50 

$\boxed{1\cdot 3-1}$

# Exponential Function

Prove $e^{ix} = \cos x + i\sin x$

We know

$$e^{x} = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} + \frac{x^5}{\lfloor 5} + \frac{x^6}{\lfloor 6} + \frac{x^7}{\lfloor 7} + \cdots$$

Put $x = ix$

$$e^{ix} = 1 + (ix) + \frac{(ix)^2}{\lfloor 2} + \frac{(ix)^3}{\lfloor 3} + \frac{(ix)^4}{\lfloor 4} + \frac{(ix)^5}{\lfloor 5} + \frac{(ix)^6}{\lfloor 6} + \cdots$$

$$= 1 + ix + (i)^2\frac{x^2}{\lfloor 2} + (i)^3\frac{x^3}{\lfloor 3} + (i)^4\frac{x^4}{\lfloor 4} + (i)^5\frac{x^5}{\lfloor 5} + (i)^6\frac{x^6}{\lfloor 6} + \cdots$$

$$= 1 + ix + i^2\frac{x^2}{\lfloor 2} + i^2(i)\frac{x^3}{\lfloor 3} + (i^2)^2\frac{x^4}{\lfloor 4} + i(i^2)^2\frac{x^5}{\lfloor 5} + i^2(i^2)^2\frac{x^6}{\lfloor 6}$$

$$= 1 + ix - \frac{x^2}{\lfloor 2} - i\frac{x^3}{\lfloor 3} + (1)\frac{x^4}{\lfloor 4} + i(1)\frac{x^5}{\lfloor 5} - (1)\frac{x^6}{\lfloor 6} + \cdots$$

$$= \left(1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \cdots\right) + \left(ix - i\frac{x^3}{\lfloor 3} + i\frac{x^5}{\lfloor 5} - \cdots\right)$$

$$= \left(1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \cdots\right) + i\left(x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \cdots\right)$$

$$= \cos x + i\sin x$$

Note: if $z = x + iy$ in Cartesian form

$z = r(\cos\theta + i\sin\theta)$ in Polar form

$z = re^{i\theta}$ in Exponential form.

Note: $a^z = e^{z\ln a}$ if $a \in R$, $a > 0$

$\because \ln a^z = \ln e^{z\ln a}$

$z\ln a = z\ln a(\ln e)$

$z\ln a = z\ln a$

## Trigonometric Functions

We know

$e^{ix} = \cos x + i\sin x$

$e^{-ix} = \cos x - i\sin x$

$e^{ix} + e^{-ix} = 2\cos x$

$\frac{e^{ix} + e^{-ix}}{2} = \cos x$

Similarly $e^{ix} - e^{-ix} = 2i\sin x$

$\frac{e^{ix} - e^{-ix}}{2i} = \sin x$

$\sec x = \frac{2}{e^{ix} + e^{-ix}}$

$\operatorname{cosec} x = \frac{2i}{e^{ix} - e^{-ix}}$

$\tan x = \frac{\sin x}{\cos x} = \frac{e^{ix} - e^{-ix}}{i(e^{ix} + e^{-ix})}$

$\cot x = \frac{i(e^{ix} + e^{-ix})}{e^{ix} - e^{-ix}}$

## Hyperbolic Functions

$\sinh x = \frac{e^{x} - e^{-x}}{2}$ $\quad$ $\operatorname{cosech} x = \frac{2}{e^{x} - e^{-x}}$

$\cosh x = \frac{e^{x} + e^{-x}}{2}$ $\quad$ $\operatorname{sech} x = \frac{e^{x} + e^{-x}}{2}$

$\tanh x = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$ $\quad$ $\coth x = \frac{e^{x} + e^{-x}}{e^{x} - e^{-x}}$

## Osborn's Rule

Relation b/w Trig Fns & Hyperbolic fns.

| | |
|---|---|
| $\sin iz = i\sinh z$ | $\sinh iz = i\sin z$ |
| $\cos iz = \cosh z$ | $\cosh iz = \cos z$ |
| $\tan iz = i\tanh z$ | $\tanh iz = i\tan z$ |
| $\cot iz = -i\coth z$ | $\coth iz = -i\cot z$ |
| $\sec iz = \operatorname{sech} z$ | $\operatorname{sech} iz = \sec z$ |
| $\operatorname{cosec} iz = -i\operatorname{cosech} z$ | $\operatorname{cosech} iz = -i\operatorname{cosec} z$ |

Note: To Prove Osborn's Rule just Put $iz = z$ & solve.

