PERMUTATIONS

Permutation

Let X be non-empty set. Let $f: X \to X$ be a bijective function, then f is called a permutation on X.

If f is a permutation on $X = \{1,2,3, \dots \dots n\}$ then f is written as

$$f = \begin{pmatrix} 1 & 2 & \dots\dots\dots & n \\ (1)f & (2)f & \dots\dots\dots & (n)f \end{pmatrix}$$

Example

Let $X = \{1,2,3\}$

Let $f : X \to X$ be defined by

$(1)f = 2$

$(2)f = 3$

$(3)f = 1$

Clearly f is a bijective function.

Thus f is a permutation on $X = \{1,2,3\}$ and is written as

$$f = \begin{pmatrix} 1 & 2 & 3 \\ (1)f & (2)f & (3)f \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

Note

(i) The total number of permutations defined on a set $X = \{1,2,3, \dots, n\}$ with n elements is $n!$

(ii) The set of all permutations defined on a set X with n elements is denoted by S_n

$\therefore$ Number of elements of $S_n = n!$

Let $X = \{1,2,3\}$

Then number of permutations defined on a set X is 3! i.e. 6 and are given by

$$I = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \ f_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \ f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix},$$

$$f_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \ f_4 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \ f_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

Thus $S_3 = \{I, f_1, f_2, f_3, f_4, f_5\}$

(iii) $I = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ is called identity permutation.

Composition of Permutations

or Multiplication

Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & \dots\dots\dots & n \\ (1)\alpha & (2)\alpha & (3)\alpha & \dots\dots\dots & (n)\alpha \end{pmatrix}$

And $\beta = \begin{pmatrix} 1 & 2 & 3 & \dots\dots\dots & n \\ (1)\beta & (2)\beta & (3)\beta & \dots\dots\dots & (n)\beta \end{pmatrix}$

The composition of α and β is denoted by $\alpha \circ \beta$ and is defined by

$$\alpha \circ \beta = \begin{pmatrix} 1 & 2 & 3 & \dots\dots\dots & n \\ (1)(\alpha \circ \beta) & (2)(\alpha \circ \beta) & (3)(\alpha \circ \beta) & \dots\dots\dots & (n)(\alpha \circ \beta) \end{pmatrix}$$

Where $(x)(\alpha \circ \beta) = ((x)\alpha)\beta$

Let $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

$$\text{Then } \alpha\circ\beta = \begin{pmatrix} 1 & 2 & 3 \\ (1)(\alpha\circ\beta) & (2)(\alpha\circ\beta) & (3)(\alpha\circ\beta) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ ((1)\alpha)\beta & ((2)\alpha)\beta & ((3)\alpha)\beta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ (2)\beta & (3)\beta & (1)\beta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

Practical Application

Let $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

Write α and β in such a way that 2nd row of α and 1st row of β are same.

i.e. $\alpha\circ\beta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 & 1 \\ 1 & 3 & 2 \end{pmatrix}$

Now ignore 2nd row of α and 1st row of β.

i.e. $\alpha\circ\beta = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$

Note

(i) Some times we write composition of α and β by $\alpha\beta$ instead of $\alpha\circ\beta$.

(ii) Multiplication or product or composition of α and β are same.

Theorem

The set S_n of all permutations defined on a set X with n elements is a group under the operation of composition of permutations.

Proof:

(I) Let $f, g \in S_n$. We shall prove that $f \circ g \in S_n$

Since f, g are permutations on X.

$\therefore$ $f: X \to X$ and $g: X \to X$ are bijective functions then $f \circ g : X \to X$ is also a bijective function.

$\therefore$ $f \circ g$ is a permutation on X.

Hence $f \circ g \in S_n$

(II) Let $f, g, h \in S_n$. We shall prove that $(f \circ g) \circ h = f \circ (g \circ h)$

$$(x)[(f \circ g) \circ h] = ((x)(f \circ g))h = (((x)f)g)h \qquad (x)[f \circ (g \circ h)] = ((x)f)(g \circ h) = (((x)f)g)h$$

Clearly $(f \circ g) \circ h = f \circ (g \circ h)$

(III) Let $I: X \to X$ be defined by $(x)I = x$

Then $I: X \to X$ is a bijective function thus I is a permutation on X.

$\therefore \quad I \in S_n$

We shall prove that I is an identity element of S_n.

Let $f \in S_n$. We have to show that $f \circ I = I \circ f = f$

$$(x)(f \circ I) = ((x)f)I = (x)f \qquad (x)(I \circ f) = ((x)I)f = (x)f$$

Clearly $f \circ I = I \circ f = f$

(IV) Let $f \in S_n$ then $f: X \to X$ is a bijective function.

$\therefore \quad f^{-1}: X \to X$ defined as $(x)f = y \Leftrightarrow (y)f^{-1} = x$ is bijective and hence a permutation on X. $\therefore \ f^{-1} \in S_n$

Now we shall prove that f^{-1} is the inverse of f and for this we have to prove that $\quad f \circ f^{-1} = f^{-1} \circ f = I$

$$(x)(f \circ f^{-1}) = ((x)f)f^{-1} = (y)f^{-1} = x = (x)I \qquad (y)(f^{-1} \circ f) = ((y)f^{-1})f = (x)f = y = (y)I$$

Clearly $f \circ f^{-1} = f^{-1} \circ f = I$

Hence $(S_n, \circ)$ is a group.

<u>Symmetric Group</u>

The group S_n of all permutation defined on a set X with n elements is called symmetric group of degree n.

Example

Show that $(S_3, \circ)$ is a non-abelian group.

Solution

Here S_3 is the set of all permutation defined on a set X with three elements and is group under the composition of permutations. If $X = \{1,2,3\}$, then all the permutations defined on the set X are

$$I = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \ f_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \ f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix},$$

$$f_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \ f_4 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \ f_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$\text{Now } f_1 \circ f_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 & 1 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

And $f_3 \circ f_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

Clearly $f_1 \circ f_3 \neq f_3 \circ f_1$

$\Rightarrow$ $(S_3, \circ)$ is a non-abelian group.

Example

Let $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$ and $g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$,

Show that $f \circ g \neq g \circ f$

Solution

Here $f \circ g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 & 4 & 1 \\ 1 & 4 & 3 & 2 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$

And $g \circ f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$

Clearly $f \circ g \neq g \circ f$

Example

Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 2 & 6 & 4 & 1 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 2 & 6 & 5 \end{pmatrix}$,

Show that $\alpha \circ \beta \neq \beta \circ \alpha$

Solution

Here $\alpha \circ \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 2 & 6 & 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 2 & 6 & 5 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 2 & 6 & 4 & 1 \end{pmatrix}\begin{pmatrix} 5 & 3 & 2 & 6 & 4 & 1 \\ 6 & 1 & 4 & 5 & 2 & 3 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 4 & 5 & 2 & 3 \end{pmatrix}$

And $\beta \circ \alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 2 & 6 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 2 & 6 & 4 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 2 & 6 & 5 \end{pmatrix}\begin{pmatrix} 3 & 4 & 1 & 2 & 6 & 5 \\ 2 & 6 & 5 & 3 & 1 & 4 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 5 & 3 & 1 & 4 \end{pmatrix}$

Clearly $\alpha \circ \beta \neq \beta \circ \alpha$

CYCLES

Cycle

Let $X = \{a_1, a_2, \dots\dots\dots, a_k, a_{k+1}, \dots\dots\dots, a_n\}$

If α is a permutation on X, such that

$$\alpha = \begin{pmatrix} a_1 & a_2 \dots\dots\dots & a_k & a_{k+1} \dots\dots\dots & a_n \\ a_2 & a_3 \dots\dots\dots & a_1 & a_{k+1} \dots\dots\dots & a_n \end{pmatrix}$$

Then α is called a cyclic permutation or simply a cycle of length k.

Thus α will be called a cycle of length k if

$(a_1)\ \alpha = a_2$

$(a_2)\ \alpha = a_3$

....

....

