Ex 3.1

① Let $A=\begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}$ and $B=\begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$

(ii) $A-B = \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix} - \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$

$$= \begin{bmatrix} 2+1 & -3-3 & -5-5 \\ -1-1 & 4+3 & 5+5 \\ 1+1 & -3-3 & -4-5 \end{bmatrix}$$

$$A-B = \begin{bmatrix} 3 & -6 & -10 \\ -2 & 7 & 10 \\ 2 & -6 & -9 \end{bmatrix}$$

(iii) $2A+3B = 2\begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix} + 3\begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$

$$= \begin{bmatrix} 4 & -6 & -10 \\ -2 & 8 & 10 \\ 2 & -6 & -8 \end{bmatrix} + \begin{bmatrix} -3 & 3 & 15 \\ 3 & -9 & -15 \\ -3 & 9 & 15 \end{bmatrix}$$

$$= \begin{bmatrix} 4-3 & -6+3 & -10+15 \\ -2+3 & 8-9 & 10-15 \\ 2-3 & -6+9 & -8+15 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 & +5 \\ 1 & -1 & -5 \\ -1 & 3 & 7 \end{bmatrix}$$

(v) $AB = \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}\begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$

$$= \begin{bmatrix} -2-3+5 & 6+9-15 & 10+15-25 \\ 1+4-5 & -3-12+15 & -5-20+25 \\ -1-3+4 & 3+9-12 & 5+15-20 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Q. (ii)
$$\begin{bmatrix} x & y & z \end{bmatrix}\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} ax+hy+gz & hx+by+fz & gx+fy+cz \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \left[x(ax+hy+gz)+y(hx+by+fz)+z(gx+fy+cz)\right]$$

(iv)
$$\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}^3 = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}^2\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+8 & 2-6 \\ 4-12 & 8+9 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -4 \\ -8 & 17 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 9-16 & 18+12 \\ -8+68 & -16-51 \end{bmatrix} = \begin{bmatrix} -7 & 30 \\ 60 & -67 \end{bmatrix}$$

(vi)
$$\begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^3 = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^2\begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & 1-1+0 & 1+0-1 \\ 0+0+0 & 0+1+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+1 \end{bmatrix}\begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(5) $A = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$ $B = \begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$

$$AB = \begin{bmatrix} \cos^2\theta\cos^2\phi + \cos\theta\sin\theta\cos\phi\sin\phi & \cos^2\theta\cos\phi\sin\phi + \cos\theta\sin\theta\sin^2\phi \\ \cos\theta\sin\theta\cos^2\phi + \sin^2\theta\cos\phi\sin\phi & \cos\theta\sin\theta\cos\phi\sin\phi + \sin^2\theta\cdot\sin^2\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\phi(\cos\theta\cos\phi + \sin\theta\sin\phi) & \cos\theta\sin\phi(\cos\theta\cos\phi + \sin\theta\sin\phi) \\ \sin\theta\cos\phi(\cos\theta\cos\phi + \sin\theta\sin\phi) & \sin\theta\sin\phi(\cos\theta\cos\phi + \sin\theta\sin\phi) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\phi\cos(\theta-\phi) & \cos\theta\sin\phi\cos(\theta-\phi) \\ \sin\theta\cos\phi\cos(\theta-\phi) & \sin\theta\sin\phi\cos(\theta-\phi) \end{bmatrix}$$

$$= \cos(\theta-\phi)\begin{bmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi \\ \sin\theta\cos\phi & \sin\theta\sin\phi \end{bmatrix}$$

(given) $\theta - \phi = K\frac{\pi}{2}$ where K is odd $K = 1, 3, 5, 7 \ldots$

$$\therefore AB = \cos\left(K\frac{\pi}{2}\right)\begin{bmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi \\ \sin\theta\cos\phi & \sin\theta\sin\phi \end{bmatrix}$$

$$= 0 \cdot \begin{bmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi \\ \sin\theta\cos\phi & \sin\theta\sin\phi \end{bmatrix}$$

$$= O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \because \cos K\frac{\pi}{2} = 0. \text{ where } K \text{ is odd.}$$