**Prove** $\sin iz = i\sinh z$

Proof: $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$

Put $z = iz$

$$\sin iz = \frac{e^{iiz} - e^{-iiz}}{2i} = \frac{e^{i^2 z} - e^{-i^2 z}}{2i} = \frac{e^{(-1)z} - e^{-(-1)z}}{2i}$$

$$= \frac{e^{-z} - e^{z}}{2i} = i\left(\frac{e^{-z} - e^{z}}{2ii}\right) = i\left(\frac{e^{-z} - e^{z}}{-2}\right)$$

$$= i\left(\frac{e^{z} - e^{-z}}{2}\right) = i\sinh z$$

**Prove** $\cos iz = \cosh z$

$\cos z = \frac{e^{iz} + e^{-iz}}{2}$

Put $z = iz$

$$\cos iz = \frac{e^{iiz} + e^{-iiz}}{2} = \frac{e^{i^2 z} + e^{-i^2 z}}{2}$$

$$= \frac{e^{-z} + e^{z}}{2} = \cosh z$$

Q1 $\boxed{\text{Ex } 1\cdot 3}$

Show that $e^z$ is never zero.

Sol: $e^z \cdot \frac{1}{e^z} = 1$ since the multiplicative inverse of $e^z$ exists so $e^z$ is never zero

$\boxed{1.2-38}$

$$L.H.S = \left(\frac{Z_1}{Z_2}\right)^n = \left[\frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}\right]^n$$

$$\Rightarrow \left(\frac{Z_1}{Z_2}\right)^n = \frac{r_1^n}{r_2^n}\left[\frac{(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)}\right]^n$$

$$= \frac{r_1^n}{r_2^n}\left(\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1)}{\cos^2\theta_2 + \sin^2\theta_2}\right)^n$$

$$= \frac{r_1^n}{r_2^n}\left[\frac{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)}{1}\right]^n$$

$$= \frac{r_1^n}{r_2^n}\left[\cos n(\theta_1 - \theta_2) + i\sin n(\theta_1 - \theta_2)\right]$$

$$= \frac{r_1^n}{r_2^n}\left[\cos(n\theta_1 - n\theta_2) + i\sin(n\theta_1 - n\theta_2)\right]$$

$$= \frac{r_1^n}{r_2^n}\left[(\cos n\theta_1\cos n\theta_2 + \sin n\theta_1\sin n\theta_2) + i(\sin n\theta_1\cos n\theta_2 - \sin n\theta_2\cos n\theta_1)\right]$$

$$= \frac{r_1^n}{r_2^n}\left[(\cos n\theta_1\cos n\theta_2 - i^2\sin n\theta_1\sin n\theta_2) + i(\sin n\theta_1\cos n\theta_2 - \sin n\theta_2\cos n\theta_1)\right]$$

$$= \frac{r_1^n}{r_2^n}\left[(\cos n\theta_1\cos n\theta_2 - i\sin n\theta_2\cos n\theta_1) + (i\sin n\theta_1\cos n\theta_2 - i^2\sin n\theta_1\sin n\theta_2)\right]$$

$$= \frac{r_1^n}{r_2^n}\left[\cos n\theta_1(\cos n\theta_2 - i\sin n\theta_2) + i\sin n\theta_1(\cos n\theta_2 - i\sin n\theta_2)\right]$$

$$= \frac{r_1^n}{r_2^n}\left[(\cos n\theta_1 + i\sin n\theta_1)(\cos n\theta_2 - i\sin n\theta_2)\right]$$

$$= \frac{r_1^n}{r_2^n} \times (\cos\theta_1 + i\sin\theta_1)^n(\cos n\theta_2 + i\sin n\theta_2)^{-1}$$

$$= \frac{r_1^n}{r_2^n}\frac{(\cos\theta_1 + i\sin\theta_1)^n}{(\cos n\theta_2 + i\sin n\theta_2)} = \frac{r_1^n}{r_2^n} \times \frac{(\cos\theta_1 + i\sin\theta_1)^n}{(\cos\theta_2 + i\sin\theta_2)^n} = \frac{Z_1^n}{Z_2^n}$$

$= R.H.S$

57 $\boxed{1.3+2}$

Q

(ii) $|e^{iz}|=1$

**Proof** Since $e^{iz} = \cos z + i\sin z$

$\Rightarrow |e^{iz}| = |\cos z + i\sin z|$

$= \sqrt{\cos^2 z + \sin^2 z} = 1$ = R.H.S.

