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Shariq Mehtab Syed
Lecturer (Maths)
GOVT. DEGREE COLLEGE
90-SB, SARGODHA
285-B, Satellite Town Sargodha.

Permutations

Let X be a non-empty set. A bijective function $f: X \to X$ is called a permutation on X. If X consists of 'n' elements then we write S_n for set of all permutations on X. $\{\forall x \in X,\ (x)f = \text{image of } x \text{ under } f\}$

Th The Set S_n of all permutations on a set X with n elements is a group under the operation of composition of permutations.

Proof To show $(S_n, \circ)$ is a group we have to prove all the axioms for a group.

Let $f, g \in S_n$ be two bijective fns $\quad f: X \to X,\ g: X \to X$

We define $x(f \circ g) = ((x)f)g$

i) Since composition (product) of two bijective fns is alway bijective fn. So $f \circ g$ is bijective fn. $\therefore f \circ g \in S_n$. hence S_n is closed under '$\circ$'.

ii) Let f, g, h be three bijective fns on X & $x \in X$

$$(f \circ g) \circ h = f \circ (g \circ h)$$
$$(x)[(f \circ g) \circ h] = (x)[f \circ (g \circ h)]$$
$$[(x)(f \circ g)]h = ((x)f)(g \circ h)$$
$$[((x)f)g]h = (((x)f)g)h$$

Hence '$\circ$' is Associative

(iii) The fn $I : X \to X$ defined by $(x)I = x \;\forall x$
is the identity element of S_n
For any $f \in S_n$

$$(x)(f \circ I) = ((x)f)I = (x)f \quad \because (x)I = x$$

$$(x)(f \circ I) = (x)f \Rightarrow f \circ I = f \quad \forall f \in S_n$$

$$(x)(I \circ f) = ((x)I)f = (x)f$$

$$(x)(I \circ f) = (x)f \Rightarrow I \circ f = f \quad \forall f \in S_n$$

Hence I is the identity element of S_n

(iv) Inverse of a bijective fn is also a bijective fn. So for any bijective fn $f \in S_n$ $(f : X \to X)$ $\exists$ its inverse $f^{-1} \in S_n$, $(f^{-1} : X \to X)$
By def of f^{-1}, for any $x, y \in X$

$$(x)f = y \Rightarrow x = (y)f^{-1}$$

$$(x)(f \circ f^{-1}) = ((x)f)f^{-1} = (y)f^{-1} = x = (x)I$$

$$(x)(f \circ f^{-1}) = (x)I \Rightarrow f \circ f^{-1} = I$$

$$y(f^{-1} \circ f) = ((y)f^{-1})f = (x)f = y = (y)I$$

$$y(f^{-1} \circ f) = (y)I \Rightarrow f^{-1} \circ f = I$$

All axioms are satisfied
Hence $(S_n, \circ)$ is a group

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Cycles

Def A Cyclic Permutation or A Cycle is a permutation of the form $\alpha = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_K \\ a_2 & a_3 & a_4 & \cdots & a_1 \end{pmatrix}$

i.e image of a_1 under $\alpha = a_2$ $\quad \alpha = (a_1 a_2 a_3 a_4 \cdots a_K)$

" " a_2 " " $= a_3$

" " a_3 " " $= a_4$

$\vdots$

image of a_K under $\alpha = a_1$ $\quad$ where K is length of cycle.

e.g $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$ cycle of length 5

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 5 & 1 & 4 \end{pmatrix} = (1\ 3\ 5\ 4)$

$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix} = (3\ 4)$

Note (i) Product of two cycles need not be a cycle.

$\alpha = (1, 2, 5)$ $\beta = (2\ 1\ 4\ 5\ 6)$ $\quad \alpha, \beta$ are cycles.

$\alpha \circ \beta = \alpha\beta = \begin{pmatrix} 1 & 2 & 4 & 5 & 6 \\ 1 & 6 & 5 & 4 & 2 \end{pmatrix}$ Not Cycle

(ii) Product of two mutually disjoint cycles is commutative.