(4)
$$\begin{bmatrix} \lambda_1^2 & \lambda_1\lambda_2 & \lambda_1\lambda_3 \\ \lambda_1\lambda_2 & \lambda_2^2 & \lambda_2\lambda_3 \\ \lambda_1\lambda_3 & \lambda_2\lambda_3 & \lambda_3^2 \end{bmatrix}\begin{bmatrix} \mu_1^2 & \mu_1\mu_2 & \mu_1\mu_3 \\ \mu_1\mu_2 & \mu_2^2 & \mu_2\mu_3 \\ \mu_1\mu_3 & \mu_2\mu_3 & \mu_3^2 \end{bmatrix}$$

$$\begin{bmatrix} \lambda_1^2\mu_1^2 + \lambda_1\lambda_2\mu_1\mu_2 + \lambda_1\lambda_3\mu_1\mu_3 & \lambda_1^2\mu_1\mu_2 + \lambda_1\lambda_2\mu_2^2 + \lambda_1\lambda_3\mu_2\mu_3 & \lambda_1^2\mu_1\mu_3 + \lambda_1\lambda_2\mu_2\mu_3 + \lambda_1\lambda_3\mu_3^2 \\ \lambda_1\lambda_2\mu_1^2 + \lambda_2^2\mu_1\mu_2 + \lambda_2\lambda_3\mu_1\mu_3 & \lambda_1\lambda_2\mu_1\mu_2 + \lambda_2^2\mu_2^2 + \lambda_2\lambda_3\mu_2\mu_3 & \lambda_1\lambda_2\mu_1\mu_3 + \lambda_2^2\mu_2\mu_3 + \lambda_2\lambda_3\mu_3^2 \\ \lambda_1\lambda_3\mu_1^2 + \lambda_2\lambda_3\mu_1\mu_2 + \lambda_3^2\mu_1\mu_3 & \lambda_1\lambda_3\mu_1\mu_2 + \lambda_2\lambda_3\mu_2^2 + \lambda_3^2\mu_2\mu_3 & \lambda_1\lambda_3\mu_1\mu_3 + \lambda_2\lambda_3\mu_2\mu_3 + \lambda_3^2\mu_3^2 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1\mu_1(\lambda_1\mu_1+\lambda_2\mu_2+\lambda_3\mu_3) & \lambda_1\mu_2(\lambda_1\mu_1+\lambda_2\mu_2+\lambda_3\mu_3) & \lambda_1\mu_3(\lambda_1\mu_1+\lambda_2\mu_2+\lambda_3\mu_3) \\ \lambda_2\mu_1(\lambda_1\mu_1+\lambda_2\mu_2+\lambda_3\mu_3) & \lambda_2\mu_2(\lambda_1\mu_1+\lambda_2\mu_2+\lambda_3\mu_3) & \lambda_2\mu_3(\lambda_1\mu_1+\lambda_2\mu_2+\lambda_3\mu_3) \\ \lambda_3\mu_1(\lambda_1\mu_1+\mu_2\lambda_2+\lambda_3\mu_3) & \lambda_3\mu_2(\lambda_1\mu_1+\lambda_2\mu_2+\lambda_3\mu_3) & \lambda_3\mu_3(\lambda_1\mu_1+\lambda_2\mu_2+\lambda_3\mu_3) \end{bmatrix}$$

$= 0$ only when $\lambda_1\mu_1+\lambda_2\mu_2+\lambda_3\mu_3 = 0$

i.e when lines are $\perp$ to each other.

(5)

$$(A+B)^2 = (A+B)(A+B)$$
$$= A^2+AB+BA+B^2$$
$$(A+B)^2 \neq A^2+2AB+B^2 \qquad \because AB \neq BA$$

Also
$$(A-B)(A+B)$$
$$= A^2+AB-BA-B^2$$
$$\neq A^2-B^2 \qquad \because AB \neq BA.$$

Equality will hold only if $AB = BA$ i.e A & B commute

(6) $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

$A^2 - 4A - 5I = 0$

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = 0$$

$$\begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = 0$$

$$\begin{bmatrix} 9-4-5 & 8-8-0 & 8-8-0 \\ 8-8-0 & 9-4-5 & 8-8-0 \\ 8-8-0 & 8-8 & 9-4-5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.$$

Some Special Types of Square Matrix.

(i) Periodic Matrix

A square matrix A is said to be a periodic matrix of period K, if $\boxed{A^{K+1} = A}$

(ii) Idempotent Matrix if $\boxed{A^2 = A}$

(iii) Nilpotent Matrix if $\boxed{A^P = 0}$ P is index

(iv) Involutory Matrix if $\boxed{A^2 = I}$

(v) Symmetric Matrix if $\boxed{A^t = A}$

(vi) Skew Symmetric Matrix if $\boxed{A^t = -A}$

(vii) Hermition Matrix if $\boxed{(\bar{A})^t = A}$

(viii) Skew Hermition Matrix if $\boxed{(\bar{A})^t = -A}$

(IX) Let A be a matrix over C (complex numbers) If the elements of A are replaced by their complex conjugate, the resulting matrix is called Conjugate of A and denoted by $\bar{A}$

$$A = \begin{bmatrix} 2+3i & -i \\ 4 & 5+3i \end{bmatrix} \qquad \bar{A} = \begin{bmatrix} 2-3i & i \\ 4 & 5-3i \end{bmatrix}$$

