Exercise 1.3 (Solutions) Mathematical Method -By S.M. Yusuf, A. Majeed and M. Amin Available at www.MathCity.org xercise # 1.3 Show that (i) e<sup>z</sup> is never zero. OR---- $e^{2} = e^{x+y} = e^{x} e^{y}$  $e^{\frac{2}{e}} \cdot \frac{1}{e^{\frac{2}{e}}} = 1$ = ex [Cosy+iSiny] Multiplicative inverse of e<sup>z</sup> is possible, so e<sup>z</sup> can never be zero .. e<sup>x</sup> ≠0 there is no angle at which Cos & Sin become zero at same time 30 eZ ≠0  $(ii) | e^{i^2} | = 1$ Euler formula 1COSZ+iSinz =1  $\cos^2 z + \sin^2 z = 1$  $e^{i\Theta} = CosO + iSinO = Cis(O)$  $\Pi = 1$ (11) e=1=e=1 iff =1-===2kxi 1=1 where k is integer. Proof:  $e^{Z_1} = e^{Z_2} \Rightarrow \frac{e^{Z_1!}}{e^{Z_2}} = 1$ iii)  $e^{z} = 1$  iff z is integral multiple of 271.  $e^{z_1-z_2} = 1$  put  $z_1-z=z$ Proof: Suppose ez e<sup>2C+iy</sup> = e<sup>z</sup> = 1  $e^{\alpha}$  [Cosy +isiny] = 1 Now question is e<sup>x</sup> Cosy + i e<sup>x</sup> Siny = 1+0i Equating real and imaginary part. Same as Q1 part (iii)  $e^{x} \cos y = 1$   $e^{x} \sin y = 0$   $e^{x} = 1; \quad \cos y = 1$   $e^{x} = 0; \quad \sin y = 0$   $\Rightarrow \pi = 0; \quad \sin y = 0$   $\sin y = 0; \quad \sin y = 0$   $\cos (k\pi) = 1$   $\Rightarrow y = k\pi \rightarrow d$   $\cosh k \in \mathbb{Z}$   $(-1)^{k} = 1$   $where \quad k \in \mathbb{Z}$ lez = e where x+iy=z (v)  $|e^{\pm}| = |e^{x+iy}|$  $= e^{\alpha} e^{i y}$ K=2n >put in i)  $y = 2n \pi$ Now  $Z = \alpha + iy = 0 + i2\pi in$   $Z = (2\pi i)n \quad \text{integral multiple}$ of  $2\pi i$   $Z = (2\pi i)n \quad \text{integral multiple}$   $Z = 2\pi n i$ = ex |eiy|  $= e^{\alpha} \int \cos^2 y + \sin^2 y$ = ex Jī  $|e^{z}| = e^{\alpha}$  $e^{\frac{1}{2}} = e^{2\pi\pi L}$ = Cos(2nx)+iSin2nx  $e^{z} = 1 + 0 \Rightarrow e^{z} = 1$ 

(Vi)  $e^{\underline{z}_1}, e^{\underline{z}_2}, \dots, e^{\underline{z}_n} = e^{\underline{z}_1 + \underline{z}_2 + \dots + \underline{z}_n} \rightarrow 1$ We'll prove this by Mathematical induction Casel put n=1 $\frac{put}{e^{z_2+z_2}} = e^{z_1} e^{z_2}$ Condition is true for n=1 & 2 CasII Suppose this is true for  $n=1 \otimes 2$   $e^{Z_1} e^{Z_2} e^{Z_K} = e^{Z_1+Z_2+\cdots+2K}$ Case III Multiply both sides with ezk+1  $e^{\frac{Z_{k}}{E}}e^{\frac{Z_{k+1}}{E}} = e^{\frac{Z_{1}+Z_{2}+\cdots+Z_{K}}{E}} + \frac{Z_{K+1}}{E}$   $= e^{\frac{Z_{1}+Z_{2}+\cdots+Z_{K}+Z_{K+1}}{E}}$  $e^{z_1} e^{z_2}$ This is true for two numbers Hence Jhes is true for n = K + ISo by Induction (1) is true.  $(e^{\frac{z}{2}})^{n} = e^{n\frac{z}{2}}$ , n being an integer. L.H.S =  $(e^{\frac{z}{2}})^{n} = (e^{\frac{\alpha+iy}{2}})^{n}$ (Mi)  $= (e^{\alpha} e^{i \theta})^{n}$  $= e^{\alpha n} (e^{i \theta})^{n}$ = exn (Cosy+i Siny)n Euler formula  $= e^{2cn} \left[ Cosny + iSinny \right]$  $= e^{2cn} e^{iny}$  $= e^{2n+iny} = e^{n(iy+x)}$  $(e^{\pm})^n = e^{n\pm}$ Formulae:  $\frac{\cosh z}{z} = \frac{e^{z} + e^{-z}}{z}, \quad \sinh z = \frac{e^{z} - e^{-z}}{z}, \\ \tanh z = \frac{e^{z} - e^{-z}}{z}, \\ \tan hz = \frac{e^{z} - e^{-z}}{z}$  $\frac{\&}{\cos z} = \frac{e^{iz} + \hat{e}^{iz}}{2}$ Q2. Prove that (YZ,,Z,ZEC) iii  $\tan^2 \pm 1 = \operatorname{Sec}^2 \pm$  $\frac{1.H.S = 1 + \tan^{2} z}{= 1 + \left(\frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}\right)^{2}}$  $\frac{\sin z}{2i} = \frac{e^{iz}}{2i}$  $\frac{\tan z}{i(e^{iz} - e^{-iz})} = \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}$  $\frac{(e^{iZ} - e^{-iZ})^{L}}{(e^{iZ} + e^{-iZ})^{2}}$ 

