Linearly dependent & linearly independent vectors:- 50

The vectors $v_1, v_2, \ldots, v_m$ of a vector space V over F are said to be linearly dependent if there exist elements $a_1, a_2, \ldots, a_m \in F$, not all zeros such that

$$a_1v_1 + a_2v_2 + \cdots + a_mv_m = 0$$

On the other hand if

$$a_1v_1 + a_2v_2 + \cdots + a_mv_m = 0 \quad \text{where all } a_i = 0$$

then vectors $v_1, v_2, \ldots, v_m$ are said to be linearly independent.

Note

① If any two vectors out of $v_1, v_2, \ldots, v_m$ are equal say $v_3 = v_4$
then $v_1, v_2, \ldots, v_m$ are linearly dependent
because $0.v_1 + 0.v_2 + 1.v_3 + (-1)v_4 + 0.v_5 + \cdots + 0v_m = 0$

② If any one of the vector out of $v_1, v_2, \ldots, v_m$ is zero, say $v_2 = 0$ then $v_1, v_2, \ldots, v_m$ are linearly dependent because

$$0.v_1 + 1.v_2 + 0v_3 + \cdots + 0v_m = 0$$

③ A single non zero vector is always linearly independent because
let $v \neq 0$ be the single vector
then $av = 0 \Rightarrow a = 0$
Thus v is linearly independent.

④ Two vectors v_1 & v_2 are linearly dependent if one of them is a multiple of other.

Theorem Let V be a vector space over a field F & $S = \{v_1, v_2, \ldots \ldots v_m\}$ be a set of vectors in V Then

(i) If S is linearly independent then any subset of S is also linearly independent.

(ii) If S is linearly dependent, then the set $\{v, v_1, v_2, \ldots, v_m\}$ is linearly dependent for all $v \in V$ i.e., every superset of S is also linearly dependent.

Proof:-

(i) Here $S = \{v_1, v_2, \ldots, v_m\}$

Let $\{v_1, v_2, \ldots, v_i\}$ where $i < m$ is a subset of S

& let $a_1v_1 + a_2v_2 + \ldots + a_iv_i = 0$ where $a_i \in F$

or $a_1v_1 + a_2v_2 + \ldots + a_iv_i + 0v_{i+1} + \ldots + 0v_m = 0$

But $S = \{v_1, v_2, \ldots, v_m\}$ is linearly dependent

So $a_1 = a_2 = a_3 = \ldots = a_i = 0$

Hence $\{v_1, v_2, \ldots, v_i\}$ is linearly dependent.

(ii) As $S = \{v_1, v_2, \ldots, v_m\}$ is linearly dependent

So $a_1v_1 + a_2v_2 + \ldots + a_mv_m = 0$ where $a_i \neq 0$ for some i

Now

$0.v + a_1v_1 + a_2v_2 + \ldots + a_mv_m = 0$ where $a_i \neq 0$ for some i

So $\{v, v_1, v_2, \ldots, v_m\}$ is linearly dependent.

Theorem A set $S = \{v_1, v_2, \ldots, v_n\}$ of n vectors $(n \geq 2)$ in a vector space V is linearly dependent iff. atleast one of the vectors in S is a linear combination of the remaining vectors of the set.

Proof:-

(i) Suppose the set

$S = \{ V_1, V_2, \text{------}, V_n \}$ is linearly independent

Then there exist scalars $a_1, a_2, \text{------}, a_n$ atleast one of them say a_i is non zero, such that

$$a_1V_1 + a_2V_2 + \text{------} + a_iV_i + \text{------} + a_nV_n = 0$$

or $a_iV_i = -a_1V_1 - a_2V_2 - \text{------} - a_{i-1}V_{i-1} - a_{i+1}V_{i+1} - \text{------} - a_nV_n$

or

$$V_i = -\frac{a_1}{a_i}V_1 - \frac{a_2}{a_i}V_2 - \text{------} - \frac{a_{i-1}}{a_i}V_{i-1} - \frac{a_{i+1}}{a_i}V_{i+1} - \text{------} - a_nV_n$$

which shows that V_i is a linear Combination of remaining vectors of the set.

Conversely

let some vector V_j of the given set is a linear Combination of the remaining vectors. i.e.,

$$V_j = a_1V_1 + a_2V_2 + \text{------} + a_{j-1}V_{j-1} + a_{j+1}V_{j+1} + \text{------} + a_nV_n$$

Then above eq. Can be written as

$$a_1V_1 + a_2V_2 + \text{------} + a_{j-1}V_{j-1} + (-1)V_j + a_{j+1}V_{j+1} + \text{------} + a_nV_n = 0$$

Here there is atleast one Cofficient namely -1 of V_j which is non zero & so the set

$\{V_1, V_2, \text{------}, V_{j-1}, V_j, V_{j+1}, \text{------}, V_n\}$ is linearly dependent.

(ii) Suppose that the set $S = \{V_1, V_2, \text{------}, V_n\}$ is linearly dependent then there exist scalars $a_1, a_2, \text{------}, a_n \in F$ not all zero such that

$$a_1V_1 + a_2V_2 + \text{------} + a_nV_n = 0 \quad \text{———} \quad (1)$$

Let a_k be the last non zero scalar in (1) then the terms $a_{k+1}V_{k+1}, a_{k+2}V_{k+2}, \text{------}, a_nV_n$ are all zero

So eq. (1) becomes

$$a_1V_1 + a_2V_2 + \text{------} + a_kV_k = 0 \quad \text{where } a_k \neq 0$$

or $a_kV_k = -a_1V_1 - a_2V_2 - \text{------} - a_{k-1}V_{k-1}$

or $$V_k = -\frac{a_1}{a_k}V_1 - \frac{a_2}{a_k}V_2 - \text{------} - \frac{a_{k-1}}{a_k}V_{k-1}$$

Which shows that V_K is a linear Combination of 53 the vectors preceding it.

Conversely

Suppose that in $S = \{V_1, V_2, \ldots\ldots, V_n\}$, some of the vectors say V_K is a linear Combination of the vectors preceding it

i.e., $V_K = b_1V_1 + b_2V_2 + \ldots\ldots\ldots + b_{K-1}V_{K-1}$

or $b_1V_1 + b_2V_2 + \ldots\ldots\ldots + b_{K-1}V_{K-1} + (-1)V_K = 0$

or $b_1V_1 + b_2V_2 + \ldots\ldots + b_{K-1}V_{K-1} + (-1)V_K + 0V_{K+1} + \ldots\ldots + 0V_n = 0$

Here atleast one cofficient -1 of V_K is non zero

Hence $S = \{V_1, V_2, \ldots\ldots, V_n\}$ is linearly dependent.

Basis of a vector space:

A linearly independent set which generates or spans a vector space V is called a basis for V.

Theorem Any finite dimensional vector space Contains a basis.

Proof:

Let V be a finite dimensional vector space then V should be linear span of some finite set. Let $\{V_1, V_2, \ldots\ldots V_r\}$ be a finite spanning set of V. In case $V_1, V_2, \ldots\ldots, V_n$ are linearly independent, then they form a basis for V & the proof is Complete.

Suppose $V_1, V_2, \ldots\ldots, V_r$ are not linearly independent i.e; they are linearly dependent, so one of the vectors V_i is a linear Combination of the preceding vectors. We drop this vector V_i from the set & obtain a set of $r-1$ vectors

$v_1, v_2, \ldots\ldots, v_{r-1}$. Clearly any linear Combination of $v_1, v_2, \ldots, v_r$ is also a linear Combination of $v_1, v_2, \ldots\ldots, v_{r-1}$. So $\{v_1, v_2, \ldots\ldots, v_{r-1}\}$ is also a spanning set for V.
Continuing in this way, we arrive at a linearly independent spanning set $\{v_1, v_2, \ldots, v_n\}; 1 \le n \le r$ & so it forms a basis for V.
Thus every finite dimensional vector space Contains a basis.

<u>Note</u> If a vector space V is generated by $v_1, v_2, \ldots\ldots, v_m$ then any linearly independent set in V Cannot have more than m no. of elements.

<u>Dimension of a vector space</u>:
The no. of elements in a basis of a vector space V over F is called dimension of V. It is denoted by $\dim V$.

<u>Theorem</u> All bases of a finite dimensional vector space Contains the same no. of elements.
<u>Proof</u>: Let a vector space V over F has two bases A & B with m & n no. of elements.
Since A spans V & B is a linearly independent subset in V, so B Cannot have more than m no. of elements.

$$\text{i.e.,} \quad n \le m \quad \text{———} \quad (1)$$

Now since B spans V & A is a linearly independent subset in V, so A Cannot have more than n no. of elements

i.e., $m \leq n$ ——— (2)

from (1) & (2)

$$m = n$$

Hence no. of elements in A = no. of elements in B which is req. proof.