Some Results (i) If A is symmetric then $A+A^t$ is Symmetric & $A-A^t$ is Skew Symmetric

(ii) Every square matrix can be written as a sum of Symmetric & skew symmetric matrix

$$A = \frac{1}{2}(A+A^t) + \frac{1}{2}(A-A^t)$$

(iii) If A is square matrix over C then $A+(\bar{A})^t$ is Hermition & $A-(\bar{A})^t$ is Skew Hermition

(iv) Every square matrix can be written as a sum of Hermition & Skew Hermition Matrix

$$A = \frac{1}{2}\left[A+(\bar{A})^t\right] + \frac{1}{2}\left[A-(\bar{A})^t\right]$$

⑦

$$A = \begin{pmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{pmatrix}$$

$A^{K+1} = A$.
K is period

$$A^2 = A \cdot A = \begin{pmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{pmatrix}\begin{pmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 1+6-12 & -2-4 & -6-18+18 \\ -3-6+18 & 6+4 & 18+18-27 \\ 2-6 & -4 & -12+9 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} -5 & -6 & -6 \\ 9 & 10 & 9 \\ -4 & -4 & -3 \end{pmatrix}$$

$$A^3 = A^2 \cdot A = \begin{pmatrix} -5 & -6 & -6 \\ 9 & 10 & 9 \\ -4 & -4 & -3 \end{pmatrix}\begin{pmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} -5+18-12 & 10-12 & 30-54+18 \\ 9-30+18 & -18+20 & -54+90-27 \\ -4+12-6 & 8-8 & 24-36+9 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{pmatrix} = A$$

$A^3 = A$ or $A^{2+1} = A$. hence period $= 2$.

⑧

$$A = \begin{pmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{pmatrix}$$

Nilpotent $\boxed{A^P = 0}$

$$A^2 = A \cdot A = \begin{pmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{pmatrix}\begin{pmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} 1+3-4 & -3-9+12 & -4-12+16 \\ -1-3+4 & 3+9-12 & 4+12-16 \\ 1+3-4 & -3-9+12 & -4-12+16 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Involutry $A^2 = I$

(9) $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$

$$A^2 = A \cdot A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0+4-3 & 0-3+3 & 0+4-4 \\ 0-12+12 & 4+9-12 & -4-12+16 \\ -12+12 & 3+9-12 & -3-12+16 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$A^2 = I$ hence Involutry

x ———————— x

(10) 2)

$$A = \frac{1}{2}(2A)$$

$$= \frac{1}{2}[A + A]$$

$$= \frac{1}{2}[A + A^t - A^t + A]$$

$$A = \frac{1}{2}[A + A^t] + \frac{1}{2}[A - A^t]$$

$$A = B + C \quad \text{—— (1)}$$

Now we prove that B is symmetric & C is skew symmetric

$$B = \frac{1}{2}[A + A^t]$$

$$B^t = \frac{1}{2}[A + A^t]^t = \frac{1}{2}[A^t + (A^t)^t] = \frac{1}{2}[A^t + A]$$

$$B^t = \frac{1}{2}[A + A^t] = B$$

So B is symmetric

Now $C = \frac{1}{2}[A - A^t]$

$$C^t = \frac{1}{2}[A - A^t]^t = \frac{1}{2}[A^t - (A^t)^t] = \frac{1}{2}[A^t - A]$$

$$C^t = -\frac{1}{2}[A - A^t] = -C$$

So C is skew symmetric

(11)

$$A = \frac{1}{2}(2A)$$
$$= \frac{1}{2}(A+A)$$
$$= \frac{1}{2}\left[A+(\bar{A})^t-(\bar{A})^t+A\right]$$
$$= \frac{1}{2}\left[A+(\bar{A})^t\right]+\frac{1}{2}\left[A-(\bar{A})^t\right]$$

$$A = B + D \quad \text{——} (1)$$

We prove that B is Hermition & D is skew Hermition :

$$B = \frac{1}{2}\left[A+(\bar{A})^t\right]$$

$$\bar{B} = \frac{1}{2}\overline{\left[A+(\bar{A})^t\right]} = \frac{1}{2}\left[\bar{A}+(\bar{\bar{A}})^t\right] = \frac{1}{2}\left[\bar{A}+(A)^t\right]$$

$$(\bar{B})^t = \frac{1}{2}\left[\bar{A}+(A)^t\right]^t = \frac{1}{2}\left[(\bar{A})^t+(A^t)^t\right]$$
$$= \frac{1}{2}\left[(\bar{A})^t+A\right] = \frac{1}{2}\left[A+(\bar{A})^t\right] = B$$