$(a_k)\ \alpha = a_1$

$(x)\ \alpha = x \qquad \forall\ x \in X - \{a_1, a_2, \dots\dots\dots, a_k\}$

The cycle α of length k can also be written as

$$\alpha = \begin{pmatrix} a_1 & a_2 \dots\dots\dots & a_k \\ a_2 & a_3 \dots\dots\dots & a_1 \end{pmatrix}$$

In short α can be written as $\alpha = (a_1 a_2 \dots\dots\dots a_k)$

Here $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \end{pmatrix}$ is a cycle of length 6 which can also be written as $\alpha = (1\,2\,3\,4\,5\,6)$

And $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 4 & 5 & 3 & 6 \end{pmatrix}$ is a cycle of length 3 which can also be written as $\beta = (3\,4\,5)$

Example

Show by means of an example that the product of two cyclic permutations need not be a cyclic permutation.

Solution

Let $\alpha = (1\,2\,5)$ and $\beta = (2\,1\,4\,5\,6)$ be two cycles.

Then $\alpha\beta = (1\,2\,5)(2\,1\,4\,5\,6)$

$$= \begin{pmatrix} 1 & 2 & 5 \\ 2 & 5 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 4 & 5 & 6 \\ 1 & 4 & 5 & 6 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 3 & 4 & 1 & 6 \end{pmatrix}\begin{pmatrix} 2 & 3 & 1 & 4 & 5 & 6 \\ 1 & 3 & 4 & 5 & 6 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 3 & 4 & 1 & 6 \end{pmatrix}\begin{pmatrix} 2 & 5 & 3 & 4 & 1 & 6 \\ 1 & 6 & 3 & 5 & 4 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 3 & 5 & 4 & 2 \end{pmatrix}, \quad \text{This is not a cycle.}$$

Thus product of two cycles need not be a cycle.

Theorem

Every permutation of degree n can be expressed as a product of cyclic permutations acting on mutually disjoint sets.

Proof:

Let α be permutation on X with n elements.

Suppose $a_1, a_2, \dots\dots, a_k \in X$, on which α acts as

$$(a_1)\,\alpha = a_2$$
$$(a_2)\,\alpha = a_3$$
$$\dots\dots\dots$$
$$\dots\dots\dots$$
$$(a_k)\,\alpha = a_1$$

Let $\alpha_1 = \begin{pmatrix} a_1 & a_2 \dots\dots\dots a_k \\ a_2 & a_3 \dots\dots\dots a_1 \end{pmatrix}$, where $k \le n$

If $k = n$ then $\alpha = \alpha_1$ Hence the proof.

If $k < n$ then let $b_1, b_2, \dots\dots\dots, b_p \in X - \{a_1, a_2, \dots\dots\dots, a_k\}$ on which α acts as

$$(b_1)\,\alpha = b_2$$
$$(b_2)\,\alpha = b_3$$
$$\dots\dots\dots$$
$$\dots\dots\dots$$
$$(b_p)\,\alpha = b_1$$

Let $\alpha_2 = \begin{pmatrix} b_1 & b_2 \dots\dots\dots b_p \\ b_2 & b_3 \dots\dots\dots b_1 \end{pmatrix}$, where $k + p \le n$

If $k + p = n$ then $\alpha = \alpha_1\alpha_2$ Hence the proof.

If $k + p < n$ then we continue this process of extracting each time a cycle. As degree of α is finite therefore this process must end at some cycle say α_q where

$$\alpha_q = \begin{pmatrix} c_1 & c_2 \dots\dots\dots c_q \\ c_2 & c_3 \dots\dots\dots c_1 \end{pmatrix}$$

Hence $\alpha = \alpha_1\alpha_2 \dots\dots\dots \alpha_q$ where $k + p + \cdots + q = n$

Where $\{a_1, a_2, \dots\dots, a_k\} \cap \{b_1, b_2, \dots\dots, b_p\} \cap \dots\dots \cap \{c_1, c_2, \dots\dots, c_q\} = \phi$

Otherwise α will not be a bijective function.

In fact

$$\alpha = \begin{pmatrix} a_1 & a_2 \dots\dots\dots a_k & b_1 & b_2 \dots\dots\dots b_p \dots\dots\dots & c_1 & c_2 \dots\dots\dots c_q \\ a_2 & a_3 \dots\dots\dots a_1 & b_2 & b_3 \dots\dots\dots b_1 \dots\dots\dots & c_2 & c_3 \dots\dots\dots c_1 \end{pmatrix}$$

TRANSPOSITIONS

Transposition

A cycle of length 2 is called a transposition. Thus the permutation of the type $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ is called a transposition.

Theorem IMPORTANT

Prove that every cyclic permutation can be expressed as a product of transpositions.

Proof:

Let α be a cyclic permutation of length n. i.e. $\alpha = (a_1\, a_2 \dots\dots\dots a_n)$

We shall prove that

$$(a_1\, a_2 \dots\dots\dots a_n) = (a_1\, a_2)(a_1\, a_3) \dots\dots\dots\dots (a_1\, a_n) \qquad \dots\dots\dots \quad (1)$$

We shall prove (1) by principle of mathematical induction.

For $n = 2$

L.H.S $= (a_1\, a_2)$ R.H.S $= (a_1\, a_2)$

Hence (1) is true for $n = 2$

For $n = 3$

L.H.S of (1) $= (a_1\, a_2\, a_3)$

R.H.S of (1) $= (a_1\, a_2)(a_1\, a_3)$

$$= \begin{pmatrix} a_1 & a_2 \\ a_2 & a_1 \end{pmatrix}\begin{pmatrix} a_1 & a_3 \\ a_3 & a_1 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 & a_2 & a_3 \\ a_2 & a_1 & a_3 \end{pmatrix}\begin{pmatrix} a_1 & a_2 & a_3 \\ a_3 & a_2 & a_1 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 & a_2 & a_3 \\ a_2 & a_1 & a_3 \end{pmatrix}\begin{pmatrix} a_2 & a_1 & a_3 \\ a_2 & a_3 & a_1 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_1 \end{pmatrix}$$

$$= (a_1\, a_2\, a_3)$$

$=$ L.H.S of (1)

For $n = k$

Suppose (1) is true for k

i.e. $(a_1\, a_2 \dots\dots\dots a_k) = (a_1\, a_2)(a_1\, a_3) \dots\dots\dots\dots (a_1\, a_k) \qquad \dots\dots\dots\dots \quad (2)$

For $n = k + 1$

We shall prove that (1) is true for $n = k + 1$

i.e. $(a_1\, a_2 \dots\dots\dots a_{k+1}) = (a_1\, a_2)(a_1\, a_3) \dots\dots\dots\dots (a_1\, a_{k+1})$

$$R.H.S = (a_1\, a_2)(a_1\, a_3) \ldots\ldots\ldots (a_1\, a_k)(a_1\, a_{k+1})$$

$$= (a_1\, a_2 \ldots\ldots a_k)(a_1\, a_{k+1}) \qquad \text{By (2)}$$

$$= \begin{pmatrix} a_1 & a_2 \ldots\ldots & a_k \\ a_2 & a_3 \ldots\ldots & a_1 \end{pmatrix} \begin{pmatrix} a_1 & a_{k+1} \\ a_{k+1} & a_1 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 & a_2 \ldots\ldots & a_k & a_{k+1} \\ a_2 & a_3 \ldots\ldots & a_1 & a_{k+1} \end{pmatrix} \begin{pmatrix} a_1 & a_2 \ldots\ldots & a_k & a_{k+1} \\ a_{k+1} & a_2 \ldots\ldots & a_k & a_1 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 & a_2 \ldots\ldots & a_k & a_{k+1} \\ a_2 & a_3 \ldots\ldots & a_1 & a_{k+1} \end{pmatrix} \begin{pmatrix} a_2 & a_3 \ldots\ldots & a_1 & a_{k+1} \\ a_2 & a_3 \ldots\ldots & a_{k+1} & a_1 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 & a_2 \ldots\ldots & a_k & a_{k+1} \\ a_2 & a_3 \ldots\ldots & a_{k+1} & a_1 \end{pmatrix}$$

~~$= (a_1\, a_2)(a_1\, a_3) \ldots\ldots\ldots (a_1\, a_{k+1})$~~

$(a_1\, a_2 \, \ldots \, a_{k+1})$

$= L.H.S$

Theorem

Every permutation of degree n can be expressed as a product of transpositions.