×————————×

(iv) $e^{z_1}=e^{z_2} \Leftrightarrow z_1 - z_2 = 2k\pi i$, where k is an integer

**Proof** Suppose that

$e^{z_1}=e^{z_2} \Rightarrow e^{z_1}\cdot e^{-z_2} = e^{z_2}\cdot e^{-z_2}$

$e^{z_1-z_2} = e^{z_2-z_2} = e^0 = 1$

Put $z = z_1 - z_2$,

$e^z = 1$, if and only if $z = 2k\pi i$

k is any integer, then

$e^{2k\pi i} = \cos 2k\pi + i\sin 2k\pi$, where $k = 0, \pm1, \pm2, \ldots$

$= 1 + 0i = 1$

(Since $\cos 2k\pi = 1$, $\sin 2k\pi = 0$)

$e^{z_1-z_2} = 1 \Rightarrow e^{z_1-z_2} = e^0 = e$

$e^{z_1}\cdot e^{-z_2} = 1 \Rightarrow e^{z_1} = e^{z_2}$

suppose that

$e^z = 1$, taking $z = x+iy$

$\Rightarrow e^{x+iy} = e^x(\cos y + i\sin y) = 1$

$= e^x\cos y + i e^x\sin y = 1 = 1+0i$

$\Rightarrow e^x\cos y = 1$ and $e^x\sin y = 0$

but $e^x \neq 0 \Rightarrow \cos y = 1$ and $\sin y = 0$

2nd Method

$e^{z_1} = e^{z_2}$

$\Rightarrow \dfrac{e^{z_1}}{e^{z_2}} = 1$ or

$\Rightarrow e^{z_1-z_2} = 1$

which is possible only if $z_1 - z_2$ is an integral multiple of $2\pi i$ (as proved in (iii))

$\Rightarrow z_1 - z_2 = 2\pi i K$ where K is integer

Conversely,

$z_1 - z_2 = 2\pi i K$

$e^{z_1-z_2} = e^{2\pi i K}$

$= \cos 2K\pi + i\sin 2K\pi$

$= 1+0$

$= 1$

$e^{z_1-z_2} = 1$

$e^{z_1} = e^{z_2}$

(58) $\boxed{1\cdot 3-3}$

Q. imp

(iii) $e^z=1 \Leftrightarrow z$ is an integral multiple of $2\pi i$

Proof Let $z = 2\pi i k$ where $k$ is any integer

i.e $k = 0, \pm 1, \pm 2, \pm 3, ---$

Then $e^z = e^{2\pi i k} = \cos 2\pi k + i \sin 2\pi k$ —— (E)

$= 1 + 0 = 1$

Since $\cos 2\pi k = 1$ and $\sin 2\pi k = 0$ when $k$ is any integer

Conversely Suppose that $e^z = 1$.

and if $z = x+iy$

$\Rightarrow e^z = e^{x+iy} = e^x e^{iy} = e^x[\cos y + i \sin y]$

$\therefore 1 = e^x \cos y + i e^x \sin y$ $\quad \because e^z = 1$ supposed

Comparing Real & Imaginary Parts

$\Rightarrow 1 = e^x \cos y \quad \& \quad 0 = e^x \sin y$

$\Rightarrow$ Since $e^x \neq 0 \therefore \cos y \neq 0$ $\quad \Rightarrow \because e^x \neq 0, \therefore \sin y = 0$

$\sin y = 0 \Rightarrow \boxed{y = n\pi}$ where $n$ is any integer

i.e $n = 0, \pm 1, \pm 2, ---$

Now $\cos y = \cos n\pi = (-1)^n$ where $n = 0, \pm 1, \pm 2 ---$

$\therefore e^x \cos y = 1$

becomes $e^x(-1)^n = 1$

$\therefore (-1)^n > 0$

$\Rightarrow n$ must be even i.e $\boxed{n = 2k}$ where $k = 0, \pm 1, \pm 2, ---$

Since $e^x > 0$ and product of two +ives or product of two −ives is positive. So $(-1)^n$ is +ive $\because$ product of $e^x$ & $(-1)^n$ is +ive 1 and $e^x > 0$ so $(-1)^n$ must be +ive

$\therefore \Rightarrow e^x(-1)^{2k} = 1 \Rightarrow e^x \cdot 1 = 1 = e^0 \Rightarrow x = 0$

$\Rightarrow z = 0 + iy = in\pi = i(2k)\pi = 2k\pi i$

where $k = 0, \pm 1, \pm 2, ---$

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36

and it only possible when ~~y is an even integer~~

i.e $y = 2k\pi$ and $x = 0$

$\Rightarrow Z = 0 + iy = 0 + 2k\pi i$

$\Rightarrow Z_1 - Z_2 = 2k\pi i$

x ——— x

(v) Show that $|e^z| = e^x$, when $Z = x + iy$

Sol $L.H.S = |e^z| = |e^{x+iy}|$ Since $z = x + iy$ (given)

$= |e^x \cdot e^{iy}| = |e^x||e^{iy}|$

$= |e^x| \, |\cos y + i\sin y|$

$= |e^x| \sqrt{\cos^2 y + \sin^2 y} = |e^x| \cdot 1 = |e^x| = R.H.S$

x ——— x

(vi) $e^{z_1} \cdot e^{z_2} \cdots\cdots e^{z_n} = e^{z_1 + z_2 + z_3 + \cdots\cdots z_n}$ where $n = 1, 2, 3 \ldots$

Proof we shall prove it by Induction.

