2)$ By def. of $S+T$

$= S(u_1) + S(u_2) + T(u_1) + T(u_2)$ $\because$ S,T are linear

$= S(u_1) + T(u_1) + S(u_2) + T(u_2)$

$= (S+T)(u_1) + (S+T)(u_2)$

Let $K \in F$ & $u \in U$

$(S+T)(Ku) = S(Ku) + T(Ku)$

$= KS(u) + KT(u)$ $\because$ S,T are linear

$= K\big(S(u) + T(u)\big)$

$= K(S+T)(u)$

Hence $S+T$ is linear & so $S+T \in L(U,V)$.

(ii) Associative law

Let $R,S,T \in L(U,V)$ then we prove

$$R + (S+T) = (R+S) + T$$

Now Consider for $u \in U$

$[R+(S+T)](u) = R(u) + (S+T)(u)$ (By def. of sum)

$= R(u) + [S(u) + T(u)]$ " " "

$= [R(u) + S(u)] + T(u)$ ∵ $R(u), S(u), T(u) \in F$

$= [(R+S)(u)] + T(u)$

$= [(R+S)(u) + T(u)]$

$= [(R+S)+T](u)$

$\Rightarrow R+(S+T) = (R+S)+T$

So $+$ is associative in $L(U,V)$.

(iii) Identity law

Clearly the zero transformation $\underline{O}$ defined by

$\underline{O}(u) = 0$ for all $u \in U$

is a linear transformation from U to V & it is the additive identity in $L(U,V)$

(iv) Inverse law

For each $T \in L(U,V)$, we define

$-T \in L(U,V)$ by

$(-T)(u) = -T(u)$

then $-T$ is the additive inverse of T.

(v) Commutative law

Let $S, T \in L(U,V)$ then we show $S+T = T+S$

Now Consider

$(S+T)(u) = S(u) + T(u)$ By def. of sum

$= T(u) + S(u)$ ∵ $S(u), T(u) \in F$

$= (T+S)(u)$

$\Rightarrow S+T = T+S$

Hence $+$ is commutative in $L(U,V)$

So $L(U,V)$ is an abelian gr. under $+$.

Now we check scalar multiplication axioms.

(i) Let $a \in F$ & $S \in L(U,V)$ then we prove $aS \in L(U,V)$.

Now
$$(aS)(u_1+u_2) = a[S(u_1+u_2)]$$
$$= a[S(u_1)+S(u_2)] \qquad \because S \text{ is linear}$$
$$= aS(u_1) + aS(u_2)$$

Suppose $K \in F$ & $u \in U$ then
$$(aS)(Ku) = a[S(Ku)]$$
$$= a[KS(u)] \qquad \because S \text{ is linear}$$
$$= (aK)S(u)$$
$$= (Ka)S(u)$$
$$= K(aS)(u)$$

Hence aS is linear & so $aS \in L(U,V)$.

(ii) Let $a,b \in F$ & $S \in L(U,V)$ then we prove $a(bS) = (ab)S$

Now
$$[a(bS)](u) = a.(bS)(u)$$
$$= a[b.S(u)]$$
$$= (ab).S(u)$$
$$= [(ab)S](u)$$

$$\Rightarrow \quad a(bS) = (ab)S$$

(iii) Let $a,b \in F$ & $S \in L(U,V)$ then we prove $(a+b)S = aS+bS$

Now
$$[(a+b)S](u) = (a+b).S(u)$$
$$= a.S(u) + b.S(u)$$
$$= (aS)(u) + (bS)(u)$$
$$= [aS+bS](u)$$

(iv) Let $a \in F$ & $S,T \in L(U,V)$ then we prove $a(S+T) = aS+aT$

Now
$$[a(S+T)](u) = a[(S+T)(u)]$$
$$= a[S(u) + T(u)]$$
$$= a.S(u) + a.T(u)$$
$$= (aS)(u) + (aT)(u)$$

So $[\alpha(S+T)](u) = [\alpha S + \alpha T](u)$

$\Rightarrow \quad \alpha(S+T) = \alpha S + \alpha T$

(v) Let $1 \in F$ & $S \in L(U,V)$ then we prove $1.S = S$

Now $(1.S)(u) = 1.S(u)$

$= S(u)$

$\Rightarrow \quad 1.S = S$

Since all the Conditions are satisfied. So $L(U,V)$ is a vector space over F.

Q15 Find a basis & dimension of each of $R(T)$ & $N(T)$, where

(i) $T : R^3 \to R^3$ is defined by

$$T(x_1, x_2, x_3) = (x_1 + 2x_2 - x_3,\ x_2 + x_3,\ x_1 + x_2 - 2x_3)$$

Sol. Given transformation is

$$T(x_1, x_2, x_3) = (x_1 + 2x_2 - x_3,\ x_2 + x_3,\ x_1 + x_2 - 2x_3)$$

Since R^3 is generated by $(1,0,0), (0,1,0)$ & $(0,0,1)$. So $R(T)$ will be generated by $T(1,0,0), T(0,1,0)$ & $T(0,0,1)$

Here $T(1,0,0) = (1,0,1)$

$T(0,1,0) = (2,1,1)$

& $T(0,0,1) = (-1,1,-2)$

Hence $R(T)$ is generated by $(1,0,1), (2,1,2)$ & $(-1,1,-2)$

Since $(2,1,1) = 3(1,0,1) + 1(-1,1,-2)$

So casting out the vector $(2,1,1)$, the set $\{(1,0,1), (-1,1,-2)\}$ also spans $R(T)$. Since none of the two vectors is a multiple of other, so the set $\{(1,0,1), (-1,1,-2)\}$ is linearly independent & so forms a basis for $R(T)$.