Proof:

Let α be a permutation of degree n. We know that α → cyclic → transp

$$\alpha = \text{Product of disjoint cycles} \ldots\ldots\ldots \quad (1)$$

(∵ *Every permutation can be expressed as a product of disjoint cycles.*)

Also, A cycle = Product of transpositions ……… (2)

(∵ *Every cyclic permutation can be expressed as a product of transpositions.*)

From (1) and (2), It is clear that

$$\alpha = \text{Product of transposition}$$

Even Permutation

A permutation α in S_n is said to be an even permutation if it can be written as a product of an even number of transpositions.

For example

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix} = (1\ 2)(3\ 4)$$

Clearly the number of transpositions in the decomposition of α is 2, which is an even number so, α is an even permutation.

Odd Permutation

A permutation β in S_n is said to be an odd permutation if it can be written as a product of an odd number of transpositions.

For example

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = (1\ 2\ 3\ 4) = (1\ 2)(1\ 3)(1\ 4)$$

Clearly the number of transpositions in the decomposition of α is 3, which is an odd number so, β is an odd permutation.

Theorem

(i) **The product of two even or two odd permutations is an even permutation.**

(ii) **The product of an even permutation and an odd permutation is an odd permutation.**

Proof:

(i) *(a)* Let α and β are two even permutations. Then α and β can be written as a product of even number of transpositions. Let α can be written as a product of $2m$ transpositions and β can be written as a product of $2n$ transpositions.

Then the product $\alpha\beta$ can be written as a product of $2m + 2n = 2(m + n)$ transpositions i.e. even number of transpositions [$\because 2(m + n)$ is an even number].

Thus the product of two even permutations is an even permutation.

(b) Let f and g are two odd permutations. Then f and g can be written as a product of odd number of transpositions. Let f can be written as a product of $2m + 1$ transpositions and g can be written as a product of $2n + 1$ transpositions.

Then the product fg can be written as a product of $(2m + 1) + (2n + 1) = 2(m + n + 1)$ transpositions i.e. even number of transpositions [$\because 2(m + n + 1)$ is an even number].

Thus the product of two odd permutations is an even permutation.

(ii) Let α be an even permutation and β be an odd permutation. Then α and β can be written as a product of an even and an odd number of transpositions respectively. Let α can be written as a product of $2m$ transpositions and β can be written as a product of $2n + 1$ transpositions.

Then the product $\alpha\beta$ can be written as a product of $2m + (2n + 1) = 2(m + n) + 1$ transpositions i.e. odd number of transpositions [$\because 2(m + n) + 1$ is an odd number].

Thus the product of an even and an odd permutation is an odd -permutation.

Theorem

Let α be any permutation of degree n and τ be a transposition then $\alpha\tau$ or $\tau\alpha$ is an even or an odd permutation according as α is odd or even respectively.

Proof:

Suppose α is an even permutation.

Then $\alpha =$ Product of m transpositions. (Where m is even.)

$\Rightarrow \tau\alpha =$ Product of $(m + 1)$ transpositions. (Where $(m + 1)$ is odd.)

$\Rightarrow \tau\alpha$ is an odd permutation.

Suppose α is an odd permutation.

Then $\alpha =$ Product of n transpositions. (Where n is odd.)

$\Rightarrow \tau\alpha =$ Product of $(n+1)$ transpositions. (Where $(n+1)$ is even.)

$\Rightarrow \tau\alpha$ is an even permutation.

Theorem

For $n \geq 2$, the number of even permutations in S_n is equal to the number of odd permutations in S_n.

Proof:

We know that S_n is called symmetric group of degree n.

Also the order of $S_n = n!$

Let $p_1, p_2, \dots\dots\dots, p_r$ be even permutations,

And $q_1, q_2, \dots\dots\dots, q_s$ be odd permutations.

Where $r + s = n!$

Let τ be a transposition.

Then $\tau p_1, \tau p_2, \dots\dots\dots, \tau p_r$ are all distinct odd permutations.

[$\because$ If $\tau p_i = \tau p_j$ for $i \neq j$ then $p_i = p_j$ (cancellation law) $\Rightarrow i = j$ a contradiction]

Thus we have " r " odd permutations in S_n.

But number of odd permutations in S_n is " s "

$\therefore \quad s \geq r \quad \dots\dots\dots (1)$

Now $\tau q_1, \tau q_2, \dots\dots\dots, \tau q_r$ are all distinct even permutations.

Thus we have " s " even permutations in S_n.

But number of even permutations in S_n is " r "

$\therefore \quad r \geq s \quad \dots\dots\dots (2)$

From (1) and (2) it is clear that $r = s$

Since $r + s = n! \Rightarrow r = s = \frac{n!}{2}$

Theorem

Prove that the set A_n of all even permutations in S_n form a subgroup of S_n. Explain why the set B_n of all odd permutations in S_n do not form a subgroup of S_n?

Proof:

Let A_n be the set of all even permutations in S_n.

We shall prove that A_n is a subgroup of S_n.

Obviously $A_n \subseteq S_n$

Let $\alpha, \beta \in A_n$. We have to prove that $\alpha\beta^{-1} \in A_n$.

Now α and β are even permutations.

Also β^{-1} is an even permutation.

($\because$ Inverse of an even permutation is an even permutation)

Hence $\alpha\beta^{-1}$ is an even permutation.

($\because$ Product of two even permutations is an even permutation)

Thus $\alpha\beta^{-1} \in A_n$

Since $\alpha, \beta \in A_n \Rightarrow \alpha\beta^{-1} \in A_n, \ \forall \ \alpha, \beta \in A_n$

Thus A_n is a subgroup of S_n.

Even x Even = Even
odd x odd = Even.

Let B_n be the set of all odd permutations in S_n.
Obviously $B_n \subseteq S_n$
Let $f,g \in B_n$.
Now f and g are odd permutations.
Also g^{-1} is an odd permutation.
($\because$ Inverse of an odd permutation is an odd permutation)
But fg^{-1} is not an odd permutation.
($\because$ Product of two odd permutations is an even permutation)
Thus $fg^{-1} \notin B_n$
Since $f,g \in B_n \nRightarrow fg^{-1} \in B_n, \ \forall f,g \in B_n$
Thus B_n is not a subgroup of S_n.

Example

Find all the subgroup of S_3

Solution

Here S_3 is set of all permutations defined on a set X with three elements also called symmetric group of degree 3.
Let $X = \{1,2,3\}$. Then all the permutations on X are

$$I = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad \alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad \alpha^2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix},$$

$$\beta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \quad \alpha\beta = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \quad \alpha^2\beta = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

Thus $S_3 = \{ I, \alpha, \alpha^2, \beta, \alpha\beta, \alpha^2\beta \}$
Where $\alpha^3 = \beta^2 = (\alpha\beta)^2 = (\alpha^2\beta)^2 = I$
Trivial subgroups of S_3 are I, S_3
Non- trivial subgroups of S_3 are
Let $H_1 = \langle \alpha : \alpha^3 = I \rangle = \{ \alpha, \alpha^2, \alpha^3 = I \} = \{ \alpha, \alpha^2, I \}$
$H_2 = \langle \beta : \beta^2 = I \rangle = \{ \beta, \beta^2 = I \} = \{ \beta, I \}$
$H_3 = \langle \alpha\beta : (\alpha\beta)^2 = I \rangle = \{ \alpha\beta, I \}$
$H_4 = \langle \alpha^2\beta : (\alpha^2\beta)^2 = I \rangle = \{ \alpha^2\beta, I \}$

Order of a permutation

Let α be a permutation. Then the least positive integer m is called order of α if $\alpha^m = I$, where I is an identity permutation.

$$\text{Let } \alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$\text{Then } \alpha^2 = \alpha.\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

Also $\alpha^3 = \alpha\,.\,\alpha^2 = \begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix}$

$= \begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix}\begin{pmatrix}2 & 3 & 1\\ 1 & 2 & 3\end{pmatrix}$

$= \begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 3\end{pmatrix} = I$

Since $\alpha^3 = I \quad \Rightarrow \quad$ Order of $\alpha = 3$

Note

The order of a transposition is 2.