Case-1 Put $n = 2$, then

$e^{z_1} \cdot e^{z_2} = e^{z_1 + z_2}$

Hence c-1 is true for $n = 2$

2nd Method

$z_1 = (x_1 + iy_1)$ $z_2 = (x_2 + iy_2)$

LHS $e^{x_1+iy_1} \cdot e^{x_2+iy_2} \cdot e^{x_3+iy_3} \cdots\cdots e^{x_n+iy_n}$

$= e^{x_1+x_2+\cdots x_n} (\cos y_1 + i\sin y_1) \cdots\cdots (\cos y_n + i\sin y_n)$

$= e^{x_1+x_2\cdots x_n} [\cos(y_1+y_2\cdots y_n) + i\sin(y_1+y_2 \cdots y_n)]$

$= e^{x_1+x_2\cdots x_n} e^{i(y_1+y_2 \cdots\cdots y_n)}$

$= e^{(x_1+iy_1)+(x_2+iy_2)+\cdots\cdots(x_n+iy_n)}$

$= e^{z_1+z_2+\cdots\cdots z_n}$

C-2 Let it is true for $n = k$.

$e^{z_1} \cdot e^{z_2} \cdot e^{z_3} \cdots\cdots e^{z_k} = e^{z_1 + z_2 + z_3 + \cdots + z_k}$ → (1)

Now we have to prove that it is true for $n = k+1$. For 'x' Eq (1) by $e^{z_{k+1}}$ both sides, we get

$$\left(e^{z_1}\cdot e^{z_2}\cdot e^{z_3}\cdots\cdots e^{z_k}\right)\cdot e^{z_{k+1}} = e^{z_1+z_2+z_3+\cdots z_k}\cdot e^{z_{k+1}}$$

$$\Rightarrow e^{z_1}\cdot e^{z_2}\cdot e^{z_3}\cdots\cdots e^{z_{k+1}} = e^{z_1+z_2+z_3+\cdots z_k+z_{k+1}}$$

$$= e^{z_1+z_2+\cdots\cdots+z_{k+1}}$$

$\Rightarrow$ given statement is true for $n = k+1$. Thus it is true for all +ive integral values of $n$.

x ———— x

**(vii)** $(e^z)^n = e^{nz}$, where $n$ is any integer

Proof: let $z = x+iy$, then

$$L.H.S = (e^z)^n = \left(e^{x+iy}\right)^n = \left(e^x\cdot e^{iy}\right)^n = \left[e^x(\cos y + i\sin y)\right]^n$$

$$= e^{nx}\left[\cos ny + i\sin ny\right] \quad \text{(De moivres Th.)}$$

$$= e^{nx}\cdot e^{iny} = e^{nx+iny} = e^{n(x+iy)}$$

$$= e^{nz} = R.H.S.$$

x ———— x

**Q.2 (i)** Prove that $\forall\, z_1, z_2, z_3 \in \mathbb{C}$

$$1+\tan^2 z = \sec^2 z$$

L.H.S $\quad 1+\tan^2 z$ $\qquad \left(\because \tan z = \dfrac{e^{iz}-e^{-iz}}{i(e^{iz}+e^{-iz})}\right)$

$$= 1+\left(\frac{e^{iz}-e^{-iz}}{i(e^{iz}+e^{-iz})}\right)^2$$

$$= 1+\frac{e^{2iz}+e^{-2iz}-2}{i^2(e^{2iz}+e^{-2iz}+2)} = 1+\frac{2-e^{2iz}-e^{-2iz}}{e^{2iz}+e^{-2iz}+2}$$

$(\because i^2=-1)$

$$= \frac{\cancel{e^{2iz}} + \cancel{e^{-2iz}} + 2 + 2 - \cancel{e^{2iz}} - \cancel{e^{-2iz}}}{e^{2iz} + e^{-2iz} + 2}$$

$$= \frac{4}{e^{2iz} + e^{-2iz} + 2} = \left(\frac{2}{e^{iz} + e^{-iz}}\right)^2 = \sec^2 z$$
$= R.H.S.$

