M. 6

(Vector spaces) (Chapter No 6)

Vector Space

Let F be a field & V a non empty set on which an operation of addition is defined. Suppose for every $a \in F$ & $v \in V$, av is an element of V then V is called a vector space over F if the following conditions are satisfied

(i) V is an abelian gr under addition

(ii) $a(bu) = (ab)u$ $a, b \in F$, $u \in V$

(iii) $(a+b)u = au + bu$ $a, b \in F$, $v \in V$

(iv) $a(u+w) = au + aw$ $a \in F$, $v, w \in V$

(v) $1v = v$ where 1 is unity of F

Note ① The elements of F are called scalars & the elements of V are called vectors

② If V is a vector space over F, we write it as $V(F)$

③ We do not multiply elements of V with any ~~and then we multiply an element~~ of F by an element of V

Theorem Let V be a vector space over F then

(i) $a0 = 0$ for $a \in F$

(ii) $0v = 0$ for $v \in V$

(iii) $(-a)v = a(-v) = -av$ for $a \in F$, $v \in V$

(iv) If $av = 0$ then either $a = 0$ or $v = 0$

(v) $a(u - v) = au - av$ for $a \in F$, $u, v \in V$

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Examples

(1) If $\lambda = 6327.9$ E

Proof

$\because 0 = 0+0$ ($\because 0$ is add... identity of V) $\forall u \in V$

$\therefore a0 = a(0+0)$

$a0 = a0 + a0$ (by iv)

$a0+0 = a0 + a0$ $\because 0$ is identity of V

$\therefore 0 = a0$ $\because$ V is a gr. so cancellation law holds

or $\boxed{a0 = 0}$

(ii) $\because 0 = 0+0$ $\because 0$ is additive identity of F

$0u = (0+0)u$ $\forall u \in V$

or $0u = 0u + 0u$

$0u + 0 = 0u + 0u$ ($\because 0$ is identity of V)

So $0 = 0u$ ($\because$ V is a gr. so cancellation law holds)

$\therefore \boxed{0u = 0}$

(iii) $\because (-a)u + au = (-a+a)u$

$= 0u$

$= 0$

So $(-a)u$ & au are additive inverses of each other

$\therefore \boxed{(-a)u = -au}$

Also $a(-u) + au = a(-u+u)$ by (iv)

$= a0$

$= 0$

So $a(-u)$ & au are additive inverses of each other

$a(-u) = -au$

So $(-a)u = a(-u) = -au$

(iv)
let $a \neq 0$ so that a^{-1} exists s.t. $aa^{-1} = a^{-1}a = 1$

Now
$u = 1 \cdot u$ (by V)

$= (a^{-1}a)u$

$= a^{-1}(au)$ (by (ii))

$= a^{-1}(0)$

$= 0$

So if $a \neq 0$ then $u = 0$

Similarly if $u \neq 0$ then $a = 0$

So $au = 0 \Rightarrow$ either $a = 0$ or $u = 0$

(v) $a(u-v) = a(u+(-v))$

$= au + a(-v)$ by (iv)

$= au + (-av)$ just proved

$= au - av$

So $a(u-v) = au - av$

Subspace
A non empty subset W of a vector space V is called a subspace of V if W itself is a vector space over F under the same operation as defined in V

·Empty

V. Therefore W is a subspace of V

Corollary A non empty subset W of a vector space V is a subspace of V iff for $w_1, w_2 \in W$ & $a, b \in F \Rightarrow aw_1 + bw_2 \in W$

Proof.

We shall establish that conditions (i) & (ii) of above theorem imply the condition of this corollary & viceversa.

For this suppose that W is a subspace of V then by (ii) condition

$aw_1, bw_2 \in W$

& therefore by (i) condition $aw_1 + bw_2 \in W$

Conversly

Let W be a non empty subset of V in which the condition

$w_1, w_2 \in W$ & $a, b \in F \Rightarrow aw_1 + bw_2 \in W$ holds

We shall prove that W is a subspace of V

Take $a = b = 1$ in above condition

$1w_1 + 1 \cdot w_2 = w_1 + w_2 \in W$

Now take $b = 0$

So $aw_1 + 0w_2 = aw_1 \in W$

Hence by previous theorem

W is a subspace of V

therum tat U of W la subspaces of a vector space V over F then $U \wedge W$ is also a

Proof,

let $a, b \in F$ a $w_1, w_2 \in U \wedge W$

$\Rightarrow w_1, w_2 \in U$ ' d $w_1, w_2 \in W$

But U + W are subspaces of V

So $aw_1 + bw_2 \in U$

+ $aw_1 + bw_2 \in W$

Hence $aw_1 + bw_2 \in U \cap W$

So $a, b \in F$ + $w_1, w_2 \in U \cap W \Rightarrow aw_1 + bw_2 \in U \wedge W$

Hence $U \wedge W$ is a subspace of V.

thareni The intersection of any no. of subspaus of a vector space V is a subspace of V

Proof

Let $\{U_\alpha : \alpha \in I\}$ be any subcallection of subspaus of a vector space V over the field F Then we have to prove that $\bigcap_{\alpha \in I} U_\alpha$ is also a subspace of V

For this let $a, b \in F$ + $u_1, u_2 \in \bigcap_{\alpha \in I} U_\alpha$

$\Rightarrow u_1, u_2 \in U_\alpha$ for all $\alpha \in I$

But each U_α is a subspace of V

So $au_1 + bu_2 \in U_\alpha$ for each $\alpha \in I$

I

$\Rightarrow au_1 + bu_2 \in \bigcap_{\alpha \in I} U_2$

So $\bigcap_{\alpha \in I} U_\alpha$ is a subspace of V.

Sum of two Subspaces

let U & W be two subspaces of a vector space V. We define U+W as

$U+W = \{u+w = u \in U + w \in W\}$

Theorem If U, W are subspaces of a vector space V then U+W is a subspace of V containing both U, W. Further U+W is the smallest subspace of V containing both U + W.

Proof Given U & W are subspaces of V then we define

$U+W = \{u+w \mid u \in U + w \in W\}$

we will prove that U+W is a subspace of V

For this let $a, b \in F$ $v_1, v_2 \in U+W$

$\Rightarrow v_1 = u_1 + w_1$

$+ \quad v_2 = u_2 + w_2$

where $u_1, u_2 \in U$ + $w_1, w_2 \in W$

Now since U is a subspace of V so $au_1 + bu_2 \in U$

+ since W is a subspace of V, so $aw_1 + bw_2 \in W$

Hence $(au_1 + bv_2) + (aw_1 + bw_2) \in U+W$

or $(au_1 + aw_1) + (bu_2 + bw_2) \in U+W$

$a(u_1+w_1) + b(u_2+w_2) \in U+W$

or $ax_1 + bx_2 \in U+W$

So for $a, b \in F$ & $x_1, x_2 \in U+W \Rightarrow ax_1 + bx_2 \in U+W$

Hence $U+W$ is a subspace of V

Next we prove that $U+W$ is a subspace of V containing both U & W. i.e, $U \subseteq U+W$ + $W \subseteq U+W$

Since $u \in U$ + $0 \in W$

$\Rightarrow u+0 = u \in U+W$ for all $u \in U$

So $U \subseteq U+W$

Similarly $W \subseteq U+W$

Hence $U+W$ is a subspace of V containing both U + W

Now we will prove that $U+W$ is the smallest subspace of V containing both U & W

let S be any subspace of V containing both $U+W$ then for every $u \in U$ + $w \in W$,

we have $u \in S$ + $w \in S$ so that $u+w \in S$

But $u+w \in U+W$

So $U+W \subseteq S$

Hence $U+W$ is the smallest subspace of V containing both U + W.

Direct sum:

A vector space V is called the direct sum of its subspaces U & W if

(i) $V = U + W$

(ii) $U \cap W = \{0\}$

Linear Combination:

Let V be a vector space over a field F & let $u_1, u_2, u_3, \dots, u_n \in V$

Any vector in V of the form.

$a_1u_1 + a_2u_2 + \dots + a_nu_n$ where $a_i \in F$

is called a linear combination of $v_1, v_2, \dots, u_n$

Linear span:

Let S be a non empty subset of a vector space V Than the set of all linear combination of finite no. of elements of S is called the linear span of S & is denoted by $\langle S \rangle$

Note: $\langle S \rangle$ is said to be spanned or generated by S a S is called a spanning set for $\langle S \rangle$

Theorem: Let S be a non empty set of vectors in a vector space V over a field F Than $\langle S \rangle$ is a subspace of V containing S & it is the smallest subspace of V containing S

Proof

Let $a, b \in F$ & $u, u \in \langle S \rangle$ then.

$u = a_1u_1 + a_2u_2 + \dots a_nu_n = \sum_{i=1}^{n} a_iu_i$

$u = b_1u_1 + b_1u_2 + \dots + b_mu_m = \sum_{j=1}^{m} b_ju_j$

where $u_i, u_j \in S$, $a_i, b_j \in F$

Now $au + bv = a\left(\sum_{i=1}^{n} a_i u_i\right) + b\left(\sum_{j=1}^{m} b_j v_j\right)$

$$= \sum_{i=1}^{n} a(a_i u_i) + \sum_{j=1}^{m} b(b_j v_j)$$

$$= \sum_{i=1}^{n} (aa_i)u_i + \sum_{j=1}^{m} (bb_j)v_j \qquad \because a(bu) = (ab)u$$

Which shows that $au+bv$ is a linear Combination of vectors in S. So $au+bv \in \langle S \rangle$

So $a, b \in F$, $u, v \in \langle S \rangle \Rightarrow au+bv \in \langle S \rangle$

Hence $\langle S \rangle$ is a subspace of V.

(Now)

Since for all $u \in S$

$$u = 1.u \quad \Rightarrow \quad u \in \langle S \rangle$$

So $S \subseteq \langle S \rangle$

Hence $\langle S \rangle$ is a subspace of V Containing S.