.. eiz. e-iz = eiz-iz = e = 1 37) 1+Cot2 = Cosec2  $= (e^{iZ} + e^{iZ})^{2} - (e^{iZ} - e^{iZ})^{2}$  $LHS = 1+Cot^2 z$  $1 + \begin{bmatrix} i(e^{iz} + \overline{e}^{iz}) \\ -(e^{iz} - e^{-iz}) \end{bmatrix}^{2} \cdot i^{2} = 1$ (ei=+e-i=)2  $= \underbrace{\underbrace{e^{2it} + e^{-2it} + 2 - (\underbrace{e^{it} + e^{-2it} - 2}_{(e^{it} \mp + e^{-it} + 2)^2}}_{(e^{it} \mp + e^{-it} \mp 2)^2}$  $1 - (e^{iZ} + e^{-iZ})^2$ (eiz-eiz) - $\frac{(e^{iZ} - \bar{e}^{iZ})^2}{(e^{iZ} - e^{-iZ})^2}$  $(e^{iZ}+e^{iZ})^{L}$  $\frac{e^{2it^2} + e^{-2it^2} - 2 - (e^{2it^2} + e^{-2it^2})}{(e^{it^2} - e^{-it^2})^2}$  $= \int \frac{2}{-\left(e^{iZ} + e^{-iZ}\right)}^{2}$  $= \frac{(-i)(2)^{L}}{(e^{i2} - e^{-i2})^{L}} = (-i) = i^{2}$  $= \begin{bmatrix} -1 \\ \cos 2 \end{bmatrix}^{2}$  $= (Sec =)^{2}$  $\frac{1}{\left(\frac{2i}{e^{i\vec{z}}-e^{-i\vec{z}}}\right)^2}$ = Sec<sup>2</sup>=  $(iii) Sin(Z_1-Z_2) = Sin Z_1 Cos Z_2 - Cos Z_1 Sin Z_1 = Cosec^2 Z_2 = R.H.S$  $\frac{R \cdot H \cdot S = Sin z_{1} Cos z_{2} - Cos z_{1} Sin z_{2}}{= \left(\frac{e^{i z_{1}} - e^{-i z_{1}}}{2i}\right) \left(\frac{e^{i z_{2}} + e^{-i z_{2}}}{2}\right) - \left(\frac{e^{i z_{1}} + e^{-i z_{1}}}{2}\right) \left(\frac{e^{i z_{2}} - e^{-i z_{2}}}{2i}\right)$  $= \frac{1}{4i} \left( \frac{e^{iZ_{1}} - e^{-iZ_{1}}}{e^{iZ_{1}} + e^{iZ_{2}}} - \frac{e^{iZ_{1}} + e^{-iZ_{1}}}{e^{iZ_{1}} + e^{-iZ_{2}}} \right) - \frac{e^{iZ_{1}} + e^{-iZ_{2}}}{e^{iZ_{1}} + e^{iZ_{2}} + e^{iZ_{1}} + e^{iZ_{1}} - e^{-iZ_{1}} + e^{iZ_{2}} - e^{iZ_{1}} + e^{iZ_{2}} + e^{iZ_{1}} - e^{iZ_{1}} + e^{iZ_{2}} - e^{iZ_{1}} + e^{iZ_{2}} - e^{iZ_{1}} + e^{iZ_{2}} + e^{iZ_{1}} - e^{iZ_{1}} + e^{iZ_{2}} + e^{iZ_{1}} - e^{iZ_{1}} - e^{iZ_{1}} + e^{iZ_{1}} - e^{iZ_{1}} - e^{iZ_{1}} + e^{iZ_{1}} - e^{iZ_{1}} + e^{iZ_{1}} - e^{iZ_{1}} + e^{iZ_{1}} - e^$  $e^{-i\overline{z}_i}e^{i\overline{z}_2}+e^{-i\overline{z}_i}e^{i\overline{z}_2}$  $= \frac{1}{4i} \left( 2e^{i\frac{2}{2}i}e^{-i\frac{2}{2}i} - 2e^{-i\frac{2}{2}i}e^{i\frac{2}{2}i} \right)$  $\begin{array}{r}
4\iota \\
= \underbrace{1}{24\iota} & 2\left( e^{i(Z_{1}-Z_{1})} - e^{-i(Z_{1}-Z_{2})} \right) \\
= \underbrace{e^{i(Z_{1}-Z_{1})} - e^{-i(Z_{1}-Z_{2})}}_{2\iota}
\end{array}$ MathCity.org Merging Man and maths =  $Sin(z_1 - z_2) = L.H.s$ (iv)  $Cos(Z_1+Z_2) = Cos Z_1 Cos Z_2 - Sin Z_1 Sin Z_2$ R.H.S = COSZ, COSZ2 - SinZ1 SinZ2  $\left(\frac{e^{iZ_{I}}+e^{-iZ_{I}}}{2}\right)\left(\frac{e^{iZ_{2}}+e^{-iZ_{2}}}{2}\right)-\left(\frac{e^{iZ_{I}}-e^{-iZ_{I}}}{2i}\right)\left(\frac{e^{iZ_{2}}-e^{iZ_{2}}}{2i}\right)$  $= \frac{1}{4} \left( (e^{iz_1} + e^{-iz_1}) (e^{iz_2} + e^{-iz_1}) + (e^{iz_1} - e^{-iz_1}) (e^{iz_2} - e^{-iz_2}) \right)$  $= \frac{1}{4} \left[ e^{i\overline{z}_{1}} e^{i\overline{z}_{2}} + e^{i\overline{z}_{1}} e^{i\overline{z}_{2}} + \frac{e^{i\overline{z}_{1}} e^{i\overline{z}_{1}}}{e^{i\overline{z}_{1}} e^{i\overline{z}_{2}} + e^{i\overline{z}_{1}} e^{i\overline{z}_{2}} + e^{i\overline{z}_{1}} e^{i\overline{z}_{2}} + e^{i\overline{z}_{1}} e^{i\overline{z}_{2}} + e^{-i\overline{z}_{2}} \right]$  $= \frac{1}{4} \left[ 2e^{iZ_1}e^{iZ_2} + 2e^{-iZ_1}e^{-iZ_2} \right]$  $= \frac{1}{2} \left[ e^{i(Z_1 + Z_2)} + e^{-i(Z_1 + Z_2)} \right]$ = Cos  $(Z_1 + Z_2)$ 