(ii) $1 + \cot^2 z = \operatorname{cosec}^2 z$



Proof $\underline{L.H.S} = 1 + \cot^2 z$

$$= 1 + \left(\frac{i(e^{iz} + e^{-iz})}{e^{iz} - e^{-iz}}\right)^2 = 1 + \frac{(-1)(e^{2iz} + e^{-2iz} + 2)}{e^{2iz} + e^{-2iz} - 2}$$

(using $i^2 = -1$)

$$= \frac{\cancel{e^{2iz}} + \cancel{e^{-2iz}} - 2 - \cancel{e^{2iz}} - \cancel{e^{-2iz}} - 2}{e^{2iz} + e^{-2iz} - 2}$$

$$= \frac{-4}{e^{2iz} + e^{-2iz} - 2} = \frac{4i^2}{(e^{iz} - e^{-iz})^2} = \left(\frac{2i}{e^{iz} - e^{-iz}}\right)^2$$

$= \operatorname{cosec}^2 z = R.H.S$



(iii) $\sin(z_1 - z_2) = \sin z_1 \cos z_2 - \cos z_1 \sin z_2$

$\underline{R.H.S.}$ $\sin z_1 \cos z_2 - \cos z_1 \sin z_2$

$$= \left(\frac{e^{iz_1} - e^{-iz_1}}{2i}\right)\left(\frac{e^{iz_2} + e^{-iz_2}}{2}\right) - \left(\frac{e^{iz_1} + e^{-iz_1}}{2}\right)\left(\frac{e^{iz_2} - e^{-iz_2}}{2i}\right)$$

$$= \frac{e^{iz_1} \cdot e^{iz_2} - e^{-iz_1} \cdot e^{-iz_2} + e^{iz_1} \cdot e^{-iz_2} - e^{-iz_1} \cdot e^{iz_2}}{4i} - \frac{e^{iz_1} \cdot e^{iz_2} - e^{iz_1} \cdot e^{-iz_2} + e^{-iz_1} \cdot e^{iz_2} - e^{-iz_1} \cdot e^{-iz_2}}{4i}$$

$$\frac{\cancel{e^{iz_1+iz_2}} + e^{iz_1-iz_2} - e^{-iz_1+iz_2} - e^{-iz_1-iz_2} - e^{iz_1+iz_2} + e^{iz_1-iz_2} - e^{-iz_1+iz_2} + \cancel{e^{-iz_1-iz_2}}}{4i}$$

$$= \frac{2\left(e^{iz_1-iz_2} - e^{-iz_1+iz_2}\right)}{4i} = \frac{e^{i(z_1-z_2)} - e^{-i(z_1-z_2)}}{2i}$$

$$= \sin(z_1 - z_2) = L.H.S.$$

× ——— ×

**(iv)** $\cos(z_1+z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$

R.H.S $= \cos z_1 \cos z_2 - \sin z_1 \sin z_2$

$$= \left(\frac{e^{iz_1}+e^{-iz_1}}{2}\right)\left(\frac{e^{iz_2}+e^{-iz_2}}{2}\right) - \left(\frac{e^{iz_1}-e^{-iz_1}}{2i}\right)\left(\frac{e^{iz_2}-e^{-iz_2}}{2i}\right)$$

$$\frac{e^{iz_1}\cdot e^{iz_2} + \cancel{e^{iz_1} e^{-iz_2}} + \cancel{e^{-iz_1}\cdot e^{iz_2}} + e^{-iz_1}\cdot e^{-iz_2}}{4}$$

$$+ \frac{e^{iz_1}\cdot e^{iz_2} - \cancel{e^{iz_1} e^{-iz_2}} - \cancel{e^{-iz_1} e^{iz_2}} + e^{-iz_1} e^{-iz_2}}{4}$$

$$= \frac{e^{iz_1}\cdot e^{iz_2} + e^{-iz_1}\cdot e^{-iz_2} + e^{iz_1} e^{iz_2} + e^{-iz_1}\cdot e^{-iz_2}}{4}$$

$$= \frac{2\left(e^{iz_1+iz_2} + e^{-iz_1-iz_2}\right)}{4} = \frac{e^{i(z_1+z_2)} + e^{-i(z_1+z_2)}}{2}$$

$$= \cos(z_1+z_2) = R.H.S.$$

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**Q.2 (vi)** $\cos 2z = \cos^2 z - \sin^2 z = 2\cos^2 z - 1 = 1 - 2\sin^2 z$