Let $\tau = \begin{pmatrix}a & b\\ b & a\end{pmatrix}$ be a transposition.

Then $\tau^2 = \tau.\tau = \begin{pmatrix}a & b\\ b & a\end{pmatrix}\begin{pmatrix}a & b\\ b & a\end{pmatrix}$

$= \begin{pmatrix}a & b\\ b & a\end{pmatrix}\begin{pmatrix}b & a\\ a & b\end{pmatrix}$

$= \begin{pmatrix}a & b\\ a & b\end{pmatrix}$

$= I$

Since $\tau^2 = I \quad \Rightarrow \quad$ Order of $\tau = 2$

Theorem

The order of cyclic permutation of length m is m.

Proof:

Let α be a cyclic permutation of length m.

i.e. $\alpha = \begin{pmatrix}a_1 & a_2 & a_3 \ldots\ldots\ldots & a_m\\ a_2 & a_3 & a_4 \ldots\ldots\ldots & a_1\end{pmatrix}$

Then $\alpha^2 = \alpha.\alpha = \begin{pmatrix}a_1 & a_2 & a_3 \ldots\ldots\ldots & a_m\\ a_2 & a_3 & a_4 \ldots\ldots\ldots & a_1\end{pmatrix}\begin{pmatrix}a_1 & a_2 & a_3 \ldots\ldots\ldots & a_m\\ a_2 & a_3 & a_4 \ldots\ldots\ldots & a_1\end{pmatrix}$

$= \begin{pmatrix}a_1 & a_2 & a_3 \ldots\ldots\ldots & a_m\\ a_2 & a_3 & a_4 \ldots\ldots\ldots & a_1\end{pmatrix}\begin{pmatrix}a_2 & a_3 & a_4 \ldots\ldots\ldots & a_1\\ a_3 & a_4 & a_5 \ldots\ldots\ldots & a_2\end{pmatrix}$

$= \begin{pmatrix}a_1 & a_2 & a_3 \ldots\ldots\ldots & a_m\\ a_3 & a_4 & a_5 \ldots\ldots\ldots & a_2\end{pmatrix}$

Now $\alpha^3 = \alpha^2.\alpha = \begin{pmatrix}a_1 & a_2 & a_3 \ldots\ldots\ldots & a_m\\ a_3 & a_4 & a_5 \ldots\ldots\ldots & a_2\end{pmatrix}\begin{pmatrix}a_1 & a_2 & a_3 \ldots\ldots\ldots & a_m\\ a_2 & a_3 & a_4 \ldots\ldots\ldots & a_1\end{pmatrix}$

$= \begin{pmatrix}a_1 & a_2 & a_3 \ldots\ldots\ldots & a_m\\ a_3 & a_4 & a_5 \ldots\ldots\ldots & a_2\end{pmatrix}\begin{pmatrix}a_3 & a_4 & a_5 \ldots\ldots\ldots & a_2\\ a_4 & a_5 & a_6 \ldots\ldots\ldots & a_3\end{pmatrix}$

$= \begin{pmatrix}a_1 & a_2 & a_3 \ldots\ldots\ldots & a_m\\ a_4 & a_5 & a_6 \ldots\ldots\ldots & a_3\end{pmatrix}$

Similarly $\alpha^4 = \begin{pmatrix}a_1 & a_2 & a_3 \ldots\ldots\ldots & a_m\\ a_5 & a_6 & a_7 \ldots\ldots\ldots & a_4\end{pmatrix}$

....

....

Also $\alpha^m = \begin{pmatrix}a_1 & a_2 & a_3 \ldots\ldots\ldots & a_m\\ a_1 & a_2 & a_3 \ldots\ldots\ldots & a_m\end{pmatrix}$

Thus Order of $\alpha = m =$ Length of α

Not in the book

Theorem

The order of a permutation is the least common multiple (L.C.M) of the orders of the disjoint cycles into whose product it is decomposed.

Proof:

Let α be any permutation of degree n.

We know that α can be written as a product of disjoint cycles say

$$\alpha_1, \alpha_2, \ldots\ldots, \alpha_k$$

Then $\quad \alpha = \alpha_1\alpha_2 \ldots\ldots \alpha_k \qquad \ldots\ldots (1)$

Suppose $\quad m_i = \text{Length of } \alpha_i \quad \text{where} \quad i = 1,2,\ldots\ldots,k$

$\Rightarrow \quad m_i = \text{Length of } \alpha_i \quad \text{where} \quad i = 1,2,\ldots\ldots,k$

($\because$ Length of a cycle = Order of a cycle)

Let $\quad L.C.M$ of $m_i = m, \quad i = 1,2,\ldots\ldots,k$

Then $\quad m = m_i q_i \quad$ where $\; i = 1,2,\ldots\ldots,k,$ and $q \in Z^{+}$

Now from equation (1)

$$\begin{aligned}
\alpha^m &= \alpha_1^{\,m}\alpha_2^{\,m} \ldots\ldots \alpha_k^{\,m} \\
&= \alpha_1^{\,m_1q_1}\alpha_2^{\,m_2q_2} \ldots\ldots \alpha_k^{\,m_kq_k} \\
&= (\alpha_1^{\,m_1})^{q_1}(\alpha_2^{\,m_2})^{q_2} \ldots\ldots (\alpha_k^{\,m_k})^{q_k} \\
&= (I)^{q_1}(I)^{q_2} \ldots\ldots (I)^{q_k} \qquad \because\ m_i = O(\alpha_i) \\
&= I.I \ldots\ldots I \\
&= I
\end{aligned}$$

$12 = L.C.M\ of\ (2,3,4)$
$\therefore\ 12 = 2q,\ q = 6$
$12 = 2q,\ q = 6$
$12 = 4q,\ q = 3$

Hence order of $\alpha = m$

$$\begin{aligned}
&= L.C.M \text{ of } \; m_1, m_2, \ldots\ldots, m_k \\
&= L.C.M \text{ of } O(\alpha_1), O(\alpha_2), \ldots\ldots O(\alpha_k)
\end{aligned}$$

Example

Find the order of $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 2 & 3 & 4 & 1 & 7 & 9 & 6 & 5 & 8 & 12 & 11 & 10 \end{pmatrix}$

Solution

Here $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 2 & 3 & 4 & 1 & 7 & 9 & 6 & 5 & 8 & 12 & 11 & 10 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}\begin{pmatrix} 5 & 7 & 6 & 9 & 8 \\ 7 & 6 & 9 & 8 & 5 \end{pmatrix}\begin{pmatrix} 11 \\ 11 \end{pmatrix}\begin{pmatrix} 10 & 12 \\ 12 & 10 \end{pmatrix}$

$$= (1\ 2\ 3\ 4)(5\ 7\ 6\ 9\ 8)(11)(10\ 12)$$

Let $\alpha_1 = (1\ 2\ 3\ 4),\ \alpha_2 = (5\ 7\ 6\ 9\ 8),\ \alpha_3 = (11),\ \alpha_4 = (10\ 12)$

Then lengths of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are $4, 5, 1, 2$

$\therefore$ Orders of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are $4, 5, 1, 2$

Thus order of $\alpha = L.C.M$ of $[\, O(\alpha_1), O(\alpha_2), O(\alpha_3), O(\alpha_4) \,]$

$$\begin{aligned}
&= L.C.M \text{ of } [\, 4, 5, 1, 2 \,] \\
&= 20
\end{aligned}$$

EXERCISE 2.3

Q. No.1

Multiply the following permutations.