$(\bar{B})^t = B$ Hence B is Hermition.

$$D = \frac{1}{2}\left[A-(\bar{A})^t\right]$$

$$\bar{D} = \frac{1}{2}\overline{\left[A-(\bar{A})^t\right]} = \frac{1}{2}\left[\bar{A}-(\bar{\bar{A}})^t\right] = \frac{1}{2}\left[\bar{A}-A^t\right]$$

$$(\bar{D})^t = \frac{1}{2}\left[\bar{A}-A^t\right]^t = \frac{1}{2}\left[(\bar{A})^t-(A^t)^t\right] = \frac{1}{2}\left[(\bar{A})^t-A\right]$$
$$= -\frac{1}{2}\left[A-(\bar{A})^t\right] = -D$$

$(\bar{D})^t = -D$ Hence D is skew Hermition.

(12) A & B are symmetric Matrices

So $A^t = A$ —(1) & $B^t = B$ —(2)

Suppose A & B commute i.e. $AB = BA$ & Prove $(AB)^t = AB$

$$(AB)^t = B^t A^t = BA = AB$$

Hence AB is symmetric using (1) & (2) supposition

Now suppose $(AB)^t = AB$

& Prove $AB = BA$

$(AB)^t = AB$

$B^t A^t = AB$

$BA = AB$ using ① & ②

Hence A & B commute.

(13) Let A is symmetric i.e $A^t = A$ —— ①

We prove $B = P^t A P$ is symmetric

$B^t = (P^t A P)^t$ $\qquad (AB)^t = B^t A^t$

$= P^t A^t (P^t)^t$

$B^t = P^t A P$ $\qquad \because A^t = A$ & $(P^t)^t = P$.

$B^t = B$ Hence B is symmetric

Now let A is skew symmetric i.e $A^t = -A$ —— ②

We prove $B = P^t A P$ is skew symmetric

$B^t = (P^t A P)^t$

$= P^t A^t (P^t)^t$

$= P^t (-A) P$ $\qquad \because$ using ②

$= -P^t A P$

$B^t = -B$

Hence B is skew symmetric.

(14) To Prove AA^t & A^tA are symmetric for a square matrix A.

Let A be a square matrix

Now $(AA^t)^t = (A^t)^t (A^t)$ $\qquad (AB)^t = B^tA^t$

$= AA^t$

So AA^t is symmetric.

Now $(A^tA)^t = A^t(A^t)^t$

$= A^tA$

So A^tA is also symmetric.

(15) $a_nA^n + a_{n-1}A^{n-1} + \dots + a_1A + a_0I$

Let A is symmetric So $A^t = A$.

First we prove that $(A^n)^t = (A^t)^n$. where n is +ve integer

Now $(A^n)^t = (A \cdot A \dots n \text{ times})^t$

$= A^t \cdot A^t \dots n \text{ times}$

$= (A^t)^n$

Now we prove $B = a_nA^n + a_{n-1}A^{n-1} + \dots a_1A + a_0I$ is symmetric

$B^t = [a_nA^n + a_{n-1}A^{n-1} + \dots a_1A + a_0I]^t$

$= (a_nA^n)^t + (a_{n-1}A^{n-1})^t + \dots (a_1A)^t + (a_0I)^t$

$= a_n(A^n)^t + a_{n-1}(A^{n-1})^t + \dots a_1(A)^t + a_0(I)^t$

$= a_n(A^t)^n + a_{n-1}(A^t)^{n-1} + \dots a_1A^t + a_0I$

$= a_nA^n + a_{n-1}(A)^{n-1} + \dots a_1A + a_0I \qquad \because A^t = A.$

$B^t = B$

Hence $a_nA^n + a_{n-1}A^{n-1} + \dots a_1A + a_0I$ is Symmetric

(16) Let $B = A + (\bar{A})^t$

$\bar{B} = \overline{A + (\bar{A})^t} = \bar{A} + (\bar{\bar{A}})^t = \bar{A} + (A)^t$

$(\bar{B})^t = [\bar{A} + (A)^t]^t = [(\bar{A})^t + (A^t)^t] = [(\bar{A})^t + A]$

$(\bar{B})^t = [A + (\bar{A})^t] = B$

Hence $B = A + (\bar{A})^t$ is Hermition.

Let $C = A(\bar{A})^t$

$\bar{C} = \overline{A(\bar{A})^t} = \bar{A}(\bar{\bar{A}})^t = \bar{A}(A)^t$

$(\bar{C})^t = (\bar{A}A^t)^t = (A^t)^t(\bar{A})^t = (A)(\bar{A})^t$

$(\bar{C})^t = C$

Hence $C = A(\bar{A})^t$ is Hermition.

