Determinants (Chapter No. 5)

Mathematical Method 5

Consider the simultaneous eqs.

$$a_1 x + b_1 = 0 \quad \text{---} \quad (1)$$

$$a_2 x + b_2 = 0 \quad \text{---} \quad (2)$$

Let us eliminate x from these two eqs.

From (1) $x = -\dfrac{b_1}{a_1}$

Put in (2)

$$a_2\left(-\frac{b_1}{a_1}\right) + b_2 = 0$$

$$-a_2 b_1 + a_1 b_2 = 0$$

or $a_1 b_2 - a_2 b_1 = 0$

The expression on the left i.e., $a_1 b_2 - a_2 b_1$ is symbolically written as

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$$ & is called a determinant

As this determinant has two rows & two columns, so it is said to be a determinant of order 2.

Again

Consider the simultaneous eqs.

$$a_1 x + b_1 y + c_1 = 0 \quad \text{---} \quad (1)$$

$$a_2 x + b_2 y + c_2 = 0 \quad \text{---} \quad (2)$$

$$a_3 x + b_3 y + c_3 = 0 \quad \text{---} \quad (3)$$

Let us eliminate x & y from these three eqs.

from (2) & (3)

$$\frac{x}{b_2 c_3 - b_3 c_2} = \frac{-y}{a_2 c_3 - a_3 c_2} = \frac{1}{a_2 b_3 - a_3 b_2}$$

$$\Rightarrow x = \frac{b_2 c_3 - b_3 c_2}{a_2 b_3 - a_3 b_2} \quad \& \quad y = -\frac{a_2 c_3 - a_3 c_2}{a_2 b_3 - a_3 b_2}$$

Put values in ①

$$a_1\left(\frac{b_2c_3-b_3c_2}{a_2b_3-a_3b_2}\right)+b_1\left(-\frac{a_2c_3-a_3c_2}{a_2b_3-a_3b_2}\right)+c_1 = 0$$

or

$$a_1(b_2c_3-b_3c_2)-b_1(a_2c_3-a_3c_2)+c_1(a_2b_3-a_3b_2) = 0$$

or

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

As it consists of three row & three columns, so it is said to be a determinant of order 3.

Properties of determinants:

Following are some important properties of determinants.

(i) The value of a determinant is the same as the value of its transpose.

(ii) The interchange of two adjacent rows or columns changes the sign of the determinant.

(iii) If a row or column of a determinant is passed over m rows or columns then its value is multiplied by $(-1)^m$.

(iv) If any two rows or columns of a determinant are identical then value of determinant is zero.

(v) If all the elements in a row or column of a determinant are zero then value of the determinant is zero.

(vi) If a non zero scalar is multiplied by a determinant then this scalar will be multiplied by any one of the rows or columns of that det.

(vii) If each element in a row or Column of a determinant is the sum of two elements then this determinant will be written as the sum of two determinants as

$$\begin{vmatrix} a_1 & b_1+t_1 & c_1 \\ a_2 & b_2+t_2 & c_2 \\ a_3 & b_3+t_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \begin{vmatrix} a_1 & t_1 & c_1 \\ a_2 & t_2 & c_2 \\ a_3 & t_3 & c_3 \end{vmatrix}$$

(viii) Addition of some scalar multiple of a row or Column to any other row or Column does not change the value of that determinant.

Minors & Cofactors:

$$\text{Let} \quad \Delta = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

be a given determinant of order n.

The minor of an element a_{ij} of Δ is the det. M_{ij} obtained by deleting the rows & Columns in which a_{ij} lies. Clearly M_{ij} is a determinant of order $n-1$.

The Cofactor A_{ij} of an element a_{ij} of Δ is

$$A_{ij} = (-1)^{i+j} \cdot M_{ij}$$

Note

(i) $\Delta = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots\cdots + a_{in}A_{in}$ for any i

(ii) $\Delta = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots\cdots + a_{nj}A_{nj}$ for any j

(iii) If the elements of a line are multiplied by the cofactors of the corresponding elements of any other parallel line & the results so obtained are added the answer will be zero.

Adjoint of a square matrix:

Let $A = [a_{ij}]_{n\times n}$ be a square matrix of order n. Denoting the co-factors by A_{ij} of the elements a_{ij} of A, we define

$$\text{Adj}\,A = [A_{ij}]^t_{n\times n} = [A_{ji}]_{n\times n}$$

Inverse of a square matrix:

Let A be a non singular square matrix of order n then inverse of A is defined as

$$A^{-1} = \frac{\text{Adj}\,A}{|A|}$$

Note If A & B are square matrices of order n then

(i) $\det(AB) = \det(A) \cdot \det(B)$

(ii) $\det(BA) = \det(B) \cdot \det(A)$

(iii) $\det(A^{-1}) = (\det(A))^{-1}$ if A is non singular

(iv) $\det(A^t) = \det(A)$

(v) $\det(A^n) = [\det(A)]^n$ where $n \in Z^+$

(vi) $\det(KA) = K^n.\det(A)$

Exercise No. 5.1

Q1 Let M_2 be the set of all 2×2 matrices. Set up the transformation $A \longrightarrow \det(A)$, $A \in M_2$. What is the range of this mapping? Is the mapping one-to-one?

Sol.

Let $f: A \longrightarrow \det(A)$; $A \in M_2$

be defined by

$$f(A) = \det(A)$$

Suppose the field for all $A \in M_2$ be the set of Complex no's. C, then the range of f is C. But if the field is taken as the set of real no's. R then range of f is also R.

This mapping f is not one-to-one as shown by the following example

Let $A = \begin{bmatrix} 2 & 2 \\ 1 & 4 \end{bmatrix}$ & $B = \begin{bmatrix} 6 & 0 \\ 9 & 1 \end{bmatrix}$

then clearly $A \neq B$

Now

$$\det(A) = \begin{vmatrix} 2 & 2 \\ 1 & 4 \end{vmatrix} = 8-2 = 6$$

& $$\det(B) = \begin{vmatrix} 6 & 0 \\ 9 & 1 \end{vmatrix} = 6-0 = 6$$

Hence $\det(A) = \det(B)$

So we have proved that

$$A \neq B \Rightarrow \det(A) = \det(B)$$

Hence by def., f is not one-to-one.

Q2 For 2×2 matrices A & B which of the following equations hold ?

(i) $\det(A+B) = \det(A) + \det(B)$

Sol.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ & $B = \begin{bmatrix} f & g \\ h & k \end{bmatrix}$

then

$$A+B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} f & g \\ h & k \end{bmatrix}$$

or

$$A+B = \begin{bmatrix} a+f & b+g \\ c+h & d+k \end{bmatrix}$$

$$\Rightarrow \det(A+B) = \begin{vmatrix} a+f & b+g \\ c+h & d+k \end{vmatrix}$$

$$= (a+f)(d+k) - (b+g)(c+h)$$

$$= ad + ak + fd + fk - bc - bh - gc - gh \quad \text{——— (1)}$$

Now

$$\det A + \det B = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} f & g \\ h & k \end{vmatrix}$$

$$= ad - bc + fk - gh \quad \text{——— (2)}$$

from (1) & (2)

$$\det(A+B) \neq \det A + \det B$$

(ii) $\det(A+B)^2 = [\det(A+B)]^2$

Sol.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ & $B = \begin{bmatrix} f & g \\ h & k \end{bmatrix}$

then

$$A+B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} f & g \\ h & k \end{bmatrix}$$

or $$A+B = \begin{bmatrix} a+f & b+g \\ c+h & d+k \end{bmatrix}$$

Now

$$(A+B)^2 = \begin{bmatrix} a+f & b+g \\ c+h & d+k \end{bmatrix}\begin{bmatrix} a+f & b+g \\ c+h & d+k \end{bmatrix}$$

$$\Rightarrow (A+B)^2 = \begin{bmatrix} (a+f)^2+(b+g)(c+h) & (a+f)(b+g)+(b+g)(d+k) \\ (c+h)(a+f)+(d+k)(c+h) & (c+h)(b+g)+(d+k)^2 \end{bmatrix}$$

So

$$\det(A+B)^2 = \begin{vmatrix} (a+f)^2+(b+g)(c+h) & (a+f)(b+g)+(b+g)(d+k) \\ (c+h)(a+f)+(d+k)(c+h) & (c+h)(b+g)+(d+k)^2 \end{vmatrix} \quad \text{——— (1)}$$

Now

$$A+B = \begin{bmatrix} a+f & b+g \\ c+h & d+k \end{bmatrix}$$

then

$$\det(A+B) = \begin{vmatrix} a+f & b+g \\ c+h & d+k \end{vmatrix}$$

&

$$[\det(A+B)]^2 = \begin{vmatrix} a+f & b+g \\ c+h & d+k \end{vmatrix}\begin{vmatrix} a+f & b+g \\ c+h & d+k \end{vmatrix}$$

$$\Rightarrow [\det(A+B)]^2 = \begin{vmatrix} (a+f)^2+(b+g)(c+h) & (a+f)(b+g)+(b+g)(d+k) \\ (c+h)(a+f)+(d+k)(c+h) & (c+h)(b+g)+(d+k)^2 \end{vmatrix} \quad \text{——— (2)}$$

from (1) & (2)

$$\det(A+B)^2 = [\det(A+B)]^2$$

(iii) $\det(A+B)^2 = \det(A^2+B^2)$

Sol. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ & $B = \begin{bmatrix} f & g \\ h & k \end{bmatrix}$

then

$$A+B = \begin{bmatrix} a+f & b+g \\ c+h & d+k \end{bmatrix}$$

Now

$$(A+B)^2 = \begin{bmatrix} a+f & b+g \\ c+h & d+k \end{bmatrix}\begin{bmatrix} a+f & b+g \\ c+h & d+k \end{bmatrix}$$

$$(A+B)^2 = \begin{bmatrix} (a+f)^2+(b+g)(c+h) & (a+f)(b+g)+(b+g)(d+k) \\ (c+h)(a+f)+(d+k)(c+h) & (c+h)(b+g)+(d+k)^2 \end{bmatrix}$$

So

$$\det(A+B)^2 = \begin{vmatrix} (a+f)^2+(b+g)(c+h) & (a+f)(b+g)+(b+g)(d+k) \\ (c+h)(a+f)+(d+k)(c+h) & (c+h)(b+g)+(d+k)^2 \end{vmatrix} \text{———} (1)$$

Now

$$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix}$$

&

$$B^2 = \begin{bmatrix} f & g \\ h & k \end{bmatrix}\begin{bmatrix} f & g \\ h & k \end{bmatrix}$$

$$= \begin{bmatrix} f^2+gh & fg+gk \\ hf+kh. & gh+k^2 \end{bmatrix}$$

So

$$A^2+B^2 = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix} + \begin{bmatrix} f^2+gh & fg+gk \\ hf+kh & gh+k^2 \end{bmatrix}$$

$$A^2+B^2 = \begin{bmatrix} a^2+f^2+bc+gh & ab+bd+fg+gk \\ ac+cd+hf+kh & d^2+k^2+bc+gh \end{bmatrix}$$