38 (V)  $Cos(Z_1-Z_2) = Cos Z_1 Cos Z_2 + Sin Z_1 Sin Z_2.$ R.H.S = Cos ZI Cos Z2 + Sin Z, Sin Z2  $\frac{\left(\frac{e^{iz_{l}}+\bar{e}^{iz_{l}}}{2}\right)\left(\frac{e^{iz_{l}}+\bar{e}^{iz_{l}}}{2}\right)+\left(\frac{e^{iz_{l}}-e^{-iz_{l}}}{2i}\right)\left(\frac{e^{iz_{l}}+\bar{e}^{iz_{l}}}{2i}\right)$  $= \frac{1}{4} \frac{(e^{iZ_{1}} + e^{iZ_{1}})(e^{iZ_{1}} + e^{iZ_{1}}) + i(e^{iZ_{1}} - e^{iZ_{1}})(e^{iZ_{2}} - e^{iZ_{2}})}{(e^{iZ_{1}}e^{iZ_{2}} + e^{iZ_{1}} + e^{-iZ_{1}}e^{iZ_{2}} + e^{iZ_{1}}e^{iZ_{2}} + e^{iZ_{1}}e^{iZ_{1}} + e^$  $2e^{iz_1}e^{-iz_2} + 2e^{-iz_1}e^{iz_2}$  $\frac{1}{2} \left( e^{i(z_1-z_2)} + e^{-i(z_1-z_2)} \right)$  $\cos(z_1 - z_2) = L \cdot H.S.$  $\frac{\cos 2z}{\cos^2 z} = \frac{\cos^2 z}{\sin^2 z} = \frac{\sin^2 z}{\sin^2 z} = \frac{2\cos^2 z}{-1} = \frac{1 - 2\sin^2 z}{1 - 2\sin^2 z}$   $\frac{\cos^2 z}{\cos^2 z} = \frac{(e^{iz} + e^{-iz})^2}{2} - \frac{(e^{iz} - e^{-iz})^2}{2i}$ Ŵ  $\frac{(e^{iZ} + e^{-iZ})^2 - (e^{iZ} - e^{-iZ})^2}{4} - \frac{(e^{iZ} - e^{-iZ})^2}{4(e^{iZ} - e^{-iZ})^2} + (e^{iZ} - e^{-iZ})^2$ \*∠=-l  $e^{2iZ} + e^{2iZ} + \chi + e^{2iZ} + e^{-2iZ} - \chi$ 2e<sup>2iZ</sup> + 2e<sup>-2iZ</sup>  $\left[e^{2i\overline{z}}+e^{-2i\overline{z}}\right]$  $\frac{e^{2iz}+e^{-2iz}}{2}$ = Cos27Prove  $2\cos^2 z - 1 = 2\left(\frac{e^{iz} + e^{iz}}{2}\right)^2$  $\frac{1-2\sin^{2} z}{2} = \frac{1}{2} \frac{2}{(e^{iz} - e^{-iz})^{2}}{\frac{2}{4i}}$  $= 1 + \frac{(e^{iz} - e^{-iz})^{2}}{2}$  $=\frac{2}{4}(e^{iz}+e^{-iz})^{2}-1$  $(e^{i\overline{z}} + e^{-i\overline{z})^2} - 2$  $= 2 + (e^{i\overline{z}} - e^{-i\overline{z}})^{\perp}$  $\frac{2}{2+e^{2L^2}+e^{-2L^2}-2}$  $e^{2i\frac{z}{t}} + \overline{e}^{2i\frac{z}{t}} + \frac{y-2i}{t}$  $= e^{2iZ} + e^{-2iZ}$  $\frac{e^{2i\overline{z}} + e^{-2i\overline{z}}}{2}$ Cos2z 2 Cos2Z

39 (VII) Sin2z = 2SinzCosz  $2Sin \neq Cos \neq = \mathcal{Z}\left(\frac{e^{i \neq} - e^{-i \neq}}{2i}\right) \left(\frac{e^{i \neq} + \bar{e}^{i \neq}}{2i}\right)$  $= \frac{1}{2i} \left( e^{i\overline{z}} - e^{i\overline{z}} \right) \left( e^{i\overline{z}} + e^{-i\overline{z}} \right)$  $\frac{1}{2i}\left(e^{2iz}-e^{-2iz}\right)$ = Sin2z  $\frac{Viii}{Cos z_1 - Cos z_2} = 2 Sin \frac{Z_1 + Z_2}{2} Gin \frac{Z_2 - Z_1}{2}$  $\begin{array}{rcl} \mathcal{R}.\mathcal{H}.\mathcal{S} &=& 2Sin\left(\frac{\mathcal{Z}_{1}+\mathcal{Z}_{2}}{2}\right)Sin\left(\frac{\mathcal{Z}_{2}-\mathcal{Z}_{1}}{2}\right) \\ &=& \mathcal{Z}\left[\frac{e^{i\left(\frac{\mathcal{Z}_{1}+\mathcal{Z}_{1}}{2}\right)}-e^{-i\left(\frac{\mathcal{Z}_{1}+\mathcal{Z}_{1}}{2}\right)}}{\mathcal{Z}_{1}}\right]\left[\frac{e^{i\left(\frac{\mathcal{Z}_{2}-\mathcal{Z}_{1}}{2}\right)}-e^{-i\left(\frac{\mathcal{Z}_{2}-\mathcal{Z}_{1}}{2}\right)}}{2i}\right] \end{array}$  $= \frac{1}{2i^{2}} \left| \left( e^{i \left( \frac{Z_{1} + Z_{2}}{2} \right)} - e^{-i \left( \frac{Z_{1} + Z_{2}}{2} \right)} \right) \left( e^{i \left( \frac{Z_{2} - Z_{1}}{2} \right)} - e^{-i \left( \frac{Z_{2} - Z_{1}}{2} \right)} \right) \right|$  $-\frac{1}{2} \left[ e^{i\left(\frac{Z_1+Z_1}{2}\right)} e^{i\left(\frac{Z_2-Z_1}{2}\right)} - e^{i\left(\frac{Z_1+Z_2}{2}\right)} e^{-i\left(\frac{Z_2-Z_1}{2}\right)} - e^{-i\left(\frac{Z_1+Z_2}{2}\right)} e^{i\left(\frac{Z_2-Z_1}{2}\right)} \right]$  $+e^{-i\left(\frac{Z_I+Z_L}{2}\right)}e^{-i\left(\frac{Z_L-Z_I}{2}\right)}$  $=\frac{1}{2}\left(e^{i\left(\frac{z_{1}+z_{2}}{2}\right)+i\left(\frac{z_{2}-z_{1}}{2}\right)}-i\left(\frac{z_{1}+z_{2}}{2}\right)+i\left(\frac{z_{2}-z_{1}}{2}\right)-i\left(\frac{z_{1}+z_{2}}{2}\right)-i\left(\frac{z$  $=\frac{-i}{2}\left[\frac{e^{iz_{2}}-e^{iz_{1}}-e^{-iz_{1}}+e^{-iz_{2}}}{2}\right]$  $=-\frac{1}{2}\left[\left(e^{iZ_{2}}+e^{-iZ_{2}}\right)-\left(e^{iZ_{1}}+e^{-iZ_{1}}\right)\right]$ MathCITY.OFS Merging Man and maths  $= -\left(\frac{e^{iZ_{2}} + e^{-iZ_{2}}}{2}\right) + \left(\frac{e^{iZ_{1}} + e^{-iZ_{1}}}{2}\right)$ = - COSZ2 + COSZI = L.H.S proved (ix)  $Sin z_1 - Sin z_2 = 2Cos(\frac{z_1+z_2}{2})Sin(\frac{z_1-z_2}{2})$  $\frac{R \cdot H \cdot S}{2} = 2\cos\left(\frac{Z_1 + Z_2}{2}\right) \sin\left(\frac{Z_1 - Z_2}{2}\right)$  $=\frac{2}{2} \underbrace{e^{i(\frac{z_{1}+z_{2}}{2})} + e^{-i(\frac{z_{1}+z_{2}}{2})}}_{2} \underbrace{e^{i(\frac{z_{2}-z_{1}}{2})} - e^{i(\frac{z_{2}-z_{1}}{2})}}_{2}$ 

 $\frac{i\left(\frac{z_1+z_2}{2}\right)}{\rho} - \frac{-i\left(\frac{z_1+z_2}{2}\right)}{2}$  $\begin{pmatrix} \frac{i\left(\frac{z_{1}-z_{1}}{2}\right)}{e} - i\left(\frac{z_{1}-z_{1}}{2}\right) \\ -e \end{pmatrix}$  $\frac{i\left(\frac{z_{l}+z_{1}}{2}\right)}{e} - i\left(\frac{z_{l}-z_{1}}{2}\right) + e} - i\left(\frac{z_{l}+z_{1}}{2}\right) e^{i\left(\frac{z_{l}-z_{1}}{2}\right)}$  $e^{-i\left(\frac{2}{2}+\frac{2}{2}\right)}$  $\int_{0}^{i} \left( \frac{\overline{z_{1}} + \overline{z_{2}} - \overline{z_{1}} + \overline{z_{1}}}{2} \right) \int_{0}^{-i} \left( \frac{\overline{z_{1}} + \overline{z_{2}} - \overline{z_{1}} + \overline{z_{1}}}{2} \right) - i \left( \frac{\overline{z_{1}} + \overline{z_{1}} + \overline{z_{1}} - \overline{z_{1}}}{2} \right)$  $F(e^{iZ_{t}}-e^{-iZ_{t}})-(e^{iZ_{2}}-e^{-iZ_{1}})$  $\frac{e^{i\overline{z}_{i}}-e^{i\overline{z}_{i}}}{2i} = \frac{e^{i\overline{z}_{i}}-e^{i\overline{z}_{i}}}{2i}$ Sin ZI - Sin ZI - L.H.S 3Sinz - 4Sin<sup>3</sup>z Sin3 = =  $\frac{4\left(\frac{e^{iZ}-e^{-iZ}}{2i}\right)}{2i}$ <u>3Sinz - 4Sin<sup>3</sup>7</u>  $3e^{2iz-iz}$   $7=3e^{iz}$ 3e<sup>(Z-2)Z</sup>  $-\frac{4}{81^3}(e^{iz}-e^{-iz})^3$  $\left(\frac{e^{iz}}{2i}-e^{-iz}\right)$  $e^{-iz} = \frac{1}{-2i} \left( \frac{3iz}{e^{-2i}} \right)$ 77  $3e^{iz} - 3e^{iz} + e^{3iz} - e^{-3iz} - 3e^{iz} + 3e^{-iz}$ <u>- e - 312</u> Sin 3z = R.H.S (XII)  $\tan(Z_1-Z_2) = \frac{\tan Z_1 - \tan Z_2}{1 + \tan Z_1 \tan Z_2}$ tan (Z1+Z2) = tanZ1+tanZ2 XŇ 1- tanz, tanz2  $\tan(z_1 - z_2) = \frac{\sin(z_1 - z_2)}{\cos(z_1 - z_2)}$  $\tan(z_1+z_2) = \sin(z_1+z_2)$  $Cos(Z_1+Z_2)$ = Sin ZiCosZ2 - CosZ, Sin ZL = SinZICOSZ2 + COSZISINZI COSZICOSZI + Sin Zi SinZi CosziCoszi - SinziSinz, Sinz, Coszi Coszi Ginzi  $\frac{Sin \neq_i Cog \neq_1}{Cos \neq_i Cos \neq_1} \leftarrow \frac{Cog \neq_1}{Cos \neq_1 Cos \neq_1}$ COSZI COSZI COSZI COSZI COSZICOSZI + SINZISINZL COSZICOSZI COSZICOSZI Cosz Coszi Sinzi Sinz2 CoszyCosZ2 Cosz, Cosz, tanz, + tanz.  $\tan(z_1 - z_1) = \tan z_1 - \tan z_2$ 1 - tanz, tanzi 1+tonzitanz2 B M.TANVEER Superior Cellege Sargodha