Hence $\dim R(T) = 2$

Now we find $\dim N(T)$.

A vector $(x_1, x_2, x_3) \in N(T)$ if $T(x_1, x_2, x_3) = 0$

i.e., if $(x_1 + 2x_2 - x_3,\ x_2 + x_3,\ x_1 + x_2 - 2x_3) = (0,0,0)$

$$\Rightarrow \quad x_1 + 2x_2 - x_3 = 0 \quad \text{——— (1)}$$

$$x_2 + x_3 = 0 \quad \text{——— (2)}$$

$$x_1 + x_2 - 2x_3 = 0 \quad \text{——— (3)}$$

Adding (1) & (2)

$$x_1 + 3x_2 = 0$$

or $\boxed{x_1 = -3x_2}$

Put in (3)

$$-3x_2 + x_2 - 2x_3 = 0$$

$$-2x_2 - 2x_3 = 0$$

$$x_2 + x_3 = 0$$

or $\boxed{x_3 = -x_2}$

If $x_2 = 1$

then $x_1 = -3$, $x_2 = 1$, $x_3 = -1$

So the vector $(-3, 1, -1)$ spans $N(T)$. Also $(-3, 1, -1)$ is linearly independent. So $\{(-3, 1, -1)\}$ forms a basis for $N(T)$.

Hence $\dim N(T) = 1$

(ii) $T : R^3 \longrightarrow R^4$ is defined by

$T(x_1, x_2, x_3) = (2x_1 + x_3, 4x_1 + x_2, x_1 + x_3, x_3 - 4x_2)$

Sol. Given transformation is

$T(x_1, x_2, x_3) = (2x_1 + x_3, 4x_1 + x_2, x_1 + x_3, x_3 - 4x_2)$

Since R^3 is generated by $(1,0,0)$, $(0,1,0)$ & $(0,0,1)$. So $R(T)$ will be generated by $T(1,0,0)$, $T(0,1,0)$ & $T(0,0,1)$

Here
$$T(1,0,0) = (2,4,1,0)$$
$$T(0,1,0) = (0,1,0,-4)$$
$$T(0,0,1) = (1,0,1,1)$$

Hence $R(T)$ will be generated by $(2,4,1,0), (0,1,0,-4), (1,0,1,1)$

Now we check whether these vectors are linearly independent. For this let

$a(2,4,1,0) + b(0,1,0,-4) + c(1,0,1,1) = (0,0,0,0)$ where $a, b, c \in F$

or $(2a + c, 4a + b, a + c, -4b + c) = (0,0,0,0)$

$$\Rightarrow \quad 2a+c=0 \quad \text{——} \quad (1)$$
$$4a+b=0 \quad \text{——} \quad (2)$$
$$a+c=0 \quad \text{——} \quad (3)$$
$$-4b+c=0 \quad \text{——} \quad (4)$$

$(1)-(3) \Rightarrow \boxed{a=0}$

$(3) \Rightarrow 0+c=0 \Rightarrow \boxed{c=0}$

$(2) \Rightarrow 0+b=0 \Rightarrow \boxed{b=0}$

Hence vectors $(2,4,1,0),(0,1,0,-4)$ & $(1,0,1,1)$ are linearly independent. Hence $\{(2,4,1,0),(0,1,0,-4),(1,0,1,1)\}$ form a basis for $R(T)$

Hence $\dim R(T) = 3$

Q16 Show that linear transformations preserve linear dependence.

Sol. Let $T: U \rightarrow V$ be a linear transformation, where U & V are vector spaces over the same field F.

Suppose a set $\{u_1, u_2, ------, u_n\}$ in U is linearly dependent we want to show that $\{T(u_1), T(u_2), -------, T(u_n)\}$ is a linearly dependent set in V.

Since $\{u_1, u_2, -----, u_n\}$ is linearly dependent, so there exist scalars $a_1, a_2, -----, a_n$, not all zero, such that

$$a_1u_1 + a_2u_2 + -------- + a_nu_n = 0$$

Applying T on both sides

$$T(a_1u_1 + a_2u_2 + ------ + a_nu_n) = T(0)$$

or $a_1T(u_1) + a_2T(u_2) + ------ + a_nT(u_n) = 0$ $\quad$ ($\because T$ is linear)

Since $a_1, a_2, -----, a_n$ are not all zero, so the above eq. shows that $\{T(u_1), T(u_2), -----, T(u_n)\}$ are linearly dependent in V. Hence T preserve linear dependence.