Let $D = (\bar{A})^t A$

$\bar{D} = \overline{(\bar{A})^t A} = (\bar{\bar{A}})^t \bar{A} = (A)^t \bar{A}$

$(\bar{D})^t = [(A)^t \bar{A}]^t = (\bar{A})^t (A^t)^t = (\bar{A})^t A$

$(\bar{D})^t = D$ Hence D is Hermition.

x——————x

Imp (17) Let A be a square Matrix

Now $A = \frac{1}{2}(2A)$

$= \frac{1}{2}(A+A)$

$= \frac{1}{2}[A + (\bar{A})^t - (\bar{A})^t + A]$

$= \frac{1}{2}[A + (\bar{A})^t] + \frac{1}{2}[A - (\bar{A})^t]$

$A = \frac{1}{2}[A + (\bar{A})^t] + i \cdot \frac{1}{2i}[A - (\bar{A})^t]$

$A = P + iQ$

Now we prove that P & Q are Hermition

$P = \frac{1}{2}(A + \bar{A}^t)$

$(\bar{P})^t = \frac{1}{2}\left[\overline{A + \bar{A}^t}\right]^t$

$= \frac{1}{2}\left[\bar{A} + (\bar{\bar{A}})^t\right]^t$

$= \frac{1}{2}\left[\bar{A} + (A)^t\right]^t = \frac{1}{2}\left[(\bar{A})^t + (A^t)^t\right]$

$= \frac{1}{2}\left[(\bar{A})^t + A\right] = \frac{1}{2}\left[A + (\bar{A})^t\right] = P$

$(\bar{P})^t = P$ Hence P is Hermition

Now $Q = \frac{1}{2i}(A - (\bar{A})^t)$

$(\bar{Q}) = \overline{\frac{1}{2i}(A - \bar{A}^t)} = -\frac{1}{2i}\left[\bar{A} - (\bar{\bar{A}})^t\right]$

$= -\frac{1}{2i}(\bar{A} - A^t)$

$(\bar{Q})^t = -\frac{1}{2i}(\bar{A} - A^t)^t = -\frac{1}{2i}\left[(\bar{A})^t - (A^t)^t\right]$

$= -\frac{1}{2i}\left[(\bar{A})^t - A\right] = \frac{1}{2i}\left[A - (\bar{A})^t\right] = Q$

$(\bar{Q})^t = Q$ Hence Q is Hermition

Now we prove the uniqueness of P & Q.

Let $A = R + iS$ — ① (where R & S are Hermition)

Now $(\bar{A})^t = \left[\overline{R + iS}\right]^t = \left[\bar{R} - i\bar{S}\right]^t = (\bar{R})^t - i(\bar{S})^t$

$(\bar{A})^t = R - iS$ — ② $\because$ R & S are Hermition.

Adding ① & ②

$A + (\bar{A})^t = 2R$

$\frac{1}{2}\left[A + (\bar{A})^t\right] = R$

$\boxed{P = R}$

Subtracting ② from ①

$2iS = A - (\bar{A})^t$

$S = \frac{1}{2i}\left[A - (\bar{A})^t\right] = Q$

Hence $P + iQ$ is unique.

(17) (Easy) $\bar{A} = A \because A$ is real
$A^t = A \because A$ is symmetric
$\bar{B} = B \because B$ is real
$B^t = -B \because B$ is skew symmetric

To Prove $A+iB$ is Hermition

Let $Q = A+iB$

$$\bar{Q} = \overline{A+iB}$$
$$= \bar{A} - i\bar{B}$$
$$\bar{Q} = A - iB \quad \because \bar{A}=A \ \& \ \bar{B}=B$$
$$(\bar{Q})^t = (A-iB)^t$$
$$= A^t - iB^t$$
$$= A - i(-B) \quad \because A^t = A, \ B^t = -B$$
$$= A + iB$$
$$(\bar{Q})^t = Q$$

2nd Method

Let P is a Hermition Matrix i.e. $(\bar{P})^t = P$
then P is a square Matrix over C

$\therefore P = [\alpha_{ij} + i\beta_{ij}]$ where $\alpha_{ij}, \beta_{ij} \in R$
$= [\alpha_{ij}] + i[\beta_{ij}]$

$P = A + iB$ where $A = [\alpha_{ij}]$, $B = [\beta_{ij}]$

$\bar{P} = \overline{A+iB}$ $\quad \because \bar{A} = A, \ \bar{B} = B$ over R

$(\bar{P})^t = (\bar{A} - i\bar{B})^t$
$= (A - iB)^t$
$(\bar{P})^t = A^t - iB^t$
$P = A^t - iB^t \quad \because (\bar{P})^t = P$
$A+iB = A^t - iB^t \quad \because P = A+iB$

Equating we get
$A = A^t$ $\therefore$ A is Symmetric
$\& \ B = -B^t$ & B is skew symmetric

Hence A is symmetric & Real
B is skew symmetric & Real.