So

$$\det(A^2+B^2) = \begin{vmatrix} a^2+f^2+bc+gh & ab+bd+fg+gk \\ ac+cd+hf+kh & d^2+k^2+bc+gh \end{vmatrix} \text{———} (2)$$

from ① & ②

$$\det(A+B)^2 \neq \det(A^2+B^2)$$

(iv) $\det(A+B)^2 = \det(A^2+2AB+B^2)$

Sol.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ & $B = \begin{bmatrix} f & g \\ h & k \end{bmatrix}$

then

$$A+B = \begin{bmatrix} a+f & b+g \\ c+h & d+k \end{bmatrix}$$

Now

$$(A+B)^2 = \begin{bmatrix} a+f & b+g \\ c+h & d+k \end{bmatrix}\begin{bmatrix} a+f & b+g \\ c+h & d+k \end{bmatrix}$$

$$(A+B)^2 = \begin{bmatrix} (a+f)^2+(b+g)(c+h) & (a+f)(b+g)+(b+g)(d+k) \\ (c+h)(a+f)+(d+k)(c+h) & (c+h)(b+g)+(d+k)^2 \end{bmatrix}$$

So

$$\det(A+B)^2 = \begin{vmatrix} (a+f)^2+(b+g)(c+h) & (a+f)(b+g)+(b+g)(d+k) \\ (c+h)(a+f)+(d+k)(c+h) & (c+h)(b+g)+(d+k)^2 \end{vmatrix} \quad \text{——— ①}$$

Now

$$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} f & g \\ h & k \end{bmatrix}\begin{bmatrix} f & g \\ h & k \end{bmatrix} = \begin{bmatrix} f^2+gh & fg+gk \\ hf+kh & gh+k^2 \end{bmatrix}$$

&

$$2AB = 2\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} f & g \\ h & k \end{bmatrix}$$

$$= 2\begin{bmatrix} af+bh & ag+bk \\ cf+dh & gc+dk \end{bmatrix}$$

$$= \begin{bmatrix} 2af+2bh & 2ag+2bk \\ 2cf+2dh & 2gc+2dk \end{bmatrix}$$

So

$$A^2+2AB+B^2 = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix} + \begin{bmatrix} 2af+2bh & 2ag+2bk \\ 2cf+2dh & 2gc+2dk \end{bmatrix} + \begin{bmatrix} f^2+gh & fg+gk \\ hf+kh & gh+k^2 \end{bmatrix}$$

$$A^2+2AB+B^2 = \begin{bmatrix} a^2+bc+2af+2bh+f^2+gh & ab+bd+2ag+2bk+fg+gk \\ ac+cd+2cf+2dh+hf+kh & bc+d^2+2gc+2dk+gh+k^2 \end{bmatrix}$$

So

$$\det(A^2+2AB+B^2) = \begin{vmatrix} a^2+bc+2af+2bh+f^2+gh & ab+bd+2ag+2bk+fg+gk \\ ac+cd+2cf+2dh+hf+kh & bc+d^2+2gc+2dk+gh+k^2 \end{vmatrix} \quad \text{---} (2)$$

from (1) & (2)

$$\det(A+B)^2 \neq \det(A^2+2AB+B^2)$$

Q3 Find the value of each of the following determinants:

(i)
$$\begin{vmatrix} 1 & 0 & 2 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{vmatrix}$$

Sol

$$\text{Let } \Delta = \begin{vmatrix} 1 & 0 & 2 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{vmatrix}$$

Expanding from R_1

$$= 1\begin{vmatrix} 4 & 5 \\ 6 & 7 \end{vmatrix} - 0\begin{vmatrix} 3 & 5 \\ 5 & 7 \end{vmatrix} + 2\begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix}$$

$$= 1(28-30) - 0 + 2(18-20)$$

$$= -2 + 2(-2)$$

$$= -2-4$$

$$\Delta = -6$$

(ii) $\begin{vmatrix} 2 & -1 & 1 \\ 3 & 2 & 4 \\ -1 & 0 & 3 \end{vmatrix}$

Sol.

Let $\Delta = \begin{vmatrix} 2 & -1 & 1 \\ 3 & 2 & 4 \\ -1 & 0 & 3 \end{vmatrix}$

Expanding from R_1

$$= 2\begin{vmatrix} 2 & 4 \\ 0 & 3 \end{vmatrix} + 1\begin{vmatrix} 3 & 4 \\ -1 & 3 \end{vmatrix} + 1\begin{vmatrix} 3 & 2 \\ -1 & 0 \end{vmatrix}$$

$$= 2(6-0) + (9+4) + (0+2)$$

$$= 2(6) + 13 + 2$$

$$= 12 + 15$$

$$\Delta = 27$$

(iii) $\begin{vmatrix} 6 & -6 & 6 \\ 2 & 4 & -6 \\ 15 & -5 & 5 \end{vmatrix}$

Sol.

Let $\Delta = \begin{vmatrix} 6 & -6 & 6 \\ 2 & 4 & -6 \\ 15 & -5 & 5 \end{vmatrix}$

Expanding from R_1

$$= 6\begin{vmatrix} 4 & -6 \\ -5 & 5 \end{vmatrix} + 6\begin{vmatrix} 2 & -6 \\ 15 & 5 \end{vmatrix} + 6\begin{vmatrix} 2 & 4 \\ 15 & -5 \end{vmatrix}$$

$$= 6(20-30) + 6(10+90) + 6(-10-60)$$

$$= 6(-10) + 6(100) + 6(-70)$$

$$= -60 + 600 - 420$$

$$\Delta = -60 + 180 = 120$$

Q4 Evaluate

(i) $$\begin{vmatrix} 2 & 3 & -2 & 4 \\ 7 & 4 & -3 & 10 \\ 3 & 2 & 3 & 4 \\ -2 & 4 & 0 & 5 \end{vmatrix}$$

Sol:

Let $$\Delta = \begin{vmatrix} 2 & 3 & -2 & 4 \\ 7 & 4 & -3 & 10 \\ 3 & 2 & 3 & 4 \\ -2 & 4 & 0 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 3 & -2 & 4 \\ 7 & 4 & -3 & 10 \\ 1 & -1 & 5 & 0 \\ -2 & 4 & 0 & 5 \end{vmatrix} \qquad R_3 - R_1$$

$$= \begin{vmatrix} 0 & 5 & -12 & 4 \\ 0 & 11 & -38 & 10 \\ 1 & -1 & 5 & 0 \\ 0 & 2 & 10 & 5 \end{vmatrix} \qquad \begin{matrix} R_1 - 2R_3 \\ R_2 - 7R_3 \\ R_4 + 2R_3 \end{matrix}$$

Expanding from C_1

$$= 0 - 0 + \begin{vmatrix} 5 & -12 & 4 \\ 11 & -38 & 10 \\ 2 & 10 & 5 \end{vmatrix} - 0$$

$$= \begin{vmatrix} 5 & -12 & 4 \\ 11 & -38 & 10 \\ 2 & 10 & 5 \end{vmatrix}$$

Expanding from R_1

$$= 5\begin{vmatrix} -38 & 10 \\ 10 & 5 \end{vmatrix} + 12\begin{vmatrix} 11 & 10 \\ 2 & 5 \end{vmatrix} + 4\begin{vmatrix} 11 & -38 \\ 2 & 10 \end{vmatrix}$$

$$= 5(-190 - 100) + 12(55 - 20) + 4(110 + 76)$$

$$= 5(-290) + 12(35) + 4(186)$$

$$= -1450 + 420 + 744$$

$$= -1450 + 1164$$

$$\Delta = -286$$

(ii) $\begin{vmatrix} 3 & 7 & 5 & 2 \\ 2 & 4 & 1 & 1 \\ -2 & 0 & 0 & 0 \\ 1 & 1 & 3 & 4 \end{vmatrix}$

Sol.

$$\text{Let } \Delta = \begin{vmatrix} 3 & 7 & 5 & 2 \\ 2 & 4 & 1 & 1 \\ -2 & 0 & 0 & 0 \\ 1 & 1 & 3 & 4 \end{vmatrix}$$

Expanding from R_3

$$= -2\begin{vmatrix} 7 & 5 & 2 \\ 4 & 1 & 1 \\ 1 & 3 & 4 \end{vmatrix}$$

Expanding from R_1

$$= -2\left\{ 7\begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix} - 5\begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} + 2\begin{vmatrix} 4 & 1 \\ 1 & 3 \end{vmatrix} \right\}$$

$$= -2\left\{ 7(4-3) - 5(16-1) + 2(12-1) \right\}$$

$$= -2\left\{ 7(1) - 5(15) + 2(11) \right\}$$

$$= -2(7 - 75 + 22)$$

$$= -2(-46)$$

$$= 92$$

(iii) $\begin{vmatrix} 1 & -2 & 3 & -4 \\ 0 & 1 & -1 & 1 \\ 1 & 3 & 0 & -3 \\ 0 & -7 & 3 & 1 \end{vmatrix}$

Soln

$$\text{Let } \Delta = \begin{vmatrix} 1 & -2 & 3 & -4 \\ 0 & 1 & -1 & 1 \\ 1 & 3 & 0 & -3 \\ 0 & -7 & 3 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -2 & 3 & -4 \\ 0 & 1 & -1 & 1 \\ 0 & 5 & -3 & 1 \\ 0 & -7 & 3 & 1 \end{vmatrix} \qquad R_3 - R_1$$

$$= \begin{vmatrix} 1 & -1 & 1 \\ 5 & -3 & 1 \\ -7 & 3 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -1 & 1 \\ 0 & 2 & -4 \\ 0 & -4 & 8 \end{vmatrix} \qquad \begin{matrix} R_2 - 5R_1 \\ R_3 + 7R_1 \end{matrix}$$

{Expanding from C_1

$$= 1 \begin{vmatrix} 2 & -4 \\ -4 & 8 \end{vmatrix}$$

$$= 16 - 16$$

$$\Delta = 0$$

(iv) $\begin{vmatrix} 9 & 93 & 12 & -6 \\ 1 & 92 & 84 & -6 \\ 2 & 185 & 100 & -12 \\ 4 & 270 & 196 & -24 \end{vmatrix}$

Soln

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Let $\Delta = \begin{vmatrix} 9 & 93 & 12 & -6 \\ 1 & 92 & 84 & -6 \\ 2 & 185 & 100 & -12 \\ 4 & 270 & 196 & -24 \end{vmatrix}$

taking -6 Common from C_4

$$= -6 \begin{vmatrix} 9 & 93 & 12 & 1 \\ 1 & 92 & 84 & 1 \\ 2 & 185 & 100 & 2 \\ 4 & 270 & 196 & 4 \end{vmatrix}$$

$$= -6 \begin{vmatrix} 9 & 93 & 12 & 1 \\ -8 & -1 & 72 & 0 \\ -16 & -1 & 76 & 0 \\ -32 & -102 & 148 & 0 \end{vmatrix} \qquad \begin{matrix} R_2 - R_1 \\ R_3 - 2R_1 \\ R_4 - 4R_1 \end{matrix}$$

Expanding from C_4

$$= -6 \begin{vmatrix} -8 & -1 & 72 \\ -16 & -1 & 76 \\ -32 & -102 & 148 \end{vmatrix}$$

taking $-8, -1, 4$ Common from C_1, C_2, C_3

$$= (-6)(-8)(-1)(4) \begin{vmatrix} 1 & 1 & 18 \\ 2 & 1 & 19 \\ 4 & 102 & 37 \end{vmatrix}$$

$$= -192 \begin{vmatrix} 1 & 1 & 18 \\ 2 & 1 & 19 \\ 4 & 102 & 37 \end{vmatrix}$$

$$= -192 \begin{vmatrix} 1 & 1 & 18 \\ 0 & -1 & -17 \\ 0 & 98 & -35 \end{vmatrix} \qquad \begin{matrix} R_2 - 2R_1 \\ R_3 - 4R_1 \end{matrix}$$

Expanding from C_1

$$= -192 \begin{vmatrix} -1 & -17 \\ 98 & -35 \end{vmatrix}$$

$\Delta = -192(35+1666)$

$= -192(1701)$

$\Delta = -326592$

(v) $\begin{vmatrix} 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 \\ -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 \\ -1 & -1 & -1 & 1 & 1 \end{vmatrix}$

Sol.