Now we prove that $\langle S \rangle$ is the smallest subspace of V Containing S.

If W is any other subspace of V Containing S. then it Contains all vectors of the form $\sum_{i=1}^{n} a_i u_i$; where $a_i \in F$ & $u_i \in S$

$\Rightarrow \langle S \rangle \subseteq W$

Thus $\langle S \rangle$ is the smallest subspace of V Containing S.

<u>Theorem</u>:- If S, T are subsets of V then $S \subset T$ implies $\langle S \rangle \subset \langle T \rangle$

<u>Proof</u>:-

Let $S = \{v_1, v_2, \ldots\ldots, v_r\}$

& $T = \{v_1, v_2, \ldots\ldots, v_r, v_{r+1}, \ldots\ldots, v_n\}$

then obviously

$$S \subset T$$

We want to show that $\langle S \rangle \subset \langle T \rangle$. For this 11

let $v \in \langle S \rangle$ then by definition of $\langle S \rangle$, v is a linear combination of vectors $v_1, v_2, \ldots\ldots, v_r$ of S

i.e., $v = a_1 v_1 + a_2 v_2 + \cdots\cdots + a_r v_r$

Now v can also be written as

$$v = a_1 v_1 + a_2 v_2 + \cdots\cdots + a_r v_r + 0 v_{r+1} + 0 v_{r+2} + \cdots + 0 v_n$$

Which shows that v is a linear combination of vectors of T

Hence $v \in \langle T \rangle$

So $\langle S \rangle \subset \langle T \rangle$

Finite dimensional vector space:

A vector space V is said to be finite dimensional if there is a finite subset S in V such that $V = \langle S \rangle$

Exercise No. 6.1

Q1 Let V be the set of all infinite sequences in a field F with addition & scalar multiplication defined as below:

For $u = \{a_n\} = a_1, a_2, \ldots\ldots, a_n, \ldots\ldots \in V$

$v = \{b_n\} = b_1, b_2, \ldots\ldots, b_n, \ldots\ldots \in V$

$u + v = \{a_n\} + \{b_n\} = a_1+b_1, a_2+b_2, \ldots\ldots, a_n+b_n, \ldots\ldots$

& $ku = k\{a_n\} = ka_1, ka_2, \ldots\ldots, ka_n, \ldots\ldots$

where a_n, b_n & k are all in F, $n = 1, 2, 3, \ldots\ldots$

Show that V is a vector space over F.

Sol.

Here $V = \{(a_1, a_2, \ldots\ldots) \mid a_i \in F\}$

First we prove that $(V, +)$ is an abelian gr.

(a)

(i) Closure law

~~Let $v_1 = (a_1, a_2, \ldots\ldots)$~~

& $v_2 = (b_1, b_2, \ldots\ldots\ldots)$

$\Rightarrow v_1 + v_2 = (a_1, a_2, \ldots\ldots) + (b_1, b_2, \ldots\ldots)$

$= (a_1 + b_1, a_2 + b_2, \ldots\ldots\ldots) \in V$

(ii) Association law

Let $v_1 = (a_1, a_2, \ldots\ldots)$

$v_2 = (b_1, b_2, \ldots\ldots)$

& $v_3 = (c_1, c_2, \ldots\ldots) \in V$

then we prove $v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3$

Now

$v_1 + (v_2 + v_3) = (a_1, a_2, \ldots\ldots) + [(b_1, b_2, \ldots\ldots) + (c_1, c_2, \ldots\ldots)]$

$= (a_1, a_2, \ldots\ldots) + [(b_1 + c_1), (b_2 + c_2), \ldots\ldots]$

$= [a_1 + (b_1 + c_1), a_2 + (b_2 + c_2), \ldots\ldots]$

$= [(a_1 + b_1) + c_1, (a_2 + b_2) + c_2, \ldots\ldots]$

$= [(a_1 + b_1), (a_2 + b_2), \ldots\ldots] + (c_1, c_2, \ldots\ldots)$

$= [(a_1, a_2, \ldots\ldots) + (b_1, b_2, \ldots\ldots)] + (c_1, c_2, \ldots\ldots)$

$= (v_1 + v_2) + v_3$

(iii) Identity law

Here $0 = (0, 0, \ldots\ldots)$ is the additive identity in V because for any $v = (a_1, a_2, \ldots\ldots) \in V$

$v + 0 = (a_1, a_2, \ldots\ldots) + (0, 0, \ldots\ldots)$

$= (a_1 + 0, a_2 + 0, \ldots\ldots)$

$= (a_1, a_2, \ldots\ldots)$

$v + 0 = v$

Similarly $0 + v = v$

(iv) Inverse law

Every $v = (a_1, a_2, \ldots\ldots) \in V$ has additive inverse

$-v = (-a_1, -a_2, \ldots\ldots) \in V$ because

$v + (-v) = (a_1, a_2, \ldots\ldots) + (-a_1, -a_2, \ldots\ldots)$

$= (a_1 - a_1, a_2 - a_2, \ldots\ldots)$

$u+(-u) = (0, 0, \ldots\ldots)$

$u+(-u) = 0$

Similarly $-u+u = 0$

(V) Commutative law

Let $u_1 = (a_1, a_2, \ldots\ldots)$

& $u_2 = (b_1, b_2, \ldots\ldots)$

then we prove $u_1+u_2 = u_2+u_1$

Now

$$u_1+u_2 = (a_1, a_2, \ldots\ldots) + (b_1, b_2, \ldots\ldots)$$
$$= (a_1+b_1, a_2+b_2, \ldots\ldots)$$
$$= (b_1+a_1, b_2+a_2, \ldots\ldots)$$
$$= (b_1, b_2, \ldots\ldots) + (a_1, a_2, \ldots\ldots)$$
$$u_1+u_2 = u_2+u_1$$

Hence $(V, +)$ is an abelian gr.

(b) Scalar multiplication:

(i) Let $a \in F$ & $u = (a_1, a_2, \ldots\ldots) \in V$ then

$$au = a(a_1, a_2, \ldots\ldots)$$
$$= (aa_1, aa_2, \ldots\ldots) \in V$$

(ii)

Let $a, b \in F$ & $u = (u_1, u_2, \ldots\ldots) \in V$

then we prove $a(bu) = (ab)u$

Now $a(bu) = a[b(a_1, a_2, \ldots\ldots)]$

$$= a[(ba_1, ba_2, \ldots\ldots)]$$
$$= [a(ba_1), a(ba_2), \ldots\ldots]$$
$$= [(ab)a_1, (ab)a_2, \ldots\ldots]$$
$$= (ab)(a_1, a_2, \ldots\ldots)$$

$\therefore \; a(bu) = (ab)u$

(iii)

Let $a, b \in F$ & $u = (a_1, a_2, \ldots\ldots)$

then we prove $(a+b)v = av + bv$

Now.

$(a+b)v = (a+b)(a_1, a_2, \ldots\ldots)$

$= [(a+b)a_1, (a+b)a_2, \ldots\ldots]$

$= [(aa_1+ba_1), (aa_2+ba_2), \ldots\ldots]$

$= (aa_1, aa_2, \ldots\ldots) + (ba_1, ba_2, \ldots\ldots)$

$= a(a_1, a_2, \ldots\ldots) + b(a_1, a_2, \ldots\ldots)$

$(a+b)v = av + bv$

(iv) Let $a \in F$ & $v_1 = (a_1, a_2, \ldots\ldots)$, $v_2 = (b_1, b_2, \ldots\ldots) \in V$

then we show $a(v_1+v_2) = av_1 + av_2$

Now

$a(v_1+v_2) = a[(a_1, a_2, \ldots\ldots) + (b_1, b_2, \ldots\ldots)]$

$= a[(a_1+b_1, a_2+b_2, \ldots\ldots]$

$= [a(a_1+b_1), a(a_2+b_2), \ldots\ldots]$

$= [(aa_1+ab_1, aa_2+ab_2, \ldots\ldots)]$

$= (aa_1, aa_2, \ldots\ldots) + (ab_1, ab_2, \ldots\ldots)$

$= a(a_1, a_2, \ldots\ldots) + a(b_1, b_2, \ldots\ldots)$

$\therefore a(v_1+v_2) = av_1 + av_2$

(v) Let $1 \in F$ & $v = (a_1, a_2, \ldots\ldots) \in V$ then we prove

$1.v = v$

Now $1.v = 1.(a_1, a_2, \ldots\ldots)$

$= (1.a_1, 1.a_2, \ldots\ldots)$

$= (a_1, a_2, \ldots\ldots)$

$1.v = v$

Since all Conditions are satisfied.

So V is a vector space over F.

Q2 Let V be the set of all ordered pairs of real no's.[15]
Check whether V is a vector space over R w.r.t. the indicated operations. If not, state the axioms which fail to hold.

(i) $(a,b)+(c,d) = (a+c, b+d)$

& $K(a,b) = (Ka, b)$

Sol:- Here $V = \{(a,b) \mid a,b \in R\}$

& $(a,b)+(c,d) = (a+c, b+d)$

$K(a,b) = (Ka, b)$

First we prove that $(V,+)$ is an abelian gr.

(a)
(i) Closure law

Let $u_1 = (a,b)$ & $u_2 = (c,d)$ $\in V$

then $u_1+u_2 = (a,b)+(c,d)$
$= (a+c, b+d) \in V$

(ii) Associative law

Let $u_1 = (a,b)$, $u_2 = (c,d)$, $u_3 = (e,f) \in V$ then
we prove $u_1+(u_2+u_3) = (u_1+u_2)+u_3$

Now

$$
\begin{aligned}
u_1+(u_2+u_3) &= (a,b)+\big((c,d)+(e,f)\big) \\
&= (a,b)+(c+e, d+f) \\
&= \big(a+(c+e), b+(d+f)\big) \\
&= \big((a+c)+e, (b+d)+f\big) \\
&= (a+c, b+d)+(e,f) \\
&= \big((a,b)+(c,d)\big)+(e,f)
\end{aligned}
$$

$$u_1+(u_2+u_3) = (u_1+u_2)+u_3$$

(iii) Identity law

Here $O = (0,0)$ is the additive identity in V because for $u = (a,b) \in V$

$$
\begin{aligned}
0+u &= (0,0)+(a,b) \\
&= (0+a, 0+b) \\
&= (a,b)
\end{aligned}
$$

$0+u = u$. Similarly $u+0 = u$.