$\alpha = (1\ 2\ 3)$ $\beta = (4\ 5\ 6)$ $\quad \beta\alpha = \begin{pmatrix} 4 & 5 & 6 & 1 & 2 & 3 \\ 5 & 6 & 4 & 2 & 3 & 1 \end{pmatrix}$

$\alpha\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \end{pmatrix}$ $\quad \beta\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \end{pmatrix}$

$\alpha\beta = \beta\alpha$

Th. Every permutation of degree n can be written as a product of cyclic permutations acting on mutually disjoint sets. OR

Every element of S_n is a product of disjoint cycles.

Proof. Let α be a permutation of degree n (i.e. permutation of n objects namely elements of X) & $X=\{1,2,3 \dots n\}$. Suppose that α acts on an element a_1. Further Suppose that under the action of α, $a_1 \to a_2$, $a_2 \to a_3$, $a_3 \to a_4$ $\dots a_{K-1} \to a_K$, $a_K \to a_1$ ($\because n$ is finite $\therefore \exists$ a natural no K s.t $a_K \to a_1$)

Thus a part of the effect of α on elements of set X is cycle.

$\alpha_1 = (a_1 a_2 a_3 \dots a_K)$

e.g $\left\{ \begin{array}{l} \alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 \end{pmatrix} \\ \alpha = (1\,8)(3\,6\,4)(5\,7) \end{array} \right.$

Now if $K=n$ then $\alpha = \alpha_1$ & since α_1 is cyclic so theorem is proved. But

if $K<n$ then $\exists$ a b_1 which is not in $(a_1, a_2, a_3 \dots a_K)$

Suppose that under the action of α, $b_1 \to b_2$, $b_2 \to b_3$ $b_3 \to b_4 \dots b_p \to b_1$. Thus a part of the effect of α on elements of Set X is cycle, $\alpha_2 = (b_1\ b_2\ b_3 \dots b_p)$

So we have extracted two cycles from α.

If $K+P = n$, then $\alpha = \alpha_1 \alpha_2$

But if $K+P<n$, then $\exists$ a c_1 which is not in α_1 & α_2. Again Suppose that under the action of α, $c_1 \to c_2$, $c_2 \to c_3 \dots$ $\dots c_q \to c_1$. Thus a part of the effect of α on elements of Set X is cycle, $\alpha_3 = (c_1\ c_2\ c_3 \dots c_q)$

Continuing in this way, this process of extracting a cycle must end after a finite no of steps because n is finite.

$\therefore \exists$ a natural no 'r' s.t $n = K+P+q+\dots r$

and a part of the effect of α is a cycle

$$\alpha_t = (u_1, u_2\ u_3 \dots u_r) \quad \text{where } u_i \notin (\alpha_1 \alpha_2 \alpha_3 \dots)$$

$$\therefore \alpha = \alpha_1 \cdot \alpha_2 \cdot \alpha_3 \dots \alpha_t$$ where each $\alpha_1, \alpha_2 \dots \alpha_t$ acts on mutually disjoint subsets of X.

Transpositions

A cycle of length 2 is called a Transposition. eg $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$

Note 1) Every cyclic permutation can be expressed as a product of transpositions. $\alpha = (a_1 a_2 a_3 \cdots a_k)$ $\alpha' = (a_1 a_2)(a_1 a_3) \cdots (a_1 a_k)$

2) Every permutation of degree n can be expressed as a product of transpositions.

3) For a given permutation the number of transpositions is always even or odd.

4) Inverse of a transposition is the same transposition. $\alpha = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$, $\alpha^{-1} = \begin{pmatrix} b & a \\ a & b \end{pmatrix} = \begin{pmatrix} a & b \\ b & a \end{pmatrix} = \alpha$

Even Permutation.

A permutation α in S_n is said to be an even permutation if it can be written as a product of an even number of transpositions.

(Inverse of even permutation is even permutation)
(& Inverse of odd permutation is odd permutation)

Odd Permutation

A permutation α in S_n is said to be an odd permutation if it can be written as a product of an odd number of transpositions.

Identity Permutation is an even permutation, because the number of transpositions in the decomposition of the identity permutation is zero which is an even integer.