Let $\Delta = \begin{vmatrix} 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 \\ -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 \\ -1 & -1 & -1 & 1 & 1 \end{vmatrix}$

$= \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 2 & 0 & 0 \\ -1 & 2 & -2 & 0 & -2 \\ 1 & -2 & 0 & 0 & 2 \\ -1 & 0 & -2 & 0 & 0 \end{vmatrix}$ $\quad C_2 - C_1$, $C_3 + C_1$, $C_4 + C_1$, $C_5 + C_1$

$= 0 \qquad (\because C_4 = 0)$

So $\Delta = 0$

Q5 Without expanding, show that

(i) $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = \begin{vmatrix} e & b & h \\ d & a & g \\ f & c & k \end{vmatrix}$

Sol.:

Sol:

Consider $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix}$

$= \begin{vmatrix} d & e & f \\ a & b & c \\ g & h & k \end{vmatrix}$ R_{12}

$= \begin{vmatrix} e & d & f \\ b & a & c \\ h & g & k \end{vmatrix}$ C_{12}

$= \begin{vmatrix} e & b & h \\ d & a & g \\ f & c & k \end{vmatrix}$ By taking transpose

So $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = \begin{vmatrix} e & b & h \\ d & a & g \\ f & c & k \end{vmatrix}$

(ii) $\begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix} = 0$

Sol:

Let $\Delta = \begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix}$

$= \begin{vmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{vmatrix}$ By taking transpose

$= (-1)^3 \begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix}$ By taking -1 Common from R_1, R_2, R_3

$\Delta = -\Delta$

$\Delta + \Delta = 0$

$2\Delta = 0$

$\Delta = 0$

$$\begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix} = 0$$

(iii) $$\begin{vmatrix} a+b & c & 1 \\ b+c & a & 1 \\ c+a & b & 1 \end{vmatrix} = 0$$

Sol.

Let $$\Delta = \begin{vmatrix} a+b & c & 1 \\ b+c & a & 1 \\ c+a & b & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a+b+c & c & 1 \\ b+c+a & a & 1 \\ c+a+b & b & 1 \end{vmatrix} \qquad C_1 + C_2$$

$$= \begin{vmatrix} a+b+c & c & 1 \\ a+b+c & a & 1 \\ a+b+c & b & 1 \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & c & 1 \\ 1 & a & 1 \\ 1 & b & 1 \end{vmatrix} \qquad \text{taking } a+b+c \text{ Common from } C_1$$

$$= (a+b+c)(0) \qquad \because C_1 = C_3$$

$\Delta = 0$

or $$\begin{vmatrix} a+b & c & 1 \\ b+c & a & 1 \\ c+a & b & 1 \end{vmatrix} = 0$$

Q6 Prove that

(i) $\begin{vmatrix} bc & ca & ab \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ a^2 & b^2 & c^2 \end{vmatrix} = 0$

Sol.

Let $\Delta = \begin{vmatrix} bc & ca & ab \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ a^2 & b^2 & c^2 \end{vmatrix}$

$= \frac{1}{abc}\begin{vmatrix} bc & ca & ab \\ bc & ca & ab \\ a^2 & b^2 & c^2 \end{vmatrix}$ Multiplying R_2 by abc

$= \frac{1}{abc}(0)$ $\because R_1 = R_2$

$\Delta = 0$

So $\begin{vmatrix} bc & ca & ab \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ a^2 & b^2 & c^2 \end{vmatrix} = 0$

(ii) $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$

Sol.

Let $\Delta = \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$

$$= \begin{vmatrix} a-b+b-c+c-a & b-c & c-a \\ b-c+c-a+a-b & c-a & a-b \\ c-a+a-b+b-c & a-b & b-c \end{vmatrix} \qquad C_1+(C_2+C_3)$$

$$= \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix}$$

$$\Delta = 0 \qquad \because C_1 = 0$$

So
$$\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$$

(iii)
$$\begin{vmatrix} a & a^2 & a/bc \\ b & b^2 & b/ca \\ c & c^2 & c/ab \end{vmatrix} = 0$$

Sol.

$$\text{let } \Delta = \begin{vmatrix} a & a^2 & a/bc \\ b & b^2 & b/ca \\ c & c^2 & c/ab \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} a & a^2 & a^2 \\ b & b^2 & b^2 \\ c & c^2 & c^2 \end{vmatrix} \qquad \text{Multiplying } C_3 \text{ by } abc$$

$$\Delta = \frac{1}{abc}(0)$$

$$\Delta = 0$$

So
$$\begin{vmatrix} a & a^2 & a/bc \\ b & b^2 & b/ca \\ c & c^2 & c/ab \end{vmatrix} = 0$$

(iv)
$$\begin{vmatrix} \sin^2\theta & 1 & \cos^2\theta \\ \sin^2\phi & 1 & \cos^2\phi \\ \sin^2\psi & 1 & \cos^2\psi \end{vmatrix} = 0$$

Sol.

$$\text{Let } \Delta = \begin{vmatrix} \sin^2\theta & 1 & \cos^2\theta \\ \sin^2\phi & 1 & \cos^2\phi \\ \sin^2\psi & 1 & \cos^2\psi \end{vmatrix}$$

$$= \begin{vmatrix} \sin^2\theta+\cos^2\theta & 1 & \cos^2\theta \\ \sin^2\phi+\cos^2\phi & 1 & \cos^2\phi \\ \sin^2\psi+\cos^2\psi & 1 & \cos^2\psi \end{vmatrix} \qquad R_1 + R_2$$

$$= \begin{vmatrix} 1 & 1 & \cos^2\theta \\ 1 & 1 & \cos^2\phi \\ 1 & 1 & \cos^2\psi \end{vmatrix}$$

$$\Delta = 0 \qquad \because C_1 = C_2$$

or
$$\begin{vmatrix} \sin^2\theta & 1 & \cos^2\theta \\ \sin^2\phi & 1 & \cos^2\phi \\ \sin^2\psi & 1 & \cos^2\psi \end{vmatrix} = 0$$

(v)
$$\begin{vmatrix} \sin^2\alpha & \cos 2\alpha & \cos^2\alpha \\ \sin^2\beta & \cos 2\beta & \cos^2\beta \\ \sin^2\gamma & \cos 2\gamma & \cos^2\gamma \end{vmatrix} = 0$$

Sol.

Let $\Delta = \begin{vmatrix} \sin^2\alpha & \cos 2\alpha & \cos^2\alpha \\ \sin^2\beta & \cos 2\beta & \cos^2\beta \\ \sin^2\gamma & \cos 2\gamma & \cos^2\gamma \end{vmatrix}$

$$= \begin{vmatrix} \sin^2\alpha & \cos 2\alpha & \cos^2\alpha - \sin^2\alpha \\ \sin^2\beta & \cos 2\beta & \cos^2\beta - \sin^2\beta \\ \sin^2\gamma & \cos 2\gamma & \cos^2\gamma - \sin^2\gamma \end{vmatrix} \qquad C_3 - C_1$$

$$= \begin{vmatrix} \sin^2\alpha & \cos 2\alpha & \cos 2\alpha \\ \sin^2\beta & \cos 2\beta & \cos 2\beta \\ \sin^2\gamma & \cos 2\gamma & \cos 2\gamma \end{vmatrix} \qquad \because \cos 2\theta = \cos^2\theta - \sin^2\theta$$

$\Delta = 0 \qquad \because C_2 = C_3$

So

$$\begin{vmatrix} \sin^2\alpha & \cos 2\alpha & \cos^2\alpha \\ \sin^2\beta & \cos 2\beta & \cos^2\beta \\ \sin^2\gamma & \cos 2\gamma & \cos^2\gamma \end{vmatrix} = 0$$

(vi)

$$\begin{vmatrix} \cos\alpha & \sin\alpha & \sin(\alpha+\delta) \\ \cos\beta & \sin\beta & \sin(\beta+\delta) \\ \cos\gamma & \sin\gamma & \sin(\gamma+\delta) \end{vmatrix} = 0$$

<u>Sol.</u>

Let $\Delta = \begin{vmatrix} \cos\alpha & \sin\alpha & \sin(\alpha+\delta) \\ \cos\beta & \sin\beta & \sin(\beta+\delta) \\ \cos\gamma & \sin\gamma & \sin(\gamma+\delta) \end{vmatrix}$

or

$$\Delta = \begin{vmatrix} \cos\alpha & \sin\alpha & \sin\alpha\cos\delta + \cos\alpha\sin\delta \\ \cos\beta & \sin\beta & \sin\beta\cos\delta + \cos\beta\sin\delta \\ \cos\gamma & \sin\gamma & \sin\gamma\cos\delta + \cos\gamma\sin\delta \end{vmatrix}$$

$$= \begin{vmatrix} \cos\alpha & \sin\alpha & \sin\alpha\cos\delta \\ \cos\beta & \sin\beta & \sin\beta\cos\delta \\ \cos\gamma & \sin\gamma & \sin\gamma\cos\delta \end{vmatrix} + \begin{vmatrix} \cos\alpha & \sin\alpha & \cos\alpha\sin\delta \\ \cos\beta & \sin\beta & \cos\beta\sin\delta \\ \cos\gamma & \sin\gamma & \cos\gamma\sin\delta \end{vmatrix}$$

$$= \cos\delta \begin{vmatrix} \cos\alpha & \sin\alpha & \sin\alpha \\ \cos\beta & \sin\beta & \sin\beta \\ \cos\gamma & \sin\gamma & \sin\gamma \end{vmatrix} + \sin\delta \begin{vmatrix} \cos\alpha & \sin\alpha & \cos\alpha \\ \cos\beta & \sin\beta & \cos\beta \\ \cos\gamma & \sin\gamma & \cos\gamma \end{vmatrix}$$

$= \cos\delta(0) + \sin\delta(0)$ ($\because$ two Columns are identical)

$= 0 + 0$

$\Delta = 0$

So
$$\begin{vmatrix} \cos\alpha & \sin\alpha & \sin(\alpha+\delta) \\ \cos\beta & \sin\beta & \sin(\beta+\delta) \\ \cos\gamma & \sin\gamma & \sin(\gamma+\delta) \end{vmatrix} = 0$$

(vii)

$$\begin{vmatrix} 1 & \cos\alpha & \cos\beta \\ \cos\alpha & 1 & \cos(\alpha+\beta) \\ \cos\beta & \cos(\alpha+\beta) & 1 \end{vmatrix} = 0$$

Sol.

$$\Delta = \begin{vmatrix} 1 & \cos\alpha & \cos\beta \\ \cos\alpha & 1 & \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \cos\beta & \cos\alpha\cos\beta - \sin\alpha\sin\beta & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ \cos\alpha & 1-\cos^2\alpha & -\sin\alpha\sin\beta \\ \cos\beta & -\sin\alpha\sin\beta & 1-\cos^2\beta \end{vmatrix} \qquad \begin{matrix} C_2 - (\cos\alpha)C_1 \\ C_3 - (\cos\beta)C_1 \end{matrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ \cos\alpha & \sin^2\alpha & -\sin\alpha\sin\beta \\ \cos\beta & -\sin\alpha\sin\beta & \sin^2\beta \end{vmatrix}$$

$$= (\sin\alpha)(-\sin\beta)\begin{vmatrix} 1 & 0 & 0 \\ \cos\alpha & \sin\alpha & \sin\alpha \\ \cos\beta & -\sin\beta & -\sin\beta \end{vmatrix}$$ taking $\sin\alpha$ Common from C_2 & $-\sin\beta$ Common from C_3

$$= -\sin\alpha\sin\beta(0) \qquad \because C_2 = C_3$$

$$\Delta = 0$$

So
$$\begin{vmatrix} 1 & \cos\alpha & \cos\beta \\ \cos\alpha & 1 & \cos(\alpha+\beta) \\ \cos\beta & \cos(\alpha+\beta) & 1 \end{vmatrix} = 0$$

(viii) $$\begin{vmatrix} (a+b)^2 & a^2+b^2 & ab \\ (c+d)^2 & c^2+d^2 & cd \\ (g+h)^2 & g^2+h^2 & gh \end{vmatrix} = 0$$

Sol.