(iv) Inverse law

Every element $v = (a,b) \in V$ has its additive inverse $-v = (-a,-b)$ in V because

$v + (-v) = (a,b) + (-a,-b) = (a-a, b-b) = (0,0) = 0$

& $-v + v = (-a,-b) + (a,b) = (-a+a, -b+b) = (0,0) = 0$

(v) Commutative law

Let $v_1 = (a,b)$, $v_2 = (c,d) \in V$ then we prove $v_1 + v_2 = v_2 + v_1$

Now

$$v_1 + v_2 = (a,b) + (c,d) = (a+c, b+d) = (c+a, d+b)$$
$$= (c,d) + (a,b) = v_2 + v_1$$

Hence $(V,+)$ is an abelian gr.

(b) Scalar multiplication

(i) Let $a \in R$ & $v_1 = (a_1,b_1) \in V$

then $av_1 = a(a_1,b_1) = (aa_1, b_1) \in V$

(ii) Let $a,b \in R$ & $v_1 = (a_1,b_1) \in V$

Then we prove $a(bv_1) = (ab)v_1$

Now $a(bv_1) = a(b(a_1,b_1)) = a(ba_1, b_1) = (a(ba_1), b_1)$

$= ((ab)a_1, b_1) = (ab)(a_1,b_1) = (ab)v_1$

(iii) Let $a,b \in R$ & $v_1 = (a_1,b_1) \in V$ then we prove

$(a+b)v_1 = av_1 + bv_1$

Now $(a+b)v_1 = (a+b)(a_1,b_1) = ((a+b)a_1, b_1) = ((aa_1+ba_1), b_1)$

& $av_1 + bv_1 = a(a_1,b_1) + b(a_1,b_1) = (aa_1,b_1) + (ba_1,b_1) = (aa_1+ba_1, 2b_1)$

So $(a+b)v_1 \neq av_1 + bv_1$

Since this condition is not satisfied.

So V is not a vector space over R.

(ii) $(a,b) + (c,d) = (a,b)$ & $k(a,b) = (ka, kb)$

Sol. Let $V = \{(a,b) \mid a,b \in R\}$

Here $(a,b) + (c,d) = (a,b)$

& $k(a,b) = (ka, kb)$

First we prove $(V,+)$ is an abelian gr.

(a)

(i) <u>closure law</u>

Let $u_1 = (a,b)$, $u_2 = (c,d) \in V$ then

$u_1 + u_2 = (a,b) + (c,d)$

$= (a,b) \in V$

(ii) <u>Associative law</u>

Let $u_1 = (a,b)$, $u_2 = (c,d)$, $u_3 = (e,f) \in V$ then

we prove $u_1 + (u_2 + u_3) = (u_1 + u_2) + u_3$

Now

$u_1 + (u_2 + u_3) = (a,b) + [(c,d) + (e,f)]$

$= (a,b) + (c,d)$

$= (a,b)$

& $(u_1 + u_2) + u_3 = [(a,b) + (c,d)] + (e,f)$

$= (a,b) + (e,f)$

$= (a,b)$

So $u_1 + (u_2 + u_3) = (u_1 + u_2) + u_3$

(iii) <u>Identity law</u>

There is no identity element in V

As this condition is not satisfied. So V is not a vector space over R.

(iii) $(a,b) + (c,d) = (a+c, b+d)$ & $k(a,b) = (k^2 a, k^2 b)$

Sol. Let $V = \{(a,b) \mid a,b \in R\}$

Here $(a,b) + (c,d) = (a+c, b+d)$

& $k(a,b) = (k^2 a, k^2 b)$

First we prove that $(V,+)$ is an abelian gr.

(a)(i) <u>closure law</u>

Let $u_1 = (a,b)$, $u_2 = (c,d) \in V$ then

$u_1 + u_2 = (a,b) + (c,d) = (a+c, b+d) \in V$

(ii) <u>Associative law</u>

Let $u_1 = (a,b)$, $u_2 = (c,d)$, $u_3 = (e,f) \in V$

then we prove $v_1+(v_2+v_3) = (v_1+v_2)+v_3$.

Now

$$\begin{aligned} v_1+(v_2+v_3) &= (a,b)+\left[(c,d)+(e,f)\right] \\ &= (a,b)+(c+e,d+f) \\ &= (a+(c+e),\ b+(d+f)) \\ &= ((a+c)+e,\ (b+d)+f) \\ &= (a+c,b+d)+(e,f) \\ &= \left[(a,b)+(c,d)\right]+(e,f) \end{aligned}$$

$\Rightarrow$ $v_1+(v_2+v_3) = (v_1+v_2)+v_3$

(iii) <u>Identity law</u>

Here $0=(0,0)$ is the additive identity in V because for $v=(a,b)\in V$

$0+v = (0,0)+(a,b) = (0+a,0+b) = (a,b) = v$

& $v+0 = (a,b)+(0,0) = (a+0,b+0) = (a,b) = v$

(iv) <u>Inverse law</u>

Each element $v=(a,b)\in V$ has its additive inverse $-v=(-a,-b)$ because

$v+(-v) = (a,b)+(-a,-b) = (a-a,b-b) = (0,0) = 0$

& $-v+v = (-a,-b)+(a,b) = (-a+a,-b+b) = (0,0) = 0$

(V) <u>Commutative law</u>

Let $v_1=(a,b),\ v_2=(c,d)\in V$ then we prove $v_1+v_2 = v_2+v_1$

Now

$$\begin{aligned} v_1+v_2 &= (a,b)+(c,d) \\ &= (a+c,b+d) \\ &= (c+a,d+b) \\ &= (c,d)+(a,b) \\ &= v_2+v_1 \end{aligned}$$

Hence $(V,+)$ is an abelian gr.

(b) <u>Scalar multiplication</u>

(i) Let $\alpha\in R$ & $v_1=(a_1,b_1)\in V$

then $\alpha v_1 = \alpha(a_1,b_1) = (\alpha^2 a_1, \alpha^2 b_1)\in V$

(ii) Let $a, b \in R$ & $u_1 = (a_1, b_1) \in V$

then we prove $a(bu_1) = (ab)u_1$

Now $a(bu_1) = a(b(a_1, b_1))$

$= a(b^2a_1, b^2b_1)$

$= (a^2(b^2a_1), a^2(b^2b_1))$

$= ((a^2b^2)a_1, (a^2b^2)b_1)$

$= (ab)(a_1, b_1)$

$\therefore a(bu_1) = (ab)u_1$

(iii) Let $a, b \in R$ & $u_1 = (a_1, b_1) \in V$ then we prove

$(a+b)u_1 = au_1 + bu_1$

Now $(a+b)u_1 = (a+b)(a_1, b_1)$

$= ((a+b)^2a_1, (a+b)^2b_1)$

&

$au_1 + bu_1 = a(a_1, b_1) + b(a_1, b_1)$

$= (a^2a_1, a^2b_1) + (b^2a_1, b^2b_1)$

$= (a^2a_1 + b^2a_1, a^2b_1 + b^2b_1)$

$= ((a^2+b^2)a_1, (a^2+b^2)b_1)$

So $(a+b)u_1 \neq au_1 + bu_1$

As this condition is not satisfied.

So V is not a vector space over R.

(iv) $(a, b) + (c, d) = (a+c, b+d)$ & $K(a, b) = (Ka, 0)$

Sol.-

Let $V = \{(a, b) \mid a, b \in R\}$

Here $(a, b) + (c, d) = (a+c, b+d)$

& $K(a, b) = (Ka, 0)$

First we prove that $(V, +)$ is an abelian gr.

(a)(i) Closure law.

Let $u_1 = (a, b)$, $u_2 = (c, d) \in V$ then

$u_1 + u_2 = (a, b) + (c, d)$

$= (a+c, b+d) \in V$

(ii) Associative law

Let $u_1 = (a,b)$, $u_2 = (c,d)$, $u_3 = (e,f) \in V$
then we prove
$$u_1+(u_2+u_3) = (u_1+u_2)+u_3$$
Now
$$\begin{aligned} u_1+(u_2+u_3) &= (a,b)+[(c,d)+(e,f)] \\ &= (a,b)+(c+e,d+f) \\ &= (a+(c+e), b+(d+f)) \\ &= ((a+c)+e, (b+d)+f) \\ &= (a+c, b+d)+(e,f) \\ &= [(a,b)+(c,d)]+(e,f) \end{aligned}$$
$\Rightarrow u_1+(u_2+u_3) = (u_1+u_2)+u_3$

(iii) Identity law

Here $O = (0,0)$ is the additive identity in V because for $u = (a,b) \in V$

$O + u = (0,0)+(a,b) = (0+a,0+b) = (a,b) = u$

& $u + O = (a,b)+(0,0) = (a+0,b+0) = (a,b) = u$

(iv) Inverse law

Each element $u = (a,b) \in V$ has its additive inverse $-u = (-a,-b)$ in V because

$u+(-u) = (a,b)+(-a,-b) = (a-a, b-b) = (0,0) = O$

& $-u+u = (-a,-b)+(a,b) = (-a+a, -b+b) = (0,0) = O$

(v) Commutative law

Let $u_1 = (a,b)$ & $u_2 = (c,d) \in V$ then we prove
$$u_1+u_2 = u_2+u_1$$
Now
$$\begin{aligned} u_1+u_2 &= (a,b)+(c,d) \\ &= (a+c, b+d) \\ &= (c+a, d+b) \\ &= (c,d)+(a,b) \\ &= u_2+u_1 \end{aligned}$$

Hence $(V,+)$ is an abelian gr.