Every transposition is an odd permutation.

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Th: (69) i) The product of two even permutations is an even permutation.

ii) The product of two odd permutations is an even permutation.

iii) The product of an even permutation and an odd permutation is an odd permutation.

Proof Let α_1 & α_2 be any two permutations of degree n i.e. $\alpha_1, \alpha_2 \in S_n$

then α_1 & α_2 can be expressed as a product of m_1 & m_2 transpositions. (by theorem)

The product $\alpha_1\alpha_2$ contains $m_1 + m_2 - 2K$ transpositions,

where $K = 0$ or K = a natural no.

{The term $2K$ occurs because of the possible cancellation (simplification) of pairs of transpositions} {i.e. K is no. of common transpositions in α_1 & α_2}
e.g. α_1 contain (p,q), α_2 contain (q,p)
then $\alpha_1 \cdot \alpha_2 = (p,q)(q,p) = \begin{pmatrix} p & q \\ p & q \end{pmatrix}$ = Identity

Case 1 No. of transpositions in $\alpha_1 \cdot \alpha_2 = m_1 + m_2 - 2K$
If α_1 & α_2 are even permutations

$\Rightarrow m_1$ & m_2 are even

$\Rightarrow m_1 + m_2 - 2K$ is an even integer.

$\Rightarrow \alpha_1\alpha_2$ is an even permutation.

Even + Even = Even
$4 + 4 = 8$

Case 2 If α_1 & α_2 are odd permutations

$\Rightarrow m_1$ & m_2 are odd

$\Rightarrow m_1 + m_2 - 2K$ is an even integer

$\Rightarrow \alpha_1\alpha_2$ is an even permutation

$2K$ is always even

odd + odd
$3 + 3 = 6$ even

Case 3 If α_1 is odd & α_2 is even permutations

$\Rightarrow m_1$ is odd & m_2 is even.

$\Rightarrow m_1 + m_2 - 2K$ is an odd integer

$\Rightarrow \alpha_1\alpha_2$ is an odd permutation

Even + odd = odd.
$4 + 5 = 9$

$\boxed{2\cdot 3-7}$

Th For $n \geq 2$ the number of even permutations in S_n is equal to the number of odd permutations in S_n.

Proof Let all even permutations in S_n be $\alpha_1, \alpha_2, \alpha_3 \cdots \alpha_K$ —— ①
and Let all odd permutations in S_n be $\beta_1, \beta_2, \beta_3 \cdots \beta_l$ —— ②
So that $K + l = n!$

Let h be a transposition, then

③ — $h\alpha_1, h\alpha_2, h\alpha_3$ —— $h\alpha_K$ are all odd permutations and

④ — $h\beta_1, h\beta_2, h\beta_3$ —— $h\beta_l$ are all even permutations

∴ from ① + ④ (even) $l \leq K$ —— ⑤

from ② + ③ (odd) $K \leq l$ —— ⑥

from ⑤ + ⑥ ∴ $K = l$

see Q11

e.g $\alpha_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 6 & 5 & 3 \end{pmatrix}$

$\alpha_1 = (1\,2\,4\,6\,3)$

$\alpha_1 = (1\,2)(1\,4)(1\,6)(1\,3)$

α_1 is even

Let $h = (7\,8)$

$h\alpha_1 = (1\,2)(1\,4)(1\,6)(1\,3)(7\,8)$

$h\alpha_1$ is odd.

Since $K + l = n!$

∴ $K + K = n!$

$2K = n!$

$K = \frac{n!}{2}$ ∴ Number of all even permutations $= \frac{n!}{2}$ ∵ $K = l$

OR Number of all odd permutations

Th Set A_n of all even permutations in S_n forms a subgp of S_n and Set B_n of all odd permutations in S_n does not form a subgp of S_n.