**PROOF** $\cos^2 z - \sin^2 z = \left(\frac{e^{iz}+e^{-iz}}{2}\right)^2 - \left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2$

$$= \frac{e^{2iz}+e^{-2iz}+2}{4} - \frac{e^{2iz}+e^{-2iz}-2}{4i^2}$$

$$= \cancel{\frac{e^{2iz}+e^{-2iz}+2-e^{2iz}-e^{-2iz}+2}{4}}$$

$$= \frac{e^{2iz}+e^{-2iz}+\cancel{2}}{4} + \frac{e^{2iz}+e^{-2iz}-\cancel{2}}{4}$$

$$= \frac{2\left(e^{2iz}+e^{-2iz}\right)}{4} = \frac{e^{2iz}+e^{-2iz}}{2}$$

$$= \cos 2z = L.H.S$$

$$2\cos^2 z - 1 = 2\left(\frac{e^{iz}+e^{-iz}}{2}\right)^2 - 1$$

$$= \cancel{2}\left(\frac{e^{2iz}+e^{-2iz}+2}{4_2}\right) - 1$$

$$= \frac{e^{2iz}+e^{-2iz}+2}{2} - 1 = \frac{e^{2iz}+e^{-2iz}+\cancel{2}-\cancel{2}}{2}$$

$$= \frac{e^{2iz}+e^{-2iz}}{2} = \cos 2z = L.H.S$$

$$1 - 2\sin^2 z = 1 - 2\left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2$$

$$= 1 - 2\left(\frac{e^{2iz}+e^{-2iz}-2}{4i^2}\right)$$

$$= 1 + \frac{e^{2iz}+e^{-2iz}-2}{2} \qquad (\because i^2 = -1)$$

$$= \frac{\cancel{2}+e^{2iz}+e^{-2iz}-\cancel{2}}{2} = \frac{e^{2iz}+e^{-2iz}}{2} = \cos 2z = L.H.S$$

2(vii) $\sin 2z = 2\sin z\cos z$

$$R.H.S. = 2\sin z\cos z$$
$$= \cancel{2}\left(\frac{e^{iz}-e^{-iz}}{2i}\right)\left(\frac{e^{iz}+e^{-iz}}{\cancel{2}}\right)$$
$$= \frac{e^{2iz}-e^{-2iz}}{2i} = \sin 2z = L.H.S.$$

x ———— x

2(viii) $\cos z_1 - \cos z_2 = 2\sin\frac{z_1+z_2}{2}\sin\frac{z_2-z_1}{2}$

$L.H.S.\ \cos z_1 - \cos z_2$

$$= \frac{e^{iz_1}+e^{-iz_1}}{2} - \frac{e^{iz_2}+e^{-iz_2}}{2}$$

$\boxed{\sin z = \frac{e^{iz}-e^{-iz}}{2i}}$

$$= \left(e^{iz_1}+e^{-iz_1}-e^{iz_2}-e^{-iz_2}\right)/2 \longrightarrow (1)$$

$R.H.S.\ 2\sin\frac{z_1+z_2}{2}\sin\frac{z_2-z_1}{2}$

$$= \cancel{2}\left(\frac{e^{i\left(\frac{z_1+z_2}{2}\right)}-e^{-i\left(\frac{z_1+z_2}{2}\right)}}{\cancel{2}i}\right)\left(\frac{e^{i\left(\frac{z_2-z_1}{2}\right)}-e^{-i\left(\frac{z_2-z_1}{2}\right)}}{2i}\right)$$

$$= \frac{1}{2i^2}\left[e^{\frac{iz_1+iz_2}{2}}\cdot e^{\frac{iz_2-iz_1}{2}} - e^{\frac{iz_1+iz_2}{2}}\cdot e^{\frac{-iz_2+iz_1}{2}} - e^{\frac{-iz_1-iz_2}{2}}\, e^{\frac{iz_2-iz_1}{2}} + e^{\frac{-iz_1-iz_2}{2}}\cdot e^{\frac{-iz_2+iz_1}{2}}\right]$$

$$= -\frac{1}{2}\left[e^{\frac{2iz_2}{2}} - e^{\frac{2iz_1}{2}} - e^{\frac{-2iz_1}{2}} + e^{\frac{-2iz_2}{2}}\right]$$

$$= \frac{e^{iz_1}+e^{-iz_1}-e^{iz_2}-e^{-iz_2}}{2} \longrightarrow (2)$$

from (1) and (2)

$$L.H.S. = R.H.S.$$

1·3–10 66

**Q.2 (ix)** $\sin z_1 + \sin z_2 = 2 \sin \frac{z_1+z_2}{2} \cos \frac{z_1-z_2}{2}$

$$L.H.S. = \sin z_1 + \sin z_2$$
$$= \frac{e^{iz_1} - e^{-iz_1}}{2i} + \frac{e^{iz_2} - e^{-iz_2}}{2i}$$
$$= \frac{e^{iz_1} - e^{-iz_1} + e^{iz_2} - e^{-iz_2}}{2i} \longrightarrow (1)$$