(b) Scalar multiplication.

(i) Let $a \in R$ & $u_1 = (a_1,b_1) \in V$ then

$a u_1 = a(a_1,b_1) = (aa_1, 0) \in V$

(ii) Let $a, b \in R$ & $u_1 = (a_1, b_1) \in V$ then we prove

$a(bu_1) = (ab)u_1$

Now

$$a(bu_1) = a\big(b(a_1, b_1)\big)$$
$$= a(ba_1, 0)$$
$$= (a(ba_1), 0)$$
$$= ((ab)a_1, 0)$$
$$= (ab)(a_1, b_1)$$

$\therefore a(bu_1) = (ab)u_1$

(iii) Let $a, b \in R$ & $u_1 = (a_1, b_1) \in V$ then we prove

$(a+b)u_1 = au_1 + bu_1$

Now

$$(a+b)u_1 = (a+b)(a_1, b_1)$$
$$= ((a+b)a_1, 0)$$
$$= (aa_1 + ba_1, 0)$$
$$= (aa_1, 0) + (ba_1, 0)$$
$$= a(a_1, b_1) + b(a_1, b_1)$$

$\therefore (a+b)u_1 = au_1 + bu_1$

(iv) Let $a \in R$ & $u_1 = (a_1, b_1)$, $u_2 = (a_2, b_2) \in V$ then we prove $a(u_1 + u_2) = au_1 + au_2$

Now

$$a(u_1 + u_2) = a\big((a_1, b_1) + (a_2, b_2)\big)$$
$$= a(a_1 + a_2, b_1 + b_2)$$
$$= (a(a_1 + a_2), 0)$$
$$= (aa_1 + aa_2, 0)$$
$$= (aa_1, 0) + (aa_2, 0)$$
$$= a(a_1, b_1) + a(a_2, b_2)$$

$\therefore a(u_1 + u_2) = au_1 + au_2$

(v) Let $1 \in R$ & $u = (a, b) \in V$ then we prove $1.u = u$

Now

$$1.u = 1.(a, b)$$
$$= (1.a, 0) = (a, 0) \neq (a, b) = u$$

So $1.u \neq u$

As this condition is not satisfied.

So V is not a vector space over R

Q3 Check whether each of the following is a real vector space. 22

(i) The set $C[a,b]$ of all continuous real valued functions defined on $[a,b]$ with the usual operations on functions as

For $f, g \in C[a,b]$ & $\alpha \in R$

$$(f+g)(x) = f(x) + g(x)$$

$$\& \quad (\alpha f)(x) = \alpha \cdot f(x).$$

Sol.

Let $C[a,b] = \{f : f \text{ is continuous real valued fn. defined on } [a,b]\}$

then clearly $C[a,b]$ is a subset of the vector space V of all real valued continuous functions defined on R.

To show that $C[a,b]$ is a vector space over R, we have to show that $C[a,b]$ is a subspace of V

For this

let $f, g \in C[a,b]$

then both f & g are real valued continuous fns. defined on $[a,b]$.

Then $(f+g)(x) = f(x) + g(x)$ for $x \in [a,b]$

Now $f+g$ being the sum of two continuous real valued functions defined on $[a,b]$ is also a continuous real valued function defined on $[a,b]$

So $f+g \in C[a,b]$.

Hence $f, g \in C[a,b] \Rightarrow f+g \in C[a,b]$

Now let $\alpha \in R$ & $f \in C[a,b]$

then f is a real valued continuous fn. defined on $[a,b]$.

& $(\alpha f)(x) = \alpha \cdot f(x)$

Clearly scalar multiple of a continuous real valued function is also a continuous real valued function defined on $[a,b]$.

So $\alpha \in R$, $f \in C[a,b] \Rightarrow \alpha f \in C[a,b]$

Hence $C[a,b]$ is a subspace of the vector space V of all real valued functions defined on R

Hence $C[a,b]$ is a vector space over R

(ii) The set of all functions $f \in C[a,b]$ such that $f(a) = f(b)$

Sol:-

Let $C'[a,b] = \{ f \mid f \in C[a,b]$ & $f(a) = f(b) \}$

then Clearly $C'[a,b]$ is a subset of $C[a,b]$

To show that $C'[a,b]$ is a vector space over R, we have to show that $C'[a,b]$ is a subspace of the vector space $C[a,b]$.

For this

let $f, g \in C'[a,b]$

then $f, g \in C[a,b]$ s.t.

$$f(a) = f(b) \quad \& \quad g(a) = g(b)$$

Now $f+g$ being sum of two real valued Continuous functions defined on $[a,b]$ is a real valued Continuous function defined on $[a,b]$.

Hence $f+g \in C[a,b]$

Moreover $(f+g)(a) = f(a) + g(a)$

$$= f(b) + g(b)$$

$$= (f+g)(b)$$

i.e. $(f+g)(a) = (f+g)(b)$

$\Rightarrow f+g \in C'[a,b]$

So $f, g \in C'[a,b] \Rightarrow f+g \in C'[a,b]$

Now

let $\alpha \in R$ & $f \in C'[a,b]$ then clearly αf is a real valued Continuous function defined on $[a,b]$,

So $\alpha f \in C'[a,b]$

Moreover

$(af)(a) = af(a)$

$= af(b)$ $\quad \because f \in C'[a,b]$ so $f(a)=f(b)$

$\therefore (af)(a) = (af)(b)$

Hence $af \in C'[a,b]$

So $a \in R, f \in C'[a,b] \Rightarrow af \in C'[a,b]$

Hence $C'[a,b]$ is a subspace of $C[a,b]$

So $C[a,b]$ is a vector space over R

(iii) The set of all solutions of the diff. eq.

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

Sol.

Let W be the set of all solutions of the given diff. eq. Then V is a subset of the vector space V of all real functions defined on R.

To show that W is a vector space over R, we have to show that W is a subspace of V over R.

For this

let $f, g \in W$ & $a, b \in R$

then f & g are solns. of diff. eq. $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

Then $\left.\begin{array}{l} \frac{df}{dx^2} - 5\frac{df}{dx} + 6f = 0 \\ \& \ \frac{d^2g}{dx^2} - 5\frac{dg}{dx} + 6g = 0 \end{array}\right]$

Now

$$\frac{d^2}{dx^2}(af+bg) - 5\frac{d}{dx}(af+bg) + 6(af+bg)$$

$$= \frac{d^2}{dx^2}(af) + \frac{d^2}{dx^2}(bg) - 5\frac{d}{dx}(af) - 5\frac{d}{dx}(bg) + 6af + 6bg$$

$$= a\frac{d^2f}{dx^2} + b\frac{d^2g}{dx^2} - 5a\frac{df}{dx} - 5b\frac{dg}{dx} + 6af + 6bg$$

$$= a\left(\frac{d^2f}{dx^2} - 5\frac{df}{dx} + 6f\right) + b\left(\frac{d^2g}{dx^2} - 5\frac{dg}{dx} + 6g\right)$$

$$= a(0) + b(0)$$

$$= 0$$

So $af+bg$ is a soln. of $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

Hence $af+bg \in W$

So W is a subspace of V over R

Hence W is a vector space over R.

(iv) The set of all 2×2 real matrices of the form $\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}$

Sol.

Let $V = \left\{ \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} : a, b \in R \right\}$

First we prove that V is an abelian gr. under matrix addition.

This set V has no additive identity

So V is not a gr. under addition.

As this condition is not satisfied.

So V is not a vector space over R.

Q4 Check whether each of the following subsets is a subspace of the indicated vector space:

(i) Q, the set of rational no's. in R

Sol.

Let $a, b \in R$ & $q_1, q_2 \in Q$

then q_1 & q_2 are rational numbers

Since $a, b \in R$ & $q_1, q_2 \in Q$

So $aq_1 + bq_2$ may not be a rational no.

Hence $aq_1 + bq_2 \notin Q$

So $a, b \in R$, $q_1, q_2 \in Q \Rightarrow aq_1 + bq_2 \notin Q$

Hence Q is not a subspace of R

(ii) All 2×2 non singular real matrices in M_{22}.

Sol.

Let V be the set of all non singular real matrices

in M_{22}.
As additive identity $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ of V does not belong to V. So V itself is not a vector space over R. Hence V is not a subspace of M_{22}.

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iii) The set $B[a,b]$ of all bounded real functions defined on $[a,b]$ in the space of all real functions defined on $[a,b]$.

Soln

Here $B[a,b]$ is the set of all bounded real functions defined on $[a,b]$

Let $a,b \in R$ & $f,g \in B[a,b]$

Then f & g are bounded real functions defined on $[a,b]$ then $af+bg$ is also a bounded real valued function defined on $[a,b]$.

So $af+bg \in B[a,b]$

Hence $a,b \in R$ & $f,g \in B[a,b] \Rightarrow af+bg \in B[a,b]$

So $B[a,b]$ is the subspace of the vector space of all real functions defined on $[a,b]$.

Q5 Show that the union of two subspaces of a vector space need not be a subspace. Let X & Y be subspaces of a vector space V. Prove that $X \cup Y$ is a subspace of V if & only if either $X \subset Y$ or $Y \subset X$.

Soln

Consider the Euclidean space R^3 where

$R^3 = \{(x,y,z) : x,y,z \in R\}$

Let $X = \{(x_1,0,0) : x_1 \in R\}$

& $Y = \{(0,x_2,0) : x_2 \in R\}$

Then clearly X & Y are subspaces of R^3.

We shall show that $X \cup Y$ is not a subspace of R^3.