Let $$\Delta = \begin{vmatrix} (a+b)^2 & a^2+b^2 & ab \\ (c+d)^2 & c^2+d^2 & cd \\ (g+h)^2 & g^2+h^2 & gh \end{vmatrix}$$

$$= \begin{vmatrix} a^2+b^2+2ab & a^2+b^2 & ab \\ c^2+d^2+2cd & c^2+d^2 & cd \\ g^2+h^2+2gh & g^2+h^2 & gh \end{vmatrix}$$

$$= \begin{vmatrix} 2ab & a^2+b^2 & ab \\ 2cd & c^2+d^2 & cd \\ 2gh & g^2+h^2 & gh \end{vmatrix} \qquad C_1 - C_2$$

$$= 2\begin{vmatrix} ab & a^2+b^2 & ab \\ cd & c^2+d^2 & cd \\ gh & g^2+h^2 & gh \end{vmatrix} \qquad \text{taking 2 common from } C_1$$

$$= 2(0) \qquad (\because C_1 = C_3)$$

$$\Delta = 0$$

So $$\begin{vmatrix} (a+b)^2 & a^2+b^2 & ab \\ (c+d)^2 & c^2+d^2 & cd \\ (g+h)^2 & g^2+h^2 & gh \end{vmatrix} = 0$$

(ix) $$\begin{vmatrix} (a^m+a^{-m})^2 & (a^m-a^{-m})^2 & abc \\ (b^n+b^{-n})^2 & (b^n-b^{-n})^2 & abc \\ (c^p+c^{-p})^2 & (c^p-c^{-p})^2 & abc \end{vmatrix} = 0$$

Soln.

Let $$\Delta = \begin{vmatrix} (a^m+a^{-m})^2 & (a^m-a^{-m})^2 & abc \\ (b^n+b^{-n})^2 & (b^n-b^{-n})^2 & abc \\ (c^p+c^{-p})^2 & (c^p-c^{-p})^2 & abc \end{vmatrix}$$

$$= abc\begin{vmatrix} a^{2m}+a^{-2m}+2 & a^{2m}+a^{-2m}-2 & 1 \\ b^{2n}+b^{-2n}+2 & a^{2m}+a^{-2m}-2 & 1 \\ c^{2p}+c^{-2p}+2 & c^{2p}+c^{-2p}-2 & 1 \end{vmatrix}$$

$$= abc\begin{vmatrix} a^{2m}+a^{-2m} & a^{2m}+a^{-2m} & 1 \\ b^{2n}+b^{-2n} & b^{2n}+b^{-2n} & 1 \\ c^{2p}+c^{-2p} & c^{2p}+c^{-2p} & 1 \end{vmatrix} \qquad \begin{matrix} C_1-2C_3 \\ C_2+2C_3 \end{matrix}$$

$= (abc)(0) \qquad \because C_1 = C_2$

$\Delta = 0$

(x) $$\begin{vmatrix} \frac{1}{2!} & 1 & 0 \\ \frac{1}{3!} & \frac{1}{2!} & 1 \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} \end{vmatrix} = 0$$

Soln.

let $$\Delta = \begin{vmatrix} \frac{1}{2!} & 1 & 0 \\ \frac{1}{3!} & \frac{1}{2!} & 1 \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} \end{vmatrix}$$

$$\Delta = \begin{vmatrix} \frac{1}{2} & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & 1 \\ \frac{1}{24} & \frac{1}{6} & \frac{1}{2} \end{vmatrix}$$

$$= \left(\tfrac{1}{2}\right)\left(\tfrac{1}{6}\right)\left(\tfrac{1}{24}\right) \begin{vmatrix} 1 & 2 & 0 \\ 1 & 3 & 6 \\ 1 & 4 & 12 \end{vmatrix}$$ taking $\frac{1}{2}, \frac{1}{6}, \frac{1}{24}$ common from R_1, R_2, R_3

$$= \frac{1}{288} \begin{vmatrix} 1 & 2 & 0 \\ 1 & 3 & 6 \\ 1 & 4 & 12 \end{vmatrix}$$

$$= \frac{1}{288} \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 6 \\ 1 & 2 & 12 \end{vmatrix}$$

$$= \frac{6}{288} \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{vmatrix}$$ taking 6 common from C_3

$$= \frac{1}{48}(0)$$ $\because C_2 = C_3$

$$\Delta = 0$$

(xi) $$\begin{vmatrix} a^2 & b\sin\alpha & c\sin\alpha \\ b\sin\alpha & 1 & \cos\alpha \\ c\sin\alpha & \cos\alpha & 1 \end{vmatrix} = 0$$

where a, b, c are the magnitudes of the sides of a triangle & α is the measure of the angle opposite to the side with magnitude a

$$= \begin{vmatrix} a^2 - c^2\sin^2\alpha & b\sin\alpha - c\sin\alpha\cos\alpha & 0 \\ b\sin\alpha - c\sin\alpha\cos\alpha & 1-\cos^2\alpha & 0 \\ c\sin\alpha & \cos\alpha & 1 \end{vmatrix} \quad \begin{matrix} R_1 - c\sin\alpha\, R_3 \\ R_2 - \cos\alpha\, R_3 \end{matrix}$$

Expanding from C_3

$$= \begin{vmatrix} a^2 - c^2\sin^2\alpha & \sin\alpha(b - c\cos\alpha) \\ \sin\alpha(b-\cos\alpha) & \sin^2\alpha \end{vmatrix}$$

$$= \sin^2\alpha\,(a^2 - c^2\sin^2\alpha) - \sin^2\alpha\,(b - c\cos\alpha)^2$$

$$= a^2\sin^2\alpha - c^2\sin^4\alpha - \sin^2\alpha\,(b^2 + c^2\cos^2\alpha - 2bc\cos\alpha)$$

$$= a^2\sin^2\alpha - c^2\sin^4\alpha - b^2\sin^2\alpha - c^2\sin^2\alpha\cos^2\alpha + 2bc\sin^2\alpha\cos\alpha$$

$$= a^2\sin^2\alpha - c^2\sin^4\alpha - b^2\sin^2\alpha - c^2\sin^2\alpha(1-\sin^2\alpha) + 2bc\sin^2\alpha\cos\alpha$$

$$= a^2\sin^2\alpha - \cancel{c^2\sin^4\alpha} - b^2\sin^2\alpha - c^2\sin^2\alpha + \cancel{c^2\sin^4\alpha} + 2bc\sin^2\alpha\cos\alpha$$

$$= [a^2 - b^2 - c^2 + 2bc\cos\alpha]\sin^2\alpha$$

$$= \left[a^2 - b^2 - c^2 + 2bc\left(\frac{b^2+c^2-a^2}{2bc}\right)\right]\sin^2\alpha$$

$$= [\cancel{a^2} - \cancel{b^2} - \cancel{c^2} + \cancel{b^2} + \cancel{c^2} - \cancel{a^2}]\sin^2\alpha$$

$$= (0)\sin^2\alpha$$

$$\Delta = 0$$

(xii)
$$\begin{vmatrix} a & b & c & d & 1 \\ b & c & d & a & 1 \\ c & d & a & b & 1 \\ d & a & b & c & 1 \\ b & a & d & c & 1 \end{vmatrix} = 0$$

Sol:-

Let $\Delta = \begin{vmatrix} a & b & c & d & 1 \\ b & c & d & a & 1 \\ c & d & a & b & 1 \\ d & a & b & c & 1 \\ b & a & d & c & 1 \end{vmatrix}$

$$= \begin{vmatrix} a+b+c+d & b & c & d & 1 \\ b+c+d+a & c & d & a & 1 \\ c+d+a+b & d & a & b & 1 \\ d+a+b+c & a & b & c & 1 \\ b+a+d+c & a & d & c & 1 \end{vmatrix} \qquad C_1 + (C_2 + C_3 + C_4)$$

$$= \begin{vmatrix} a+b+c+d & b & c & d & 1 \\ a+b+c+d & c & d & a & 1 \\ a+b+c+d & d & a & b & 1 \\ a+b+c+d & a & b & c & 1 \\ a+b+c+d & a & d & c & 1 \end{vmatrix}$$

$$= (a+b+c+d) \begin{vmatrix} 1 & b & c & d & 1 \\ 1 & c & d & a & 1 \\ 1 & d & a & b & 1 \\ 1 & a & b & c & 1 \\ 1 & a & d & c & 1 \end{vmatrix}$$

$= (a+b+c+d)(0) \qquad \because C_1 = C_5$

$= 0$

So $\Delta = 0$

(xiii) $\begin{vmatrix} a^2 & (a+1)^2 & (a+2)^2 & (a+3)^2 \\ b^2 & (b+1)^2 & (b+2)^2 & (b+3)^2 \\ c^2 & (c+1)^2 & (c+2)^2 & (c+3)^2 \\ d^2 & (d+1)^2 & (d+2)^2 & (d+3)^2 \end{vmatrix}$

Sol.