Proof Let $\alpha_1, \alpha_2 \in A_n$ then we prove $\alpha_1\alpha_2^{-1} \in A_n$
since the inverse of an even permutation is also even permutation
So α_2^{-1} is an even permutation.
Now Since the product of two even permutations is also an even permutation (by theorem)
∴ $\alpha_1\alpha_2^{-1}$ is an even permutation
$\Rightarrow \alpha_1\alpha_2^{-1} \in A_n$. Hence A_n is subgp of S_n

Now Let $\beta_1, \beta_2 \in B_n$ then we prove $\beta_1\beta_2^{-1} \notin B_n$
Since the inverse of an odd permutation is also odd permutation.
So β_2^{-1} is odd permutation.
Also since the product of two odd permutations is not odd, but even. (by theorem)
∵ $\beta_1\beta_2^{-1}$ is an even permutation
$\Rightarrow \beta_1\beta_2^{-1} \notin B_n$. Hence B_n is not subgp of S_n

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Order of a Permutation

Let α be a permutation in S_n. The order of α is the least positive integer 'm' such that $\boxed{\alpha^m = I}$ Identity permutation

e.g $\alpha = (1\,2\,3)$

$$\alpha^2 = (1\,2\,3)(1\,2\,3) = \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix} = \begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$$

$$\alpha^3 = \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix} = \begin{pmatrix}1&2&3\\1&2&3\end{pmatrix} = I \quad \therefore \text{order} = 3$$

The order of transposition is 2. $\begin{pmatrix}a&b\\b&a\end{pmatrix}\begin{pmatrix}a&b\\b&a\end{pmatrix} = \begin{pmatrix}a&b\\a&b\end{pmatrix} = I$

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Th The order of a cyclic permutation of length 'm' is 'm'

Proof Let $\alpha = (a_1\, a_2 \cdots\cdots a_m)$ be a cyclic permutation of length m. Then under action of α^2

$$\alpha^2 = \alpha\alpha = \begin{pmatrix}a_1 & a_2 & a_3 & \cdots & a_m\\ a_2 & a_3 & a_4 & \cdots & a_1\end{pmatrix}\begin{pmatrix}a_1 & a_2 & a_3 & \cdots & a_m\\ a_2 & a_3 & a_4 & \cdots & a_1\end{pmatrix} = \begin{pmatrix}a_1 & a_2 & a_3 & \cdots & a_m\\ a_3 & a_4 & a_5 & \cdots & a_2\end{pmatrix}$$

$$\alpha^3 = \alpha^2\alpha = \begin{pmatrix}a_1 & a_2 & \cdots & a_m\\ a_3 & a_4 & \cdots & a_2\end{pmatrix}\begin{pmatrix}a_1 & a_2 & a_3 & \cdots & a_m\\ a_2 & a_3 & a_4 & \cdots & a_1\end{pmatrix} = \begin{pmatrix}a_1 & a_2 & \cdots & a_m\\ a_4 & a_5 & \cdots & a_3\end{pmatrix}$$

Similarly continuing we get

$$\alpha^m = \begin{pmatrix}a_1 & a_2 & a_3 & \cdots & a_m\\ a_1 & a_2 & a_3 & \cdots & a_m\end{pmatrix} = I$$

and m is least +ive integer for which $\alpha^m = I$

Hence order of 'α' is 'm'. $\therefore$ A cycle of length m has order 'm'.

To find the order of a permutation 'α'

First decompose α as a product of cyclic permutations $\alpha_1, \alpha_2, \alpha_3 \cdots \alpha_K$ of lengths $m_1, m_2, m_3 \cdots m_K$ resp ignoring identity permutation. Then take LCM of $m_1, m_2, \cdots m_K$: we get order of α.

$\boxed{2\cdot 3-9}$

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Ex 2·3

① (a) $\begin{pmatrix}1&2&3&4&5\\2&3&5&4&1\end{pmatrix}\begin{pmatrix}1&2&3&4&5\\3&1&4&2&5\end{pmatrix}=\begin{pmatrix}1&2&3&4&5\\1&4&5&2&3\end{pmatrix}$

(b) $\begin{pmatrix}1&2&3&4&5&6\\2&3&5&6&4&1\end{pmatrix}\begin{pmatrix}1&2&3&4&5&6\\6&3&2&5&1&4\end{pmatrix}=\begin{pmatrix}1&2&3&4&5&6\\3&2&1&4&5&6\end{pmatrix}$