Then $x = (x_1, 0, 0)$ & $y = (0, x_2, 0) \in Y$

$\Rightarrow x, y \in X \cup Y$

But $x + y = (x_1, 0, 0) + (0, x_2, 0)$

$= (x_1, x_2, 0) \notin X \cup Y$

As closure property under addition does not hold in $X \cup Y$

So $X \cup Y$ is not a subspace of R^3.

<u>Next</u>

Suppose $X \cup Y$ is a subspace of V & suppose neither $X \subset Y$ nor $Y \subset X$

Then there are elements x & y such that

$x \in X$ but $x \notin Y$ & $y \in Y$ but $y \notin X$.

Now $x, y \in X \cup Y$ & since $X \cup Y$ is a vector space

So $x + y \in X \cup Y$

$\Rightarrow$ either $x + y \in X$ or $x + y \in Y$

Suppose $x + y \in X$

then $y = (x+y) - x \in X$ (Since X is a vector space)

viz Contradiction

Similarly if $x + y \in Y$

Then $x = (x+y) - y \in Y$ (Since Y is a vector space)

viz again Contradiction

Hence our supposition is wrong.

Hence either $X \subset Y$ or $Y \subset X$

Conversely

Let $X \subset Y$ or $Y \subset X$

$\Rightarrow X \cup Y = Y$ or $X \cup Y = X$

Since X & Y are subspaces of V

Hence $X \cup Y$ is also a subspace of V

Q6 Which of the following are subspaces of R^3?

(i) $W = \{(x,y,z) : x+y+z = 0\}$

Sol.

$W = \{(x,y,z) : x+y+z = 0\}$

Let $w_1, w_2 \in W$

$\Rightarrow w_1 = (x_1, y_1, z_1)$

& $w_2 = (x_2, y_2, z_2)$ where $x_1+y_1+z_1 = 0$ & $x_2+y_2+z_2 = 0$

Now let $a, b \in R$ then

$$aw_1 + bw_2 = a(x_1, y_1, z_1) + b(x_2, y_2, z_2)$$
$$= (ax_1, ay_1, az_1) + (bx_2, by_2, bz_2)$$
$$= (ax_1+bx_2, ay_1+by_2, az_1+bz_2)$$

Now $aw_1 + bw_2 \in W$ if $ax_1+bx_2+ay_1+by_2+az_1+bz_2 = 0 \quad \forall a, b \in R$

Now

$$ax_1+bx_2+ay_1+by_2+az_1+bz_2 = ax_1+ax_2+ax_3+bx_2+by_2+bz_2$$
$$= a(x_1+y_1+z_1) + b(x_2+y_2+z_2)$$
$$= a(0) + b(0)$$
$$= 0$$

So $aw_1 + bw_2 \in W$

Hence for $a, b \in R$, $w_1, w_2 \in W \Rightarrow aw_1 + bw_2 \in W$

Hence W is a subspace of R^3.

(ii) $W = \{(x,y,z) : x \geq 0\}$

Sol. Let $w_1, w_2 \in W$

$\Rightarrow w_1 = (x_1, y_1, z_1)$

& $w_2 = (x_2, y_2, z_2)$ where $x_1 \geq 0$ & $x_2 \geq 0$

Let $a, b \in R$ then

$$aw_1 + bw_2 = a(x_1, y_1, z_1) + b(x_2, y_2, z_2)$$
$$= (ax_1, ay_1, az_1) + (bx_2, by_2, bz_2)$$
$$= (ax_1+bx_2, ay_1+by_2, az_1+bz_2)$$

Now $aw_1 + bw_2 \in W$ if $ax_1 + bx_2 \geq 0$ $\forall a, b \in R$

As $a, b \in R$ & $x_1, x_2 \geq 0$

So $ax_1 + bx_2$ may not be ≥ 0

Hence $aw_1 + bw_2 \notin W$ $\forall a, b \in R$

So W is not a subspace of R^2

(iii) $W = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$

Sol:

Let $w_1, w_2 \in W$

$\Rightarrow$ $w_1 = (x_1, y_1, z_1)$

& $w_2 = (x_2, y_2, z_2)$ where $x_1^2 + y_1^2 + z_1^2 \leq 1$ & $x_2^2 + y_2^2 + z_2^2 \leq 1$

Now let $a, b \in R$ then

$aw_1 + bw_2 = a(x_1, y_1, z_1) + b(x_2, y_2, z_2)$

$= (ax_1, ay_1, az_1) + (bx_2, by_2, bz_2)$

$= (ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2)$

Now $aw_1 + bw_2 \in W$ if $(ax_1 + bx_2)^2 + (ay_1 + by_2)^2 + (az_1 + bz_2)^2 \leq 1$

As $(ax_1 + bx_2)^2 + (ay_1 + by_2)^2 + (az_1 + bz_2)^2$

$= a^2(x_1^2 + y_1^2 + z_1^2) + b^2(x_2^2 + y_2^2 + z_2^2) + 2ab(x_1x_2 + y_1y_2 + z_1z_2)$

If we take

$a^2 = \dfrac{1}{x_1^2 + y_1^2 + z_1^2}$ & $b^2 = \dfrac{1}{x_2^2 + y_2^2 + z_2^2}$ &

$a, b, x_1, y_1, z_1, x_2, y_2, z_2$ are all +ve then above expression is $\not\leq 1$

So $aw_1 + bw_2 \notin W$ $\forall a, b \in R$

Hence W is not a subspace of R^3

(iv) $W = \{(x, y, z) : x, y, z$ are rationals$\}$

Sol:

Let $w_1, w_2 \in W$ then

$w_1 = (x_1, y_1, z_1)$

& $w_2 = (x_2, y_2, z_2)$ where x_1, y_1, z_1 & x_2, y_2, z_2 are rationals.

Let $a, b \in R$ then

$$aw_1 + bw_2 = a(x_1, y_1, z_1) + b(x_2, y_2, z_2)$$
$$= (ax_1, ay_1, az_1) + (bx_2, by_2, bz_2)$$
$$= (ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2)$$

Now $aw_1 + bw_2 \in W$ if $ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2$ are rationals

Since $a, b \in R$, so a, b may not be rationals. Hence

$ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2$ may not be rationals

So $aw_1 + bw_2 \notin W \quad \forall a, b \in R$

Hence W is not a subspace of R^3

(V): $W = \{(x, 0, z) : x, z \in R\}$

Sol.

Let $w_1, w_2 \in W$ then

$w_1 = (x_1, 0, z_1)$

& $w_2 = (x_2, 0, z_2)$ where $x_1, z_1, x_2, z_2 \in R$

Let $a, b \in R$ then

$$aw_1 + bw_2 = a(x_1, 0, z_1) + b(x_2, 0, z_2)$$
$$= (ax_1, 0, az_1) + (bx_2, 0, bz_2)$$
$$= (ax_1 + bx_2, 0, az_1 + bz_2)$$

Now $aw_1 + bw_2 \in W$ if $ax_1 + bx_2, az_1 + bz_2 \in R$

But as $a, b, x_1, z_1, x_2, z_2 \in R$.

So $ax_1 + bx_2, az_1 + bz_2 \in R. \quad \forall a, b \in R$

Hence $aw_1 + bw_2 \in W$

So W is a subspace of R^3

(Vi) $W = \{(x, y, z) : y^2 = x + z\}$

Sol. Let $w_1, w_2 \in W$.

$\Rightarrow w_1 = (x_1, y_1, z_1)$

& $w_2 = (x_2, y_2, z_1)$ where $y_1^2 = x_1^2 + z_1^2$ & $y_2^2 = x_2^2 + z_2^2$

Let $a, b \in R$ then

$$aw_1 + bw_2 = a(x_1, y_1, z_1) + b(x_2, y_2, z_2)$$
$$= (ax_1, ay_1, az_1) + (bx_2, by_2, bz_2)$$
$$= (ax_1 + bx_2,\ ay_1 + by_2,\ az_1 + bz_2)$$

Now $aw_1 + bw_2 \in W$ if $(ay_1 + by_2)^2 = (ax_1 + bx_2)^2 + (az_1 + bz_2)^2$

Now

$$(ay_1 + by_2)^2 = a^2y_1^2 + b^2y_2^2 + 2aby_1y_2$$
$$= a^2(x_1^2 + z_1^2) + b^2(x_2^2 + z_2^2) + 2aby_1y_2$$
$$= (a^2x_1^2 + b^2x_2^2) + (a^2z_1^2 + b^2z_2^2) + 2ab\sqrt{x_1^2 + z_1^2}\sqrt{x_2^2 + z_2^2}$$
$$= (a^2x_1^2 + b^2x_2^2) + (a^2z_1^2 + b^2z_2^2) + 2ab\sqrt{(x_1^2 + z_1^2)(x_2^2 + z_2^2)}$$

&

$$(ax_1 + bx_2)^2 + (az_1 + bz_2)^2 = a^2x_1^2 + b^2x_2^2 + 2abx_1x_2 + a^2z_1^2 + b^2z_2^2 + 2abz_1z_2$$
$$= (a^2x_1^2 + b^2x_2^2) + (a^2z_1^2 + b^2z_2^2) + 2ab(x_1x_2 + z_1z_2)$$

So $(ay_1 + by_2)^2 \neq (ax_1 + bx_2)^2 + (az_1 + bz_2)^2$

Hence $aw_1 + bw_2 \in W$

So W is not a subspace of R^3

Q7 Let V be the vector space of all real valued functions defined on R. State which of the following are subspaces of V.

(i) The set of all even functions.

(ii) The set of all differentiable functions.

(iii) The set $W = \{ f \mid f(x) = Kf(-x),\ K \in R \text{ fixed} \}$

(iv) The set $W = \{ f \in V : \int_0^1 f(x)\,dx = 0 \}$

Sol.