Q17 Find the rank of each matrix in problem 8 of exercise 3.2 by the method of 6.42

(i) $\begin{bmatrix} 1 & 3 \\ 0 & -2 \\ 5 & -1 \\ -2 & 3 \end{bmatrix}$

Sol:-

Let $A = \begin{bmatrix} 1 & 3 \\ 0 & -2 \\ 5 & -1 \\ -2 & 3 \end{bmatrix}$ then

$$\text{rank } A = 1 + \text{rank} \begin{bmatrix} \begin{vmatrix} 1 & 3 \\ 0 & -2 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ 5 & -1 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ -2 & 3 \end{vmatrix} \end{bmatrix}$$

$$= 1 + \text{rank} \begin{bmatrix} -2-0 \\ -1-15 \\ 3+6 \end{bmatrix}$$

$$= 1 + \text{rank} \begin{bmatrix} -2 \\ -16 \\ 9 \end{bmatrix}$$

$$= 2 + \text{rank} \begin{bmatrix} \begin{vmatrix} -2 & 0 \\ -16 & 0 \end{vmatrix} \\ \begin{vmatrix} -2 & 0 \\ 9 & 0 \end{vmatrix} \end{bmatrix}$$

$$= 2 + \text{rank} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{rank } A = 2$$

(ii) $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & 0 \\ -2 & -1 & 3 \\ -1 & 4 & -2 \end{bmatrix}$

Sol:.

Let $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & 0 \\ -2 & -1 & 3 \\ -1 & 4 & -2 \end{bmatrix}$ then

$$\text{rank } A = 1 + \text{rank}\begin{bmatrix} \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -3 \\ 2 & 6 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ -2 & -1 \end{vmatrix} & \begin{vmatrix} 1 & -3 \\ -2 & 3 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix} & \begin{vmatrix} 1 & -3 \\ -1 & -2 \end{vmatrix} \end{bmatrix}$$

$$= 1 + \text{rank}\begin{bmatrix} -3 & 6 \\ 3 & -3 \\ 6 & -5 \end{bmatrix}$$

$$= 2 + \text{rank}\begin{bmatrix} \begin{vmatrix} -3 & 6 \\ 3 & -3 \end{vmatrix} \\ \begin{vmatrix} -3 & 6 \\ 6 & -5 \end{vmatrix} \end{bmatrix}$$

$$= 2 + \text{rank}\begin{bmatrix} 9-18 \\ 15-36 \end{bmatrix}$$

$$= 2 + \text{rank}\begin{bmatrix} -9 \\ -21 \end{bmatrix}$$

$$= 3 + \text{rank}\begin{bmatrix} \begin{vmatrix} -9 & 0 \\ -21 & 0 \end{vmatrix} \end{bmatrix}$$

$$= 3 + \text{rank}\begin{bmatrix} 0 \end{bmatrix}$$

$$= 3$$

(iii) $\begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix}$

Sol.

Let $A = \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix}$ then

$$\text{rank } A = 1 + \text{rank}\begin{bmatrix} \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & -3 \\ 1 & -4 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & -4 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 2 & -7 \end{vmatrix} & \begin{vmatrix} 1 & -3 \\ 2 & -3 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ 3 & 8 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 3 & -7 \end{vmatrix} & \begin{vmatrix} 1 & -3 \\ 3 & -8 \end{vmatrix} \end{bmatrix}$$

$$= 1 + \text{rank}\begin{bmatrix} 1 & 2 & 1 & -1 \\ -3 & -6 & -3 & 3 \\ -1 & -2 & -1 & 1 \end{bmatrix}$$

$$= 2 + \text{rank}\begin{bmatrix} \begin{vmatrix} 1 & 2 \\ -3 & -6 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ -3 & -3 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ -3 & 3 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ -1 & -2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} \end{bmatrix}$$

$$= 2 + \text{rank}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= 2 + 0$$

$$= 2$$

(iv) $$\begin{bmatrix} 1 & 3 & -2 & 5 & 4 \\ 1 & 4 & 1 & 3 & 5 \\ 1 & 4 & 2 & 4 & 3 \\ 2 & 7 & -3 & 6 & 13 \end{bmatrix}$$

Sol.

Let $$A = \begin{bmatrix} 1 & 3 & -2 & 5 & 4 \\ 1 & 4 & 1 & 3 & 5 \\ 1 & 4 & 2 & 4 & 3 \\ 2 & 7 & -3 & 6 & 13 \end{bmatrix}$$ then

$$\text{rank } A = 1 + \text{rank}\begin{bmatrix} \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 5 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ 1 & 5 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 5 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 2 & -3 \end{vmatrix} & \begin{vmatrix} 1 & 5 \\ 2 & 6 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ 2 & 13 \end{vmatrix} \end{bmatrix}$$

$$= 1 + \text{rank}\begin{bmatrix} 1 & 3 & -2 & 1 \\ 1 & 4 & -1 & -1 \\ 1 & 1 & -4 & 5 \end{bmatrix}$$

$$= 2 + \text{rank}\begin{bmatrix} \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 1 & -4 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} \end{bmatrix}$$

$$= 2 + \text{rank}\begin{bmatrix} 1 & 1 & -2 \\ -2 & -2 & 4 \end{bmatrix}$$

$$= 3 + \text{rank}\begin{bmatrix} \begin{vmatrix} 1 & 1 \\ -2 & -2 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ -2 & 4 \end{vmatrix} \end{bmatrix}$$

$$= 3 + \text{rank}\begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$= 3 + 0$$