Formulae: \* Sin (LZ)=i Sinhz 3. Show that. \* Cos(iz) = Coshz \* tan(1=) = itanh= Sinz = Sin Z U) Sin = Sin(x+iy) Sinz = Sin(x+iy) = SinxCos(iy) + Cosx Sin(iy) =  $Sin(x - \dot{y})$ =Sina Cos(iy) - Cosx Sin(iy) = Sinx Coshy + Cosx i Sinhy = SinxCoshy - Cosxisinhy Sinz = Sinx Coshy + i Cosx Sinhy Sin = coshy sinx - i cosx sinhy = Sinx Coshy - i Cosx sinhy - iù trom is u iii Sinz - Sin Z (ii) Cosz = CosZ Cosz =  $\cos(\pi + m)$ Cosz = Cos(x+y) = Cos(x-iij) = Cosoc Cosiy - Size Sin (iy) = Cosoc Cosiy + Sinx Sintuy = Cosx Coshy - Sinx i Sinhy Cos= cosxCoshy +iSinxSinhy Cosz = Cosz Coshy - iSina Sinhy = Cosx Coshy +i Sinx Sinhy +is. from i) & il, Cosz = Cosz (111) ----- Tanz = Tanz tanz = tan(x+iy)  $\tan \frac{1}{2} = \tan(\frac{1}{2} + \frac{1}{4})$  $= \frac{\tan x + \tan(iy)}{1 - \tan(x) + \tan(iy)}$  $= \tan(x - iy)$ tanx +itanhy tanz = tanx - tan(iy) tanxitanhy 1 + tan(x)tan(iy) ZI ZI (tan x + itanhy tanz tanz = tanx -itanhy I-itanztanhy 1+ i tanxt anhi tanx+itanhy 1-i-tanxtanhy A U (2) From tanx -itanhy tanz tanz. 1+itanxtanhy -Sint - E) = -Sin = Sin(-z) = -Sinz(iv)\_ -e-iz = e<sup>iz</sup> Sinz etz <u>– e</u>iZ Sin(=) = $e^{iZ} - e^{iZ}$ Sinz From  $\frac{Sin(-z)}{z} = 1$ 

(42 (vi) tan(-z) = -tan z  $tan(z) = e^{iz} - e^{-iz}$   $i(e^{iz} + e^{-iz})$  $(V) \quad (OS(-Z) = + COS(Z))$  $\frac{\cos(z) = e^{iz} + e^{iz}}{\cos(z) = e^{iz} + e^{iz}}$  $\frac{2}{\cos(-z)} = e^{-iz} + e^{iz}$ by (-Z) replace Z  $\tan(-z) = e^{-iz} - e^{iz}$  $\overline{i(e^{-i\overline{z}}+e^{i\overline{z}})}$ Cos(-z) = Cosz from (1)  $\frac{\tan(-z) - - (e^{iz} - e^{-iz})}{i(e^{iz} + e^{-iz})}$  $\tan(-z) = \tan(z)$  from (i) (vii) Sigh(-z) = - Sighz Sighz =  $e^{z} - e^{z} \rightarrow 0$ (vii) Cosh(z) = Cosh z $\frac{\cosh z}{2} = \frac{e^{z} + e^{-z}}{2} \xrightarrow{\text{(1)}} \frac{2}{\cosh(-z)} = \frac{e^{-z} + e^{-z}}{e^{-z} + e^{-z}}$ reptate z by  $\frac{1}{2}$ -z Sinh(-z) =  $e^{-2} - e^{2}$  $\left(\frac{e^2-e^{-2}}{e^{-2}}\right)$ Cosh(-z) = Coshz from ① tanhz = tanh Z (X) Sinh(-z) = - Sinhz from () By formula itanhz = tanliz) (x) tanh(-z) = tanhz  $\frac{tanhz}{e^2+e^{-z}}$ tanhz = tan(iz) $\frac{-\text{replace} - z - by - z}{-tanh(-z)} = \frac{e^{-z} - e^{-z}}{-e^{-z} - e^{-z}}$  $= (\frac{\tan(iz)}{i}) = \frac{\tan(iz)}{i}$ tanhz tan (iz) = tanz = tanz  $tanh(-z) = \left(\frac{e^{z} - e^{-z}}{e^{z} + e^{-z}}\right)$ -1 tan (-1, ₹) tanh(-z) = - tanhz from i <u>tan(i=)</u> -i <u>= tan ((Z)</u> Prove the following = ÿtanh(₹) identities tanhz = tanh = proved Cosh'z - Sinh'z = 1 Ò  $\frac{\cos^2 z}{\left[\cos\left(iz\right)\right]^2 + \sin^2 z} = 1$   $\left[\cos\left(iz\right)\right]^2 + \left[\sin\left(iz\right)\right]^2 = 1$ replace Z by iz  $(Coshz)^2 + (iSinhz)^2$ Cosh'z + 12 Sinh'z = 1 = -Cosh<sup>2</sup>z - Sinh<sup>2</sup>z = 1 proved.