Let $\Delta = \begin{vmatrix} a^2 & (a+1)^2 & (a+2)^2 & (a+3)^2 \\ b^2 & (b+1)^2 & (b+2)^2 & (b+3)^2 \\ c^2 & (c+1)^2 & (c+2)^2 & (c+3)^2 \\ d^2 & (d+1)^2 & (d+2)^2 & (d+3)^2 \end{vmatrix}$

$$= \begin{vmatrix} a^2 & a^2+2a+1 & a^2+4a+4 & a^2+6a+9 \\ b^2 & b^2+2b+1 & b^2+4b+4 & b^2+6b+9 \\ c^2 & c^2+2c+1 & c^2+4c+4 & c^2+6c+9 \\ d^2 & d^2+2d+1 & d^2+4d+4 & d^2+6d+9 \end{vmatrix}$$

$$= \begin{vmatrix} a^2 & 2a+1 & 4a+4 & 6a+9 \\ b^2 & 2b+1 & 4b+4 & 6b+9 \\ c^2 & 2c+1 & 4c+4 & 6c+9 \\ d^2 & 2d+1 & 4d+4 & 6d+9 \end{vmatrix} \quad \begin{matrix} C_2-C_1 \\ C_3-C_1 \\ C_4-C_1 \end{matrix}$$

$$= \begin{vmatrix} a^2 & 2a+1 & 2 & 6 \\ b^2 & 2b+1 & 2 & 6 \\ c^2 & 2c+1 & 2 & 6 \\ d^2 & 2d+1 & 2 & 6 \end{vmatrix} \quad \begin{matrix} C_3-2C_2 \\ C_4-3C_2 \end{matrix}$$

$$= 3\begin{vmatrix} a^2 & 2a+1 & 2 & 2 \\ b^2 & 2b+1 & 2 & 2 \\ c^2 & 2c+1 & 2 & 2 \\ d^2 & 2d+1 & 2 & 2 \end{vmatrix} \quad \text{taking 3 Common from } C_4$$

$\Delta = 0 \qquad \because C_3 = C_4$

Q7 Without expansion, prove that

$$\begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

Sol.

$$\text{Let } \Delta = \begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} abc & a^2 & a^3 \\ abc & b^2 & b^3 \\ abc & c^2 & c^3 \end{vmatrix} \quad \text{Multiplying } R_1, R_2, R_3 \text{ by } a, b, c$$

$$= \frac{\cancel{abc}}{\cancel{abc}}\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} \quad \text{taking } abc \text{ common from } C_1$$

$$\Delta = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}.$$

So

$$\begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

Q8 Prove that

$$\begin{vmatrix} 1 & \alpha & \beta\gamma \\ 1 & \beta & \gamma\alpha \\ 1 & \gamma & \alpha\beta \end{vmatrix} = \begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{vmatrix}$$

Sol.

$$\text{Let } \Delta = \begin{vmatrix} 1 & \alpha & \beta\gamma \\ 1 & \beta & \gamma\alpha \\ 1 & \gamma & \alpha\beta \end{vmatrix}$$

Q9 Prove that

$$\begin{vmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{vmatrix} = (a^2+b^2+c^2+d^2)^2$$

Sol.

$$\text{Let } \Delta = \begin{vmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{vmatrix}$$

$$= \frac{1}{abcd}\begin{vmatrix} a^2 & b^2 & c^2 & d^2 \\ -ab & ab & -cd & cd \\ -ac & bd & ac & -bd \\ -ad & -bc & bc & ad \end{vmatrix}$$ Multiplying C_1, C_2, C_3, C_4 by a, b, c, d resp.

$$= \frac{1}{abcd}\begin{vmatrix} a^2+b^2+c^2+d^2 & b^2 & c^2 & d^2 \\ 0 & ab & -cd & cd \\ 0 & bd & ac & -bd \\ 0 & -bc & bc & ad \end{vmatrix}$$ $C_1 + (C_2 + C_3 + C_4)$

Expanding from C_1

$$= \frac{(a^2+b^2+c^2+d^2)}{abcd}\begin{vmatrix} ab & -cd & cd \\ bd & ac & -bd \\ -bc & bc & ad \end{vmatrix}$$

$$= \frac{(a^2+b^2+c^2+d^2)\cancel{bcd}}{a\cancel{bcd}}\begin{vmatrix} a & -d & c \\ d & a & -b \\ -c & b & a \end{vmatrix}$$ taking b, c, d common from C_1, C_2, C_3

$$= \frac{(a^2+b^2+c^2+d^2)}{a}\begin{vmatrix} a & -d & c \\ d & a & -b \\ -c & b & a \end{vmatrix}$$

Expanding from R_1

$$\Delta = \frac{(a^2+b^2+c^2+d^2)}{a}\left\{a(a^2+b^2)+d(ad-bc)+c(bd+ac)\right\}$$

$$= \frac{(a^2+b^2+c^2+d^2)}{a}\left\{a^3+ab^2+ad^2-\cancel{bcd}+\cancel{bcd}+ac^2\right\}$$

$$= (a^2+b^2+c^2+d^2)(a^2+b^2+c^2+d^2)$$

$$\Delta = (a^2+b^2+c^2+d^2)^2$$

Q10 Prove that

(i) $$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3$$

Sol.

Let $$\Delta = \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$$

$$= \begin{vmatrix} (b+c)^2-a^2 & 0 & a^2 \\ 0 & (c+a)^2-b^2 & b^2 \\ c^2-(a+b)^2 & c^2-(a+b)^2 & (a+b)^2 \end{vmatrix} \qquad \begin{matrix} C_1-C_3 \\ C_2-C_3 \end{matrix}$$

$$= \begin{vmatrix} (b+c+a)(b+c-a) & 0 & a^2 \\ 0 & (c+a+b)(c+a-b) & b^2 \\ (c+a+b)(c-a-b) & (c+a+b)(c-a-b) & (a+b)^2 \end{vmatrix}$$

$$\Delta = (a+b+c)^2\begin{vmatrix}(b+c-a) & 0 & a^2\\ 0 & (c+a-b) & b^2\\ (c-a-b) & (c-a-b) & (a+b)^2\end{vmatrix}$$ taking $(a+b+c)$ common from C_1 & C_2

$$= (a+b+c)^2\begin{vmatrix}b+c-a & 0 & a^2\\ 0 & c+a-b & b^2\\ -2b & -2a & 2ab\end{vmatrix}$$ $R_3-(R_1+R_2)$

$$= -2(a+b+c)^2\begin{vmatrix}b+c-a & 0 & a^2\\ 0 & c+a-b & b^2\\ b & a & -ab\end{vmatrix}$$

Expanding from R_1

$$= -2(a+b+c)^2\left[(b+c-a)\left\{(-ab)(c+a-b)-ab^2\right\}-0+a^2\left\{0-b(c+a-b)\right\}\right]$$

$$= -2(a+b+c)^2\left\{(b+c-a)(-abc-a^2b+\cancel{ab^2}-\cancel{ab^2})+a^2(-bc-ab+b^2)\right\}$$

$$= -2(a+b+c)^2\left\{(-ab)(b+c-a)(c+a)+(-ab)(ac+a^2-ab)\right\}$$

$$= -2(a+b+c)^2(-ab)\left\{(b+c-a)(c+a)+ac+a^2-ab\right\}$$

$$= 2ab(a+b+c)^2\left\{bc+\cancel{ab}+c^2+\cancel{ac}-\cancel{ac}-\cancel{a^2}+ac+\cancel{a^2}-\cancel{ab}\right\}$$

$$= 2ab(a+b+c)^2(bc+c^2+ac)$$

$$= 2abc(a+b+c)^2(b+c+a)$$

$$\Delta = 2abc(a+b+c)^3$$

(ii) $$\begin{vmatrix} \frac{a^2+b^2}{c} & c & c \\ a & \frac{b^2+c^2}{a} & a \\ b & b & \frac{c^2+a^2}{b} \end{vmatrix} = 4abc$$

Sol:

$$\text{Let } \Delta = \begin{vmatrix} \frac{a^2+b^2}{c} & c & c \\ a & \frac{b^2+c^2}{a} & a \\ b & b & \frac{c^2+a^2}{b} \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} a^2+b^2 & c^2 & c^2 \\ a^2 & b^2+c^2 & a^2 \\ b^2 & b^2 & c^2+a^2 \end{vmatrix}$$ Multiplying R_1, R_2, R_3 by c, a, b resp.

$$= \frac{1}{abc}\begin{vmatrix} a^2+b^2-c^2 & 0 & c^2 \\ 0 & b^2+c^2-a^2 & a^2 \\ b^2-c^2-a^2 & b^2-c^2-a^2 & c^2+a^2 \end{vmatrix}$$ C_1-C_3, C_2-C_3

$$= \frac{1}{abc}\begin{vmatrix} a^2+b^2-c^2 & 0 & c^2 \\ 0 & b^2+c^2-a^2 & a^2 \\ -2a^2 & -2c^2 & 0 \end{vmatrix}$$ $R_3-(R_1+R_2)$

$$= \frac{-2}{abc}\begin{vmatrix} a^2+b^2-c^2 & 0 & c^2 \\ 0 & b^2+c^2-a^2 & a^2 \\ a^2 & c^2 & 0 \end{vmatrix}$$ taking -2 common from R_3

Expanding from R_1

$$= -\frac{2}{abc}\left\{(a^2+b^2-c^2)(0-a^2c^2) - 0 + c^2\left(0 - a^2(b^2+c^2-a^2)\right)\right\}$$

$$-\frac{2}{abc}\left\{-a^2c^2(a^2+b^2-c^2) - a^2c^2(b^2+c^2-a^2)\right\}$$

$$= -\frac{2}{abc}(-a^2c^2)\left\{\cancel{a^2}+b^2-\cancel{c^2}+b^2+\cancel{c^2}-\cancel{a^2}\right\}$$

$$= \frac{2ac}{b}(2b^2)$$

$$\Delta = 4abc$$

Q11 Prove that

$$\begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix} = (x+3a)(x-a)^3$$

Sol

Let $\Delta = \begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix}$

$$= \begin{vmatrix} x+3a & a & a & a \\ x+3a & x & a & a \\ x+3a & a & x & a \\ x+3a & a & a & x \end{vmatrix} \qquad C_1 + (C_2 + C_3 + C_4)$$

$$= (x+3a)\begin{vmatrix} 1 & a & a & a \\ 1 & x & a & a \\ 1 & a & x & a \\ 1 & a & a & x \end{vmatrix} \qquad \text{taking } (x+3a) \text{ Common from } C_1$$

$$= (x+3a)\begin{vmatrix} 1 & a & a & a \\ 0 & x-a & 0 & 0 \\ 0 & 0 & x-a & 0 \\ 0 & 0 & 0 & x-a \end{vmatrix} \qquad \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{matrix}$$

$$\Delta = (x+3a)(x-a)^3 \begin{vmatrix} 1 & a & a & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$ taking $(x-a)$ Common from R_2, R_3, R_4

Expanding from C_1

$$= (x+3a)(x-a)^3 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\Delta = (x+3a)(x-a)^3$$

Q12 Show that

$$\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \beta\gamma & \gamma\alpha & \alpha\beta \end{vmatrix} = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$$

Sol.

Let $$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \beta\gamma & \gamma\alpha & \alpha\beta \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ \alpha-\gamma & \beta-\gamma & \gamma \\ \beta(\gamma-\alpha) & \alpha(\gamma-\beta) & \alpha\beta \end{vmatrix}$$ $C_1 - C_3$, $C_2 - C_3$

Expanding from R_1

$$= \begin{vmatrix} \alpha-\gamma & \beta-\gamma \\ \beta(\gamma-\alpha) & \alpha(\gamma-\beta) \end{vmatrix}$$

$$= (\alpha-\gamma)(\beta-\gamma) \begin{vmatrix} 1 & 1 \\ -\beta & -\alpha \end{vmatrix}$$ taking $\alpha-\gamma$, $\beta-\gamma$ Common from C_1, C_2 resp.

$$\Delta = (\alpha-\gamma)(\beta-\gamma)(-\alpha+\beta)$$

$$\Delta = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$$

Q13 Show that

$$\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^3 & \beta^3 & \gamma^3 \end{vmatrix} = (\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\alpha+\beta+\gamma)$$

Sol:-

Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^3 & \beta^3 & \gamma^3 \end{vmatrix}$

$$= \begin{vmatrix} 0 & 0 & 1 \\ \alpha-\gamma & \beta-\gamma & \gamma \\ \alpha^3-\gamma^3 & \beta^3-\gamma^3 & \gamma^3 \end{vmatrix} \qquad \begin{matrix} C_1 - C_3 \\ C_2 - C_3 \end{matrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ (\alpha-\gamma) & \beta-\gamma & \gamma \\ (\alpha-\gamma)(\alpha^2+\alpha\gamma+\gamma^2) & (\beta-\gamma)(\beta^2+\beta\gamma+\gamma^2) & \gamma^3 \end{vmatrix}$$

$$= (\alpha-\gamma)(\beta-\gamma)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & \gamma \\ \alpha^2+\alpha\gamma+\gamma^2 & \beta^2+\beta\gamma+\gamma^2 & \gamma^3 \end{vmatrix}$$ taking $(\alpha-\gamma)$, $(\beta-\gamma)$ Common from C_1 & C_2 resp.