$$\text{rank } A = 3$$

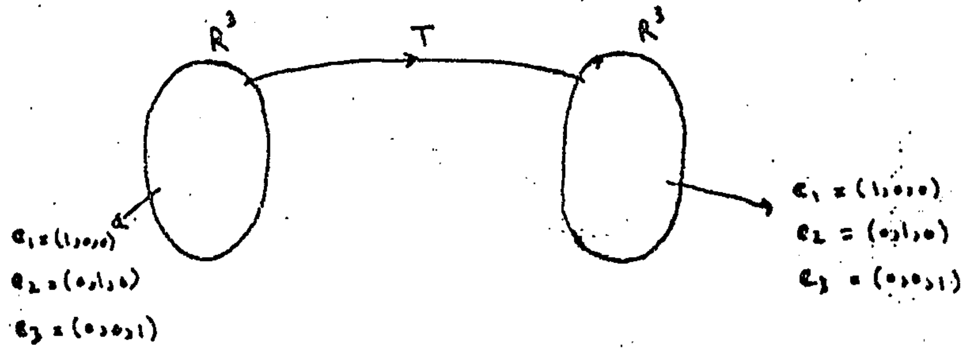
✳ Exercise No. 6.4 ✳

Q1 Find the matrix of each of the following linear transformations from R^3 to R^3 with respect to the standard basis for R^3:

(i) $T(x_1, x_2, x_3) = (x_1, x_2, 0)$

Sol:- Given linear transformation is

$$T(x_1, x_2, x_3) = (x_1, x_2, 0)$$



Now

$$T(1,0,0) = (1,0,0) = 1(1,0,0) + 0(0,1,0) + 0(0,0,1)$$

$$T(0,1,0) = (0,1,0) = 0(1,0,0) + 1(0,1,0) + 0(0,0,1)$$

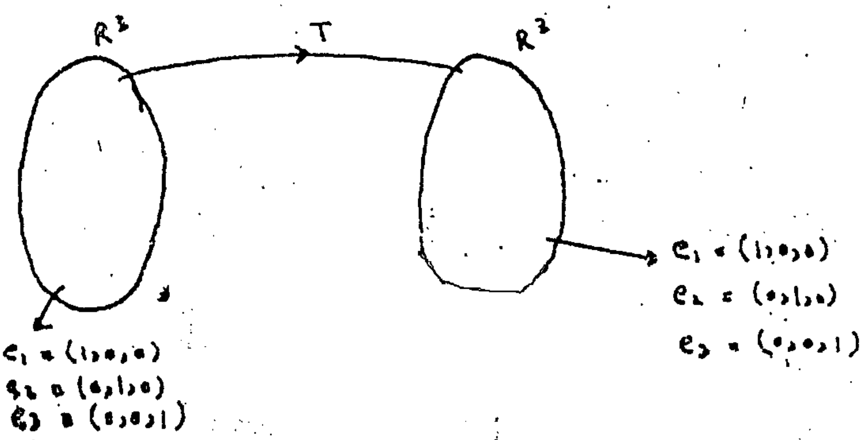
$$T(0,0,1) = (0,0,0) = 0(1,0,0) + 0(0,1,0) + 0(0,0,1)$$

Hence matrix of linear transformation T is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(ii) $T(x_1, x_2, x_3) = (x_1 + x_2, -x_1 - x_2, x_3)$

Sol:-



Here $\quad T(x_1,x_2,x_3) = (x_1+x_2,\ -x_1-x_2,\ x_3)$

then. $\quad T(1,0,0) = (1,-1,0) = 1(1,0,0)+(-1)(0,1,0)+0(0,0,1)$

$\quad T(0,1,0) = (1,-1,0) = 1(1,0,0)+(-1)(0,1,0)+0(0,0,1)$

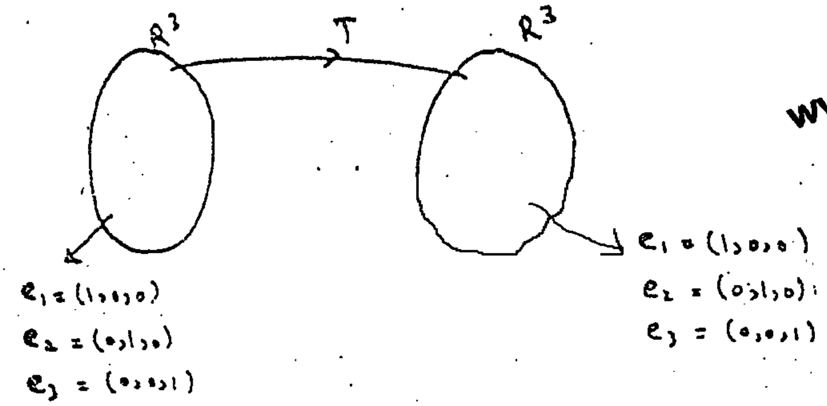
$\quad T(0,0,1) = (0,0,1) = 0(1,0,0)+0(0,1,0)+1(0,0,1)$

Hence matrix of linear transformation T is

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(iii) $\quad T(x_1,x_2,x_3) = (x_2,\ -x_1,\ -x_3)$

Sol.