(i) Here $V = \{ f : f \text{ is a real valued fn. defined on } R \}$

& $W = \{ f : f \text{ is an even function} \}$

Let $f, g \in W$

then both f & g are even functions.

i.e.,

$f(-x) = f(x)$

& $g(-x) = g(x)$

Let $a, b \in R$ then we prove $af+bg \in W$

Now

$$
\begin{aligned}
(af+bg)(-x) &= (af)(-x) + (bg)(-x) \\
&= a.f(-x) + b g(-x) \\
&= af(x) + bg(x) \qquad \because f, g \text{ are even fns.} \\
&= (af)(x) + (bg)(x) \\
&= (af+bg)(x)
\end{aligned}
$$

So $af+bg$ is an even fn. Hence $af+bg \in W$

Hence

$a, b \in R$, $f, g \in W \Rightarrow af+bg \in W$

So W is a subspace of V.

(ii) $W = \{ f : f \text{ is a differentiable function} \}$

Sol. Let $f, g \in W$

Then both f & g are differentiable functions. i.e., f' & g' exists.

Now let $a, b \in R$ then we prove $af+bg \in W$

As

$$
\begin{aligned}
(af+bg)' &= (af)' + (bg)' \\
&= af' + bg'
\end{aligned}
$$

Since f', g' exists. Hence $(af+bg)'$ exists.

So $af+bg$ is differentiable. Hence $af+bg \in W$

Hence $a, b \in R$ & $f, g \in W \Rightarrow af+bg \in W$

So W is a subspace of V

(iii) $W = \{ f : f(x) = Kf(-x), K \in R \text{ fixed} \}$

Sol.

Let $f, g \in W$ then

$f(x) = Kf(-x)$

& $g(x) = Kg(-x)$ where $K \in R$ is fixed

Let $a, b \in R$ then we prove $af+bg \in W$
Now

$$(af+bg)(x) = (af)(x) + (bg)(x)$$
$$= af(x)$$
$$= aKf(-x) + bKg(-x)$$
$$= Kaf(-x) + Kbg(-x) \qquad \because a, K \in R$$
$$= K\left[(af)(-x) + (bg)(-x)\right] \qquad \text{So } aK = Ka$$
$$= K\left[(af+bg)(-x)\right]$$

So $af+bg \in W$

Hence $a, b \in R$, $f, g \in W \Rightarrow af+bg \in W$

So W is a subspace of V.

(iv) $W = \left\{ f \in V : \int_0^1 f(x)\,dx = 0 \right\}$

Sol. Let $f, g \in W$

Then $\int_0^1 f(x)\,dx = 0$ & $\int_0^1 g(x)\,dx = 0$

Now for $a, b \in R$, we show that $af+bg \in W$

As

$$\int_0^1 (af+bg)(x)\,dx = \int_0^1 \left[(af)(x) + (bg)(x)\right] dx$$
$$= \int_0^1 \left[af(x) + bg(x)\right] dx$$
$$= \int_0^1 af(x)\,dx + \int_0^1 bg(x)\,dx$$
$$= a\int_0^1 f(x)\,dx + b\int_0^1 g(x)\,dx$$
$$= a.0 + b.0$$
$$= 0 \quad \text{, so } af+bg \in W$$

Hence for $a, b \in R$ & $f, g \in W \Rightarrow af+bg \in W$

So W is a subspace of V.

Q8 Let V be the vector space of all real polynomials of degree $\le n$ together with the zero polynomial. Determine whether or not W is a subspace of V, where W consists of the zero polynomial and all polynomials

(i) with integral coefficients & of degree $\le n$.

(ii) of degree ≤ 3

(iii) with only even powers of x & of degree $\le n$.

Sol.

(i) Here $V = \{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n : a_i\text{'s} \in R\}$

& $W = \{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n : a_i\text{'s} \in Z\}$

Let $w_1, w_2 \in W$ then

$w_1 = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$

& $w_2 = b_0 + b_1x + b_2x^2 + \cdots + b_mx^m$ where $a_i, b_i \in Z$ & $m < n$

Let $a, b \in R$ then

$aw_1 + bw_2 = a(a_0 + a_1x + \cdots + a_nx^n) + b(b_0 + b_1x + \cdots + b_mx^m)$

$= aa_0 + aa_1x + \cdots + aa_nx^n + bb_0 + bb_1x + \cdots + bb_mx^m$

$= (aa_0 + bb_0) + (aa_1 + bb_1)x + \cdots + (aa_m + bb_m)x^m + \cdots + aa_nx^n$

Since $a, b \in R$ & $a_i, b_i \in Z$

So $aa_i + bb_i$ may not be integers

Hence $aw_1 + bw_2 \notin W \quad \forall a, b \in R$

So W is not a subspace of V.

(ii) $W = \{a_0 + a_1x + \cdots + a_nx^n : n \le 3\}$

Sol.

Let $w_1, w_2 \in W$ then

$w_1 = a_0 + a_1x + a_2x^2 + a_3x^3$

& $w_2 = b_0 + b_1x + b_2x^2$

Let $a, b \in R$ then

$aw_1 + bw_2 = a(a_0 + a_1x + a_2x^2 + a_3x^3) + b(b_0 + b_1x + b_2x^2)$

$aw_1+bw_2 = aa_0+aa_1x+aa_2x^2+aa_3x^3+bb_0+bb_1x+bb_2x^2$

$= (aa_0+bb_0)+(aa_1+bb_1)x+(aa_2+bb_2)x^2+aa_3x^3$

which is a polynomial of degree ≤ 3

Hence $aw_1+bw_2 \in W$

So W is a subspace of V

(iii) $W = \{a_0+a_1x^2+a_2x^4+ \cdots\cdots + a_nx^{2n} : n \in Z^+$

Sol.

Let $w_1, w_2 \in W$ then

$w_1 = a_0+a_1x^2+a_2x^4+\cdots\cdots+a_nx^{2n}$

& $w_2 = b_0+b_1x^2+b_2x^4+\cdots\cdots+b_mx^{2m}$ where $m<n$ & $m,n \in Z^+$

Let $a,b \in R$ then

$aw_1+bw_2 = a(a_0+a_1x^2+a_2x^4+\cdots+a_nx^{2n})+b(b_0+b_1x^2+b_2x^4+\cdots+b_mx^{2m})$

$= aa_0+aa_1x^2+\cdots+aa_nx^{2n}+bb_0+bb_1x^2+\cdots+bb_mx^{2m}$

$= (aa_0+bb_0)+(aa_1+bb_1)x^2+\cdots+(aa_m+bb_m)x^{2m}+\cdots+aa_nx^{2n}$

which is a polynomial with only even powers of x.

Hence $aw_1+bw_2 \in W$

So W is a subspace of V.

Q9 Express the vector $(2,-5,3)$ in R^3 as a linear Combination of the vectors $(1,-3,2)$, $(2,-4,-1)$ & $(1,-5,7)$.

Sol.

Let $(2,-5,3) = a(1,-3,2)+b(2,-4,-1)+c(1,-5,7)$

$=(a,-3a,2a)+(2b,-4b,-b)+(c,-5c,7c)$

So $(2,-5,3) = (a+2b+c, -3a-4b-5c, 2a-b+7c)$

$\Rightarrow$

$a+2b+c = 2$ ——— (1)

$-3a-4b-5c = -5$ ——— (2)

$2a-b+7c = 3$ ——— (3)

Multiplying (1) by 3 & adding in (2)

$$\cancel{3a} + 6b + 3C = 6$$
$$-\cancel{3a} - 4b - 5C = -5$$
$$2b - 2C = 1$$

or $\quad b - C = 1/2 \quad$ ———— (4)

Now multiplying (1) by 2 & subt. (3) from (1)

$$\cancel{2a} + 4b + 2C = 4$$
$$\cancel{2a} - b + 7C = 3$$
$$- \quad + \quad - \quad -$$
$$5b - 5C = 1.$$

or $\quad b - C = \frac{1}{5} \quad$ ———— (5)

From (4) & (5), we cannot find values of b & c

Thus $(2,-5,3)$ cannot be expressed as a linear combination of $(1,-3,2)$, $(2,-4,-1)$ & $(1,-5,7)$

Q1. For what value of K will the vector $(1,-2,K)$ in R^3 be a linear combination of the vectors $(3,0,-2)$ & $(2,-1,-5)$?

Sol.

Let $\quad (1,-2,K) = a(3,0,-2) + b(2,-1,-5)$

$$= (3a, 0, -2a) + (2b, -b, -5b)$$

or $\quad (1,-2,K) = (3a+2b, -b, -2a-5b)$

$\Rightarrow \quad 3a + 2b = 1 \quad$ ———— (1)

$-b = -2 \quad$ ———— (2)

& $\quad -2a - 5b = K \quad$ ———— (3)

from (2) $\quad \boxed{b = 2}$

Put in (1)

$$3a + 2(2) = 1$$
$$3a + 4 = 1$$
$$3a = -3$$
$$\boxed{a = -1}$$

Putting values of a & b in ③

$$-2(-1) - 5(2) = K$$
$$2 - 10 = K$$
$$-8 = K$$
or $\boxed{K = -8}$

So for $K = -8$, the vector $(1, -2, K)$ is a linear Combination of $(3, 0, -2)$ & $(2, -1, -5)$.

Q11 Let U & W be the subspaces of R^3 defined by $U = \{(x, y, z) : x = y = z\}$

& $W = \{(0, y, z) : y, z \in R\}$

Show that

$$R^3 = U \oplus W$$

Sol.