(i) $\begin{pmatrix}1&2&3&4&5\\2&3&5&4&1\end{pmatrix}$ and $\begin{pmatrix}1&2&3&4&5\\3&1&4&2&5\end{pmatrix}$

(ii) $\begin{pmatrix}1&2&3&4&5&6\\2&3&5&6&4&1\end{pmatrix}$ and $\begin{pmatrix}1&2&3&4&5&6\\6&3&2&5&1&4\end{pmatrix}$

(iii) $\begin{pmatrix}2&3&6&7&8&9\\8&6&7&9&2&3\end{pmatrix}$ and $\begin{pmatrix}2&3&6&7&8&9\\8&6&7&3&9&2\end{pmatrix}$

Solution

(i) Here $\begin{pmatrix}1&2&3&4&5\\2&3&5&4&1\end{pmatrix}\begin{pmatrix}1&2&3&4&5\\3&1&4&2&5\end{pmatrix}=\begin{pmatrix}1&2&3&4&5\\2&3&5&4&1\end{pmatrix}\begin{pmatrix}2&3&5&4&1\\1&4&5&2&3\end{pmatrix}$

$$=\begin{pmatrix}1&2&3&4&5\\1&4&5&2&3\end{pmatrix}$$

(ii) $\begin{pmatrix}1&2&3&4&5&6\\2&3&5&6&4&1\end{pmatrix}\begin{pmatrix}1&2&3&4&5&6\\6&3&2&5&1&4\end{pmatrix}=\begin{pmatrix}1&2&3&4&5&6\\2&3&5&6&4&1\end{pmatrix}\begin{pmatrix}2&3&5&6&4&1\\3&2&1&4&5&6\end{pmatrix}$

$$=\begin{pmatrix}1&2&3&4&5&6\\3&2&1&4&5&6\end{pmatrix}$$

(iii) $\begin{pmatrix}2&3&6&7&8&9\\8&6&7&9&2&3\end{pmatrix}\begin{pmatrix}2&3&6&7&8&9\\8&6&7&3&9&2\end{pmatrix}=\begin{pmatrix}2&3&6&7&8&9\\8&6&7&9&2&3\end{pmatrix}\begin{pmatrix}8&6&7&9&2&3\\9&7&3&2&8&6\end{pmatrix}$

$$=\begin{pmatrix}2&3&6&7&8&9\\9&7&3&2&8&6\end{pmatrix}$$

Q. No.2

Find the inverse of the permutation $\begin{pmatrix}1&2&3&4\\3&1&4&2\end{pmatrix}$

Solution

Let $\alpha=\begin{pmatrix}1&2&3&4\\3&1&4&2\end{pmatrix}$

Then $\alpha^{-1}=\begin{pmatrix}3&1&4&2\\1&2&3&4\end{pmatrix}$

Q. No.3

Let $\alpha=\begin{pmatrix}1&2&3&4&5&6&7\\3&5&4&1&7&2&6\end{pmatrix}$, find the inverse of α.

Solution

Here $\alpha=\begin{pmatrix}1&2&3&4&5&6&7\\3&5&4&1&7&2&6\end{pmatrix}$

Then $\alpha^{-1}=\begin{pmatrix}3&5&4&1&7&2&6\\1&2&3&4&5&6&7\end{pmatrix}$

Q. No.4

Let $f=\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}$, $g=\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix}$

Show that $f\circ g=\begin{pmatrix}1&2&3&4\\1&4&3&2\end{pmatrix}$, $f^2\circ g=\begin{pmatrix}1&2&3&4\\4&3&2&1\end{pmatrix}$

$f^3\circ g=\begin{pmatrix}1&2&3&4\\3&2&1&4\end{pmatrix}$, Also $f^4=g^2=(f\circ g)^2$

Solution

Here $f\circ g=\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix}=\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}\begin{pmatrix}2&3&4&1\\1&4&3&2\end{pmatrix}=\begin{pmatrix}1&2&3&4\\1&4&3&2\end{pmatrix}$

$f^2=f\circ f=\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}=\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}\begin{pmatrix}2&3&4&1\\3&4&1&2\end{pmatrix}=\begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}$

$f^3=f^2\circ f=\begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}=\begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}\begin{pmatrix}3&4&1&2\\4&1&2&3\end{pmatrix}=\begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix}$

$f^2\circ g=\begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix}=\begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}\begin{pmatrix}3&4&1&2\\4&3&2&1\end{pmatrix}=\begin{pmatrix}1&2&3&4\\4&3&2&1\end{pmatrix}$

$f^3\circ g=\begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix}=\begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix}\begin{pmatrix}4&1&2&3\\3&2&1&4\end{pmatrix}=\begin{pmatrix}1&2&3&4\\3&2&1&4\end{pmatrix}$

$(f\circ g)^2=(f\circ g)(f\circ g)=\begin{pmatrix}1&2&3&4\\1&4&3&2\end{pmatrix}\begin{pmatrix}1&2&3&4\\1&4&3&2\end{pmatrix}=\begin{pmatrix}1&2&3&4\\1&4&3&2\end{pmatrix}\begin{pmatrix}1&4&3&2\\1&2&3&4\end{pmatrix}$

$=\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}=I$ ____(1)

$f^4=f^3\circ f=\begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}=\begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix}\begin{pmatrix}4&1&2&3\\1&2&3&4\end{pmatrix}=\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}$

$=I$ _____(2)

$g^2=g\circ g=\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix}=\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix}\begin{pmatrix}2&1&4&3\\1&2&3&4\end{pmatrix}=I$ ______(3)

From (1), (2) and (3) we get

$f^4=g^2=(f\circ g)^2$

Q. No.5

Given the permutations

$f=\begin{pmatrix}1&2&3&4&5&6\\2&4&6&5&3&1\end{pmatrix},g=\begin{pmatrix}1&2&3&4&5&6\\6&5&4&1&2&3\end{pmatrix},h=\begin{pmatrix}1&2&3&4&5&6\\5&4&1&2&6&3\end{pmatrix},$

Find $f\circ g, g\circ f, g\circ h, h\circ g, h^2, g^2$. Also find the orders of f, g and h

Solution

Here $f\circ g=\begin{pmatrix}1&2&3&4&5&6\\2&4&6&5&3&1\end{pmatrix}\begin{pmatrix}1&2&3&4&5&6\\6&5&4&1&2&3\end{pmatrix}$

$=\begin{pmatrix}1&2&3&4&5&6\\2&4&6&5&3&1\end{pmatrix}\begin{pmatrix}2&4&6&5&3&1\\5&1&3&2&4&6\end{pmatrix}$

$=\begin{pmatrix}1&2&3&4&5&6\\5&1&3&2&4&6\end{pmatrix}$

And $g\circ f=\begin{pmatrix}1&2&3&4&5&6\\6&5&4&1&2&3\end{pmatrix}\begin{pmatrix}1&2&3&4&5&6\\2&4&6&5&3&1\end{pmatrix}$

$=\begin{pmatrix}1&2&3&4&5&6\\6&5&4&1&2&3\end{pmatrix}\begin{pmatrix}6&5&4&1&2&3\\1&3&5&2&4&6\end{pmatrix}$

$=\begin{pmatrix}1&2&3&4&5&6\\1&3&5&2&4&6\end{pmatrix}$

Also $g\circ h=\begin{pmatrix}1&2&3&4&5&6\\6&5&4&1&2&3\end{pmatrix}\begin{pmatrix}1&2&3&4&5&6\\5&4&1&2&6&3\end{pmatrix}$

$=\begin{pmatrix}1&2&3&4&5&6\\6&5&4&1&2&3\end{pmatrix}\begin{pmatrix}6&5&4&1&2&3\\3&6&2&5&4&1\end{pmatrix}$

$=\begin{pmatrix}1&2&3&4&5&6\\3&6&2&5&4&1\end{pmatrix}$

Now $h \circ g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 1 & 2 & 6 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 1 & 2 & 3 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 1 & 2 & 6 & 3 \end{pmatrix} \begin{pmatrix} 5 & 4 & 1 & 2 & 6 & 3 \\ 2 & 1 & 6 & 5 & 3 & 4 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 6 & 5 & 3 & 4 \end{pmatrix}$

And $h^2 = h \circ h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 1 & 2 & 6 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 1 & 2 & 6 & 3 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 1 & 2 & 6 & 3 \end{pmatrix} \begin{pmatrix} 5 & 4 & 1 & 2 & 6 & 3 \\ 6 & 2 & 5 & 4 & 3 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 2 & 5 & 4 & 3 & 1 \end{pmatrix}$

Also $g^2 = g \circ g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 1 & 2 & 3 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 6 & 5 & 4 & 1 & 2 & 3 \\ 3 & 2 & 1 & 6 & 5 & 4 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 1 & 6 & 5 & 4 \end{pmatrix}$.