Theorem Let V be a vector space such that $\dim V = n < \infty$.

A set of vectors $\{v_1, v_2, \ldots\ldots, v_n\} \subset V$ is a basis for V iff. each vector in V is uniquely expressible as a linear combination of vectors $v_1, v_2, \ldots\ldots, v_n$.

Proof:- Let the set $\{v_1, v_2, \ldots\ldots, v_n\}$ be a basis for V. Then every vector $v \in V$ can be expressed in atleast one way as a linear combination of the basis vectors. i.e.,

$v = a_1v_1 + a_2v_2 + \ldots\ldots + a_nv_n$ where $a_i \in F$ $1 \leq i \leq n$

Suppose v can also be expressed as

$v = b_1v_1 + b_2v_2 + \ldots\ldots + b_nv_n$ where $b_i \in F$ $1 \leq i \leq n$

Comparing above eqs.

$a_1v_1 + a_2v_2 + \ldots\ldots + a_nv_n = b_1v_1 + b_2v_2 + \ldots\ldots + b_nv_n$

or $(a_1 - b_1)v_1 + (a_2 - b_2)v_2 + \ldots\ldots + (a_n - b_n)v_n$

Since $v_1, v_2, \ldots\ldots, v_n$ are linearly independent

So $a_1 - b_1 = 0$, $a_2 - b_2 = 0$, $\ldots\ldots$, $a_n - b_n = 0$

$\Rightarrow$ $a_1 = b_1$, $a_2 = b_2$, $\ldots\ldots$, $a_n = b_n$

Hence every vector $v \in V$ can be expressed in a unique way as a linear combination of $v_1, v_2, \ldots\ldots, v_n$.

Conversely

Let every vector $v \in V$ is uniquely expressible as a linear combination of $v_1, v_2, \ldots\ldots, v_n$.

Then these vectors span V. We will prove that they are linearly independent.

Suppose that for scalars $a_1, a_2, \ldots\ldots, a_n$

$$a_1v_1 + a_2v_2 + \ldots\ldots\ldots + a_nv_n = 0 \quad \text{———} (1)$$

As $$0v_1 + 0v_2 + \ldots\ldots\ldots + 0v_n = 0 \quad \text{———} (2)$$

Since the representation is unique

So $a_1 = 0, a_2 = 0, \ldots\ldots\ldots, a_n = 0$

Hence $v_1, v_2, \ldots\ldots, v_n$ are linearly independent.

Since they also span V.

Hence $\{v_1, v_2, \ldots\ldots, v_n\}$ form a basis for V

<u>Theorem</u> Let $S = \{v_1, v_2, \ldots\ldots, v_n\}$ be a basis for an n-dimensional vector space V over a field F. Then every set with more than n vectors is linearly dependent.

<u>Proof:</u>

Let $B = \{u_1, u_2, \ldots\ldots, u_r\}$ be a set of r vectors in V where $r > n$.

We shall show that B is linearly dependent.

To show that B is linearly dependent, we must find scalars $c_1, c_2, \ldots\ldots, c_r$, not all zero, such that

$$c_1u_1 + c_2u_2 + \ldots\ldots\ldots + c_ru_r = 0 \quad \text{———} (1)$$

Since

the set $S = \{v_1, v_2, \ldots\ldots, v_n\}$ is linearly independent

So each u_i can be uniquely expressed as a linear combination of $v_1, v_2, \ldots\ldots, v_n$. Hence

$$\left.\begin{aligned} u_1 &= a_{11}v_1 + a_{12}v_2 + \ldots\ldots\ldots + a_{1n}v_n \\ u_2 &= a_{21}v_1 + a_{22}v_2 + \ldots\ldots\ldots + a_{2n}v_n \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ u_r &= a_{r1}v_1 + a_{r2}v_2 + \ldots\ldots\ldots + a_{rn}v_n \end{aligned}\right\} \text{———} (2) \quad \text{where } a_{ij} \in F$$

Putting values of $u_1, u_2, \ldots, u_r$ from ② in ① so

$$c_1(a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n) + c_2(a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n) + \cdots + c_r(a_{r1}v_1 + a_{r2}v_2 + \cdots + a_{rn}v_n) = 0$$

or

$$(c_1a_{11} + c_2a_{21} + \cdots + c_ra_{r1})v_1 + (c_1a_{12} + c_2a_{22} + \cdots + c_ra_{r2})v_2 + \cdots + (c_1a_{1n} + c_2a_{2n} + \cdots + c_ra_{rn})v_n = 0$$

Since $v_1, v_2, \ldots, v_n$ are linearly independent

$$\Rightarrow \left.\begin{array}{l} a_{11}c_1 + a_{21}c_2 + \cdots + a_{r1}c_r = 0 \\ a_{12}c_1 + a_{22}c_2 + \cdots + a_{r2}c_r = 0 \\ \cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots \\ a_{1n}c_1 + a_{2n}c_2 + \cdots + a_{rn}c_r = 0 \end{array}\right\}$$

Which is a homogeneous system of n eqs. in r unknowns $c_1, c_2, \ldots, c_r$.

Since $n < r$, so this system has a non trivial soln.

Hence atleast one of $c_1, c_2, \ldots, c_r$ is non zero & so from eq. ① set $B = \{u_1, u_2, \ldots, u_r\}$ is linearly dependent.

<u>Theorem</u> Let $v_1, v_2, \ldots, v_n$ be linearly independent in a vector space V over a field F. If v is any non zero vector in V. Then the set $\{v_1, v_2, \ldots, v_n, v\}$ is linearly independent iff. v is not in the linear span $\langle v_1, v_2, \ldots, v_n\rangle$.

<u>Proof.</u> Let $v \notin \langle v_1, v_2, \ldots, v_n\rangle$

then we prove $\{v_1, v_2, \ldots, v_n, v\}$ is linearly independent.

Consider

$$a_1v_1 + a_2v_2 + \cdots + a_nv_n + av = 0 \text{ ——— ①}$$

where $a_1, a_2, \ldots, a_n, a \in F$

Suppose that $a \neq 0$ then from ①, we have

$$V = -\frac{1}{a}(a_1V_1 + a_2V_2 + \dots\dots + a_nV_n)$$

Which shows that $V \in \langle V_1, V_2, \dots\dots, V_n\rangle$

which is a contradiction.

Hence $a = 0$

Also since $V_1, V_2, \dots\dots, V_n$ are linearly independent

So eq. ① with $a = 0$ implies $a_1, a_2, \dots a_n = 0$

Hence $\{V_1, V_2, \dots\dots, V_n, V\}$ is linearly independent set.

Conversely

let $\{V_1, V_2, \dots\dots, V_n, V\}$ be linearly independent set

then we prove $V \notin \langle V_1, V_2, \dots\dots, V_n\rangle$

Suppose $V \in \langle V_1, V_2, \dots\dots, V_n\rangle$

then V can be expressed as a linear combination of $V_1, V_2, \dots\dots, V_n$. i.e.,

$$V = a_1V_1 + a_2V_2 + \dots\dots + a_nV_n \qquad \text{where } a_i \in F$$

or

$$a_1V_1 + a_2V_2 + \dots\dots + a_nV_n + (-1)V = 0$$

Which shows that $\{V_1, V_2, \dots\dots, V_n, V\}$ is linearly dependent set.

which is a contradiction

Hence $V \notin \langle V_1, V_2, \dots\dots, V_n\rangle$

Theorem Let V be an n-dimensional vector space over a field F. Then every set $S = \{V_1, V_2, \dots\dots, V_n\}$ of n linearly independent vectors in V is a basis for V.

Proof.

Let $v \in V$ be any non zero vector then the set $\acute{S} = \{V_1, V_2, \dots\dots, V_n, V\}$ is a linearly dependent set. So we can find scalars $a_1, a_2, \dots, a_n, a$ not all zeros such that

$a_1v_1 + a_2v_2 + \cdots\cdots + a_nv_n + av = 0$ ——— (1)

Since $S = \{v_1, v_2, \ldots\ldots, v_n\}$ is linearly independent

So $a_i = 0$ for $i = 1, 2, 3, \ldots\ldots, n$

Hence $a \neq 0$

And so eq. (1) can be written as

$av = -a_1v_1 - a_2v_2 - \cdots\cdots - a_nv_n$

or

$$v = \left(-\frac{a_1}{a}\right)v_1 + \left(-\frac{a_2}{a}\right)v_2 + \cdots\cdots + \left(-\frac{a_n}{a}\right)v_n$$

Which shows that v is a linear combination of vectors $v_1, v_2, \ldots\ldots, v_n$.