To show that $R^3 = U \oplus W$, we have to prove that $R^3 = U + W$
& $U \cap W = \{0\}$

Here

$$U = \{(x, y, z) : x = y = z\}$$
& $W = \{(0, y, z) : y, z \in R\}$

Let

$$(x, y, z) \in R^3$$

then $(x, y, z) = (x, x, x) + (0, y-x, z-x) \in U + W$

So $(x, y, z) \in U + W$

Hence $R^3 \subseteq U + W$ ——— ①

Conversly

let $u \in U$ & $w \in W$ then $u + w \in U + W$

$\Rightarrow u = (x, x, x)$ & $w = (0, y, z)$

then $u + w = (x, x, x) + (0, y, z)$
$$= (x, x+y, x+z) \in R^3$$

So $u+w \in R^3$

Hence $U+W \subseteq R^3$ ——— (2)

From (1) & (2)

$$R^3 = U+W$$

Now we want to show $U \cap W = \{0\}$

let $\alpha \in U \cap W$

$\Rightarrow \alpha \in U$ & $\alpha \in W$

$\Rightarrow \alpha = (x,x,x)$ & $\alpha = (0,y,z)$

$\Rightarrow (x,x,x) = (0,y,z)$

So $x = 0$, $x = y$, $x = z$

$\Rightarrow x = y = z = 0$

So $\alpha = (0,0,0)$

Hence $U \cap W = \{0\}$

So

$$R^3 = U \oplus W$$

Q12 Show that each of the following sets of vectors generate R^3

(i) $\{(1,2,3), (0,1,2), (0,0,1)\}$

(ii) $\{(1,1,1), (0,1,1), (0,1,-1)\}$

Sol:-

(i) Given set is $\{(1,2,3), (0,1,2), (0,0,1)\}$

Let $(x,y,z) \in R^3$ & suppose

$(x,y,z) = a(1,2,3) + b(0,1,2) + c(0,0,1)$

$= (a,2a,3a) + (0,b,2b) + (0,0,c)$

So $(x,y,z) = (a, 2a+b, 3a+2b+c)$

$\Rightarrow a = x$ ——— (1)

$2a+b = y$ ——— (2)

$3a+2b+c = z$ ——— (3)

from (1) $\boxed{a = x}$

Put in (2)

$2x + b = y$

$\boxed{b = y - 2x}$

Put Values of a & b in (3)

$3x + 2(y-2x) + C = z$

$3x + 2y - 4x + C = z$

or $-x + 2y + C = z$

or $\boxed{C = x - 2y + z}$

So

$(x, y, z) = x(1,2,3) + (y-2x)(0,1,2) + (x-2y+z)(0,0,1)$

Hence given vectors generate R^3.

(ii) Given set is $\{(1,1,1), (0,1,1), (0,1,-1)\}$

Let $(x, y, z) \in R^3$ & suppose

$(x, y, z) = a(1,1,1) + b(0,1,1) + c(0,1,-1)$

$= (a,a,a) + (0,b,b) + (0,c,-c)$

or $(x, y, z) = (a, a+b+c, a+b-c)$

$\Rightarrow$

$a = x$ ——— (1)

$a + b + C = y$ ——— (2)

$a + b - C = z$ ——— (3)

from (1) $\boxed{a = x}$

Adding (2) & (3)

$2a + 2b = y + z$

$2x + 2b = y + z$

or $2b = y + z - 2x$

or $\boxed{b = \frac{y + z - 2x}{2}}$

Put values of a & b in (2)

$x + \frac{y + z - 2x}{2} + C = y$

or $c = y - x - \dfrac{y+z-2x}{2}$

$$= \frac{2y - \cancel{2x} - y - z + \cancel{2x}}{2}$$

$$\boxed{c = \frac{y-z}{2}}$$

So

$(x,y,z) = x(1,1,1) + \left(\frac{y+z-2x}{2}\right)(0,1,1) + \left(\frac{y-z}{2}\right)(0,1,-1)$

Hence given vectors generate R^3.

Q13 Determine whether the set $S = \{(1,1,2),(1,0,1),(2,1,3)\}$ spans R^3

Sol: Given set is $S = \{(1,1,2),(1,0,1),(2,1,3)\}$

Let $(x,y,z) \in R^3$ & suppose

$(x,y,z) = a(1,1,2) + b(1,0,1) + c(2,1,3)$

$= (a,a,2a) + (b,0,b) + (2c,c,3c)$

So $(x,y,z) = (a+b+2c,\ a+c,\ 2a+b+3c)$

$\Rightarrow$

$a + b + 2c = x$ ——— (1)

$a + c = y$ ——— (2)

$2a + b + 3c = z$ ——— (3)

Subt. (3) from (1)

$-a - c = x - z$

or $a + c = z - x$ ——— (4)

& $a + c = y$ ——— (2)

The eqs. (2) & (4) cannot be solved for a & c

Hence we cannot find values of a, b & c.

So the set $S = \{(1,1,2),(1,0,1),(2,1,3)\}$ does not span R^3

Q.14 Show that the yz-plane $W = \{(0,y,z) : y, z \in R\}$ is spanned by

(i) $(0,1,1)$ and $(0,2,-1)$

(ii) $(0,1,2)$, $(0,2,3)$ & $(0,3,1)$

Sol:-

(i) Let $W = \{(0,y,z) : y, z \in R\}$

Suppose

$$(0,y,z) \in W \quad \&$$

$$(0,y,z) = a(0,1,1) + b(0,2,-1)$$

$$= (0,a,a) + (0,2b,-b)$$

So $(0,y,z) = (0, a+2b, a-b)$

$$\Rightarrow \quad a+2b = y \quad \text{———} \; (1)$$

$$a-b = z \quad \text{———} \; (2)$$

Subt. (2) from (1)

$$3b = y-z$$

$$\boxed{b = \frac{y-z}{3}}$$

Put value in (2)

$$a - \frac{y-z}{3} = z$$

or $$a = \frac{y-z}{3} + z$$

$$= \frac{y-z+3z}{3}$$

So $$\boxed{a = \frac{y+2z}{3}}$$

So

$$(0,y,z) = \left(\frac{y+2z}{3}\right)(0,1,1) + \left(\frac{y-z}{3}\right)(0,2,-1)$$

Hence yz-plane is spanned by $(0,1,1)$ & $(0,2,-1)$

(ii) Sol:- Given vectors are $(0,1,2)$, $(0,2,3)$ & $(0,3,1)$

let $(0,y,z) \in W$ & suppose

$$(0,y,z) = a(0,1,2) + b(0,2,3) + c(0,3,1)$$

$= (0, a, 2a) + (0, 2b, 3b) + (0, 3c, c)$

So $(0, y, z) = (0, a+2b+3c, 2a+3b+c)$

$\Rightarrow$

$$a + 2b + 3c = y \quad \text{——— (1)}$$

& $$2a + 3b + c = z \quad \text{——— (2)}$$

Put $a = 0$

So $$2b + 3c = y \quad \text{——— (3)}$$
$$3b + c = z \quad \text{——— (4)}$$

Multiplying (4) by 3 & subt. from (3)

$$2b + \cancel{3c} = y \quad \text{——— (3)}$$
$$\underline{-}9b \underline{+} \cancel{3c} = \underline{-}3z \quad \text{——— (5)}$$

$$-7b = y - 3z$$

$$\boxed{b = \frac{3z - y}{7}}$$

Put in (3)

$$2\left(\frac{3z-y}{7}\right) + 3c = y$$

$$\frac{6z-2y}{7} + 3c = y$$

$$3c = y - \frac{6z-2y}{7}$$

$$= \frac{7y - 6z + 2y}{7}$$

$$3c = \frac{9y - 6z}{7}$$

$$\boxed{c = \frac{3y - 2z}{7}}$$

Hence $(0, y, z) = 0(0,1,2) + \left(\frac{3z-y}{7}\right)(0,2,3) + \left(\frac{3y-2z}{7}\right)(0,3,1)$

So yz - plane is spanned by $(0,1,2), (0,2,3)$ & $(0,3,1)$

Q15 Find an eq. (or equations) of the subspace W of R^3 generated by each of following sets of vectors.

(i) $\{(1,-3,5), (-2,6,-10)\}$

(ii) $\{(1,-3,2), (-2,0,3)\}$

(iii) $\{(1,-2,1), (-2,0,3), (3,-2,-2)\}$

(i) $\{(1,-3,5),(-2,6,-10)\}$

Sol. Since W is spanned by the vectors $(1,-3,5)$, $(-2,6,-10)$.

So each vector $(x,y,z) \in W$ is a linear Combination of these vectors. i.e., there exist scalars a,b s.t.

$$(x,y,z) = a(1,-3,5) + b(-2,6,-10)$$
$$= (a,-3a,5a) + (-2b,6b,-10b)$$

or $(x,y,z) = (a-2b, -3a+6b, 5a-10b)$

$$\Rightarrow \left.\begin{aligned} a-2b &= x \\ -3a+6b &= y \\ 5a-10b &= z \end{aligned}\right\} \text{———} (1)$$

We reduce the augmented matrix of system (1) to echelon form as:

$$A_b = \begin{bmatrix} 1 & -2 & x \\ -3 & 6 & y \\ 5 & -10 & z \end{bmatrix}$$

$$\sim \left[\begin{array}{cc:c} 1 & -2 & x \\ 0 & 0 & y+3x \\ 0 & 0 & z-5x \end{array}\right] \qquad \begin{array}{l} R_2+3R_1 \\ R_3-5R_1 \end{array}$$

The system (1) is Consistent if

rank A = rank A_b

$\Rightarrow y+3x = 0$ & $z-5x = 0$

$$\text{or } \left.\begin{aligned} x &= t \\ y &= -3t \\ z &= 5t \end{aligned}\right\} \qquad t \in R$$

These are the req. eqs. of the subspace W of R^3

(ii) $\{(1,-3,2),(-2,0,3)\}$

Sol. Since W is spanned by the vectors $(1,-3,2)$ & $(-2,0,3)$. So each vector $(x,y,z) \in W$ is a

linear Combination of these vectors.
i.e., there exist scalars $a, b \in R$ s.t.

$$(x, y, z) = a(1, -3, 2) + b(-2, 0, 3)$$
$$= (a, -3a, 2a) + (-2b, 0, 3b)$$
$$\text{or } (x, y, z) = (a-2b, -3a, 2a+3b)$$

$\Rightarrow$

$$\left.\begin{aligned} a-2b &= x \\ -3a &= y \\ 2a+3b &= z \end{aligned}\right\} \quad \text{———} \quad (1)$$

We reduce the augmented matrix of system (1) to echelon form as follows:

$$A_b = \begin{bmatrix} 1 & -2 & x \\ -3 & 0 & y \\ 2 & 3 & z \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & x \\ 0 & -6 & y+3x \\ 0 & 7 & z-2x \end{bmatrix} \qquad \begin{matrix} R_2 + 3R_1 \\ R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -2 & x \\ 0 & 1 & -\frac{1}{6}(y+3x) \\ 0 & 1 & \frac{1}{7}(z-2x) \end{bmatrix} \qquad -\frac{1}{6}R_2, \ \frac{1}{7}R_3$$

$$\sim \left[\begin{array}{cc|c} 1 & -2 & x \\ 0 & 1 & -\frac{1}{6}(y+3x) \\ 0 & 0 & \frac{1}{7}(z-2x)+\frac{1}{6}(y+3x) \end{array}\right] \qquad R_3 - R_2$$

The system (1) is Consistent if the rank A = rank A_b

$$\Rightarrow \quad \frac{1}{7}(z-2x) + \frac{1}{6}(y+3x) = 0$$
$$\text{or} \quad 6(z-2x) + 7(y+3x) = 0$$
$$6z - 12x + 7y + 21x = 0$$
$$9x + 7y + 6z = 0$$

Which is the req. eq. of the subspace W of $\mathbb{R}^3$.