Expanding from R_1

$$= (\alpha-\gamma)(\beta-\gamma)\begin{vmatrix} 1 & 1 \\ \alpha^2+\alpha\gamma+\gamma^2 & \beta^2+\beta\gamma+\gamma^2 \end{vmatrix}$$

$$= (\alpha-\gamma)(\beta-\gamma)\left\{ \beta^2+\beta\gamma+\cancel{\gamma^2} - \alpha^2 - \alpha\gamma - \cancel{\gamma^2} \right\}$$

$$= (\alpha-\gamma)(\beta-\gamma)\left\{ \beta^2-\alpha^2 + \beta\gamma - \gamma\alpha \right\}$$

$$\Delta = (\alpha-\gamma)(\beta-\gamma)\{(\beta-\alpha)(\beta+\alpha)+\gamma(\beta-\alpha)\}$$

$$= (\alpha-\gamma)(\beta-\gamma)(\beta-\alpha)(\beta+\alpha+\gamma)$$

$$\Delta = (\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\alpha+\beta+\gamma)$$

Q14 Show that

$$\begin{vmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{vmatrix} = (a+3)(a-1)^3$$

Sol.

Let $\Delta = \begin{vmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{vmatrix}$

$$= \begin{vmatrix} a+3 & 1 & 1 & 1 \\ a+3 & a & 1 & 1 \\ a+3 & 1 & a & 1 \\ a+3 & 1 & 1 & a \end{vmatrix} \qquad C_1+(C_2+C_3+C_4)$$

$$= (a+3)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{vmatrix} \qquad \text{taking } a+3 \text{ common from } C_1$$

$$= (a+3)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & a-1 & 0 & 0 \\ 0 & 0 & a-1 & 0 \\ 0 & 0 & 0 & a-1 \end{vmatrix} \qquad \begin{matrix} R_2-R_1 \\ R_3-R_1 \\ R_4-R_1 \end{matrix}$$

$$= (a+3)\begin{vmatrix} a-1 & 0 & 0 \\ 0 & a-1 & 0 \\ 0 & 0 & a-1 \end{vmatrix} \qquad \text{Expanding from } C_1$$

$$\Delta = (a+3)(a-1)^3 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$ taking $a-1$ common from R_1, R_2, R_3

$$\Delta = (a+3)(a-1)^3$$

(ii) $$\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix} = abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

Sol.

$$\text{Let } \Delta = \begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix}$$

$$= abcd \begin{vmatrix} 1+\frac{1}{a} & \frac{1}{a} & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} & \frac{1}{c} \\ \frac{1}{d} & \frac{1}{d} & \frac{1}{d} & 1+\frac{1}{d} \end{vmatrix}$$ taking a, b, c, d common from R_1, R_2, R_3, R_4 resp.

Adding R_2, R_3, R_4 in R_1

$$= abcd \begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} & \frac{1}{c} \\ \frac{1}{d} & \frac{1}{d} & \frac{1}{d} & 1+\frac{1}{d} \end{vmatrix}$$

$$= (abcd)\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) \begin{vmatrix} 1 & 1 & 1 & 1 \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} & \frac{1}{c} \\ 1/d & 1/d & 1/d & 1+1/d \end{vmatrix}$$ taking $\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$ common from R_1

$$\Delta = abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1/b & 1 & 0 & 0 \\ 1/c & 0 & 1 & 0 \\ 1/d & 0 & 0 & 1 \end{vmatrix} \quad \begin{matrix} C_2 - C_1 \\ C_3 - C_1 \\ C_4 - C_1 \end{matrix}$$

$$= abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \quad \text{Expanding from } R_1$$

$$\Delta = abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

(iii)
$$\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1-y \end{vmatrix} = x^2y^2$$

Sol:-

Let
$$\Delta = \begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1-y \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 0 & 1 \\ -x & -x & 0 & 1 \\ -x & 0 & y & 1 \\ (-x+y+xy) & y & y & 1-y \end{vmatrix} \quad \begin{matrix} C_1 - (1+x)C_4 \\ C_2 - C_4 \\ C_3 - C_4 \end{matrix}$$

Expanding from R_1

$$= -\begin{vmatrix} -x & -x & 0 \\ -x & 0 & y \\ -x+y+xy & y & y \end{vmatrix}$$

$$= -\begin{vmatrix} 0 & -x & 0 \\ -x & 0 & y \\ -x+xy & y & y \end{vmatrix}$$

$$= x\begin{vmatrix} 0 & 1 & 0 \\ -x & 0 & y \\ -x+xy & y & y \end{vmatrix}. \qquad \text{taking } -x \text{ common from } R_1$$

Expanding from R_1

$$= -x\begin{vmatrix} -x & y \\ -x+xy & y \end{vmatrix}$$

$$= -x\left(-\cancel{x}y + \cancel{x}y - xy^2\right)$$

$$= -x\left(-xy^2\right)$$

$$\Delta = x^2y^2$$

Q15
$$\begin{vmatrix} a^3 & 3a^2 & 3a & 1 \\ a^2 & a^2+2a & 2a+1 & 1 \\ a & 2a+1 & a+2 & 1 \\ 1 & 3 & 3 & 1 \end{vmatrix}$$

Sol.

$$\text{Let } \Delta = \begin{vmatrix} a^3 & 3a^2 & 3a & 1 \\ a^2 & a^2+2a & 2a+1 & 1 \\ a & 2a+1 & a+2 & 1 \\ 1 & 3 & 3 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a^3-1 & 3(a^2-1) & 3(a-1) & 0 \\ a^2-1 & a^2+2a-3 & 2(a-1) & 0 \\ a-1 & 2(a-1) & a-1 & 0 \\ 1 & 3 & 3 & 1 \end{vmatrix} \qquad \begin{matrix} R_1 - R_4 \\ R_2 - R_4 \\ R_3 - R_4 \end{matrix}$$

Expanding from C_4

$$\Delta = \begin{vmatrix} a^3-1 & 3(a^2-1) & 3(a-1) \\ a^2-1 & a^2+2a-3 & 2(a-1) \\ a-1 & 2(a-1) & a-1 \end{vmatrix}$$

$$= \begin{vmatrix} (a-1)(a^2+a+1) & 3(a-1)(a+1) & 3(a-1) \\ (a-1)(a+1) & (a+3)(a-1) & 2(a-1) \\ a-1 & 2(a-1) & a-1 \end{vmatrix}$$

$$= (a-1)^3 \begin{vmatrix} a^2+a+1 & 3a+3 & 3 \\ a+1 & a+3 & 2 \\ 1 & 2 & 1 \end{vmatrix}$$ taking $a-1$ Common from C_1, C_2, C_3

$$= (a-1)^3 \begin{vmatrix} a^2+a-2 & 3a-3 & 0 \\ a-1 & a-1 & 0 \\ 1 & 2 & 1 \end{vmatrix}$$ $C_1 - 3C_3$, $C_2 - 2C_3$

Expanding from C_3

$$= (a-1)^3 \begin{vmatrix} (a+2)(a-1) & 3(a-1) \\ a-1 & a-1 \end{vmatrix}$$

$$= (a-1)^5 \begin{vmatrix} a+2 & 3 \\ 1 & 1 \end{vmatrix}$$ taking $a-1$ Common from C_1 & C_2

$$= (a-1)^5 (a+2-3)$$

$$= (a-1)(a-1)$$

$$\Delta = (a-1)^6$$ —— Ans.

Q16 If
$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

& A, B, C, ... are Cofactors of a, b, c, ... in Δ, then show that

(i) $BC - F^2 = a\Delta$

(ii) $CA - G^2 = b\Delta$

(iii) $AB - H^2 = c\Delta$

(iv) $GH - AF = f\Delta$

(v) $HF - BG = g\Delta$

(vi) $FG - CH = h\Delta$

(vii) $aG + hF + gC = 0$

(viii) $hG + bF + fC = 0$

(ix) $gG + fF + cC = \Delta$

Sol.

Here $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

Expanding from R_1

$$= a(bc - f^2) - h(ch - gf) + g(hf - gb)$$

$$= abc - af^2 - ch^2 + ghf + ghf - g^2 b$$

$$\Delta = abc + 2ghf - af^2 - g^2 b - ch^2 \qquad \text{———— } (1)$$

Now

$$A = (-1)^{1+1}\begin{vmatrix} b & f \\ f & c \end{vmatrix} = (-1)^2(bc - f^2) = bc - f^2$$

$$B = (-1)^{2+2}\begin{vmatrix} a & g \\ g & c \end{vmatrix} = (-1)^4(ac - g^2) = ac - g^2$$

$$C = (-1)^{3+3}\begin{vmatrix} a & h \\ h & b \end{vmatrix} = (-1)^6(ab - h^2) = ab - h^2.$$

$$F = (-1)^{2+3}\begin{vmatrix} a & h \\ g & f \end{vmatrix} = (-1)^5(af - gh) = gh - af$$

$$G = (-1)^{1+3}\begin{vmatrix} h & b \\ g & f \end{vmatrix} = (-1)^4(hf-gb) = hf-gb$$

$$H = (-1)^{1+2}\begin{vmatrix} h & f \\ g & c \end{vmatrix} = (-1)^3(ch-gf) = gf-ch$$

Now

(i) $BC - F^2 = (ac-g^2)(ab-h^2) - (gh-af)^2$

$= a^2bc - ach^2 - abg^2 + \cancel{g^2h^2} - \cancel{g^2h^2} - a^2f^2 + 2afgh$

$= a^2bc - ach^2 - abg^2 - a^2f^2 + 2afgh$

$= a(abc + 2fgh - af^2 - g^2b^2 - ch^2)$

So $BC - F^2 = a\Delta$

(ii) $CA - G^2 = b\Delta$

Sol.

$CA - G^2 = (ab-h^2)(bc-f^2) - (hf-gb)^2$

$= ab^2c - abf^2 - h^2bc + \cancel{h^2f^2} - \cancel{h^2f^2} - g^2b^2 + 2fghb$

$= b(abc + 2ghf - af^2 - g^2b - ch^2)$

$CA - G^2 = b\Delta$

(iii) $AB - H^2 = c\Delta$

Sol.

$AB - H^2 = (bc-f^2)(ac-g^2) - (gf-hc)^2$

$= abc^2 - g^2bc - f^2ac + \cancel{g^2f^2} - \cancel{g^2f^2} - c^2h^2 + 2ghfc$

$= c(abc + 2ghf - af^2 - g^2b - ch^2)$

$AB - H^2 = c\Delta$

(iv) $GH - AF = f\Delta$

Sol.