Here $\quad T(x_1,x_2,x_3) = (x_2,\ -x_1,\ -x_3)$

then $\quad T(1,0,0) = (0,-1,0) = 0(1,0,0)+(-1)(0,1,0)+0(0,0,1)$

$\quad T(0,1,0) = (1,0,0) = 1(1,0,0)+0(0,1,0)+0(0,0,1)$

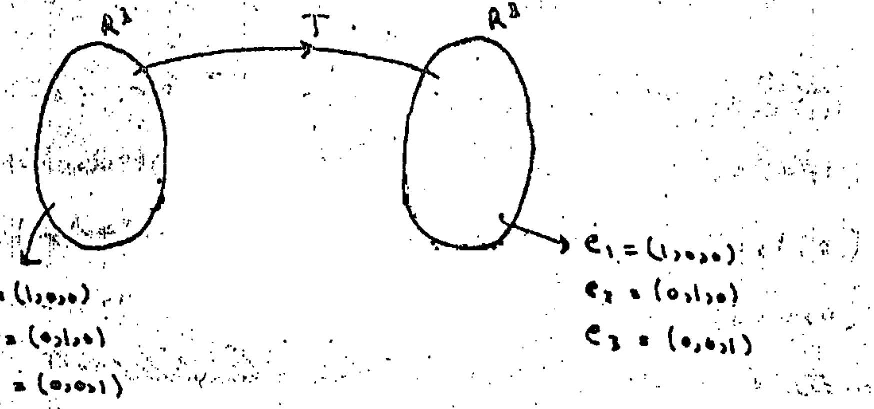
$\quad T(0,0,1) = (0,0,-1) = 0(1,0,0)+0(0,1,0)+(-1)(0,0,1)$

Hence matrix of linear transformation T is

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(iv) $\quad T(x_1,x_2,x_3) = (x_1,\ x_2+x_3,\ x_1+x_2+x_3)$

Sol.



Here $T(x_1, x_2, x_3) = (x_1, x_2 + x_3, x_1 + x_2 + x_3)$

then $T(1,0,0) = (1,0,1) = 1(1,0,0) + 0(0,1,0) + 1(0,0,1)$

$T(0,1,0) = (0,1,1) = 0(1,0,0) + 1(0,1,0) + 1(0,0,1)$

$T(0,0,1) = (0,1,1) = 0(1,0,0) + 1(0,1,0) + 1(0,0,1)$

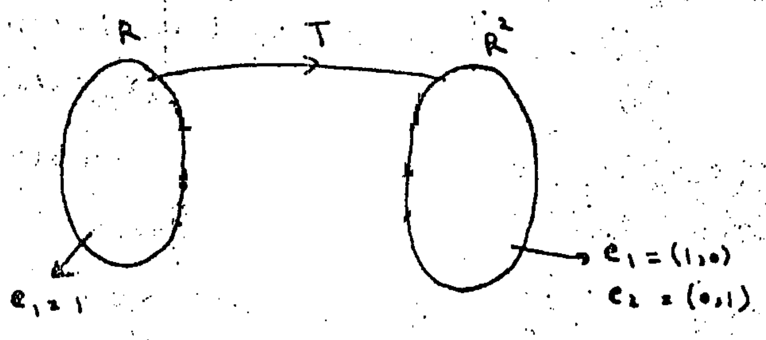
Hence matrix of linear transformation T is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Q2 Find the matrix of each of the following linear transformations with respect to the standard bases of the given spaces:

(i) $T : R \longrightarrow R^2$ defined by $T(x) = (3x, 5x)$

Sol.



Here $T(x) = (3x, 5x)$

then $T(1) = (3,5) = 3(1,0) + 5(0,1)$

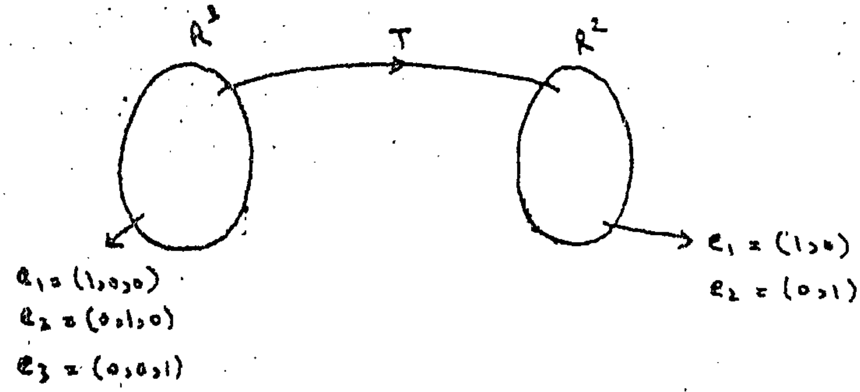
Hence matrix of linear transformation T is

$$\begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

(ii) $T : R^3 \longrightarrow R^2$ defined by

$$T(x_1, x_2, x_3) = (3x_1 - 4x_2 + 9x_3 \,,\, 5x_1 + 3x_2 - 2x_3)$$

Sol.