43 (ii) Sech'z = 1-tanh'z (iii)  $\operatorname{Cosech}^2 z = \operatorname{Coth}^2 z - I$ -  $\operatorname{Cosec}^{2} = 1 + \operatorname{Cosec}^{2} = 1 + \operatorname{Cosecc}^{2} = 1 + \operatorname{Cos$  $\therefore$  Sec<sup>2</sup>z = 1+tan<sup>2</sup>z replace Z by iz replace z by iz  $\left[\operatorname{Cosec}(iz)\right]^{2} = 1 + \left[\operatorname{Cot}(iz)\right]^{2}$  $[Sec(iz)]^{2} = 1 + [tan(iz)]^{2}$ (Sechz)^{2} = 1 + (itanhz)^{2}  $[i Cosec hz]^2 = 1 + [i Cothz]^2$ i<sup>2</sup>Cosech<sup>2</sup>Z = 1 + i<sup>2</sup>CotHZ  $\operatorname{Sech}^2 = 1 - \tanh^2 \Xi$  $-Cosech^2 z = 1 - Cot h^2 z$  $Cosech^2 z = Coth^2 z - 1$  proved. (iv)  $\cosh 2z = \cosh^2 z + \sinh^2 z = 2\cosh^2 z - 1 = 1 + 2 \sinh^2 z$  $\frac{\cos 2z}{\operatorname{replace}} = \frac{\cos^2 z}{\operatorname{by}} = \frac{\sin^2 z}{z}$  $Cos(2iz) = [Cos(iz)]^2 - [Sin(iz)]$  $\cosh(2z) = (\cosh z)^2 - (i \sinh z)^2$ \* i2=-1 Proved  $\cosh 2z = \cosh^2 z + \sinh^2 z \rightarrow i$  $\frac{\cosh^2 z}{(\operatorname{put} \text{ in } (i))}$ Cosh ZZ = 1+ Sinh Z + Sinh Z Proved  $= 1 + 2 \text{Sinh}^2 z$ Sinh'z =-1+ Cosh'z .  $\cosh 2z = \cosh^2 z + \cosh^2 z - 1$ (put in ii)  $= 2 \cosh^2 \Xi - 1$ Proved (VI) Sinh 3z = 3Sinhz+4Sinh3z (1) Sin2Z = 2SinhZ CoshZ . Sin3z = 3Sinz - 4Sin3z = Sin 27 = 2 Sin Z Cosz replace Z by iz replace z by iz  $Sin(3iz) = 3Sin(iz) - 4(Sin(iz))^3$ Sin(217) = 2 Sin(17)Cosiz i Sinh3z = i3 Sinhz -4 (i Sinhz)3 ESinh2z = 2. ESinhz Cohz isinh3z = i3sinhz - 4i2i Sinhz Sinh2z = 2 SinhzCoshz Sinh 3Z = 3Sinh z + 4Sinhz (vil) Cosh3z = 4 Cosh3z - 3 Coshz Proved  $\frac{\cos 3z}{\operatorname{replace}} = \frac{4\cos^3 z}{\cos^3 z} - 3\cos^2 z$  $\cos(3iz) = 4(\cos(iz))^3 - 3\cos(iz)$  $\cosh 3z = 4 (\cosh z)^3 - 3 \cosh z$ Merging Man and math  $= 4 \cosh^3 z = 3 \cosh z$ 

45 if z = x+iy, prove that 5. i) Sin Z = SinxCoshy +iCosxSinhy Sinz = Sin(x+iy) = Sinx Cos(iy) + Cosx Sin(iy) Sinz = Sinx Coshy + i Cosx Sinhy proved Sin2x +i Sinh2y Cos2x + Cosh2y (ii) tanz -----Formula used ton = tan(x+iy)  $\frac{Sin(x+iy)}{Cos(x+iy)} \times \frac{Cos(x-iy)}{Cos(x-iy)}$ 2 Sin AcosB = Sin(A+B) + Sin(A-B)  $2 \cos A \cos B = \cos(A + B) + \cos(A - B)$ 2 Sin(x+y)Cos(x-iy) 2Cos(x+iy)Cos(x-iy) Sin (x+y+x-y) + Sin (x+y-x+y) Cos(x+ig + x-ig)+Cos(x+ig-x+ig) Sin(2x) + Sin(2iy)= Cos(2x) + cos(24) Superior College Sargodha Sin 2x + i Sinh2y proved Cos 2x + i Cosh 24 Sin (A+iB) = x+ iy, show that 6. if y2 X<sup>2</sup> Cosh<sup>2</sup>B <u>y</u><sup>2</sup> Cos<sup>2</sup>A  $\mathbf{x}^2$ Sinh<sup>2</sup>B Sin<sup>2</sup>A Sin(A+iB) = x+iy SinACos(iB) + CosASin(iB) = x+iy MathCity.org <u>SinACoshB + iCosASinhB = x+iy</u> equating real and imaginary parts Cos A Sin h B = y SinACoshB = x  $SinhB = \frac{9}{\cos A} \rightarrow i$  $\frac{\text{CoshB}}{\text{SinA}} = \frac{x}{\Rightarrow 0}$  $\frac{\delta}{SinA} = \frac{x}{2} \rightarrow (ii)$  $\cos A = \frac{g}{\sinh B} \rightarrow (iv)$ CoshB Squaring & adding (11) & (11) Squaring w  $\frac{x^{\mu}}{Cosh^{2}B} + \frac{g^{2}}{Sinh^{2}B} = Cos^{2}A + Sin^{2}A$ Subtracting is & ville x2\_\_\_\_  $\frac{y^2}{\cos^2 A} = \cosh^2 B - \sinh^2 B$ Sin<sup>2</sup>A  $\frac{\alpha^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = 1$  $\frac{y^2}{\cos^2 A} = 1$ 322 Proved Sin<sup>2</sup>A Proved