(c) $\begin{pmatrix}2&3&6&7&8&9\\8&6&7&9&2&3\end{pmatrix}\begin{pmatrix}2&3&6&7&8&9\\8&6&8&3&9&2\end{pmatrix}=\begin{pmatrix}2&3&6&7&8&9\\9&8&3&2&8&6\end{pmatrix}$

②

$\alpha=\begin{pmatrix}1&2&3&4\\1&3&4&2\end{pmatrix}$ $\qquad \alpha\circ\alpha^{-1}=I$

$\alpha^{-1}=\begin{pmatrix}1&3&4&2\\1&2&3&4\end{pmatrix}=\begin{pmatrix}1&2&3&4\\1&4&2&3\end{pmatrix}$

③

$\alpha=\begin{pmatrix}1&2&3&4&5&6&7\\3&5&4&1&7&2&6\end{pmatrix}$ $\qquad \alpha\circ\alpha^{-1}=\alpha^{-1}\circ\alpha=I$

$\alpha^{-1}=\begin{pmatrix}3&5&4&1&7&2&6\\1&2&3&4&5&6&7\end{pmatrix}=\begin{pmatrix}1&2&3&4&5&6&7\\4&6&1&3&2&7&5\end{pmatrix}$

④

$f=\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}$ $\qquad g=\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix}$

$f\circ g=\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix}=\begin{pmatrix}1&2&3&4\\1&4&3&2\end{pmatrix}$

$f^2\circ g=?$

$f^2=f\circ f=\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}$

$f^2=\begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}$

$f^2\circ g=\begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix}$

$f^2\circ g=\begin{pmatrix}1&2&3&4\\4&3&2&1\end{pmatrix}$

$f^3 o g = ?$ $\quad f^3 = f^2 o f = \begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}$

$$f^3 = \begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix}$$

$$f^3 o g = \begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix} = \begin{pmatrix}1&2&3&4\\3&2&1&4\end{pmatrix}$$

$$f^4 = f^3 o f = \begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix} = \begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix} =$$

$$g^2 = g o g = \begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix} = \begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix} = I$$

$$(f o g)^2 = (f o g) o (f o g) = \begin{pmatrix}1&2&3&4\\1&4&3&2\end{pmatrix}\begin{pmatrix}1&2&3&4\\1&4&3&2\end{pmatrix} = \begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix} =$$

Q5 Find $f o g$, $g o f$, $g o h$, $h o g$, h^2 g^2 Do yourself

$f = \begin{pmatrix}1&2&3&4&5&6\\2&4&6&5&3&1\end{pmatrix} = (1\ 2\ 4\ 5\ 3\ 6)$

Length of $f = 6$. $\quad O(f) = 6$. i.e $f^6 =$ Identity

$g = \begin{pmatrix}1&2&3&4&5&6\\6&5&4&1&2&3\end{pmatrix} = (1\,6\,3\,4)\,(2\,5)$

order 4 $\quad$ order 2

LCM of 2 & 4 is 4

So $O(g) = 4$ $\quad$ i.e $g^4 =$ Identity

$h = \begin{pmatrix}1&2&3&4&5&6\\5&4&1&2&6&3\end{pmatrix} = (1\,5\,6\,3)\,(2\,4)$

order 4 $\quad$ order 2

LCM of 2 & 4 is 4 So $O(h) = 4$ $\quad$ i.e $h^4 =$ Identity

Q6 $a = \begin{pmatrix}1&2&3&4&5\\2&1&4&5&3\end{pmatrix} = (1\,2)(3\,4\,5)$

order 2 $\quad$ order 3

LCM of 2 & 3 is 6

So $O(a) = 6$ $\quad$ i.e $a^6 =$ Identity

$\boxed{2 \cdot 3 - 11}$

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(7) $I = (1) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$ $\quad a = (1\,2\,3\,4) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$

$b = (1\,4\,3\,2) = \begin{pmatrix} 1 & 4 & 3 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$

$c = (1\,3)(2\,4) = \begin{pmatrix} 1 & 3 & 2 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$