So S spans V.

Hence S is a linearly independent spanning set for V.

So $S = \{v_1, v_2, \ldots\ldots, v_n\}$ is a basis for V.

Theorem (i) Any linearly independent set of vectors in a finite dimensional vector space V can be extended to a basis for V.

(ii) If W is a subspace of a finite dimensional vector space V then

$\dim W \leq \dim V$.

Moreover if $\dim W = \dim V$ then $W = V$

Proof:

Since V is finite dimensional so let $\dim V = n$

Let $S = \{v_1, v_2, \ldots\ldots, v_r\}$ (where $r < n$) be a linearly independent set of vectors in V.

Since $\dim V = n$, so the set S cannot span V.

There is a vector say $v_{r+1} \in V$ such that $v_{r+1} \notin \langle S \rangle$ & the set $S \cup \{v_{r+1}\}$ is linearly independent. This process can be repeated $n-r$ times to get a larger set $\{v_1, v_2, \ldots, v_r, v_{r+1}, \ldots, v_n\}$ which is linearly independent & so this will form a basis for V.

(ii) Proof:-

Since V is finite dimensional suppose $\dim V = n$

Then any set of $n+1$ or more vectors is linearly dependent.

Moreover since a basis of W consists of linearly independent vectors, so it cannot contain more than n elements

Hence $\dim W \le n = \dim V$

$\Rightarrow \dim W \le \dim V$

If $\dim W = \dim V$

Then every basis of W is also a basis for V.

Hence $W = V$

Theorem A vector space V is the direct sum of its subspaces U & W iff. each $v \in V$ can be uniquely written as

$$v = u + w \qquad \text{for } u \in U,\ w \in W$$

Proof:-

Suppose that V is the direct sum of its subspaces U & W then by def.

(i) $V = U + W$

(ii) $U \cap W = \{0\}$

We want to show that each $v \in V$ can be uniquely written as

$$v = u + w \qquad \text{for } u \in U,\ w \in W$$

Let $v \in V$ then

$v = u + w$ —— ① for $u \in U, w \in W$ by (i)

If possible let

$v = u_1 + w_1$ for $u_1 \in U, w_1$

then

$$u + w = u_1 + w_1$$

or $u - u_1 = w_1 - w \in U \cap W$

But $U \cap W = \{0\}$ by (ii)

So $u - u_1 = w_1 - w = 0$

$\Rightarrow u = u_1$ & $w = w_1$

So the expression for v in ① is unique.

Hence each $v \in V$ can be uniquely written as

$v = u + w$ for $u \in U, w \in W$

<u>Conversely</u>

let each $v \in V$ is uniquely written as

$v = u + w$ for $u \in U, w \in W$

So (i) is satisfied. Now we prove condition (ii)

For this let $v \in U \cap W$ then v can be written as

$v = u + 0$ $u \in U$

$v = 0 + w$ $w \in W$

Since the expression for v is unique

So $u + 0 = 0 + w$

$\Rightarrow u = 0, w = 0$

Hence $v = u + w = 0$

So $U \cap W = \{0\}$

Hence condition (ii) is satisfied.

So V is the direct sum of U & W

Theorem If U & W are finite dimensional subspaces of a vector space V over a field F then

(i) $\dim(U+W) = \dim U + \dim W - \dim(U\cap W)$

(ii) If $U\cap W = \{0\}$ then $V = U \oplus W$

and $\dim V = \dim U + \dim W$

Proof.

Suppose that $U\cap W \neq \{0\}$.

Let $\{v_1, v_2, ------, v_r\}$ be a basis for $U\cap W$,

$\{v_1, v_2, ------, v_r, u_1, u_2, -----, u_s\}$ be a basis for U

& $\{v_1, v_2, ------, v_r, w_1, w_2, ----, w_t\}$ be a basis for W

Thus dimensions of $U\cap W$, U & W are r, $r+s$ & $r+t$ resp.

Clearly

$\{v_1, v_2, -----, v_r, u_1, u_2, ----, u_s, w_1, w_2, ----, w_t\}$ spans $U+W$.

Now we show that $\{v_1, v_2, ---, v_r, u_1, u_2, ----, u_s, w_1, ---, w_t\}$ is linearly independent. Suppose that

$$a_1v_1 + a_2v_2 + ----- + a_rv_r + b_1u_1 + b_2u_2 + ---- + b_su_s + c_1w_1 + ----- + c_tw_t = 0 \quad (1)$$

where a's, b's, c's $\in F$

or $$a_1v_1 + a_2v_2 + ----- + a_rv_r + b_1u_1 + b_2u_2 + ----- + b_su_s = -(c_1w_1 + ----- + c_tw_t) \quad (2)$$

from eq. (2) we see that

$-(c_1w_1 + c_2w_2 + ------ + c_tw_t) \in U\cap W$

But $\{v_1, v_2, -------, v_r\}$ is a basis for $U\cap W$. So

$$-(c_1w_1 + c_2w_2 + ----- + c_tw_t) = d_1v_1 + d_2v_2 + ------ + d_rv_r$$

where d's $\in F$

or

$$d_1v_1 + d_2v_2 + ------ + d_rv_r + c_1w_1 + c_2w_2 + ----- + c_tw_t = 0$$

Since $\{v_1, v_2, ------, v_r, w_1, w_2, ----, w_t\}$ being basis of W is linearly independent

$\Rightarrow d_1 = d_2 = ------ = d_r = 0$, $c_1 = c_2 = ------ = c_t = 0$

Then eq. (1) becomes

$$a_1v_1 + a_2v_2 + ------ + a_rv_r + b_1u_1 + ------- + b_su_s = 0$$

Again $\{v_1, v_2, \ldots, v_r, u_1, u_2, \ldots, u_s\}$ being basis of U is linearly independent.

So $a_1 = a_2 = \ldots = a_r = 0$, $b_1 = b_2 = \ldots = b_s = 0$

Hence eq. ① shows that the set

$\{v_1, v_2, \ldots, v_r, u_1, u_2, \ldots u_s, w_1, w_2, \ldots, w_t\}$ is linearly independent set. & so it forms a basis for U+W. So

$$\dim(U+W) = r+s+t$$
$$= (r+s)+(r+t) - r$$

So $\boxed{\dim(U+W) = \dim U + \dim W - \dim(U \cap W)}$

(ii) Now Suppose that $U \cap W = \{0\}$

Let $\{u_1, u_2, \ldots, u_s\}$ be a basis for U

& $\{w_1, w_2, \ldots, w_t\}$ be a basis for W

Then clearly the set $\{u_1, u_2, \ldots, u_s, w_1, w_2, \ldots, w_t\}$ spans $U+W = U \oplus W$

Consider $a_1u_1 + \ldots + a_su_s + b_1w_1 + \ldots + b_tw_t = 0$ — ③

then $a_1u_1 + \ldots + a_su_s = -(b_1w_1 + \ldots + b_tw_t)$

This shows that each vector is in $U \cap W$.

Hence $a_1 = a_2 = \ldots = a_s = 0$, $b_1 = b_2 = \ldots = b_t = 0$

So ③ shows that $\{u_1, \ldots, u_s, w_1, \ldots, w_t\}$ is linearly independent & so forms a basis for $U \oplus W$

Hence $\dim(U \oplus W) = s + t$

or $\dim(V) = \dim U + \dim W$ ∵ $V = U \oplus W$

Exercise No. 6.2

Q1 Determine whether the following vectors in R^4 are linearly independent or linearly dependent:

(i) $(1,3,-1,4),(3,8,-5,7),(2,9,4,23)$

Sol.

Let $a(1,3,-1,4)+b(3,8,-5,7)+c(2,9,4,23)=0 \quad a,b,c\in F$

or $(a,3a,-a,4a)+(3b,8b,-5b,7b)+(2c,9c,4c,23c)=0$

$(a+3b+2c,\ 3a+8b+9c,\ -a-5b+4c,\ 4a+7b+23c)=0$

$\Rightarrow$

$a+3b+2c=0$ ——— (1)

$3a+8b+9c=0$ ——— (2)

$-a-5b+4c=0$ ——— (3)

$4a+7b+23c=0$ ——— (4)

From (1) & (2)

$$\frac{a}{27-16}=\frac{-b}{9-6}=\frac{c}{8-9}$$

or $\dfrac{a}{11}=\dfrac{b}{-3}=\dfrac{c}{-1}=K$

$\Rightarrow \quad a=11K$

$b=-3K$

$c=-K$

Putting these values in (3) & (4), we see eqs. (3) & (4) are satisfied.