$GH - AF = (hf-bg)(fg-ch) - (bc-f^2)(gh-af)$

$= f^2gh - ch^2f - bfg^2 + \cancel{bcgh} - \cancel{bcgh} + abcf + f^2gh - af^3$

$$GH - AF = 2f^2gh - ch^2f - bfg^2 + abcf - af^3$$
$$= f(abc + 2fgh - af^2 - g^2b - ch^2)$$
$$GH - AF = f\Delta$$

(v) $HF - BG = g\Delta$

Sol.

$$HF - BG = (fg - ch)(gh - af) - (ac - g^2)(hf - gb)$$
$$= g^2fh - af^2g - gch^2 + a\cancel{cf}h - a\cancel{cf}h + acgb + g^2hf - g^3b$$
$$= abcg + 2g^2hf - af^2g - gch^2 - g^3b$$
$$= g(abc + 2ghf - af^2 - ch^2 - g^2b)$$
$$HF - BG = g\Delta$$

(vi) $FG - CH = h\Delta$

Sol.

$$FG - CH = (gh - af)(hf - gb) - (ab - h^2)(gf - ch)$$
$$= gh^2f - g^2hb - ahf^2 + a\cancel{b}gf - a\cancel{b}gf + abch + h^2gf - ch^3$$
$$= abch + 2gh^2f - ahf^2 - g^2hb - ch^3$$
$$= h(abc + 2ghf - af^2 - g^2b - ch^2)$$
$$FG - CH = h\Delta$$

(vii) $aG + hF + gC = 0$

Sol.

$$aG + hF + gC = a(hf - bg) + h(gh - af) + g(ab - h^2)$$
$$= a\cancel{h}f - a\cancel{b}g + \cancel{g}h^2 - a\cancel{h}f + \cancel{g}ab - \cancel{g}h^2$$
$$= 0$$

(viii) $hG + bF + fC = 0$

Sol.

$$hG + bF + fC = h(hf - bg) + b(gh - af) + f(ab - h^2)$$
$$= \cancel{h^2}f - \cancel{b}gh + \cancel{b}gh - a\cancel{b}f + a\cancel{b}f - h^2\cancel{f}$$
$$= 0$$

(ix) $gG + fF + cC = \Delta$

Sol:-

$$gG + fF + cC = g(hf - bg) + f(gh - af) + c(ab - h^2)$$
$$= ghf - g^2b + ghf - af^2 + abc - ch^2$$
$$= abc + 2ghf - af^2 - g^2b - ch^2$$
$$gG + fF + cC = \Delta$$

Q17 If Δ of problem 16 is zero, show that

(i) $F^2 + G^2 = C(A+B)$

(ii) $G^2 + H^2 = A(B+C)$

(iii) $H^2 + F^2 = B(A+C)$

(iv) $ABC = FGH$

Sol:-

(i) As we have proved that

$$\left.\begin{aligned} BC - F^2 &= a\Delta \\ \& \quad CA - G^2 &= b\Delta \end{aligned}\right]$$

But $\Delta = 0$

So

$$\left.\begin{aligned} BC - F^2 &= 0 \\ CA - G^2 &= 0 \end{aligned}\right]$$

or

$$\left.\begin{aligned} F^2 &= BC \\ G^2 &= CA \end{aligned}\right]$$

or

$$F^2 + G^2 = BC + CA$$
$$F^2 + G^2 = C(B + A)$$

or $\quad F^2 + G^2 = C(A+B)$

(ii) $\quad G^2 + H^2 = A(B+C)$

Soln. As we have proved that

$$\left.\begin{aligned} CA - G^2 &= b\Delta \\ \& \quad AB - H^2 &= c\Delta \end{aligned}\right]$$

But $\Delta = 0$

So

$$\left.\begin{aligned} CA - G^2 &= 0 \\ AB - H^2 &= 0 \end{aligned}\right]$$

or

$$\left.\begin{aligned} G^2 &= CA \\ H^2 &= AB \end{aligned}\right]$$

Adding

$$G^2 + H^2 = CA + AB$$
$$= A(C+B)$$
$$G^2 + H^2 = A(B+C)$$

(iii) $\quad H^2 + F^2 = B(A+C)$

Soln. As we have proved that

$$\left.\begin{aligned} AB - H^2 &= c\Delta \\ \& \quad BC - F^2 &= a\Delta \end{aligned}\right]$$

But $\Delta = 0$

So

$$\left.\begin{aligned} AB - H^2 &= 0 \\ BC - F^2 &= 0 \end{aligned}\right]$$

or

$$\left.\begin{aligned} H^2 &= AB \\ \& \quad F^2 &= BC \end{aligned}\right]$$

Adding $\quad H^2 + F^2 = B(A+C)$

(iv) $ABC = FGH$

Sol. As we know that

$$\left.\begin{aligned} BC &= F^2 \\ CA &= G^2 \\ AB &= H^2 \end{aligned}\right\}$$

Multiplying these eqs.

$$(BC)(CA)(AB) = F^2G^2H^2$$

$$A^2B^2C^2 = F^2G^2H^2$$

or $$ABC = FGH$$

Q18 Prove that

$$\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = \begin{vmatrix} b^2+c^2 & ab & ac \\ ba & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix}$$

Sol.

$$\text{Let } \Delta = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2$$

$$= \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix} \cdot \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0+c^2+b^2 & 0+0+ab & 0+ac+0 \\ 0+0+ab & c^2+0+a^2 & bc+0+0 \\ 0+ac+0 & bc+0+0 & b^2+a^2+0 \end{vmatrix}$$

$$= \begin{vmatrix} b^2+c^2 & ab & ac \\ ba & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix}$$

Q19 Show that

$$\begin{vmatrix} 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega \\ 1 & \omega^3 & \omega & \omega^4 \\ 1 & \omega^4 & \omega^3 & \omega^2 \end{vmatrix}^2 = 125$$

where ω is a fifth root of 1.

Sol:-

Since ω is the fifth root of 1

So $\omega^5 = 1$ ——— ①

& $1+\omega+\omega^2+\omega^3+\omega^4 = 0$ ——— ②

Now

$$\text{let } \Delta = \begin{vmatrix} 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega \\ 1 & \omega^3 & \omega & \omega^4 \\ 1 & \omega^4 & \omega^3 & \omega^2 \end{vmatrix}^2$$

$$= \begin{vmatrix} 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega \\ 1 & \omega^3 & \omega & \omega^4 \\ 1 & \omega^4 & \omega^3 & \omega^2 \end{vmatrix} \cdot \begin{vmatrix} 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega \\ 1 & \omega^3 & \omega & \omega^4 \\ 1 & \omega^4 & \omega^3 & \omega^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1+\omega+\omega^2+\omega^3 & \omega+\omega^3+\omega^5+\omega^7 & \omega^2+\omega^5+\omega^3+\omega^6 & \omega^3+\omega^2+\omega^6+\omega^5 \\ 1+\omega^2+\omega^4+\omega & \omega+\omega^4+\omega^7+\omega^5 & \omega^2+\omega^6+\omega^5+\omega^4 & \omega^3+\omega^3+\omega^8+\omega^3 \\ 1+\omega^3+\omega+\omega^4 & \omega+\omega^5+\omega^4+\omega^8 & \omega^2+\omega^7+\omega^2+\omega^7 & \omega^3+\omega^4+\omega^5+\omega^6 \\ 1+\omega^4+\omega^3+\omega^2 & \omega+\omega^6+\omega^6+\omega^6 & \omega^2+\omega^8+\omega^4+\omega^5 & \omega^3+\omega^5+\omega^7+\omega^4 \end{vmatrix}$$

$$= \begin{vmatrix} 1+\omega+\omega^2+\omega^3 & 1+\omega+\omega^2+\omega^3 & 1+\omega+\omega^2+\omega^3 & 1+\omega+\omega^2+\omega^3 \\ 1+\omega+\omega^2+\omega^4 & 1+\omega+\omega^2+\omega^4 & 1+\omega+\omega^2+\omega^4 & 4\omega^3 \\ 1+\omega+\omega^3+\omega^4 & 1+\omega+\omega^3+\omega^4 & 4\omega^2 & 1+\omega+\omega^3+\omega^4 \\ 1+\omega^2+\omega^3+\omega^4 & 4\omega & 1+\omega^2+\omega^3+\omega^4 & 1+\omega^2+\omega^3+\omega^4 \end{vmatrix} \quad \text{By ①}$$

$$\Delta = \begin{vmatrix} -\omega^4 & -\omega^4 & -\omega^4 & -\omega^4 \\ -\omega^3 & -\omega^3 & -\omega^3 & 4\omega^3 \\ -\omega^2 & -\omega^2 & 4\omega^2 & -\omega^2 \\ -\omega & 4\omega & -\omega & -\omega \end{vmatrix}$$

$$= (-1)^4 \omega^4 . \omega^3 . \omega^2 . \omega \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -4 \\ 1 & 1 & -4 & 1 \\ 1 & -4 & 1 & 1 \end{vmatrix}$$

$$= \omega^{10} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & -5 & 0 \\ 0 & -5 & 0 & 0 \end{vmatrix} \qquad \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{matrix}$$

$$= (\omega^5)^2 \begin{vmatrix} 0 & 0 & -5 \\ 0 & -5 & 0 \\ -5 & 0 & 0 \end{vmatrix} \qquad \text{expanding from } C_1$$

$$= (1)^2 \begin{vmatrix} 0 & 0 & -5 \\ 0 & -5 & 0 \\ -5 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & -5 \\ 0 & -5 & 0 \\ -5 & 0 & 0 \end{vmatrix}$$

{Expanding from R_1

$$= 0 - 0 + (-5) \begin{vmatrix} 0 & -5 \\ -5 & 0 \end{vmatrix}$$

$$= -5(0 - 25)$$

$$\Delta = 125$$

Q20 Prove that the determinant

$$\begin{vmatrix} 2b_1+c_1 & c_1+3a_1 & 2a_1+3b_1 \\ 2b_2+c_2 & c_2+3a_2 & 2a_2+3b_2 \\ 2b_3+c_3 & c_3+3a_3 & 2a_3+3b_3 \end{vmatrix}$$

is a multiple of the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$ & find the other factor.

Sol.

Consider $\begin{vmatrix} 2b_1+c_1 & c_1+3a_1 & 2a_1+3b_1 \\ 2b_2+c_2 & c_2+3a_2 & 2a_2+3b_2 \\ 2b_3+c_3 & c_3+3a_3 & 2a_3+3b_3 \end{vmatrix}$

$$= \begin{vmatrix} 0a_1+2b_1+1.c_1 & 3a_1+0b_1+1.c_1 & 2a_1+3b_1+0c_1 \\ 0a_2+2b_2+1.c_2 & 3a_2+0b_2+1.c_2 & 2a_2+3b_2+0c_2 \\ 0a_3+2b_3+1.c_3 & 3a_3+0b_3+1.c_3 & 2a_3+3b_3+0c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \cdot \begin{vmatrix} 0 & 3 & 2 \\ 2 & 0 & 3 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \cdot \left\{ 0 - 3(0-3) + 2(2-0) \right\}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} (9+4)$$

$$= 13 \cdot \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Hence given determinant is a multiple of $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

& other factor is 13.

<u>Q21</u> 35,282 ; 44,759 ; 58,916 ; 80,652 & 92,469 are all multiples of 13. Show that the determinant

$$\begin{vmatrix} 3 & 5 & 2 & 8 & 2 \\ 4 & 4 & 7 & 5 & 9 \\ 5 & 8 & 9 & 1 & 6 \\ 8 & 0 & 6 & 5 & 2 \\ 9 & 2 & 4 & 6 & 9 \end{vmatrix}$$ is also a multiple of 13.