$I \cdot I = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = I$

$I \cdot a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = a$

$I \cdot b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} = b$

$I \cdot c = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} = c$

Multiplication Table

$\cdot$	I	a	b	c
I	I	a	b	c
a	a	c	I	b
b	b	I	c	a
c	c	b	a	I

$a \cdot I = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = a$

Similarly $b \cdot I = b \quad c \cdot I = c$

$a \cdot a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = c$

$a \cdot b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = I$

$a \cdot c = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} = b$

$b \cdot a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = I$

$c \cdot a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} = b$

$b \cdot b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = c$

$b \cdot c = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = a$

$c \cdot b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = a$

$c \cdot c = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = I$

(75)

(8) Find all elements of cyclic group generated by a.

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 2 & 6 & 1 \end{pmatrix}$$

$$a^2 = a\cdot a = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 2 & 6 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 2 & 6 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 2 & 6 & 4 & 1 & 3 \end{pmatrix}$$

$$a^3 = a^2\cdot a = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 2 & 6 & 4 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 2 & 6 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 1 & 2 & 3 & 5 \end{pmatrix}$$

$$a^4 = a^3\cdot a = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 1 & 2 & 3 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 2 & 6 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$$

$$= I$$

Hence distinct elements of the cyclic group generated by a are I, a, a^2, a^3

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(9) (i) $(1\ 2\ 4)(1\ 3\ 6\ 5\ 4)$

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 & 6 & 5 & 4 \\ 3 & 6 & 5 & 4 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 3 & 1 & 5 & 6 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 6 & 1 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 6 & 3 & 4 & 5 \end{pmatrix}$$

$$= (1\ 2)(3\ 6\ 5\ 4)$$

(ii) $(1\ 2\ 3\ 4)(2\ 5\ 3\ 4\ 1)$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}\begin{pmatrix} 2 & 5 & 3 & 4 & 1 \\ 5 & 3 & 4 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 1 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 5 & 3 \\ 5 & 3 & 1 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix} = (1\ 5\ 3)(2\ 4)$$

(iii) $(1\ 4)(2\ 3\ 5)(3\ 5)(4\ 5)$ $\quad \begin{pmatrix} 1 & 4 & 2 & 3 & 5 \\ 4 & 1 & 3 & 5 & 2 \end{pmatrix}\begin{pmatrix} 3 & 5 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} 4 & 5 \\ 5 & 4 \end{pmatrix}$

$$= \left[\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 3 & 1 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 4 & 2 \end{pmatrix}\right]\left[\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 5 & 4 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 5 & 4 \end{pmatrix}\right]$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 1 & 2 \end{pmatrix}\circ\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 5 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 5 & 2 & 4 & 3 \\ 5 & 2 & 4 & 1 & 3 \end{pmatrix}$$

(10) (i) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 \end{pmatrix}$ Express Permutation as product of disjoint cycles.

$= \begin{pmatrix} 1 & 8 \\ 8 & 1 \end{pmatrix}\begin{pmatrix} 3 & 6 & 4 \\ 6 & 4 & 3 \end{pmatrix}\begin{pmatrix} 5 & 7 \\ 7 & 5 \end{pmatrix} = (1\,8)(3\,6\,4)(5\,7)$

(ii) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 4 & 1 & 8 & 2 & 5 & 7 \end{pmatrix}$

$= \begin{pmatrix} 1 & 3 & 4 \\ 3 & 4 & 1 \end{pmatrix}\begin{pmatrix} 2 & 6 \\ 6 & 2 \end{pmatrix}\begin{pmatrix} 5 & 8 & 7 \\ 8 & 7 & 5 \end{pmatrix} = (1\,3\,4)(2\,6)(5\,8\,7)$

(11) Express permutation as product of transpositions

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 1 & 6 & 5 & 3 & 8 & 9 & 7 \end{pmatrix}$

First we express permutation as product of disjoint cycles.