Hence given vectors in R^4 are linearly dependent.

(ii) $(1,-2,4,1),(2,1,0,-3),(1,-6,1,4)$

Sol.

Let $a(1,-2,4,1)+b(2,1,0,-3)+c(1,-6,1,4)=0$ where $a,b,c\in F$

or $(a,-2a,4a,a)+(2b,b,0,-3b)+(c,-6c,c,4c)=0$

or $(a+2b+c,\ -2a+b-6c,\ 4a+c,\ a-3b+4c)=0$

$\Rightarrow$

$$a+2b+c = 0 \quad \text{——— (1)}$$
$$-2a+b-6c = 0 \quad \text{——— (2)}$$
$$4a+c = 0 \quad \text{——— (3)}$$
$$a-3b+4c = 0 \quad \text{——— (4)}$$

From (1) & (2)

$$\frac{a}{-12-1} = \frac{-b}{-6+2} = \frac{c}{1+4}$$

$$\frac{a}{-13} = \frac{b}{4} = \frac{c}{5} = K$$

$\Rightarrow$ $a = -13K$
$b = 4K$
$c = 5K$

Putting these values in (3) & (4), we see that eqs. are not satisfied. They are satisfied only when $K = 0$

$\Rightarrow$ $a = b = c = 0$.

Hence given vectors in R^4 are linearly independent.

Q2 Let $V = P_3(x)$ be the vector space of all polynomials of degree ≤ 3 over R together with the zero polynomial. Determine whether $u, v, w \in V$ are linearly dependent or linearly independent.

(i) $u = x^3-4x^2+2x+3$, $v = x^3+2x^2+4x-1$, $w = 2x^3-x^2-3x+3$

Sol.-

Let $au+bv+cw = 0$ where $a,b,c \in F$

or $a(x^3-4x^2+2x+3)+b(x^3+2x^2+4x-1)+c(2x^3-x^2-3x+3) = 0$

or $(a+b+2c)x^3+(-4a+2b-c)x^2+(2a+4b-3c)x+(3a-b+3c) = 0$

$$\Rightarrow \quad a+b+2c = 0 \quad \text{——— (1)}$$
$$-4a+2b-c = 0 \quad \text{——— (2)}$$
$$2a+4b-3c = 0 \quad \text{——— (3)}$$
$$3a-b+3c = 0 \quad \text{——— (4)}$$

From (1) & (2)

$$\frac{a}{-1-4} = \frac{-b}{-1+8} = \frac{c}{2+4}$$

$$\frac{a}{-5} = \frac{b}{-7} = \frac{c}{6} = K$$

$\Rightarrow$ $a = -5K$
$b = -7K$
$c = 6K$

Putting these values in ③ & ④ we see eqs. are not satisfied. They are satisfied only when $K=0$

$\Rightarrow$ $a = b = c = 0$

Hence vectors u, v, w are linearly independent.

(ii) $u = x^3-3x^2-2x+3$, $v = x^3-4x^2-3x+4$, $w = 2x^3-7x^2-7x+9$

Sol.

Let $au + bv + cw = 0$ where $a, b, c \in F$

or $a(x^3-3x^2-2x+3)+b(x^3-4x^2-3x+4)+c(2x^3-7x^2-7x+9) = 0$

or $(a+b+2c)x^3+(-3a-4b-7c)x^2+(-2a-3b-7c)x+(3a+4b+9c)=0$

$\Rightarrow$ $a+b+2c = 0$ ——— ①

$-3a-4b-7c = 0$ ——— ②

$-2a-3b-7c = 0$ ——— ③

$3a+4b+9c = 0$ ——— ④

From ① & ②

$$\frac{a}{-7+8} = \frac{-b}{-7+6} = \frac{c}{-4+3}$$

or $$\frac{a}{1} = \frac{b}{1} = \frac{c}{-1} = K$$

$\Rightarrow$ $a = K$
$b = K$
$c = -K$

Putting these values in ③ & ④, we see eqs. are not satisfied. They are satisfied only when $K = 0$

$\Rightarrow$ $a = b = c = 0$.

Hence given vectors u, v, w are linearly independent.

Q3 Show that the vectors $(1-i, i)$ & $(2, -1+i)$ in C^2 are linearly dependent over C but linearly independent over R.

Sol.

Let $a(1-i, i) + b(2, -1+i) = 0$

or $(a(1-i), ai) + (2b, b(-1+i)) = 0$

or $(a(1-i)+2b, ai + b(-1+i)) = 0$

$\Rightarrow \quad a(1-i) + 2b = 0 \quad \text{———} \quad (1)$

$\quad\quad ai + (-1+i)b = 0 \quad \text{———} \quad (2)$

These eqs. are satisfied in R only when $a=b=0$

Hence given vectors are linearly independent over R.

Now we find the values of a & b from the set C which satisfy eqs. (1) & (2)

From (1)

$$a(1-i) = -2b$$

$$\frac{a}{b} = \frac{-2}{1-i}$$

$$= \frac{-2}{1-i} \times \frac{1+i}{1+i}$$

$$= \frac{-2(1+i)}{1+1}$$

$$= \frac{-2(1+i)}{2}$$

$$\frac{a}{b} = -(1+i)$$

or $$\frac{a}{1+i} = \frac{b}{-1} = K$$

$$\Rightarrow \quad a = (1+i)K$$

$$b = -K$$

Putting these values in eq. (2), we see eq. (2) is satisfied.

Hence given vectors are linearly dependent over C

Q4 Show that the vectors $(3+\sqrt{2}, 1+\sqrt{2})$ & $(7, 1+2\sqrt{2})$ in R^2 are linearly dependent over R but linearly independent over Q.

Sol.

Let $a(3+\sqrt{2}, 1+\sqrt{2}) + b(7, 1+2\sqrt{2}) = 0$

or $\left(a(3+\sqrt{2}), a(1+\sqrt{2})\right) + \left(7b, b(1+2\sqrt{2})\right) = 0$

or $\left(a(3+\sqrt{2})+7b, a(1+\sqrt{2}) + b(1+2\sqrt{2})\right) = 0$

$\Rightarrow$

$$(3+\sqrt{2})a + 7b = 0 \qquad (1)$$

& $$(1+\sqrt{2})a + (1+2\sqrt{2})b = 0 \qquad (2)$$

These eqs. are satisfied in Q only when $a=b=0$

Hence the given vectors are linearly independent over Q.

Now we will find values of a & b in R which satisfy the eqs. (1) + (2)

From (1)

$$(3+\sqrt{2})a = -7b$$

or $$\frac{a}{b} = \frac{-7}{3+\sqrt{2}}$$

$$\Rightarrow \frac{a}{7} = \frac{-b}{3+\sqrt{2}} = K$$

$$\Rightarrow a = 7K$$

& $$b = -(3+\sqrt{2})K$$

Putting these values in eq. (2), we see eq. (2) is satisfied.

Hence given vectors are dependent over R.

Q5 Suppose that u, v & w are linearly independent vectors. Prove that

(i) $u+v-2w$, $u-v-w$, $u+w$ are linearly independent.

(ii) $u+v-3w$, $u+3v-w$, $u+w$ are linearly independent.

(i) Sol.

Let $a(u+v-2w)+b(u-v-w)+c(u+w)=0$

or $(a+b+c)u+(a-b)v+(-2a-b+c)w=0$

But u, v & w are linearly independent

So
$$a+b+c=0 \quad \text{---} (1)$$
$$a-b=0 \quad \text{---} (2)$$
$$-2a-b+c=0 \quad \text{---} (3)$$

From (1) & (3)

$$\frac{a}{\begin{vmatrix}1 & 1\\ -1 & 1\end{vmatrix}} = \frac{-b}{\begin{vmatrix}1 & 1\\ -2 & 1\end{vmatrix}} = \frac{c}{\begin{vmatrix}1 & 1\\ -2 & -1\end{vmatrix}}$$

or

$$\frac{a}{1+1} = \frac{-b}{1+2} = \frac{c}{-1+2}$$

$$\frac{a}{2} = \frac{b}{-3} = \frac{c}{1} = K$$

$$\Rightarrow \quad a = 2K$$
$$b = -3K$$
$$\& \quad c = K$$

Putting these values in (2), we see that eq. (2) is not satisfied. It is satisfied only when $K=0$

$\Rightarrow a=b=c=0$

Hence given vectors are linearly independent.