(18) Let $A = [a_{ij} + i\alpha_{ij}]$.

$B = [b_{ij} + i\beta_{ij}]$ & $K = p + iq$

(i) To Prove $\bar{\bar{A}} = A$

$$\bar{A} = [a_{ij} - i\alpha_{ij}]$$
$$\bar{\bar{A}} = [a_{ij} + i\alpha_{ij}]$$

So $\bar{\bar{A}} = A$.

(ii) To Prove $\overline{KA} = \bar{K}\bar{A}$

$$KA = [(p+iq)(a_{ij} + i\alpha_{ij})]$$
$$\overline{KA} = [\overline{(p+iq)(a_{ij}+i\alpha_{ij})}] = [\overline{(p+iq)}\,\overline{(a_{ij}+i\alpha_{ij})}]$$
$$= [(p-iq)(a_{ij} - i\alpha_{ij})] = (p-iq)[a_{ij} - i\alpha_{ij}]$$
$$\overline{KA} = \bar{K}\cdot\bar{A}$$

(iii) $\overline{A+B} = \bar{A} + \bar{B}$ To Prove

Let $A+B = [(a_{ij}+i\alpha_{ij})+(b_{ij}+i\beta_{ij})]$

$\overline{A+B} = \overline{[(a_{ij}+i\alpha_{ij})+(b_{ij}+i\beta_{ij})]} = \overline{(a_{ij}+i\alpha_{ij})}+\overline{(b_{ij}+i\beta_{ij})}$

$= (a_{ij}-i\alpha_{ij})+(b_{ij}-i\beta_{ij})$ $\because \overline{z_1+z_2} = \bar{z}_1+\bar{z}_2$

$\overline{A+B} = \bar{A}+\bar{B}$

(iv) $A = [a_{ij}+i\alpha_{ij}]$

then $\bar{A} = [\overline{a_{ij}+i\alpha_{ij}}] = [a_{ij}-i\alpha_{ij}]$

$(\bar{A})^t = [a_{ij}-i\alpha_{ij}]^t$

$(\bar{A})^t = [a_{ji}-i\alpha_{ji}]$ ——— ①

Now $A^t = [a_{ij}+i\alpha_{ij}]^t$

$A^t = [a_{ji}+i\alpha_{ji}]$

$\overline{(A^t)} = [\overline{a_{ji}+i\alpha_{ji}}]$

$\overline{(A^t)} = [a_{ji}-i\alpha_{ji}]$ ——— ②

$(\bar{A})^t = \overline{(A^t)}$

(19) If A is a matrix over the field of real numbers and $AA^t = 0$, show that $A = 0$

Sol Let $A = [a_{ij}]_{m\times n}$, $A^t = [a_{ji}]_{n\times m}$, $AA^t = 0$ (given)

Now i^{th} row of A is $[a_{i1} \;\; a_{i2} \;\; a_{i3} \cdots a_{in}]$

j^{th} column of A^t is $\begin{bmatrix} a_{j1} \\ a_{j2} \\ a_{j3} \\ \vdots \\ a_{jn} \end{bmatrix}$

$(i,j)^{th}$ element of $AA^t = a_{i1}a_{j1} + a_{i2}a_{j2} + a_{i3}a_{j3} + \cdots a_{in}a_{jn}$

$(i,i)^{th}$ element of $AA^t = a_{i1}^2 + a_{i2}^2 + a_{i3}^2 + \cdots a_{in}^2$ $\quad \because$ Put $i=j$ for Diagonal element

$0 = a_{i1}^2 + a_{i2}^2 + a_{i3}^2 + \cdots a_{in}^2$ $\quad \because AA^t = 0 \Rightarrow (i,i)^{th}$ element $= 0$

$\therefore$ each $a_{i1} = 0$, $a_{i2} = 0$, $\cdots a_{in} = 0$
$i = 1, 2, 3, \cdots m$

Hence each element of $A = 0$

So $A = [a_{ij}] = 0$

Ruff

$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots\cdots & a_{2n} \\ a_{m1} & a_{m2} & a_{m3} \cdots\cdots & a_{mn} \end{bmatrix}_{m\times n}$

$A^t = \begin{bmatrix} a_{11} & a_{21} \cdots\cdots & a_{m1} \\ a_{12} & a_{22} & a_{m2} \\ a_{13} & a_{23} & a_{m3} \\ \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & a_{mn} \end{bmatrix}_{n\times m}$