$= \begin{pmatrix} 1 & 2 & 4 & 6 & 3 \\ 2 & 4 & 6 & 3 & 1 \end{pmatrix}\begin{pmatrix} 7 & 8 & 9 \\ 8 & 9 & 7 \end{pmatrix}$

$= (1\,2\,4\,6\,3)(7\,8\,9)$

$= (1\,2)(1\,4)(1\,6)(1\,3)(7\,8)(7\,9)$

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 1 & 5 & 6 & 7 & 8 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = (1\,2\,3\,4) = (1\,2)(1\,3)(1\,4)$

(13) $(1\,2\,3)(2\,5\,6)(4\,3\,5\,1)$ Permutations even or odd.

First we express permutations as product of disjoint cycles.

$\left[\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 5 & 3 & 4 & 6 & 2 \end{pmatrix}\right]\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 2 & 5 & 3 & 1 & 6 \end{pmatrix}$

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 1 & 4 & 6 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 2 & 5 & 3 & 1 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 5 & 4 & 3 & 6 & 2 \end{pmatrix}$

$= \begin{pmatrix} 2 & 5 & 6 \\ 5 & 6 & 2 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix} = (2\,5\,6)(3\,4)$

Express disjoint cycles into Transposition $= (2\,5)(2\,6)(3\,4)$ 3 Transposition So Odd.

$(147)(345)(87)(8345)$ Even or odd.

$$\begin{pmatrix}1&4&2&3&5&6&7&8\\4&7&2&3&5&6&1&8\end{pmatrix}\begin{pmatrix}1&2&3&4&5&6&7&8\\1&2&4&5&3&6&7&8\end{pmatrix}\begin{pmatrix}1&2&3&4&5&6&7&8\\1&2&3&4&5&6&8&7\end{pmatrix}$$

$$\begin{pmatrix}1&2&3&4&5&6&7&8\\1&2&4&5&8&6&7&3\end{pmatrix}$$

$$=\begin{pmatrix}1&2&3&4&5&6&7&8\\5&2&4&7&3&6&1&8\end{pmatrix}\begin{pmatrix}1&2&3&4&5&6&7&8\\1&2&4&5&8&6&3&7\end{pmatrix}$$

$$=\begin{pmatrix}1&2&3&4&5&6&7&8\\8&2&5&3&4&6&1&7\end{pmatrix} = \begin{pmatrix}1&8&7\\8&7&1\end{pmatrix}\begin{pmatrix}3&5&4\\5&4&3\end{pmatrix}$$

$$=(187)(354) = (18)(17)(35)(34) \quad \because 4 \text{ So even}$$

$\therefore$ No of Transpositions are 4 so even.

(14) Find the order of each permutation.

(i) $a=\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix} = \begin{pmatrix}1&2\\2&1\end{pmatrix}\begin{pmatrix}3&4\\4&3\end{pmatrix} = \underbrace{(12)}_{\text{order } 2}\underbrace{(34)}_{\text{order } 2}$

The LCM of the orders of the cycles on disjoint sets is 2.

So $O(a)=2$

(ii) $a=\begin{pmatrix}1&2&3&4\\4&2&1&3\end{pmatrix} = \begin{pmatrix}1&4&3\\4&3&1\end{pmatrix}\begin{pmatrix}2\\2\end{pmatrix} = \underbrace{(143)}_{\text{order } 3}$ $\therefore O(a)=3$

(iii) $a=\begin{pmatrix}1&2&3&4&5\\4&3&2&5&1\end{pmatrix} = \begin{pmatrix}1&4&5\\4&5&1\end{pmatrix}\begin{pmatrix}2&3\\3&2\end{pmatrix} = \underbrace{(145)}_{\text{order } 3}\underbrace{(23)}_{\text{order } 2}$

The LCM of the orders of the cycles on disjoint sets i,e $3 \& 2 = 6$

So $O(a)=6$

(v) $a=\begin{pmatrix}1&2&3&4&5&6\\2&3&1&5&6&4\end{pmatrix} = \underbrace{\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}}_{\text{order } 3}\underbrace{\begin{pmatrix}4&5&6\\5&6&4\end{pmatrix}}_{\text{order } 3}$

The LCM of the orders of the cycles on disjoint sets i,e $3 \& 3 = 3$

So $O(a)=3$