(ii) Sol.

Let $a(u+v-3w)+b(u+3v-w)+c(u+w)=0$

or $(a+b+c)u+(a+3b)v+(-3a-b+c)w=0$

But u, v & w are linearly independent

$\Rightarrow$
$$a+b+c=0 \quad \text{---} (1)$$
$$a+3b=0 \quad \text{---} (2)$$
$$-3a-b+c=0 \quad \text{---} (3)$$

From ① & ③

$$\frac{a}{\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}} = \frac{-b}{\begin{vmatrix} 1 & 1 \\ -3 & 1 \end{vmatrix}} = \frac{c}{\begin{vmatrix} 1 & 1 \\ -3 & -1 \end{vmatrix}}$$

$$\frac{a}{1+1} = \frac{-b}{1+3} = \frac{c}{-1+3}$$

$$\frac{a}{2} = \frac{b}{-4} = \frac{c}{2} = K$$

$$\Rightarrow \quad a = 2K$$
$$b = -4K$$
$$c = 2K$$

Putting these values in eq. ②, we see eq. ② is not satisfied. It is satisfied only if $K=0$

$$\Rightarrow \quad a = b = c = 0$$

Hence given vectors are linearly independent.

Q6 Determine K so that the vectors $(1,-1,K-1),(2,K,-4)$ $(0,2+K,-8)$ in R^3 are linearly dependent.

Sol.- Suppose that given vectors are linearly dependent then one of them must be a linear combination of the other two

Let $(1,-1,K-1) = a(2,K,-4) + b(0,2+K,-8)$ where $a,b \in F$

$$= (2a, aK, -4a) + (0, b(2+K), -8b)$$

or $(1,-1,K-1) = (2a, aK + b(2+K), -4a - 8b)$

$$\Rightarrow \quad 2a = 1 \quad \text{——— ①}$$
$$aK + b(2+K) = -1 \quad \text{——— ②}$$
$$-4a - 8b = K-1 \quad \text{——— ③}$$

from ① $\quad a = \frac{1}{2}$

Put in ② & ③

② $\Rightarrow \quad \frac{K}{2} + b(2+K) = -1$ ——— ④

③ $\Rightarrow \quad -4\left(\frac{1}{2}\right) - 8b = K-1$

$$-2 - 8b = K-1$$

$$-8b = K-1+2$$

$$-8b = K+1$$

$$b = -\frac{K+1}{8}$$

Put this value in (4)

$$\frac{K}{2} + (2+K)\left(-\frac{K+1}{8}\right) = -1$$

$$4K - (2+K)(K+1) = -8$$

$$4K - 2K - 2 - K^2 - K + 8 = 0$$

$$-K^2 + K + 6 = 0$$

or $K^2 - K - 6 = 0$

$$K^2 - 3K + 2K - 6 = 0$$

$$K(K-3) + 2(K-3) = 0$$

$$(K-3)(K+2) = 0$$

$$\Rightarrow \boxed{K = 3, -2}$$

So for $K = 3, -2$; given vectors are linearly dependent.

Q7 Using the technique of casting out vectors which are linear combination of others, find a linearly independent subset of the given set spanning the same subspace:

(i) $\{(1,-3,1), (2,1,-4), (-2,6,-2), (-1,10,-7)\}$ in R^3

Sol.

Given set is $\{(1,-3,1), (2,1,-4), (-2,6,-2), (-1,10,-7)\}$

We see that

$$(-2,6,-2) = -2(1,-3,1)$$

So we cast out $(-2,6,-2)$ & obtain the subset

$$\{(1,-3,1), (2,1,-4), (-1,10,-7)\}$$

Suppose, $(-1,10,-7) = a(1,-3,1) + b(2,1,-4)$ for $a, b \in R$

or $(-1,10,-7) = (a+2b, -3a+b, a-4b)$

$$\Rightarrow a + 2b = -1 \quad \text{——— (1)}$$

$$-3a + b = 10 \quad \text{——— (2)}$$

$$a - 4b = -7 \quad \text{——— (3)}$$

Multiplying ① by 2 & adding in ③

$$2a+\cancel{4b} = -2 \quad \text{———} ④$$
$$a-\cancel{4b} = -7 \quad \text{———} ③$$

$$3a = -9$$
$$\boxed{a=-3}$$

Put in ③

$$-3-4b = -7$$
$$-4b = -7+3$$
$$-4b = -4$$
$$\boxed{b=1}$$

So $(-1,10,-7) = -3(1,-3,1)+1(2,1,-4)$

Hence $(-1,10,-7)$ is a linear combination of $(1,-3,1)$ & $(2,1,-4)$

We cast out $(-1,10,-7)$ & obtain a subset

$A=\{(1,-3,1),(2,1,-4)\}$

Since $(2,1,-4)$ is not a multiple of $(1,-3,1)$ so the set $A=\{(1,-3,1),(2,1,-4)\}$ is req. linearly independent set which spans the same subspace as the given set of four vectors.

(ii) $\{1, \sin^2 x, \cos 2x, \cos^2 x\}$ in the space of all functions from R to R.

Sol

Given set is $\{1, \sin^2 x, \cos 2x, \cos^2 x\}$

As

$$\cos 2x = \cos^2 x - \sin^2 x$$

or

$$\cos 2x = 1.\cos^2 x + (-1).\sin^2 x$$

So $\cos 2x$ is a linear combination of $\cos^2 x$ & $\sin^2 x$

We cast out $\cos 2x$ & obtain the subset

$$\{1, \sin^2 x, \cos^2 x\}$$

Also $\cos^2 x = 1-\sin^2 x$

or $\cos^2 x = 1.1 + (-1)\sin^2 x$

So $\cos^2 x$ is a linear Combination of 1 & $\sin^2 x$

We cast out $\cos^2 x$ & obtain the subset

$\{1, \sin^2 x\}$

Since none of 1 & $\sin^2 x$ is multiple of other

So $\{1, \sin^2 x\}$ is req. linearly independent set which spans the same subspace as the given set of four vectors.

(iii) $\{1, 3x-4, 4x+3, x^2+2, x-x^2\}$ in the space $P_2(x)$ of polynomials.

Sol.

Given set is $\{1, 3x-4, 4x+3, x^2+2, x-x^2\}$

We see that

$$3x-4 = \frac{3}{4}(4x+3) - \frac{25}{4}(1)$$

So $3x-4$ is a linear Combination of $4x+3$ & 1

We cast out $3x-4$ & obtain the subset

$\{1, 4x+3, x^2+2, x-x^2\}$

Now

Let $x-x^2 = a(1) + b(4x+3) + c(x^2+2)$

or $x-x^2 = (a+3b+2c) + (4b)x + cx^2$

$$\Rightarrow \quad a+3b+2c = 0 \quad \text{——— } (1)$$

$$4b = 1 \quad \text{——— } (2)$$

$$c = -1 \quad \text{——— } (3)$$

(3) $\Rightarrow$ $\boxed{c = -1}$

(2) $\Rightarrow$ $\boxed{b = \frac{1}{4}}$

(1) $\Rightarrow$ $a + \frac{3}{4} - 2 = 0$

$$a - \frac{5}{4} = 0$$

$$\boxed{a = 5/4}$$

So $x - x^2 = \frac{5}{4}(1) + \frac{1}{4}(4x+3) - 1(x^2+2)$

Hence $x - x^2$ is a linear Combination of $1, 4x+3$ & x^2+2

We Cast out $x - x^2$ & obtain the subset

$A = \{1, 4x+3, x^2+2\}$

We check whether A is linearly independent or not

Let $a(1) + b(4x+3) + c(x^2+2) = 0$

or $(a+3b+2c) + 4bx + cx^2 = 0$

$$\Rightarrow \quad a+3b+2c = 0 \quad \text{——— (1)}$$

$$4b = 0 \quad \text{——— (2)}$$

$$c = 0 \quad \text{——— (3)}$$

(3) $\Rightarrow$ $\boxed{c = 0}$

(2) $\Rightarrow$ $\boxed{b = 0}$

Put in (1)

$a + 0 + 0 = 0 \Rightarrow \boxed{a = 0}$

Hence the set $A = \{1, 4x+3, x^2+2\}$ is linearly independent set which spans the same subspace as the given set of five vectors.

Q8 verify that the polynomials $2 - x^2$, $x^3 - x$, $2 - 3x^2$ & $3 - x^3$ form a basis for $P_3(x)$. Express each of (i) $1 + x$ & (ii) $x + x^2$ as a linear Combination of these basis vectors.

Sol.