(iii) $\{(1,-2,1),(-2,0,3),(3,-2,-2)\}$

Sol. Since W is spanned by the vectors $(1,-2,1)$, $(-2,0,3)$ & $(3,-2,-2)$. So each vector $(x,y,z) \in W$ is a linear combination of these vectors.

i.e., there exist scalars $a,b,c \in R$ s.t.

$$(x,y,z) = a(1,-2,1)+b(-2,0,3)+c(3,-2,-2)$$
$$= (a,-2a,a)+(-2b,0,3b)+(3c,-2c,-2c)$$

or $(x,y,z) = (a-2b+3c,\ -2a-2c,\ a+3b-2c)$

$$\Rightarrow \left.\begin{aligned} a-2b+3c &= x \\ -2a-2c &= y \\ a+3b-2c &= z \end{aligned}\right\} \text{------ (1)}$$

We reduce the augmented matrix of this system to echelon form as:

$$A_b = \begin{bmatrix} 1 & -2 & 3 & x \\ -2 & 0 & -2 & y \\ 1 & 3 & -2 & z \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & x \\ 0 & -4 & 4 & y+2x \\ 0 & 5 & -5 & z-x \end{bmatrix} \quad \begin{matrix} R_2+2R_1 \\ R_3-R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & x \\ 0 & 1 & -1 & -\frac{1}{4}(y+2x) \\ 0 & 1 & -1 & \frac{1}{5}(z-x) \end{bmatrix} \quad -\frac{1}{4}R_2,\ \frac{1}{5}R_3$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & x \\ 0 & 1 & -1 & -\frac{1}{4}(y+2x) \\ 0 & 0 & 0 & \frac{1}{5}(z-x)+\frac{1}{4}(y+2x) \end{bmatrix} \quad R_3-R_2$$

The system (1) is consistent if

rank A = rank A_b

$$\Rightarrow \frac{1}{5}(z-x)+\frac{1}{4}(y+2x) = 0$$

$\Rightarrow \quad 4(z-x)+5(y+2x) = 0$

or $\quad 4z-4x+5y+10x = 0$

or $\quad 6x+5y+4z = 0$

Which is the req. eq. of subspace W of R^3

Q16 Show that the complex no's. $2+3i$ & $1-2i$ generate the vector space C over R.

Sol. Here $C = \{x+iy : x,y \in R\}$

Any vector of C has the form $x+iy$; $x,y \in R$

Suppose

$$x+iy = a(2+3i)+b(1-2i)$$

$$= 2a+3ai+b-2bi$$

or $$x+iy = (2a+b)+i(3a-2b)$$

$\Rightarrow$

$$2a+b = x \quad \text{---} (1)$$

$$3a-2b = y \quad \text{---} (2)$$

Multiplying (1) by 2 & adding in (2)

$$4a+\cancel{2b} = 2x \quad \text{---} (1)$$

$$3a-\cancel{2b} = y \quad \text{---} (2)$$

$$7a = 2x+y$$

$$\boxed{a = \frac{2x+y}{7}}$$

Put in (1)

$$2\left(\frac{2x+y}{7}\right)+b = x$$

$$\frac{4x+2y}{7}+b = x$$

$$b = x-\frac{4x+2y}{7}$$

$$= \frac{7x-4x-2y}{7}$$

$$\boxed{b = \frac{3x-2y}{7}}$$

Hence

$$x+iy = \left(\frac{2x+y}{7}\right)(2+3i) + \left(\frac{3x-2y}{7}\right)(1-2i)$$

So given vectors $2+3i$ & $1-2i$ generate $\mathbb{C}$ over $\mathbb{R}$.

Q17 Let S & T be subsets of a vector space V. Show that

(i) $\langle S \rangle \cup \langle T \rangle \subset \langle S \cup T \rangle$

(ii) $\langle S \cap T \rangle \subset \langle S \rangle \cap \langle T \rangle$

Give an example to show that equality need not hold in either case.

Sol.

(i) Let $S = \{u_1, u_2, \ldots, u_r\}$ &

$T = \{v_1, v_2, \ldots, v_t\}$

$\Rightarrow$ $S \cup T = \{u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_t\}$

We want to show $\langle S \rangle \cup \langle T \rangle \subset \langle S \cup T \rangle$

Let $v \in \langle S \rangle \cup \langle T \rangle$

$\Rightarrow$ $v \in \langle S \rangle$ or $v \in \langle T \rangle$

Let $v \in \langle S \rangle$ then v is a linear combination of vectors of S

i.e., $v = k_1 u_1 + k_2 u_2 + \cdots + k_r u_r$

We can also write v as

$v = k_1 u_1 + k_2 u_2 + \cdots + k_r u_r + 0v_1 + 0v_2 + \cdots + 0v_t$

Which shows that v is a linear combination of vectors of $S \cup T$

$\Rightarrow v \in \langle S \cup T \rangle$

So $v \in \langle S \rangle \Rightarrow v \in \langle S \cup T \rangle$

Similarly $v \in \langle T \rangle \Rightarrow v \in \langle S \cup T \rangle$

Hence

$$\langle S \rangle \cup \langle T \rangle \subset \langle S \cup T \rangle$$

Now we give an example to show that equality does not hold in above result.

Example

In R^2, Let $S = \{(1,0)\}$, $T = \{(0,1)\}$ then

$\langle S \rangle = \{K(1,0) : K \in R\} = x$-axis

& $\langle T \rangle = \{l(0,1) : l \in R\} = y$-axis

Therefore

$$\langle S \rangle \cup \langle T \rangle = \{K(1,0) : K \in R\} \cup \{l(0,1) : l \in R\} \neq R^2$$

Now

$S \cup T = \{(1,0),(0,1)\}$ and

Any vector $(x,y) \in R^2$ is a linear Combination of $(1,0)$ & $(0,1)$, because

$$(x,y) = x(1,0) + y(0,1)$$

Therefore

$$\langle S \cup T \rangle = R^2$$

This shows that $\langle S \rangle \cup \langle T \rangle \neq \langle S \cup T \rangle$

(ii) $\langle S \cap T \rangle \subset \langle S \rangle \cap \langle T \rangle$

Soln

First we prove that if

$S \subset T$ then $\langle S \rangle \subset \langle T \rangle$

Let $S = \{v_1, v_2, \ldots\ldots, v_r\}$

& $T = \{v_1, v_2, \ldots\ldots, v_r, v_{r+1}, \ldots\ldots, v_n\}$

then obviously $S \subset T$

we want to show that $\langle S \rangle \subset \langle T \rangle$

Let $v \in \langle S \rangle$

then v is a linear Combination of vectors of S.

i.e., $v = a_1v_1 + a_2v_2 + \ldots\ldots + a_rv_r$

Now v can also be written as

$$v = a_1v_1 + a_2v_2 + \ldots\ldots + a_rv_r + 0v_{r+1} + \ldots\ldots + 0v_n$$

Which shows v is a linear Combination of vectors of T. Hence $v \in \langle T \rangle$

So $\langle S \rangle \subset \langle T \rangle$

Now as $S \cap T \subset S$ & $S \cap T \subset T$

So $\langle S \cap T \rangle \subset \langle S \rangle$ & $\langle S \cap T \rangle \subset \langle T \rangle$

$\Rightarrow \langle S \cap T \rangle \subset \langle S \rangle \cap \langle T \rangle$

Now we will give an example to show that equality does not hold in above result.

<u>Example</u>

In R^2, let $S = \{(0,0),(1,0)\}$ & $T = \{(0,0),(0,3)\}$

then $S \cap T = \{(0,0)\}$

So $\langle S \cap T \rangle = \{ k(0,0) : k \in R \}$

$= \{(0,0)\}$ ———— (i)

Now

$$\langle S \rangle = \{ a(0,0) + b(0,1) : a, b \in R \}$$
$$= \{ (0,b) : b \in R \}$$

&

$$\langle T \rangle = \{ p(0,0) + q(0,3) : p, q \in R \}$$
$$= \{ (0,3q) : q \in R \}$$
$$= \{ (0,c) : c \in R \}$$

Now

$$\langle S \rangle \cap \langle T \rangle = \{ (0,b) : b \in R \} \cap \{ (0,c) : c \in R \}$$
$$= \{ (0,y) : y \in R \} \quad \text{———— (ii)}$$

From (i) & (ii)

$$\langle S \cap T \rangle \neq \langle S \rangle \cap \langle T \rangle$$

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