(W) -Sinh(Z1-Z2) = Sinhz, CoshZ2 - CoshZ2 SinhZ2 Sin(Z, Z) = SinZ, COSZ, - COSZ, SinZL replace Z1 by iz, and Z, by iz.  $Sin(iZ_1 - iZ_2) = Sin(iZ_1) Cos(iZ_2) - Cos(iZ_2) Sin(iZ_2)$ Sini(Z1-Z1) = iSinhz, Coshz, \_ Coshz, iSinhz, iSinh(z,-z) = i [Sinhz, Coshz, - Coshz, Sinhz] Sinh(ZI-ZI) = Sinhz, Coshz, - Coshz, Sinhz, proved (IX) tanh(Z, ±Z2) tanh Zit tanh Zi Ŧ  $1 \mp \tanh z_1 \tanh z_L$  $\tan(z_1 \pm z_2)$ tanzi ± tanzz 17 tanzitanzi replace and ZL by LZL Z,  $\tan(\iota z_1 + \iota z_1) =$  $\tan(iz_1) \pm \tan(iz_1)$  $I \mp \tan(iz_i) \tan(iz_i)$  $tani(z_1+z_2)$ itanhzı + itanhzı itanhz, +itanhzi I = i tanhziltanhzi 1 Filtanhz, tanhz,  $\frac{1}{1}$   $\frac{1}$ = k (tanh  $z_1$  + tanh  $z_2$ ) + tanhz, tanhz, tanh(21 = 22)tanhzi +tanhzi \_\_\_\_ proved 1 ± tanhzitanhz (\*) tanh 37 = 3tanh 2 + tanh 32 1+3tamh27 tan 3z 3tanz -tan3z 1-3+an2z replace = = by 3tan (iz) - (tam(iz)) tan (3iz) 1 - 3 (tan(12))2 i 3tanhz - i<sup>3</sup>tanh<sup>3</sup>z 1- 3i<sup>2</sup>tanh<sup>2</sup>z  $i^{3} = i^{2} \cdot i = -i$ i 3tanhz +itanh3z 1 + 3tanh2z M. TANVEER Superior College Sargodha ±tanh3z + ( 3tanhz + tanh3z) 1+ 3tanh27 1anh3z <u> 3tanhz + tanh3z</u> VNSV CA 1+ 3tam h17

7. if  $tan(\alpha + i\beta) = x + iy$ , show that  $\frac{x^{2}+y^{2}+2x\cot 2d=1}{\operatorname{Sol}} \quad and \quad \underline{x^{2}+y^{2}-2y\coth 2B}=-1}{\frac{x^{2}+y^{2}}{\tan(\alpha+i\beta)}=x+y} \quad \exists \alpha+i\beta=\tan^{-1}(x+y) \rightarrow (i)}$  $\tan(\overline{\alpha+i\beta}) = \overline{\alpha-i\beta} = \tan(\alpha-i\beta)$  $\alpha - i\beta = \tan^{-1}(\alpha - i\gamma) \longrightarrow (ii)$ Adding is & ois  $\alpha + i\beta + \alpha - i\beta = tan^{-1}(x + iy) + tan^{-1}(x - iy)$  $\frac{dan'(x+iy) + tan'(x-iy)}{[x+iy] + x-iy]} = tan'(A) + tan'B$ 2d =  $\frac{x+iy+x-iy}{1-(x+iy)(x-iy)}$ 2a = tan'  $\frac{2x}{1-(x^2+y^2)}$  $\frac{(x+iy)(x-iy) = x^{2}-i^{2}y^{2}}{= x^{2}+y^{2}}$  $\tan(2\alpha) =$  $\frac{1}{\cot 2d} = \frac{2\pi}{1 - (\pi^2 + y^2)}$  $1 - (x^2 + y^2) = 2x \cot 2x$ proved.  $\Rightarrow (x^2 + y^2 + 2x \cot 2a = 1)$ Subtracting is & iii x+ip-x+ip = tan (x+iy)-tan (x-iy)  $\frac{2i\beta}{1+(x+iy)(x-iy)} = \frac{1}{(1+(x+iy)(x-iy))}$ 2iy 1+x2+42  $\tan(2i\beta) =$ Merging Man and mathe  $jitanh(2\beta) = j^{2}\frac{2y}{1+x^{2}+y^{2}}$  $\frac{1}{\cot h(2\beta)} = \frac{2y}{1+\chi^2+y_1}$ M. TANVEER Superior College Sargodha  $1 + x^2 + y^2 = 2y \operatorname{Coth}(2\beta)$  $\frac{\chi^2 + y^2 - 2y \operatorname{Coth}(2\beta)}{2\beta} = -1 \quad \text{proved}$ if  $Sin(\Theta + i\phi) = x + i Sind$   $Cos^2 \Theta = \pm Sind$ 8. Sol  $\frac{\sin(0 + i\phi)}{\sin(0 + i\phi)} = x + \frac{i\psi}{\cos(0 + i)} \cos(1 \phi) = x + \frac{i\psi}{\cos(0 + i)} \cos(1 \phi)$  $Sin \Theta Cosh \phi + i Cos \Theta Sinh \phi = \alpha + i \psi Cosa + i Sina$ 