We want to show that $\{2 - x^2, x^3 - x, 2 - 3x^2, 3 - x^3\}$ form a basis for $P_3(x)$.

First we prove that this set is linearly independent.

Let $a(2-x^2) + b(x^3-x) + c(2-3x^2) + d(3-x^3) = 0 \quad a, b, c, d \in F$

or $(2a+2c+3d) - bx + (-a-3c)x^2 + (b-d)x^3 = 0$

$$\Rightarrow \quad 2a + 2c + 3d = 0 \quad \text{——— (1)}$$

$$-b = 0 \quad \text{——— (2)}$$

$$-a - 3c = 0 \quad \text{——— (3)}$$

$$b - d = 0 \quad \text{——— (4)}$$

(2) $\Rightarrow$ $\boxed{b = 0}$

(4) $\Rightarrow$ $\boxed{b = d}$

(3) $\Rightarrow$ $a = -3c$

Put in (1)

$2.(-3c) + 2c + 3(0) = 0$

$-6c + 2c = 0$

$-4c = 0 \Rightarrow \boxed{c = 0}$

Put in (3)

$-a - 3(0) = 0 \Rightarrow \boxed{a = 0}$

Hence the given set of polynomials is linearly independent

As dimension of $P_3(x)$ is 4

& no. of vectors in set $\{2-x^2, x^3-x, 2-3x^2, 3-x^3\}$

So the given set $\{2-x^2, x^3-x, 2-3x^2, 3-x^3\}$ form a basis for $P_3(x)$.

Now we express $1+x$ as a linear combination of given vectors.

Let $1+x = a(2-x^2) + b(x^3-x) + c(2-3x^2) + d(3-x^3)$

or $1+x = (2a+2c+3d) - bx + (-a-3c)x^2 + (b-d)x^3$

where $a, b, c, d \in R$

$\Rightarrow$ $2a + 2c + 3d = 1$ ——— (1)

$-b = 1$ ——— (2)

$-a - 3c = 0$ ——— (3)

$b - d = 0$ ——— (4)

(2) $\Rightarrow$ $\boxed{b = -1}$

Put in (4)

$-1 - d = 0 \Rightarrow \boxed{d = -1}$

(3) $\Rightarrow$ $a = -3c$

Put in (1)

$2(-3c) + 2c + 3(-1) = 1$

$-4c - 3 = 1$

$-4c = 4$

$\Rightarrow \boxed{c = -1}$

Put in (3)

$-a - 3(-1) = 0$

$-a + 3 = 0 \Rightarrow \boxed{a = 3}$

So $1 + x = 3(2 - x^2) - (x^3 - x) - (2 - 3x^2) - (3 - x^3)$

(ii) Now we express $x + x^2$ as a linear combination of given vectors

Let $x + x^2 = a(2 - x^2) + b(x^3 - x) + c(2 - 3x^2) + d(3 - x^3)$

where $a, b, c, d \in R$

or $x + x^2 = (2a + 2c + 3d) - bx + (-a - 3c)x^2 + (b - d)x^3$

$\Rightarrow 2a + 2c + 3d = 0$ ——— (1)

$-b = 1$ ——— (2)

$-a - 3c = 1$ ——— (3)

$b - d = 0$ ——— (4)

(2) $\Rightarrow \boxed{b = -1}$

(4) $\Rightarrow -1 - d = 0$ or $\boxed{d = -1}$

(3) $\Rightarrow a = -1 - 3c$

Put in (1)

$2(-1 - 3c) + 2c + 3(-1) = 0$

$-2 - 6c + 2c - 3 = 0$

$-4c - 5 = 0$

$4c = -5 \Rightarrow \boxed{c = -\frac{5}{4}}$

Put in (3)

$-a - 3(-5/4) = 1$

$-a + \frac{15}{4} = 1$

$a = \frac{15}{4} - 1$

$\boxed{a = \frac{11}{4}}$

So $x + x^2 = \frac{11}{4}(2 - x^2) - (x^3 - x) - \frac{5}{4}(2 - 3x^2) - (3 - x^3)$

Q9 Determine whether or not the given set of vectors is a basis for R^n.

(i) $\{(1,1),(3,1)\}$

Sol. Given set is $\{(1,1),(3,1)\}$
First we will check their independency
For scalars $a,b \in R$
Let $a(1,1)+b(3,1) = 0$
or $(a+3b, a+b) = 0$
$\Rightarrow$

$$a+3b = 0 \quad \text{---} \quad (1)$$
$$a+b = 0 \quad \text{---} \quad (2)$$

Subt. (2) from (1)

$$2b = 0 \Rightarrow \boxed{b=0}$$

Put in (1)

$$a+0 = 0 \Rightarrow \boxed{a=0}$$

Hence given set of vectors is linearly independent.
Since dimension of R^2 is 2
& the linearly independent vectors are also 2.
So the given set of vectors $\{(1,1),(3,1)\}$ form a basis for R^2

(ii) $\{(2,1),(1,-1)\}$

Sol. Given set is $\{(2,1),(1,-1)\}$
First we will check their independency.
For scalars $a,b \in R$
Let $a(2,1)+b(1,-1) = 0$
or $(2a+b, a-b) = 0$

$$\Rightarrow \quad 2a+b = 0 \quad \text{---} \quad (1)$$
$$\& \quad a-b = 0 \quad \text{---} \quad (2)$$

Adding (1) + (2)

$$3a = 0 \Rightarrow \boxed{a=0}$$

Put in ②

$0 - b = 0 \Rightarrow \boxed{b = 0}$

Hence given set of vectors $\{(2,1),(1,-1)\}$ is linearly independent.

Since dimension of R^2 is 2

& no. of linearly independent vectors are also 2

So $\{(2,1),(1,-1)\}$ forms a basis for R^2

Q10 Determine whether or not the given set of vectors is a basis for R^3:

(i) $\{(1,2,-1),(0,3,1),(1,-5,3)\}$

Sol. Given set is $\{(1,2,-1),(0,3,1),(1,-5,3)\}$

First we will check their independency.

For scalars $a,b,c \in R$

Let $a(1,2,-1) + b(0,3,1) + c(1,-5,3) = 0$

or $(a+c,\ 2a+3b-5c,\ -a+b+3c) = 0$

$$\Rightarrow \quad a + c = 0 \quad \text{—— ①}$$

$$2a + 3b - 5c = 0 \quad \text{—— ②}$$

$$-a + b + 3c = 0 \quad \text{—— ③}$$

From ② & ③

$$\frac{a}{9+5} = \frac{-b}{6-5} = \frac{c}{2+3}$$

$$\frac{a}{14} = \frac{b}{-1} = \frac{c}{5} = K$$

$$\Rightarrow \quad a = 14K$$

$$b = -K$$

$$c = 5K$$

Putting these values in ①, we see eq. ① is not satisfied. It is satisfied only when $K = 0$

$\Rightarrow a = 0,\ b = 0,\ c = 0$

Hence given set of vectors $\{(1,2,-1),(0,3,1),(1,-5,3)\}$ is linearly independent.

Since dimension of R^3 is 3

& the no. of linearly independent vectors in R^3 is also 3.

So the set $\{(1,2,-1),(0,3,1),(1,-5,3)\}$ forms a basis for R^3.

(ii) $\{(2,4,-3),(0,1,1),(0,1,-1)\}$

Soln

Given set is $\{(2,4,-3),(0,1,1),(0,1,-1)\}$

First we will check their independency.

For scalars $a, b, c \in R$

Let $a(2,4,-3)+b(0,1,1)+c(0,1,-1)=0$

or $(2a, 4a+b+c, -3a+b-c) = 0$

$\Rightarrow$ $2a = 0$ —— ①

$4a+b+c = 0$ —— ②

$-3a+b-c = 0$ —— ③

① $\Rightarrow$ $\boxed{a=0}$

Put in ② & ③

$b+c = 0$ —— ②

$b-c = 0$ —— ③

Adding ② & ③

$2b=0 \Rightarrow \boxed{b=0}$

Put in ②

$0+c=0 \Rightarrow \boxed{c=0}$

Hence given set of vectors $\{(2,4,-3),(0,1,1),(0,1,-1)\}$ is linearly independent.