Let $i = 1$, $j = 2$

i^{th} row of $A = [a_{11} \;\; a_{12} \;\; a_{13} \cdots a_{1n}]$

j^{th} col of $A^t = \begin{bmatrix} a_{21} \\ a_{22} \\ a_{23} \\ \vdots \\ a_{2n} \end{bmatrix}$

$(i,j)^{th}$ element of $AA^t = a_{11}a_{21} + a_{12}a_{22} + \cdots a_{1n}a_{2n}$

$\because a_{i1} = 0$, $a_{i2} = 0$, $\cdots a_{in} = 0$ $\quad i = 1,2,3,\cdots m$

i.e

$a_{11} = 0 \quad a_{12} = 0 \cdots a_{1n} = 0$

$a_{21} = 0 \quad a_{22} = 0 \quad a_{2n} = 0$

$a_{31} = 0 \quad a_{32} = 0 \quad a_{3n} = 0$

$\vdots$

$a_{m1} = 0 \quad a_{m2} = 0 \quad a_{mn} = 0$

Note $AA^t = 0 \Rightarrow AA^t =$ zero Matrix

(20) If A is Matrix over C and $A(\bar{A})^t = 0$, show that $\bar{A} = 0 = A$

Sol Let $A = [\alpha_{ij} + i\beta_{ij}]_{m\times n}$, $\bar{A} = [\alpha_{ij} - i\beta_{ij}]$, $(\bar{A})^t = [\alpha_{ji} - i\beta_{ji}]_{n\times m}$

Now i^{th} row of A is $[\alpha_{i1} + i\beta_{i1} \quad \alpha_{i2} + i\beta_{i2} \cdots \alpha_{in} + i\beta_{in}]$

j^{th} col of $(\bar{A})^t$ is $\begin{bmatrix} \alpha_{j1} - i\beta_{j1} \\ \alpha_{j2} - i\beta_{j2} \\ \vdots \\ \alpha_{jn} - i\beta_{jn} \end{bmatrix}$

$(i,j)^{th}$ element of $A(\bar{A})^t = (\alpha_{i1} + i\beta_{i1})(\alpha_{j1} - i\beta_{j1}) + (\alpha_{i2} + i\beta_{i2})(\alpha_{j2} - i\beta_{j2}) + \cdots + (\alpha_{in} + i\beta_{in})(\alpha_{jn} - i\beta_{jn})$

$(i,i)^{th}$ element of $A(\bar{A})^t = (\alpha_{i1} + i\beta_{i1})(\alpha_{i1} - i\beta_{i1}) + (\alpha_{i2} + i\beta_{i2})(\alpha_{i2} - i\beta_{i2}) + \cdots + (\alpha_{in} + i\beta_{in})(\alpha_{in} - i\beta_{in})$

$\because$ putting $i = j$ for Diagonal elements

$0 = (\alpha_{i1}^2 + \beta_{i1}^2) + (\alpha_{i2}^2 + \beta_{i2}^2) + \cdots + (\alpha_{in}^2 + \beta_{in}^2)$ $\quad \because A(\bar{A})^t = 0 \Rightarrow (i,i)^{th}$ element $= 0$

$\therefore$ each $\alpha_{i1} = 0 = \alpha_{i2} = \alpha_{i3} \cdots \alpha_{in}$ $\quad i = 1,2,3,\cdots m$

$\beta_{i1} = 0 = \beta_{i2} = \beta_{i3} \cdots \beta_{in}$ $\quad$ i.e each

$\therefore$ each element of $A = 0$

$A = [0 + i0] = 0$, and $\bar{A} = [0 - i0] = 0$

Q23

$$A = \left[\begin{array}{cc|ccc} 1 & 2 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \qquad B = \left[\begin{array}{ccc|cc} 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & -2 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right]$$

Find AB using indicated partitioning.

Sol: Let $A_{11} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$ and $B_{11} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$A_{12} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad B_{12} = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$

$A_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad B_{21} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$A_{22} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad B_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\therefore AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix} \quad \text{—(1)}$$