Here $T(x_1, x_2, x_3) = (3x_1 - 4x_2 + 9x_3 \,,\, 5x_1 + 3x_2 - 2x_3)$

Then $T(1,0,0) = (3,5) = 3(1,0) + 5(0,1)$

$T(0,1,0) = (-4,3) = -4(1,0) + 3(0,1)$

$T(0,0,1) = (9,-5) = 9(1,0) + (-5)(0,1)$

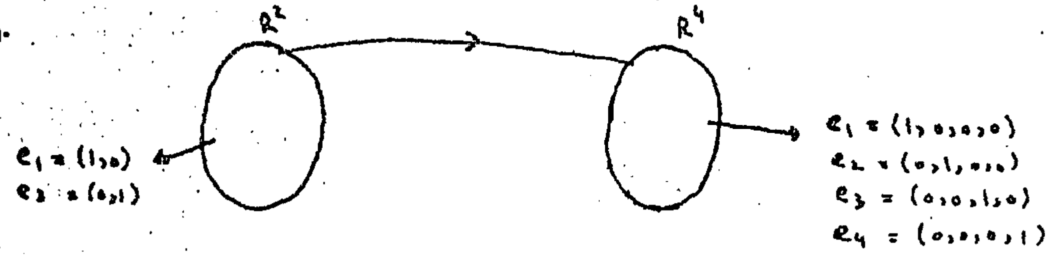
Hence matrix of linear transformation T is

$$\begin{bmatrix} 3 & -4 & 9 \\ 5 & 3 & -5 \end{bmatrix}$$

(iii) $T : R^2 \longrightarrow R^4$ defined by

$$T(x_1, x_2) = (3x_1 + 4x_2 \,,\, 5x_1 - 2x_2 \,,\, x_1 + 7x_2 \,,\, 4x_1)$$

Sol.



Here $T(x_1, x_2) = (3x_1 + 4x_2 \,,\, 5x_1 - 2x_2 \,,\, x_1 + 7x_2 \,,\, 4x_1)$

Then $T(1,0) = (3,5,1,4) = 3(1,0,0,0) + 5(0,1,0,0) + 1(0,0,1,0) + 4(0,0,0,1)$

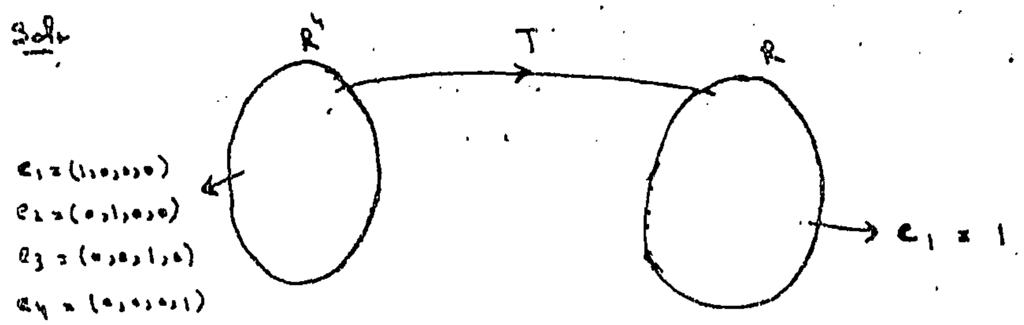
$T(0,1) = (4,-2,7,0) = 4(1,0,0,0) - 2(0,1,0,0) + 7(0,0,1,0) + 0(0,0,0,1)$

Hence matrix of linear transformation T is $\begin{bmatrix} 3 & 4 \\ 5 & -2 \\ 1 & 7 \\ 4 & 0 \end{bmatrix}$

(iv) $T : R^4 \longrightarrow R$ defined by

$T(x_1, x_2, x_3, x_4) = 2x_1 + 3x_2 - 7x_3 + x_4$

Sol:



Here $T(x_1, x_2, x_3, x_4) = 2x_1 + 3x_2 - 7x_3 + x_4$

then
$$T(1,0,0,0) = 2 = 2 \cdot 1$$
$$T(0,1,0,0) = 3 = 3 \cdot 1$$
$$T(0,0,1,0) = -7 = -7 \cdot 1$$
$$T(0,0,0,1) = 1 = 1 \cdot 1$$

Hence matrix of linear transformation T is

$$\begin{bmatrix} 2 & 3 & -7 & 1 \end{bmatrix}$$

Q3 Each of the following is the matrix of a linear transformation $T : R^n \longrightarrow R^m$. Determine m, n & express T in terms of Co-ordinates.

(i)
$$\begin{bmatrix} 3 & 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 & 1 \end{bmatrix}$$

Sol: Given linear transformation is

$T : R^n \longrightarrow R^m$

Here n = no. of Columns = 5

& m = no. of rows = 3

Now

let $x = (x_1, x_2, x_3, x_4, x_5) \in R^5$

then T is defined as

$$= \begin{bmatrix} 3 & 1 & 0 & 2 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$$T(x) = \begin{bmatrix} 3x_1 + x_2 + 0x_3 + 2x_4 + x_5 \\ x_1 + 0x_2 + 0x_3 + x_4 + x_5 \\ 0x_1 - x_2 + x_3 + x_4 + x_5 \end{bmatrix}$$

or

$$T(x_1, x_2, x_3, x_4, x_5) = (3x_1 + x_2 + 2x_4 + x_5,\ x_1 + x_4 + x_5,\ -x_2 + x_3 + x_4 + x_5)$$

which is T in terms of Co-ordinates.