Since dimension of R^3 is 3

& no. of linearly independent vectors in R^3 is also 3

So the set $\{(2,4,-3),(0,1,1),(0,1,-1)\}$ forms a basis for R^3

Q11 Let V be the real vector space of all functions defined on R into R. Determine whether the given vectors are linearly independent or linearly dependent in V:

(i) x, $\cos x$

Sol:- Given vectors are $x, \cos x$

Suppose that for scalars $a, b \in R$

$a x + b\cos x = 0$ ——— ① for all $x \in R$

Put $x = 0$ in ①

$a(0) + b\cos 0 = 0$

$0 + b = 0 \Rightarrow \boxed{b = 0}$

Now Put $x = \pi/2$ in ①

$a(\pi/2) + b\cos \pi/2 = 0$

$a(\pi/2) + 0 = 0$

$\boxed{a = 0}$

Hence given vectors are linearly independent.

(ii) $\sin^2 x$, $\cos^2 x$, $\cos 2x$

Sol:- Given vectors are $\sin^2 x$, $\cos^2 x$, $\cos 2x$

Suppose that for scalars $a, b, c \in R$.

$a\sin^2 x + b\cos^2 x + c\cos 2x = 0$ ——— ① for all $x \in R$

Put $x = 0$ in ①

$a\sin^2(0) + b\cos^2 0 + c\cos 0 = 0$

$b + c = 0$ ——— ②

Put $x = \pi/2$ in ①

$a\sin^2\frac{\pi}{2} + b\cos^2\pi/2 + c\cos\pi = 0$

$a - c = 0$ ——— ③

Put $x = \pi/4$ in ①

$a\sin^2\pi/4 + b\cos^2\pi/4 + c\cos\pi/2 = 0$

or $\frac{a}{2} + \frac{b}{2} = 0$

or $\quad a+b=0$ ———— (4)

(ii) $\Rightarrow$ $b=-c$

(i) $\Rightarrow$ $a=c$

So non zero soln. of above eqs. is

$a=c$

$b=-c$

$c=c$ $\quad$ or $(c,-c,c)$ $\quad$ where $c\in R$

So given vectors are linearly dependent

(iii) $\sin x, \cos x, \sinh x, \cosh x$

Sol.

Given vectors are $\sin x, \cos x, \sinh x, \cosh x$

Suppose that for scalars $a,b,c,d\in R$

$a\sin x+b\cos x+c\sinh x+d\cosh x=0$ ———— (A)

Put $x=0$

$a\sin 0+b\cos 0+c\sinh 0+d\cosh 0=0$

$b+d=0$ ———— (1)

Diff. (A) w.r.t. x

$a\cos x-b\sin x+c\cosh x+d\sinh x=0$ ———— (B)

Put $x=0$

$a\cos 0-b\sin 0+c\cosh 0+d\sinh 0=0$

$a+c=0$ ———— (2)

Diff. (B) w.r.t. x

$-a\sin x-b\cos x+c\sinh x+d\cosh x=0$ ———— (C)

Put $x=0$

$-a\sin 0-b\cos 0+c\sinh 0+d\cosh 0=0$

or $\quad -b+d=0$ ———— (3)

Diff. (C) w.r.t. x

$-a\cos x+b\sin x+c\cosh x+d\sinh x=0$

Put $x=0$

$-a\cos 0+b\sin 0+c\cosh 0+d\sinh 0=0$

$$-a + c = 0 \quad \text{------} (4)$$

Adding (1) & (3)

$$2d = 0 \Rightarrow \boxed{d = 0}$$

Put in (1)

$$b + 0 = 0 \Rightarrow \boxed{b = 0}$$

Adding (2) & (4)

$$2c = 0 \Rightarrow \boxed{c = 0}$$

Put in (4)

$$-a + 0 = 0 \Rightarrow \boxed{a = 0}$$

Hence given vectors are linearly independent.

(iv) $\sin x$, $\sin x + \cos x$, $\sin x - \cos x$

Sol. Given vectors are $\sin x$, $\sin x + \cos x$, $\sin x - \cos x$

Suppose that for scalars $a, b, c \in R$

$$a \sin x + b(\sin x + \cos x) + c(\sin x - \cos x) = 0$$

or $(a+b+c)\sin x + (b-c)\cos x = 0 \quad \text{------} (A)$

Put $x = 0$ in (A)

$$(a+b+c)\sin 0 + (b-c)\cos 0 = 0$$

or $b - c = 0 \quad \text{------} (1)$

Put $x = \pi/2$ in (A)

$$(a+b+c)\sin \pi/2 + (b-c)\cos \pi/2 = 0$$

or $a + b + c = 0 \quad \text{------} (2)$

Put $x = \pi$ in (A)

$$(a+b+c)\sin \pi + (b-c)\cos \pi = 0$$

or $(b-c)(-1) = 0$

or $b - c = 0 \quad \text{------} (3)$

(3) $\Rightarrow$ $b = c$

Put in (2)

$$a + c + c = 0 \Rightarrow \boxed{a = -2c}$$

So a non zero soln. of here eqs. is $(-2c, c, c)$

Cont.

Hence the given vectors are linearly dependent. 83

(v) e^{ax}, e^{bx}, e^{cx} ; a, b, c being distinct real no's.

Sol.

Given vectors are e^{ax}, e^{bx}, e^{cx}

Suppose that for scalars $\alpha, \beta, \gamma \in R$

$$\alpha e^{ax} + \beta e^{bx} + \gamma e^{cx} = 0$$

Q12. Determine a basis for each of the following subspace of R^3:

(i) The plane $x - 2y + 5z = 0$

Sol.

Given eq. of plane is $x - 2y + 5z = 0$

or $x = 2y - 5z$ where y, z are free variables

The above eq. in vector form can be written as

$$(x, y, z) = (2y - 5z, y, z)$$
$$= (2y - 5z, y + 0, 0 + z)$$
$$= (2y, y, 0) + (-5z, 0, z)$$
$$\Rightarrow (x, y, z) = y(2, 1, 0) + z(-5, 0, 1)$$

Thus given plane is spanned by vectors $(2,1,0)$ & $(-5,0,1)$. Since none of the vector is multiple of other. So the set $\{(2,1,0), (-5,0,1)\}$ is linearly independent.

Hence $\{(2,1,0), (-5,0,1)\}$ forms a basis for given subspace of R^3.

(ii) The line $\frac{x}{-2} = \frac{y}{1} = \frac{z}{6}$

Sol.

Given eq. of line is

$$\frac{x}{-2} = \frac{y}{1} = \frac{z}{6} = t$$

$\Rightarrow$ $x = -2t$

$y = t$

& $z = 6t$

The above eq. in vector form can be written as

$$(x, y, z) = (-2t, t, 6t)$$

or $(x, y, z) = t(-2, 1, 6)$

Hence the given line is spanned by the vector $(-2, 1, 6)$. Also $(-2,1,6)$ being a non zero single vector is linearly independent

So $\{(-2,1,6)\}$ forms a basis for the given subspace of $\mathbb{R}^3$.

(iii) All vectors of the form (a,b,c) where $3a-2b+c=0$

Soln.

Given eq. is

$$3a-2b+c=0$$

or $\quad c = -3a+2b \quad$ where a, b are free variables

The above eq. can be written in vector form as

$$(a,b,c) = (a,b,-3a+2b)$$
$$= (a+0,\ 0+b,\ -3a+2b)$$
$$= (a,0,-3a) + (0,b,2b)$$

or $(a,b,c) = a(1,0,-3) + b(0,1,2)$

So given subspace is spanned by $(1,0,-3)$ & $(0,1,2)$.

Now since none of the vectr is multiple of other. So the vectors $(1,0,-3)$ & $(0,1,2)$ are linearly independent

So the set $\{(1,0,-3),(0,1,2)\}$ forms a basis for the given subspace of $\mathbb{R}^3$.

Q13 Find the dimension of the subspace $\{(x_1,x_2,x_3,x_4) : x_2 = x_3\}$ of $\mathbb{R}^4$. Also determine a basis.

Soln.

Let $W = \{(x_1,x_2,x_3,x_4) : x_2 = x_3\}$

Suppose (x_1,x_2,x_2,x_4) be a general vector of W then we can write it as

$$(x_1,x_2,x_2,x_4) = (x_1,0,0,0) + (0,x_2,x_2,0) + (0,0,0,x_4)$$
$$= x_1(1,0,0,0) + x_2(0,1,1,0) + x_4(0,0,0,1)$$

Which shows that $(x_1, x_2, x_2, x_4) \in W$ is a linear Combination of vectors $(1,0,0,0), (0,1,1,0)$ & $(0,0,0,1)$.
So the set $S = \{(1,0,0,0), (0,1,1,0), (0,0,0,1)\}$ spans W.
Now we check the independency of set S.
Suppose that for scalars $a, b \in R$

$$a(1,0,0,0) + b(0,1,1,0) + c(0,0,0,1) = (0,0,0,0)$$

or $(a, b, b, c) = (0,0,0,0)$

$$\Rightarrow \quad a = 0$$
$$b = 0$$
$$c = 0$$

Hence the set $S = \{(1,0,0,0), (0,1,1,0), (0,0,0,1)\}$ is also linearly independent.
Hence S is a basis for W
So dimension of W is 3

Q14 A subspace U of R^4 is spanned by the vectors $(1,0,2,3)$ & $(0,1,-1,2)$ & a subspace W is spanned by $(1,2,3,4), (-1,-1,5,0)$ & $(0,0,0,1)$. Find the dimensions of U & W.