$$R.H.S. = 2 \sin \frac{z_1+z_2}{2} \cos \frac{z_1-z_2}{2}$$
$$= \cancel{2}\left(\frac{e^{\frac{i(z_1+z_2)}{2}} - e^{-i\left(\frac{z_1+z_2}{2}\right)}}{2i}\right)\left(\frac{e^{i\left(\frac{z_1-z_2}{2}\right)} + e^{-i\left(\frac{z_1-z_2}{2}\right)}}{\cancel{2}}\right)$$
$$= \frac{1}{2i}\left[\begin{array}{l} e^{\frac{iz_1+iz_2}{2}} \cdot e^{\frac{iz_1-iz_2}{2}} + e^{\frac{iz_1+iz_2}{2}} \cdot e^{\frac{-iz_1+iz_2}{2}} \\ - e^{\frac{-iz_1-iz_2}{2}} \cdot e^{\frac{iz_1-iz_2}{2}} - e^{\frac{-iz_1-iz_2}{2}} \cdot e^{\frac{-iz_1+iz_2}{2}} \end{array}\right]$$
$$= \frac{1}{2i}\left[e^{\frac{2iz_1}{2}} + e^{\frac{2iz_2}{2}} - e^{-\frac{2iz_2}{2}} - e^{-\frac{2iz_1}{2}}\right]$$
$$= \frac{1}{2i}\left(e^{iz_1} - e^{-iz_1} + e^{iz_2} - e^{-iz_2}\right) \longrightarrow (2)$$

from (1) and (2) $\Rightarrow$ L.H.S. = R.H.S.

× ——— ×

**Q.2 (x)** $\sin 3z = 3 \sin z - 4 \sin^3 z$

L.H.S. $3 \sin z - 4 \sin^3 z$
$$= 3\left(\frac{e^{iz} - e^{-iz}}{2i}\right) - 4\left(\frac{e^{iz} - e^{-iz}}{2i}\right)^3$$
$$= 3\left(\frac{e^{iz} - e^{-iz}}{2i}\right) - 4\left(\frac{e^{3iz} - 3e^{2iz}e^{-iz} + 3e^{iz}e^{-2iz} - e^{-3iz}}{8i^3}\right)$$
$$= \frac{3e^{iz} - 3e^{-iz}}{2i} - \cancel{4}\left(\frac{e^{3iz} - 3e^{iz} + 3e^{-iz} - e^{-3iz}}{-\cancel{4}(2i)}\right)$$

$$= \frac{3e^{iz} - 3e^{-iz} + e^{3iz} - 3e^{iz} + 3e^{-iz} - e^{-3iz}}{2i}$$

$$= \frac{e^{3iz} - e^{-3iz}}{2i} = \sin 3z = L.H.S.$$

**Q.2(xi)** $\tan(z_1+z_2) = \dfrac{\tan z_1 + \tan z_2}{1 - \tan z_1 \tan z_2}$

**Sol** If $z_1, z_2$ are complex numbers, then

$\sin(z_1+z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$

and $\cos(z_1+z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$

$\Rightarrow L.H.S. = \tan(z_1+z_2)$

$$= \frac{\sin(z_1+z_2)}{\cos(z_1+z_2)}$$ , putting values we get

$$= \frac{\sin z_1 \cos z_2 + \cos z_1 \sin z_2}{\cos z_1 \cos z_2 - \sin z_1 \sin z_2}$$

$\div$ each term of numer. and denome- by $\cos z_1 \cdot \cos z_2$, we get

$$= \frac{\frac{\sin z_1 \cos z_2}{\cos z_1 \cos z_2} + \frac{\cos z_1 \sin z_2}{\cos z_1 \cos z_2}}{1 - \frac{\sin z_1}{\cos z_1} \frac{\sin z_2}{\cos z_2}}$$

$$= \frac{\frac{\sin z_1}{\cos z_1} + \frac{\sin z_2}{\cos z_2}}{1 - \frac{\sin z_1}{\cos z_1} \cdot \frac{\sin z_2}{\cos z_2}}$$

$$= \frac{\tan z_1 + \tan z_2}{1 - \tan z_1 \tan z_2} = R.H.S$$

**Q·2(xii)** $\tan(z_1-z_2) = \dfrac{\tan z_1 - \tan z_2}{1+\tan z_1 \tan z_2}$

L.H.S $\tan(z_1-z_2)$

$$= \frac{\sin(z_1-z_2)}{\cos(z_1-z_2)} \quad \text{---}\to (i)$$

we know that if $z_1$ and $z_2$ are any complex numbers then

$$\sin(z_1-z_2) = \sin z_1 \cos z_2 - \cos z_1 \sin z_2$$

and $\cos(z_1-z_2) = \cos z_1 \cos z_2 + \sin z_1 \sin z_2$

putting these values in (i) we get

$$\tan(z_1-z_2) = \frac{\sin z_1 \cos z_2 - \cos z_1 \sin z_2}{\cos z_1 \cos z_2 + \sin z_1 \sin z_2}$$