(ii) $\begin{bmatrix} 6 & -1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$

Sol. Given linear transformation is

$$T : R^n \longrightarrow R^m$$

Here n = no. of Columns = 2

& m = no. of rows = 3

Now let $x = (x_1, x_2) \in R^2$

then T is defined as

$$T(x) = Ax$$

$$= \begin{bmatrix} 6 & -1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

or $T(x) = \begin{bmatrix} 6x_1 - x_2 \\ x_1 + 2x_2 \\ x_1 + 3x_2 \end{bmatrix}$

or $T(x_1, x_2) = (6x_1 - x_2,\ x_1 + 2x_2,\ x_1 + 3x_3)$
which is T in terms of Co-ords.

(iii) $$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 5 & 6 \\ -2 & 3 & -1 \end{bmatrix}$$

Sol. Given linear transformation is
$$T : R^n \longrightarrow R^m$$
Here n = no. of Columns = 3
& m = no. of rows = 3
Let $x = (x_1, x_2, x_3) \in R^3$ then
T is defined as

$$T(x) = Ax$$
$$= \begin{bmatrix} 1 & 1 & 2 \\ 2 & 5 & 6 \\ -2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$
$$T(x) = \begin{bmatrix} x_1 + x_2 + 2x_3 \\ 2x_1 + 5x_2 + 6x_3 \\ -2x_1 + 3x_2 - x_3 \end{bmatrix}$$
or
$$T(x_1, x_2, x_3) = (x_1 + x_2 + 2x_3,\ 2x_1 + 5x_2 + 6x_3,\ -2x_1 + 3x_2 - x_3)$$
Which is T in terms of Co-ords.

Q4 The matrix of a linear transformation
$T : R^3 \longrightarrow R^3$ is
$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$$ Find T in terms of Co-ords.
& its matrix w.r.t. the basis
$$V_1 = (0, 1, 2),\ V_2 = (1, 1, 1),\ V_3 = (1, 0, -2).$$

Sol. Given linear transformation is

$T : R^3 \longrightarrow R^3$

Let $x = (x_1, x_2, x_3) \in R^3$ then T is defined as

$T(x) = Ax$

or

$$= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

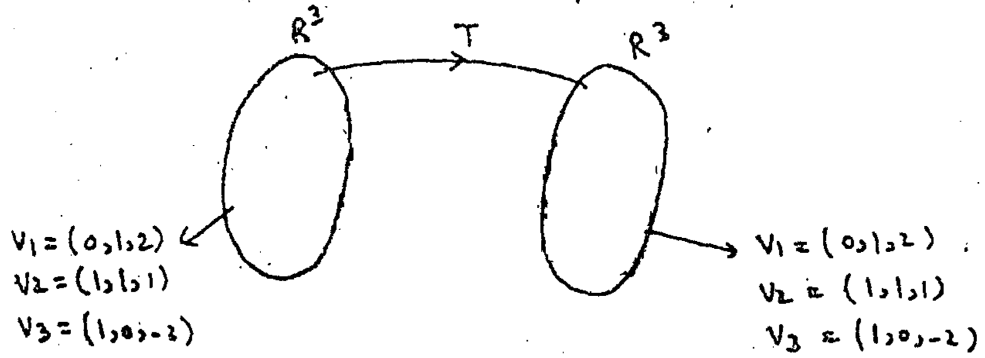
$$T(x) = \begin{bmatrix} 0x_1 + x_2 + x_3 \\ x_1 + 0x_2 - x_3 \\ -x_1 - x_2 + 0x_3 \end{bmatrix}$$

or

$$T(x_1, x_2, x_3) = (x_2 + x_3,\ x_1 - x_3,\ -x_1 - x_2)$$

which is T in terms of co-ords.

Now we find matrix of T



Here $T(x_1, x_2, x_3) = (x_2 + x_3,\ x_1 - x_3,\ -x_1 - x_2)$

then $T(0,1,2) = (3,-2,-1) = \overset{-11}{a}(0,1,2) + \overset{9}{b}(1,1,1) + \overset{-6}{c}(1,0,-2)$

$$(3,-2,-1) = (b+c,\ a+b,\ 2a+b-2c)$$

$$\Rightarrow \quad b + c = 3 \quad \text{——} \ (1)$$

$$a + b = -2 \quad \text{——} \ (2)$$

$$2a + b - 2c = -1 \quad \text{——} \ (3)$$

$(1) - (2) \Rightarrow \quad c - a = 5 \quad \text{——} \ (A)$

$(2) - (3) \Rightarrow \quad -a + 2c = -1 \quad \text{——} \ (B)$

Subt. (B) from (A)