Sol.

Let $S = \{(1,0,2,3), (0,1,-1,2)\}$
Since these vectors span U
So we only check their independency.
Suppose that for scalars $a, b \in R$

$$a(1,0,2,3) + b(0,1,-1,2) = 0$$

or $(a, b, 2a-b, 3a+2b) = (0,0,0,0)$

$$\Rightarrow \left.\begin{aligned} a &= 0 \\ b &= 0 \\ 2a-b &= 0 \\ 3a+2b &= 0 \end{aligned}\right\} \Rightarrow a = b = 0$$

Which shows that set S is linearly independent

Hence $S = \{(1,0,2,3),(0,1,-1,2)\}$ forms a basis for U

So dimension of $U = 2$

(ii) $(1,2,3,4),(-1,-1,5,0),(0,0,0,1)$

Sol.

Let $S = \{(1,2,3,4),(-1,-1,5,0),(0,0,0,1)\}$

Since these vectors span W (given)

So we only check their independency.

Suppose that for scalars $a, b \in R$

$a(1,2,3,4)+b(-1,-1,5,0)+c(0,0,0,1) = 0$

or $(a-b, 2a-b, 3a+5b, 4a+c) = (0,0,0,0)$

$\Rightarrow$

$a - b = 0$ ——— ①

$2a - b = 0$ ——— ②

$3a + 5b = 0$ ——— ③

$4a + c = 0$ ——— ④

Subt. ② from ①

$-a = 0 \Rightarrow \boxed{a = 0}$

① $\Rightarrow \boxed{b = 0}$

④ $\Rightarrow \boxed{c = 0}$

Hence the set $S = \{(1,2,3,4),(-1,-1,5,0),(0,0,0,1)\}$ is linearly independent.

So $S = \{(1,2,3,4),(-1,-1,5,0),(0,0,0,1)\}$ forms a basis for W

Hence dimension of $W = 3$

Q15 Suppose that U & W are distinct four dimensional subspaces of a vector space V of dimension six. Find the possible dimension of $U \cap W$.

Sol. We are given that

$\dim U = 4$

$\dim W = 4$

& $\dim V = 6$

Since $U+W$ is a subspace of V

So $\dim(U+W) \leq \dim V = 6$

$\Rightarrow \dim(U+W) \leq 6$

Now as $U \subseteq U+W$ & $W \subseteq U+W$

So $\dim U \leq \dim(U+W) \leq 6$

or $4 \leq \dim(U+W) \leq 6$

Hence $\dim(U+W)$ is 4 or 5 or 6

Since U & W are distinct, so they must be different by atleast one generator

So $\dim(U+W) > 4$

Hence $\dim(U+W)$ is 5 or 6

As we know

$$\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$$

or $$\dim(U \cap W) = \dim U + \dim W - \dim(U+W)$$

(i) If $\dim(U+W) = 5$

Then $\dim(U \cap W) = 4+4-5 = 3$

(ii) If $\dim(U+W) = 6$

then $\dim(U \cap W) = 4+4-6 = 2$

Hence the possible dimensions of $U \cap W$ are 2 or 3

Q16 Find a basis & dimension of the subspace W of R^4 spanned by

(i) $(1,4,-1,3),(2,1,-3,-1)$ & $(0,2,1,-5)$

(ii) $(1,-4,-2,1),(1,-3,-1,2)$ & $(3,-8,-2,7)$

Sol:-

As the subspace W of R^4 is spanned by $(1,4,-1,3)$, $(2,1,-3,-1)$ & $(0,2,1,-5)$

Now we only check their independency

For this let for $a,b,c \in F$

$$a(1,4,-1,3)+b(2,1,-3,-1)+c(0,2,1,-5)=(0,0,0,0)$$

$$(a+2b,\ 4a+b+2c,\ -a-3b+c,\ 3a-b-5c)=(0,0,0,0)$$

$$\Rightarrow \quad a+2b=0 \text{ ——— } (1)$$

$$4a+b+2c=0 \text{ ——— } (2)$$

$$-a-3b+c=0 \text{ ——— } (3)$$

$$3a-b-5c=0 \text{ ——— } (4)$$

from (1) & (2)

$$\frac{a}{4-0}=\frac{-b}{2-0}=\frac{c}{1-8}$$

$$\frac{a}{4}=\frac{b}{-2}=\frac{c}{-7}=k$$

$$\Rightarrow \left.\begin{aligned} a &= 4k \\ b &= -2k \\ c &= -7k \end{aligned}\right]$$

Put in (3) & (4), we see eqs. are not satisfied & they are satisfied only when $k=0$

i.e., when $a=b=c=0$

Which shows that given vectors are linearly independent

So $\{(1,4,-1,3),(2,1,-3,-1),(0,2,1,-5)\}$ form a basis of W

Hence $\dim W = 3$

(ii) $(1,-4,-2,1),(1,-3,-1,2),(3,-8,-2,7)$

Sol. Given vectors are

$(1,-4,-2,1)$, $(1,-3,-1,2)$ & $(3,-8,-2,7)$

As the subspace W of R^4 is spanned by $(1,-4,-2,1)$, $(1,-3,-1,2)$ & $(3,-8,-2,7)$.

So we only check their independency.

For this let for scalars a, b, c

$a(1,-4,-2,1) + b(1,-3,-1,2) + c(3,-8,-2,7) = (0,0,0,0)$

or $(a+b+3c, -4a-3b-8c, -2a-b-2c, a+2b+7c) = (0,0,0,0)$

$$\Rightarrow \quad a+b+3c = 0 \quad \text{——— (1)}$$

$$-4a-3b-8c = 0 \quad \text{——— (2)}$$

$$-2a-b-2c = 0 \quad \text{——— (3)}$$

$$a+2b+7c = 0 \quad \text{——— (4)}$$

from (1) & (2)

$$\frac{a}{-8+9} = \frac{-b}{-8+12} = \frac{c}{-3+4}$$

$$\frac{a}{1} = \frac{b}{-4} = \frac{c}{1} = k$$

$$\Rightarrow \left.\begin{aligned} a &= k \\ b &= -4k \\ c &= k \end{aligned}\right]$$

Put these values in (3) & (4), we see these eqs. are satisfied so the given vectors are linearly dependent.

Now we take only first two vectors $(1,-4,-2,1)$ & $(1,-3,-1,2)$ & check their independency.

Suppose for $a, b \in F$

$a(1,-4,-2,1) + b(1,-3,-1,2) = (0,0,0,0)$

or $(a+b, -4a-3b, -2a-b, a+2b) = (0,0,0,0)$

$$\Rightarrow \quad a+b = 0 \quad \text{——— (1)}$$

$$-4a-3b = 0 \quad \text{——— (2)}$$

$$-2a-b = 0 \quad \text{——— (3)}$$

$$a+2b = 0 \quad \text{——— (4)}$$

Adding ① & ③

$-a = 0$ or $\boxed{a = 0}$

Put in ①

$0 + b = 0 \Rightarrow \boxed{b = 0}$

Hence the vectors are linearly independent & so the set $\{(1,-4,-2,1),(1,-3,-1,2)\}$ form a basis of W

Hence $\dim W = 2$

Q17 Let U & W be 2-dimensional subspaces of R^3. Show that $U \cap W \neq \{0\}$

Sol. Given that

$\dim U = 2$

& $\dim W = 2$

If $U = W$ then $\dim U = \dim W = 2$

So $U \cap W \neq \{0\}$

Hence we suppose that $U \neq W$. This means that U & W are not spanned by the same set.

So $\dim(U+W) > 2$

Since U & W are subspaces of R^3

Hence $\dim(U+W) \leq 3$

So $2 < \dim(U+W) \leq 3$

Hence $\dim(U+W) = 3$

As $\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$

So $\dim(U \cap W) = \dim U + \dim W - \dim(U+W)$

$= 2 + 2 - 3$

$\therefore \dim(U \cap W) = 1$

It shows that $U \cap W$ contains a non zero element & so $U \cap W \neq \{0\}$