Equating real & imaginary parts  $\frac{SinOCosh\phi = Cosa}{SinO} = \frac{Cosa}{SinO} = \frac{Cosa}{Cosh\phi} = \frac{Sina}{Cosa} \rightarrow (i)$ Squaring & subtracting is & (ii)  $\frac{\cosh^2 \phi - \sinh^2 \phi}{\sinh^2 \phi} = \frac{\cos^2 \alpha}{\sin^2 \phi} = \frac{\sin^2 \alpha}{\cos^2 \phi}$ Costo  $1 = \frac{\cos^2 \lambda \cos^2 \Theta - \sin^2 \alpha \sin^2 \Theta}{\sin^2 \Theta \cos^2 \Theta}$  $4 = (1 - \sin^2 \alpha) \cos^2 \Theta = \sin^2 \alpha (1 - \cos^2 \Theta)$ Sint & Costo Sinto Costo = Costo - Costo Sinta - Sinta + Sinta Costo  $(1 - \cos^4 \Theta) \cos^4 \Theta = \cos^4 \Theta = \cos^4 \Theta = \cos^4 \Theta = \sin^4 \Theta$  $-\cos 4\theta = -\sin^2 \alpha$  $\cos^4 \Theta = \sin^2 \alpha$ Cost 0 = + Sina Proved Cos Sol tan(0+id) = tand + iseca  $O + i \phi = tan^{-1} (tan \alpha + i sec \alpha) \rightarrow i$  $tan(O-i\phi) = tan\alpha - iSec\alpha$  :  $tanz = tan\overline{z}$  $\partial -i\phi = tan'(tan \alpha - iSec \alpha) \rightarrow ii$ adding is and (ii) 20- = tan' (tanx+i Secx)+tan' (tanx-isecx) taň' | tana+i séa + tana- i séca 20 1 - (tana+iseca)(tana-iseca)] = tan'A + tan'B = -tan-' [A+B tan' 2tanz 1- (tan'd-i<sup>2</sup>secd) tan' 2tan x 1 - (tan'a + Sec'a) tam<sup>1</sup> 2tand 1 - tanta - Secta Ztand X-tan's 1-tan's 20 = tan'

48 2tana -2tana tan" 20 = tan' (- tame  $: \tan\left(\frac{\pi}{2} + \varkappa\right) = - \operatorname{Cot}_{\varkappa}$ tan' (- Cota)  $tan(\underline{x} + \alpha + \pi n)$ = tan" 20 <u>T</u>+ x + nx 20- = (ii) Subtracting is a in 2i \$ = tan (tana + i Seca) - tan (tana - i Seca) lanatiSeca - (tana-iseca) 2id = tan 1+ (tana+iseca)(tana-iseca) toma +iSecd - toma + iSeca = tan" 1 + tan'a + Sec'a 2iSec x tañ Sec' + Sec' x Zisecz Zseczd tan tan (2i\$) = secd Ċ. itanh20 = iCosa Merging Man and Coso tanh20-COSX e-20 Σø e<sup>2¢</sup> .2Ø ezø e<sup>-2</sup>9 Coso Using componendo-clividendo Theorem  $z\phi_{+}e^{2\phi}-\bar{e}^{2\phi}$ 1+COSX e20+e-20 1-Cosa +0 2, COS2 (x/2)  $2e^{2\phi}$ 2/Sin 2 (a/2) -2Ø 2e 2¢+2¢ Cot 2 x/2 e 40 Cot<sup>2</sup> ×/2  $e^{2\phi}$ + Cot (2/2) Proved M. TANVEER Superior Cellege Sargodha

10. Prove that.  $\frac{\sinh(\frac{x}{2})}{2} = \int Coshx - 1}{2}$ if xyo Coshx-1 2.<0 <u>Sol</u> :  $e^{x}+e^{-x}-2$ <u>Coshx - 1</u>  $(e^{\frac{x_{12}}{2}} - e^{-\frac{x_{12}}{2}})$  $(2)^{2}$  $e^{\alpha/2} - e^{\alpha/2}$ (Sinh(<u>x</u>) = <u>+</u> Sinh(¥) mor College Sargodha Sinh (34) Coshx-1 when Sinh <del>य</del>ू Coshx-1 when - 2 <0 11 Show multiplication that vector = by of where no. rotates is real the rector through an angle Z counterclockwise of measure α Sol r Coso +i Sino Ζ=  $= re^{iQ}$ Z(e<sup>ca</sup>) re2(0+a) Z(eix)  $\left[\cos(0+\alpha)+i\sin(0+\alpha)\right]$ proved. r 12 Show that (11)  $-3-4i = 5e^{i}(x+tan'(\frac{4}{3}))$ Z = 3-4i,  $r = \sqrt{9+16} = \sqrt{25}$  $2+i = \sqrt{5} e^{i \tan^2(1/2)}$ <u>e</u>)- $\frac{[r=5]}{s}$ そ=2+i )「=|王|=「年+1  $\cos \theta = -\frac{3}{2}$  $t_{2000} = T + d$   $t_{and} = 4/3$  $x = t_{an}^{-1}(\frac{4}{3})$ Coso = 2 Sind = Ist quardrant. 0 = \* + tan' (4)  $\tan 0 = 115 = 1/2$  $Z = r (\cos \theta + i \sin \theta)$  $\theta = tan^{-1}(1/2)$  $\frac{z}{Z} = 5 \left[ \cos(\pi + \tan^{2}(\frac{4}{3}) + i\sin(\pi + \tan^{2}(\frac{4}{3})) + i\sin(\pi +$  $Z = r (\cos \theta + i \sin \theta)$  $= J5 \left[ Cos(tan'(\frac{1}{2})) + i Sin(tan'(\frac{1}{2})) \right]$  $= J5 e^{itan'(\frac{1}{2})}$