$-c = 6$

or $\boxed{c = -6}$

Put in ①

$b - 6 = 3$

or $\boxed{b = 9}$

Put in ②

$a + 9 = -2$

$\boxed{a = -11}$

Now

$$T(1,1,1) = (2,0,-2) = \overset{-2}{a}(0,1,2) + \overset{2}{b}(1,1,1) + \overset{0}{c}(1,0,-2)$$

$$= (b+c,\ a+b,\ 2a+b-2c)$$

$\Rightarrow$ $b + c = 2$ ——— ①

$a + b = 0$ ——— ②

$2a + b - 2c = -2$ ——— ③

① − ② $\Rightarrow$ $c - a = 2$

$-a + 2c = 2$

~~Put~~ we get

$-c = 0 \Rightarrow \boxed{c = 0}$

Put in ①

$b + 0 = 2 \Rightarrow \boxed{b = 2}$

Put in ②

$a + 2 = 0 \Rightarrow \boxed{a = -2}$

Now

$$T(1,0,-2) = (-2,3,-1) = \overset{14}{a}(0,1,2) + \overset{-11}{b}(1,1,1) + \overset{9}{c}(1,0,-2)$$

$$= (b+c,\ a+b,\ 2a+b-2c)$$

$\Rightarrow$ $b + c = -2$ ——— ①

$a + b = 3$ ——— ②

$2a + b - 2c = -1$ ——— ③

① − ② $\Rightarrow$

Adding ① & ②

$2a = 0 \Rightarrow \boxed{a = 0}$

Put in ①

$0 - b = 0 \Rightarrow \boxed{b = 0}$

Hence vectors $(1,1)$ & $(-1,1)$ are linearly independent & since they are two in number, so they form a basis for R^2.

Let (x_1, x_2) be an arbitrary vector of R^2

Then (x_1, x_2) can be expressed as

$$(x_1, x_2) = a(1,1) + b(-1,1)$$
$$= (a-b, a+b)$$

$\Rightarrow$

$$a - b = x_1 \quad \text{———— Ⓐ}$$
$$a + b = x_2 \quad \text{———— Ⓑ}$$

$$2a = x_1 + x_2$$

or $\boxed{a = \frac{x_1 + x_2}{2}}$

Put in ①

$$\frac{x_1 + x_2}{2} - b = x_1$$

$$b = \frac{x_1 + x_2}{2} - x_1$$
$$= \frac{x_1 + x_2 - 2x_1}{2}$$

$$\boxed{b = \frac{x_2 - x_1}{2}}$$

So $(x_1, x_2) = \left(\frac{x_1 + x_2}{2}\right)(1,1) + \left(\frac{x_2 - x_1}{2}\right)(-1,1)$

Applying T on both sides

$$T(x_1, x_2) = T\left\{\left(\frac{x_1 + x_2}{2}\right)(1,1) + \left(\frac{x_2 - x_1}{2}\right)(-1,1)\right\}$$
$$= \left(\frac{x_1 + x_2}{2}\right)T(1,1) + \left(\frac{x_2 - x_1}{2}\right)T(-1,1) \qquad \because T \text{ is linear}$$

$c - a = -5$

$① - ③ \Rightarrow -a + 2c = 4$

Subt. we get

$-c = -9 \Rightarrow \boxed{c = 9}$

Put in ①

$b + 9 = -2 \Rightarrow \boxed{b = -11}$

Put in ③

$a - 11 = 3 \Rightarrow \boxed{a = 14}$

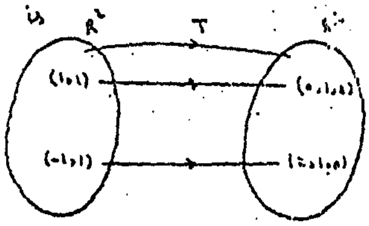
Hence matrix of linear transformation T

$$\begin{bmatrix} -11 & -2 & 14 \\ 9 & 2 & -11 \\ -6 & 0 & 9 \end{bmatrix}$$

Q5 A linear transformation $T : R^2 \longrightarrow R^3$ maps the vector $(1,1)$ into $(0,1,2)$ & the vector $(-1,1)$ into $(2,1,0)$. What matrix does T represent with respect to the standard bases for R^2 & R^3?

Sol: Given linear transformation is

$T : R^2 \longrightarrow R^3$



We are given that

$T(1,1) = (0,1,2)$

& $T(-1,1) = (2,1,0)$

First we check linear independency of vectors $(1,1)$ & $(-1,1)$

suppose for scalars a, b

$a(1,1) + b(-1,1) = 0$

$(a-b, a+b) = (0,0)$

$\Rightarrow a - b = 0$ ——— ①

$a + b = 0$ ——— ②

$T(x_1,x_2) = \left(\frac{x_1+x_2}{2}\right)(0,1,2) + \left(\frac{x_2-x_1}{2}\right)(2,1,0)$

$= \left(x_2 - x_1, \frac{x_1+x_2}{2} + \frac{x_2-x_1}{2}, x_1+x_2\right)$

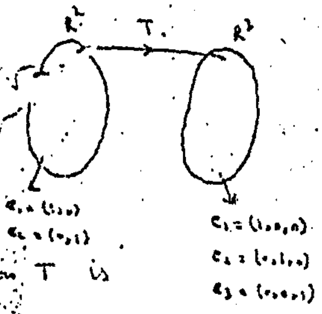
or $T(x_1,x_2) = (x_2 - x_1, x_2, x_1+x_2)$

which is T in terms of co-ords.

Now as $T(x_1,x_2) = (x_2-x_1, x_2, x_1+x_2)$

then $T(1,0) = (-1,0,1)$.

$= -1(1,0,0) + 0(0,1,0) + 1(0,0,1)$



& $T(0,1) = (1,1,1)$

$= 1(1,0,0) + 1(0,1,0) + 1(0,0,1)$

Hence matrix of linear transformation T is

$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

GOOD LUCK