Now

$$A_{11}B_{11} + A_{12}B_{21} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0 & 0+0 & 0+0 \\ 0+0 & 0+0 & 0+0 \end{bmatrix} + \begin{bmatrix} 0+0+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A_{11}B_{12} + A_{12}B_{22} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2+2 & 1-4 \\ -4+1 & -2-2 \end{bmatrix} + \begin{bmatrix} 0+0+0 & 0+0+0 \\ 0+0+0 & 0+0+0 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -3 & -4 \end{bmatrix}$$

$$A_{21}B_{11} + A_{22}B_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0 & 0+0 & 0+0 \\ 0+0 & 0+0 & 0+0 \\ 0+0 & 0+0 & 0+0 \end{bmatrix} + \begin{bmatrix} 0+0+0 & 0+0+0 & 1+0+0 \\ 0+0+0 & 0+1+0 & 0+0+0 \\ 0+0+1 & 0+0+0 & 0+0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A_{21}B_{12} + A_{22}B_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix} + \begin{bmatrix} 0+0+0 & 0+0+0 \\ 0+0+0 & 0+0+0 \\ 0+0+0 & 0+0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

from (1)
$$AB = \begin{bmatrix} 0 & 0 & 0 & 4 & -3 \\ 0 & 0 & 0 & -3 & -4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

31-17

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Inverse of a square matrix

Let A be a square matrix of order n. If there exists a matrix B of same order n s.t

$$AB = I_n = BA$$

then matrix B is called the inverse of A i.e

$$B = A^{-1} \quad \text{so} \quad AA^{-1} = A^{-1}A = I$$

Theorem Inverse of a square matrix if its exists is unique.

Proof Let B & C are two inverses of a square matrix 'A'

So, by def $BA = AB = I$

& $CA = AC = I$

Now (Associative Law) $(BA)C = B(AC)$

$(I)C = B(I)$

$C = B$

Hence inverse is unique.

Q2: $A = \left[\begin{array}{cc:c} 1 & 2 & 1 \\ 3 & 4 & 0 \\ \hdashline 0 & 0 & 2 \end{array}\right]$ $\quad B = \left[\begin{array}{ccc:c} 1 & 2 & 3 & 1 \\ 4 & 3 & 6 & 1 \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right]$

Let $A_{11} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\quad A_{12} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad A_{21} = \begin{bmatrix} 0 & 0 \end{bmatrix}$ $\quad A_{22} = [2]$

and $B_{11} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 6 \end{bmatrix}$, $B_{12} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad B_{21} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$ $\quad B_{22} = [1]$

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix} \quad \text{—(1)}$$

Now

$$A_{11}B_{11} + A_{12}B_{21} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 6 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 8 & 15 \\ 19 & 18 & 33 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 15 \\ 19 & 18 & 33 \end{bmatrix}$$

$$A_{11}B_{12} + A_{12}B_{22} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 7 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

$$A_{21}B_{11} + A_{22}B_{21} = \begin{bmatrix} 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 6 \end{bmatrix} + [2]\begin{bmatrix} 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

$$A_{21}B_{12} + A_{22}B_{22} = \begin{bmatrix} 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + [2][1] = [0] + [2] = [2]$$

Hence
$$AB = \left[\begin{array}{ccc:c} 9 & 8 & 15 & 4 \\ 19 & 18 & 33 & 7 \\ \hdashline 0 & 0 & 0 & 2 \end{array}\right] = \begin{bmatrix} 9 & 8 & 15 & 4 \\ 19 & 18 & 33 & 7 \\ 0 & 0 & 0 & 2 \end{bmatrix} \text{ Ans.}$$

Theorem

Let A & B be non-singular matrices of same order then AB is non-singular & $(AB)^{-1} = B^{-1}A^{-1}$

Proof:- If we show that $(AB)(B^{-1}A^{-1}) = I = (B^{-1}A^{-1})(AB)$ then obviously AB is non-singular (∵ inverse exists) & that its inverse is $B^{-1}A^{-1}$.

Now
$$\begin{aligned} (AB)(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1} \quad \because \text{Associative Law} \\ &= A(I)A^{-1} \\ &= AA^{-1} \\ &= I \quad \text{--- (1)} \end{aligned}$$

Also
$$\begin{aligned} (B^{-1}A^{-1})(AB) &= B^{-1}(A^{-1}A)B \quad \because \text{Associative Law} \\ &= B^{-1}(I)B \\ &= B^{-1}B \\ &= I \quad \text{--- (2)} \end{aligned}$$

So from (1) & (2) $(AB)^{-1} = B^{-1}A^{-1}$ & AB is non-singular

Ex 4.2

Q1. Inverse of a Diagonal Matrix is a Diagonal Matrix.

Let $D = \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}$ & $D^{-1} = \begin{bmatrix} d_1^{-1} & 0 & 0 & \cdots & 0 \\ 0 & d_2^{-1} & 0 & \cdots & 0 \\ 0 & 0 & d_3^{-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & d_n^{-1} \end{bmatrix}$

If D^{-1} is inverse of D, then we should have

$$DD^{-1} = I_n = D^{-1}D$$

where I_n is Identity matrix of order n.