Now we shall find the orders of f, g & h

Here $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 5 & 3 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 4 & 5 & 3 & 6 \\ 2 & 4 & 5 & 3 & 6 & 1 \end{pmatrix}$, This is a cycle of length 6.

$\therefore$ Order of $f = 6$ ($\because$ Length of a cycle = Order of cycle)

And $g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 1 & 2 & 3 \end{pmatrix}$

$= \begin{pmatrix} 1 & 6 & 3 & 4 \\ 6 & 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 5 & 2 \end{pmatrix} = (1\,6\,3\,4)(2\,5)$

Let $\alpha_1 = (1\,6\,3\,4)$, and $\alpha_2 = (2\,5)$

Then lengths of α_1, α_2 are $4, 2$

$\therefore$ Orders of α_1, α_2 are $4, 2$

Thus order of $g = L.C.M$ of $[\,O(\alpha_1), O(\alpha_2)\,]$

$= L.C.M$ of $[\,4, 2\,]$

$= 4$

Also $h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 1 & 2 & 6 & 3 \end{pmatrix}$

$= \begin{pmatrix} 1 & 5 & 6 & 3 \\ 5 & 6 & 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix} = (1\,5\,6\,3)(2\,4)$

Let $\beta_1 = (1\,5\,6\,3)$, and $\beta_2 = (2\,4)$

Then lengths of β_1, β_2 are $4, 2$

$\therefore$ Orders of β_1, β_2 are $4, 2$

Thus order of $h = L.C.M$ of $[\,O(\beta_1), O(\beta_2)\,]$

$= L.C.M$ of $[\,4, 2\,]$

$= 4$

$= L.C.M \text{ of } [4,2]$
$= 4$

Q. No.6

Given the permutation $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 5 & 3 \end{pmatrix}$, verify that $\alpha^6 = I$

Solution

Here $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 5 & 3 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 3 & 4 & 5 \\ 4 & 5 & 3 \end{pmatrix} = (1\,2)(3\,4\,5)$

Let $\alpha_1 = (1\,2)$, and $\alpha_2 = (3\,4\,5)$

Then lengths of α_1, α_2 are $2, 3$

$\therefore$ Orders of α_1, α_2 are $2, 3$

Thus order of $\alpha = L.C.M \text{ of } [O(\alpha_1), O(\alpha_2)]$
$= L.C.M \text{ of } [2,3]$
$= 6$

$\Rightarrow \alpha^6 = I$ (where I is an identity permutation.)

Q. No.7

Write the multiplication table for the permutations $(1), (1\,2\,3\,4), (1\,4\,3\,2), (1\,3)(2\,4)$.

Solution

Let $\alpha_\circ = (1) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

$\alpha_1 = (1\,2\,3\,4) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$

$\alpha_2 = (1\,4\,3\,2) = \begin{pmatrix} 1 & 4 & 3 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$

$\alpha_3 = (1\,3)(2\,4) = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$

We see that

$\circ$	$\alpha_\circ$	α_1	α_2	α_3
$\alpha_\circ$	$\alpha_\circ$	α_1	α_2	α_3
α_1	α_1	α_3	$\alpha_\circ$	α_2
α_2	α_2	$\alpha_\circ$	α_3	α_1
α_3	α_3	α_2	α_1	$\alpha_\circ$

$\alpha_\circ\alpha_\circ = \alpha_\circ$

$\alpha_\circ\alpha_1 = \alpha_1$

$\alpha_\circ\alpha_2 = \alpha_2$

$\alpha_\circ\alpha_3 = \alpha_3$

$\alpha_1\alpha_\circ = \alpha_1$

$\alpha_1\alpha_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = \alpha_3$

$\alpha_1\alpha_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 & 4 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = \alpha_\circ$

$\alpha_1\alpha_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 & 4 & 1 \\ 4 & 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} = \alpha_2$

$\alpha_2\alpha_\circ = \alpha_2$

$\alpha_2\alpha_1 = \begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix} = \begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix}\begin{pmatrix}4&1&2&3\\1&2&3&4\end{pmatrix} = \begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix} = \alpha_\circ$

$\alpha_2\alpha_2 = \begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix}\begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix} = \begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix}\begin{pmatrix}4&1&2&3\\3&4&1&2\end{pmatrix} = \begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix} = \alpha_3$

$\alpha_2\alpha_3 = \begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix}\begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix} = \begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix}\begin{pmatrix}4&1&2&3\\2&3&4&1\end{pmatrix} = \begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix} = \alpha_1$

$\alpha_3\alpha_\circ = \alpha_3$

$\alpha_3\alpha_1 = \begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix} = \begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}\begin{pmatrix}3&4&1&2\\4&1&2&3\end{pmatrix} = \begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix} = \alpha_2$

$\alpha_3\alpha_2 = \begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}\begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix} = \begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}\begin{pmatrix}3&4&1&2\\2&3&4&1\end{pmatrix} = \begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix} = \alpha_1$

$\alpha_3\alpha_3 = \begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}\begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix} = \begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}\begin{pmatrix}3&4&1&2\\1&2&3&4\end{pmatrix} = \begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix} = \alpha_\circ$

Q. No.8

Let $\alpha = \begin{pmatrix}1&2&3&4&5&6\\3&4&5&2&6&1\end{pmatrix}$, find all the elements of the cyclic group generated by α.

Solution

Here $\alpha = \begin{pmatrix}1&2&3&4&5&6\\3&4&5&2&6&1\end{pmatrix}$

Now $\alpha^2 = \alpha\alpha = \begin{pmatrix}1&2&3&4&5&6\\3&4&5&2&6&1\end{pmatrix}\begin{pmatrix}1&2&3&4&5&6\\3&4&5&2&6&1\end{pmatrix}$

$= \begin{pmatrix}1&2&3&4&5&6\\3&4&5&2&6&1\end{pmatrix}\begin{pmatrix}3&4&5&2&6&1\\5&2&6&4&1&3\end{pmatrix}$

$= \begin{pmatrix}1&2&3&4&5&6\\5&2&6&4&1&3\end{pmatrix}$

& $\alpha^3 = \alpha^2\alpha = \begin{pmatrix}1&2&3&4&5&6\\5&2&6&4&1&3\end{pmatrix}\begin{pmatrix}1&2&3&4&5&6\\3&4&5&2&6&1\end{pmatrix}$

$= \begin{pmatrix}1&2&3&4&5&6\\5&2&6&4&1&3\end{pmatrix}\begin{pmatrix}5&2&6&4&1&3\\6&4&1&2&3&5\end{pmatrix}$

$= \begin{pmatrix}1&2&3&4&5&6\\6&4&1&2&3&5\end{pmatrix}$

Also $\alpha^4 = \alpha^3\alpha = \begin{pmatrix}1&2&3&4&5&6\\6&4&1&2&3&5\end{pmatrix}\begin{pmatrix}1&2&3&4&5&6\\3&4&5&2&6&1\end{pmatrix}$

$= \begin{pmatrix}1&2&3&4&5&6\\6&4&1&2&3&5\end{pmatrix}\begin{pmatrix}6&4&1&2&3&5\\1&2&3&4&5&6\end{pmatrix}$

$= \begin{pmatrix}1&2&3&4&5&6\\1&2&3&4&5&6\end{pmatrix}$

Clearly $\alpha^4 = I$

Let G be a cyclic group generated by α.