<u>Sol.</u>

$$\text{Let } \Delta = \begin{vmatrix} 3 & 5 & 2 & 8 & 2 \\ 4 & 4 & 7 & 5 & 9 \\ 5 & 8 & 9 & 1 & 6 \\ 8 & 0 & 6 & 5 & 2 \\ 9 & 2 & 4 & 6 & 9 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 5 & 2 & 8 & 35,282 \\ 4 & 4 & 7 & 5 & 44,759 \\ 5 & 8 & 9 & 1 & 58,916 \\ 8 & 0 & 6 & 5 & 80,652 \\ 9 & 2 & 4 & 6 & 92,469 \end{vmatrix} \qquad R_5 + 10R_4 + 100R_3 + 1000R_2 + 10000R_1$$

Since 35,282, 44,759, 58,916, 80,652 & 92,469 are all multiples of 13, so 13 is a common factor of these numbers.

Hence

$$\Delta = 13\begin{vmatrix} 3 & 5 & 2 & 8 & 2714 \\ 4 & 4 & 7 & 5 & 3443 \\ 5 & 8 & 9 & 1 & 4532 \\ 8 & 0 & 6 & 5 & 6204 \\ 9 & 2 & 4 & 6 & 7113 \end{vmatrix} \quad \text{taking 13 Common from } C_5$$

So given determinant Δ is a multiple of 13.

Q22 Prove that

$$\begin{vmatrix} (a-x)^2 & (a-y)^2 & (a-z)^2 \\ (b-x)^2 & (b-y)^2 & (b-z)^2 \\ (c-x)^2 & (c-y)^2 & (c-z)^2 \end{vmatrix} = 2(a-b)(b-c)(c-a)(x-y)(y-z)(z-x)$$

Sol.

$$\text{Let } \Delta = \begin{vmatrix} (a-x)^2 & (a-y)^2 & (a-z)^2 \\ (b-x)^2 & (b-y)^2 & (b-z)^2 \\ (c-x)^2 & (c-y)^2 & (c-z)^2 \end{vmatrix}$$

$$= \begin{vmatrix} a^2-2ax+x^2 & a^2-2ay+y^2 & a^2-2az+z^2 \\ b^2-2bx+x^2 & b^2-2by+y^2 & b^2-2bz+z^2 \\ c^2-2cx+x^2 & c^2-2cy+y^2 & c^2-2cz+z^2 \end{vmatrix}$$

$$= \begin{vmatrix} a^2 & -2a & 1 \\ b^2 & -2b & 1 \\ c^2 & -2c & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$\Delta = \Delta_1 . \Delta_2 \quad \text{———— (1)}$$

Now

$$\Delta_1 = \begin{vmatrix} a^2 & -2a & 1 \\ b^2 & -2b & 1 \\ c^2 & -2c & 1 \end{vmatrix}$$

$$= -2\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}$$

$$= -2\begin{vmatrix} a^2-c^2 & a-c & 0 \\ b^2-c^2 & b-c & 0 \\ c^2 & c & 1 \end{vmatrix} \qquad \begin{matrix} R_1-R_3 \\ R_2-R_3 \end{matrix}$$

Expanding from C_3

$$= -2\begin{vmatrix} a^2-c^2 & a-c \\ b^2-c^2 & b-c \end{vmatrix}$$

$$= -2(a-c)(b-c)\begin{vmatrix} a+c & 1 \\ b+c & 1 \end{vmatrix}$$

taking $a-c$ Common from R_1
& $b-c$ Common from R_2

$$= -2(a-c)(b-c)(a+\cancel{c}-b-\cancel{c})$$

$$\Delta_1 = 2(a-b)(b-c)(c-a)$$

&

$$\Delta_2 = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ x-z & y-z & z \\ x^2-z^2 & y^2-z^2 & z^2 \end{vmatrix} \qquad \begin{matrix} C_1-C_3 \\ C_2-C_3 \end{matrix}$$

Expanding from R_1

$$= \begin{vmatrix} x-z & y-z \\ x^2-z^2 & y^2-z^2 \end{vmatrix}$$

$$\Delta_2 = (x-z)(y-z)\begin{vmatrix} 1 & 1 \\ x+z & y+z \end{vmatrix}$$ taking $x-z$, $y-z$ Common from R_1 & R_2.

$$= (x-z)(y-z)(y+\cancel{z}-x-\cancel{z})$$

$$= (x-z)(y-z)(y-x)$$

$$\Delta_2 = (x-y)(y-z)(z-x)$$

Putting values of Δ_1, Δ_2 in ①

$$\Delta = 2(a-b)(b-c)(c-a)(x-y)(y-z)(z-x)$$

Q23 Show that

$$\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \cdot \begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix} = (a^3+b^3+c^3-3abc)^2$$

Sol.

$$\text{Let } \Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \cdot \begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \cdot \begin{vmatrix} -a & -b & -c \\ c & a & b \\ b & c & a \end{vmatrix}$$ taking transpose of second det.

$$= \begin{vmatrix} -a^2+bc+bc & -\cancel{ab}+\cancel{ab}+c^2 & -\cancel{ac}+b^2+\cancel{ac} \\ -\cancel{ab}+c^2+\cancel{ab} & -b^2+ca+ca & -\cancel{bc}+\cancel{bc}+a^2 \\ -\cancel{ac}+\cancel{ac}+b^2 & -\cancel{bc}+a^2+\cancel{bc} & -c^2+ab+ab \end{vmatrix}$$

$$\Delta = \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}$$ ———— Ans I

Again Consider

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \cdot \begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix}$$

$$\Delta = \Delta_1 \cdot \Delta_2 \quad \text{———} \quad (1)$$

Now

$$\Delta_1 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

Expanding from R_1

$$= a(bc - a^2) - b(b^2 - ac) + c(ab - c^2)$$

$$= abc - a^3 - b^3 + abc + abc - c^3$$

$$= -a^3 - b^3 - c^3 + 3abc$$

$$\Delta_1 = -(a^3 + b^3 + c^3 - 3abc)$$

&

$$\Delta_2 = \begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix}$$

Expanding from R_1

$$= (-a)(a^2 - bc) - c(-ab + c^2) + b(-b^2 + ac)$$

$$= -a^3 + abc + abc - c^3 - b^3 + abc$$

$$= -a^3 - b^3 - c^3 + 3abc$$

$$\Delta_2 = -(a^3 + b^3 + c^3 - 3abc)$$

Put in (1)

$$\Delta = (a^3 + b^3 + c^3 - 3abc)^2$$

Hence

$$\begin{vmatrix} 2bc - a^2 & c^2 & b^2 \\ c^2 & 2ca - b^2 & a^2 \\ b^2 & a^2 & 2ab - c^2 \end{vmatrix} = \underline{\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix}} = (a^3 + b^3 + c^3 - 3abc)^2$$

Q2.4 Find, by the adjoint method, the inverse of each of the following matrices:

(i) $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

Sol.

Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

We know that

$$A^{-1} = \frac{\text{Adj}\,A}{|A|} \qquad \text{———— (1)}$$

Now

$$|A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$

Expanding from R_1

$$= 2(4-1) - 1(2-1) + 1(1-2)$$

$$= 2(3) - (1) + (-1)$$

$$= 6 - 1 - 1$$

$$|A| = 4 \neq 0 \quad \text{So } A^{-1} \text{ exists}$$

Now

$$\text{Adj}\,A = \begin{bmatrix} \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} \end{bmatrix}^t$$

$$= \begin{bmatrix} 4-1 & -(2-1) & 1-2 \\ -(2-1) & 4-1 & -(2-1) \\ 1-2 & -(2-1) & 4-1 \end{bmatrix}^t$$

$$= \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}^t$$

$$\text{Adj}A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

So from (1)

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

$$\text{So } A^{-1} = \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

(ii) $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

Sol:-

Let $D = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

we know that

$$D^{-1} = \frac{\text{Adj}D}{|D|} \qquad \text{——— (1)}$$

Now

$$|D| = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

Expanding from R_1

$$= a(bc-f^2) - h(ch-gf) + g(hf-gb)$$

$$= abc - af^2 - ch^2 + ghf + ghf - g^2b$$

$$|D| = abc + 2ghf - af^2 - ch^2 - g^2b \neq 0 \text{ (say) then } D^{-1} \text{ exists.}$$

Now

$$\text{Adj } D = \begin{bmatrix} \begin{vmatrix} b & f \\ f & c \end{vmatrix} & -\begin{vmatrix} h & f \\ g & c \end{vmatrix} & \begin{vmatrix} h & b \\ g & f \end{vmatrix} \\ -\begin{vmatrix} h & g \\ f & c \end{vmatrix} & \begin{vmatrix} a & g \\ g & c \end{vmatrix} & -\begin{vmatrix} a & h \\ g & f \end{vmatrix} \\ \begin{vmatrix} h & g \\ b & f \end{vmatrix} & -\begin{vmatrix} a & g \\ h & f \end{vmatrix} & \begin{vmatrix} a & h \\ h & b \end{vmatrix} \end{bmatrix}^t$$

$$= \begin{bmatrix} bc-f^2 & -(ch-gf) & hf-gb \\ -(ch-gf) & ac-g^2 & -(af-gh) \\ hf-gb & -(af-gh) & ab-h^2 \end{bmatrix}^t$$

$$= \begin{bmatrix} bc-f^2 & gf-ch & hf-gb \\ gf-ch & ac-g^2 & gh-af \\ hf-gb & gh-af & ab-h^2 \end{bmatrix}^t$$

$$\text{Adj } D = \begin{bmatrix} bc-f^2 & gf-ch & hf-gb \\ gf-ch & ac-g^2 & gh-af \\ hf-gb & gh-af & ab-h^2 \end{bmatrix}$$

So from ①

$$\bar{D}^{1} = \frac{1}{(abc+2ghf-af^2-ch^2-g^2b)}\begin{bmatrix} bc-f^2 & gf-ch & hf-gb \\ gf-ch & ac-g^2 & gh-af \\ hf-gb & gh-af & ab-h^2 \end{bmatrix}$$

(iii) $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & -1 \\ 3 & 2 & -2 \end{bmatrix}$

Sol.

Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & -1 \\ 3 & 2 & -2 \end{bmatrix}$

we know that

$$\bar{A}^{1} = \frac{\text{Adj} A}{|A|} \qquad \text{———} \quad ①$$

Now

$$|A| = \begin{vmatrix} 1 & -1 & 2 \\ 2 & 1 & -1 \\ 3 & 2 & -2 \end{vmatrix}$$

Expanding from R_1

$$= 1(-2+2)+1(-4+3)+2(4-3)$$

$$= 0-1+2$$

$|A| = 1 \neq 0$, so $\bar{A}^{1}$ exists.

Now

$$\text{Adj} A = \begin{bmatrix} \begin{vmatrix} 1 & -1 \\ 2 & -2 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 3 & -2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} \\ -\begin{vmatrix} -1 & 2 \\ 2 & -2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} \\ \begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} \end{bmatrix}^t$$

$$\text{Adj } A = \begin{bmatrix} -2+2 & -(-4+3) & 4-3 \\ -(2-4) & -2-6 & -(2+3) \\ 1-2 & -(-1-4) & 1+2 \end{bmatrix}^t$$

$$= \begin{bmatrix} 0 & 1 & 1 \\ 2 & -8 & -5 \\ -1 & 5 & 3 \end{bmatrix}^t$$

$$\text{Adj } A = \begin{bmatrix} 0 & 2 & -1 \\ 1 & -8 & 5 \\ 1 & -5 & 3 \end{bmatrix}$$

So from (1)

$$A^{-1} = \frac{1}{1}\begin{bmatrix} 0 & 2 & -1 \\ 1 & -8 & 5 \\ 1 & -5 & 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 0 & 2 & -1 \\ 1 & -8 & 5 \\ 1 & -5 & 3 \end{bmatrix}$$