Then $G = \langle\, \alpha : \alpha^4 = I \,\rangle$

$= \{ \alpha, \alpha^2, \alpha^3, I \}$

Q. No.9

Find the following products and express them as a product of cyclic permutations on mutually disjoint sets

(124)(13654), (1234)(52341), (14)(235)(35)(45)

Solution

Here $(1\,2\,4)(1\,3\,6\,5\,4) = \begin{pmatrix}1&2&4\\2&4&1\end{pmatrix}\begin{pmatrix}1&3&6&5&4\\3&6&5&4&1\end{pmatrix}$

$= \begin{pmatrix}1&2&3&4&5&6\\2&4&3&1&5&6\end{pmatrix}\begin{pmatrix}1&2&3&4&5&6\\3&2&6&1&4&5\end{pmatrix}$

$= \begin{pmatrix}1&2&3&4&5&6\\2&4&3&1&5&6\end{pmatrix}\begin{pmatrix}2&4&3&1&5&6\\2&1&6&3&4&5\end{pmatrix}$

$= \begin{pmatrix}1&2&3&4&5&6\\2&1&6&3&4&5\end{pmatrix}$

$= \begin{pmatrix}1&2\\2&1\end{pmatrix}\begin{pmatrix}3&6&5&4\\6&5&4&3\end{pmatrix}$

$= (1\,2)(3\,6\,5\,4)$

And $(1\,2\,3\,4)(2\,5\,3\,4\,1) = \begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}\begin{pmatrix}2&5&3&4&1\\5&3&4&1&2\end{pmatrix}$

$= \begin{pmatrix}1&2&3&4&5\\2&3&4&1&5\end{pmatrix}\begin{pmatrix}2&5&3&4&1\\5&3&4&1&2\end{pmatrix}$

$= \begin{pmatrix}1&2&3&4&5\\2&3&4&1&5\end{pmatrix}\begin{pmatrix}2&3&4&1&5\\5&4&1&2&3\end{pmatrix}$

$= \begin{pmatrix}1&2&3&4&5\\5&4&1&2&3\end{pmatrix}$

$= \begin{pmatrix}1&5&3&2&4\\5&3&1&4&2\end{pmatrix}$

$= \begin{pmatrix}1&5&3\\5&3&1\end{pmatrix}\begin{pmatrix}2&4\\4&2\end{pmatrix}$

$= (1\,5\,3)(2\,4)$

Now $(1\,4)(2\,3\,5) = \begin{pmatrix}1&4\\4&1\end{pmatrix}\begin{pmatrix}2&3&5\\3&5&2\end{pmatrix}$

$= \begin{pmatrix}1&2&3&4&5\\4&2&3&1&5\end{pmatrix}\begin{pmatrix}1&2&3&4&5\\1&3&5&4&2\end{pmatrix}$

$= \begin{pmatrix}1&2&3&4&5\\4&2&3&1&5\end{pmatrix}\begin{pmatrix}4&2&3&1&5\\4&3&5&1&2\end{pmatrix}$

$= \begin{pmatrix}1&2&3&4&5\\4&3&5&1&2\end{pmatrix}$

& $(1\,4)(2\,3\,5)(3\,5) = \begin{pmatrix}1&2&3&4&5\\4&3&5&1&2\end{pmatrix}\begin{pmatrix}3&5\\5&3\end{pmatrix}$

$= \begin{pmatrix}1&2&3&4&5\\4&3&5&1&2\end{pmatrix}\begin{pmatrix}1&2&3&4&5\\1&2&5&4&3\end{pmatrix}$

$= \begin{pmatrix}1&2&3&4&5\\4&3&5&1&2\end{pmatrix}\begin{pmatrix}4&3&5&1&2\\4&5&3&1&2\end{pmatrix}$

$= \begin{pmatrix}1&2&3&4&5\\4&5&3&1&2\end{pmatrix}$

So, $(1\,4)(2\,3\,5)(3\,5)(4\,5) = \begin{pmatrix}1&2&3&4&5\\4&5&3&1&2\end{pmatrix}\begin{pmatrix}4&5\\5&4\end{pmatrix}$

$= \begin{pmatrix}1&2&3&4&5\\4&5&3&1&2\end{pmatrix}\begin{pmatrix}1&2&3&4&5\\1&2&3&5&4\end{pmatrix}$

$= \begin{pmatrix}1&2&3&4&5\\4&5&3&1&2\end{pmatrix}\begin{pmatrix}4&5&3&1&2\\5&4&3&1&2\end{pmatrix}$

$$= \begin{pmatrix}1&2&3&4&5\\5&4&3&1&2\end{pmatrix} = \begin{pmatrix}1&5&2&4\\5&2&4&1\end{pmatrix}\begin{pmatrix}3\\3\end{pmatrix} = (1\,2\,3\,4)(3)$$

Q. No.10

Express the following permutations as a product of disjoint cycles

(i) $\begin{pmatrix}1&2&3&4&5&6&7&8\\8&2&6&3&7&4&5&1\end{pmatrix}$

(ii) $\begin{pmatrix}1&2&3&4&5&6&7&8\\3&6&4&1&8&2&5&7\end{pmatrix}$

Solution

Here $\begin{pmatrix}1&2&3&4&5&6&7&8\\8&2&6&3&7&4&5&1\end{pmatrix} = \begin{pmatrix}1&8&2&3&6&4&5&7\\8&1&2&6&4&3&7&5\end{pmatrix}$

$$= \begin{pmatrix}1&8\\8&1\end{pmatrix}\begin{pmatrix}2\\2\end{pmatrix}\begin{pmatrix}3&6&4\\6&4&3\end{pmatrix}\begin{pmatrix}5&7\\7&5\end{pmatrix}$$

$$= (1\,8)(2)(3\,6\,4)(5\,7)$$

And $\begin{pmatrix}1&2&3&4&5&6&7&8\\3&6&4&1&8&2&5&7\end{pmatrix} = \begin{pmatrix}1&3&4&2&6&5&8&7\\3&4&1&6&2&8&7&5\end{pmatrix}$

$$= \begin{pmatrix}1&3&4\\3&4&1\end{pmatrix}\begin{pmatrix}2&6\\6&2\end{pmatrix}\begin{pmatrix}5&8&7\\8&7&5\end{pmatrix}$$

$$= (1\,3\,4)(2\,6)(5\,8\,7)$$

Q. No.11

Express the following permutations as a product of transpositions

(i) $\begin{pmatrix}1&2&3&4&5&6&7&8&9\\2&4&1&6&5&3&8&9&7\end{pmatrix}$

(ii) $\begin{pmatrix}1&2&3&4&5&6&7&8\\2&3&4&1&5&6&7&8\end{pmatrix}$.

Solution

Here $\begin{pmatrix}1&2&3&4&5&6&7&8&9\\2&4&1&6&5&3&8&9&7\end{pmatrix} = \begin{pmatrix}1&2&4&6&3\\2&4&6&3&1\end{pmatrix}\begin{pmatrix}5\\5\end{pmatrix}\begin{pmatrix}7&8&9\\8&9&7\end{pmatrix}$

$$= (1\,2\,4\,6\,3)(5)(7\,8\,9)$$

$$= (1\,2)(1\,4)(1\,6)(1\,3)(7\,8)(7\,9)$$

And $\begin{pmatrix}1&2&3&4&5&6&7&8\\2&3&4&1&5&6&7&8\end{pmatrix} = \begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}\begin{pmatrix}5\\5\end{pmatrix}\begin{pmatrix}6\\6\end{pmatrix}\begin{pmatrix}7\\7\end{pmatrix}\begin{pmatrix}8\\8\end{pmatrix}$

$$= (1\,2\,3\,4)(5)(6)(7)(8)$$

$$= (1\,2)(1\,3)(1\,4)$$

Q. No.12

Write (i) all even permutations in S_3

(ii) all odd permutations in S_3

Solution

We know that S_3 is the set of all permutations defined on a set with 3 elements.

Let $X = \{1,2,3\}$

Then all the permutations defined on the set X are

$$I = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix}\begin{pmatrix} 3 \\ 3 \end{pmatrix} = (1)(2)(3)$$

$$f_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (1\ 2\ 3) = (1\ 2)(1\ 3)$$

$$f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \end{pmatrix} = (1\ 3\ 2) = (1\ 3)(1\ 2)$$

$$f_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = (1\ 2)(3) = (1\ 2)$$

$$f_4 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} = (1)(2\ 3) = (2\ 3)$$

$$f_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = (1\ 3)(2) = (1\ 3)$$

We see that in the decomposition of identity permutation I, number of transpositions is zero, which is an even number.

Since the permutations I, f_1, f_2 can be written as a product of even number of transpositions therefore, I, f_1, f_2 are even permutations.

Since the permutations f_3, f_4, f_5 can be written as a product of odd number of transpositions therefore, f_3, f_4, f_5 are odd permutations.

Q. N[redacted]

Are the permutations

$(1\ 2\ 3)(2\ 5\ 6)(4\ 3\ 5\ 1)$ and $(1\ 4\ 7)(3\ 4\ 5)(8\ 7)(8\ 3\ 4\ 5)$

even or odd?

Solution

Since $(1\ 2\ 3)(2\ 5\ 6)(4\ 3\ 5\ 1) = (1\ 2)(1\ 3)(2\ 5)(2\ 6)(4\ 3)(4\ 5)(4\ 1)$

(Odd number of transpositions)

$\therefore$ $(1\ 2\ 3)(2\ 5\ 6)(4\ 3\ 5\ 1)$ is an odd permutation.

& $(1\ 4\ 7)(3\ 4\ 5)(8\ 7)(8\ 3\ 4\ 5) = (1\ 4)(1\ 7)(3\ 4)(3\ 5)(8\ 7)(8\ 3)(8\ 4)(8\ 5)$

(Even number of transpositions)

$\therefore$ $(1\ 4\ 7)(3\ 4\ 5)(8\ 7)(8\ 3\ 4\ 5)$ is an even permutation.

Q. No.14

Find the orders of each of the following permutations.

(i) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$

(iii) $\begin{pmatrix}1&2&3&4&5\\4&3&2&5&1\end{pmatrix}$ (iv) $\begin{pmatrix}1&2&3&4&5&6\\2&3&1&5&6&4\end{pmatrix}$

Solution

(i) Let $\alpha = \begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix} = \begin{pmatrix}1&2\\2&1\end{pmatrix}\begin{pmatrix}3&4\\4&3\end{pmatrix} = (1\ 2)(3\ 4)$

Suppose $\alpha_1 = (1\ 2)$ and $\alpha_2 = (3\ 4)$

Then lengths of α_1 and α_2 are $2, 2$ respectively.

$\therefore$ Orders of α_1 and α_2 are $2, 2$ respectively.

($\because$ Length of cycle = Order of cycle)

Thus order of $\alpha = L.C.M$ of $[\ O(\alpha_1), O(\alpha_2)\ ]$

$= L.C.M$ of $[\ 2, 2\ ]$

$= 2$

(ii) Let $\beta = \begin{pmatrix}1&2&3&4\\4&2&1&3\end{pmatrix} = \begin{pmatrix}1&4&3\\4&3&1\end{pmatrix}\begin{pmatrix}2\\2\end{pmatrix} = (1\ 4\ 3)(2)$

Suppose $\beta_1 = (1\ 4\ 3)$

Then length of β_1 is 3

$\therefore$ Order of β_1 is 3

($\because$ Length of cycle = Order of cycle)

Thus order of β = order of β_1

$= 3$

(iii) Let $\gamma = \begin{pmatrix}1&2&3&4&5\\4&3&2&5&1\end{pmatrix} = \begin{pmatrix}1&4&5\\4&5&1\end{pmatrix}\begin{pmatrix}2&3\\3&2\end{pmatrix} = (1\ 4\ 5)(2\ 3)$

Suppose $\gamma_1 = (1\ 4\ 5)$ and $\gamma_2 = (2\ 3)$

Then lengths of γ_1 and γ_2 are $3, 2$ respectively.

$\therefore$ Orders of γ_1 and γ_2 are $3, 2$ respectively.

($\because$ Length of cycle = Order of cycle)

Thus order of $\gamma = L.C.M$ of $[\ O(\gamma_1), O(\gamma_2)\ ]$

$= L.C.M$ of $[\ 3, 2\ ]$

$= 6$

(iv) Let $\delta = \begin{pmatrix}1&2&3&4&5&6\\2&3&1&5&6&4\end{pmatrix} = \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}\begin{pmatrix}4&5&6\\5&6&4\end{pmatrix} = (1\ 2\ 3)(4\ 5\ 6)$

Suppose $\delta_1 = (1\ 2\ 3)$ and $\delta_2 = (4\ 5\ 6)$

Then lengths of δ_1 and δ_2 are $3, 3$ respectively.

$\therefore$ Orders of δ_1 and δ_2 are $3, 3$ respectively.

Thus order of $\delta = L.C.M$ of $[\ O(\delta_1), O(\delta_2)\ ]$

$= L.C.M$ of $[3,3]$

$= 3$

Q. No.15

Determine whether the given permutations are even or odd.

(i) $\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 4 & 3\end{pmatrix}$ (ii) $\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 5 & 3 & 2 & 4 & 1\end{pmatrix}$

(iii) $\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6\\ 3 & 5 & 2 & 1 & 6 & 4\end{pmatrix}$ (iv) $\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7\\ 4 & 3 & 1 & 2 & 6 & 7 & 5\end{pmatrix}$

Solution

(i) Let $\alpha = \begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 4 & 3\end{pmatrix} = \begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}\begin{pmatrix}3 & 4\\ 4 & 3\end{pmatrix} = (1\ 2)(3\ 4)$

(Even number of transpositions)

Since α can be written as a product of even number of transpositions,

Therefore α is an even permutation.

(ii) Let $\beta = \begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 5 & 3 & 2 & 4 & 1\end{pmatrix} = \begin{pmatrix}1 & 5\\ 5 & 1\end{pmatrix}\begin{pmatrix}2 & 3\\ 3 & 2\end{pmatrix}\begin{pmatrix}4\\ 4\end{pmatrix} = (1\ 5)(2\ 3)(4)$

(Even number of transpositions)

Since β can be written as a product of even number of transpositions,

Therefore β is an even permutation.

(iii) Let $\gamma = \begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6\\ 3 & 5 & 2 & 1 & 6 & 4\end{pmatrix} = \begin{pmatrix}1 & 3 & 2 & 5 & 6 & 4\\ 3 & 2 & 5 & 6 & 4 & 1\end{pmatrix}$

$= (1\ 3\ 2\ 5\ 6\ 4)$

$= (1\ 3)(1\ 2)(1\ 5)(1\ 6)(1\ 4)$

(Odd number of transpositions)

Since γ can be written as a product of even number of transpositions,

Therefore γ is an odd permutation.

(iv) Let $\delta = \begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7\\ 4 & 3 & 1 & 2 & 6 & 7 & 5\end{pmatrix} = \begin{pmatrix}1 & 4 & 2 & 3\\ 4 & 2 & 3 & 1\end{pmatrix}\begin{pmatrix}5 & 6 & 7\\ 6 & 7 & 3\end{pmatrix}$

$= (1\ 4\ 2\ 3)(5\ 6\ 7)$

$= (1\ 4)(1\ 2)(1\ 3)(5\ 6)(5\ 7)$

(Odd number of transpositions)

Since δ can be written as a product of even number of transpositions,

Therefore δ is an odd permutation.