÷ nume- and denome- by $\cos z_1 \cos z_2$

we get

$$\tan(z_1-z_2) = \frac{\frac{\sin z_1 \cos z_2}{\cos z_1 \cos z_2} - \frac{\cos z_1 \sin z_2}{\cos z_1 \cos z_2}}{1 + \frac{\sin z_1}{\cos z_1}\cdot\frac{\sin z_2}{\cos z_2}}$$

$$= \frac{\frac{\sin z_1}{\cos z_1} - \frac{\sin z_2}{\cos z_2}}{1+\frac{\sin z_1}{\cos z_1}\cdot\frac{\sin z_2}{\cos z_2}}$$

$$= \frac{\tan z_1 - \tan z_2}{1+\tan z_1 \cdot \tan z_2} = \text{R.H.S.}$$

X ——————————— X

**1.3 (i)** Show that

$$\overline{\sin z} = \sin \bar{z}$$

L.H.S $\overline{\sin z}$

let $z = x+iy$, then

$$\sin z = \sin(x+iy)$$
$$= \sin x \cos(iy) + \cos x \sin(iy)$$
$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

$\because \cos iz = \cosh z$ and $\sin iz = i \sinh z$

$$\Rightarrow \overline{\sin z} = \overline{\sin x \cosh y + i \cos x \sinh y}$$
$$= \sin x \cosh y - i \cos x \sinh y \longrightarrow (1)$$

and $\sin \bar{z} = \sin(\overline{x+iy}) = \sin(x-iy)$
$$= \sin x \cos(iy) - \sin(iy) \cos x$$
$$= \sin x \cosh y - i \sinh y \cos x \longrightarrow (2)$$

from (1) and (2)
$$\overline{\sin z} = \sin \bar{z}$$

x ———————— x

**Q.3(ii)** $\overline{\cos z} = \cos \bar{z}$

$\because \cos iz = \cosh z$ and $\sin iz = i \sinh z$

L.H.S: $\overline{\cos z}$

let $z = x+iy$, then
$$\cos z = \cos(x+iy) = \cos x \cos iy - \sin x \sin iy$$
$$= \cos x \cosh y - i \sin x \sinh y$$
$$\Rightarrow \overline{\cos z} = \cos x \cosh y + i \sin x \sinh y \longrightarrow (1)$$

and $\cos \bar{z} = \cos(\overline{x+iy}) = \cos(x-iy)$
$$= \cos x \cos iy + \sin x \sin iy$$
$$= \cos x \cosh y + i \sin x \sinh y \longrightarrow (2)$$

from (1) and (2)
$$\overline{\cos z} = \cos \bar{z}$$

x ———————— x

**Q.3(iii)** $\overline{\tan z} = \tan \bar{z}$

Sol L.H.S $= \overline{\tan z}$

(20) $\boxed{1\cdot 3-14}$

let $z = x+iy$

then $\tan z = \tan(x+iy)$

$= \dfrac{\tan x + \tan iy}{1-\tan x \tan iy}$ $\quad \because \tan iz = i\tanh z$

$\tan z = \dfrac{\tan x + i\tanh y}{1 - i\tan x \tanh y}$

$\Rightarrow \overline{\tan z} = \overline{\left(\dfrac{\tan x + i\tanh y}{1 - i\tan x \tanh y}\right)}$ $\quad \boxed{\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}}$

$= \dfrac{\overline{\tan x + i\tanh y}}{\overline{1 - i\tan x \tanh y}}$

$= \dfrac{\tan x - i\tanh y}{1 + i\tan x \tanh y}$ $\quad \cancel{\tan} \tan x$

$= \dfrac{\tan x - \tan iy}{1+\tan x \tan iy} = \tan(x-iy) = \tan \bar{z}$

$= R.H.S.$

x———x

**Q.3(iv)** $\sin(-z) = -\sin z$

L.H.S $\sin(-z) = \dfrac{e^{i(-z)} - e^{-i(-z)}}{2i}$

$= \dfrac{e^{-iz} - e^{iz}}{2i} = -\left(\dfrac{e^{iz} - e^{-iz}}{2i}\right)$

$= -\sin z = R.H.S$

x———x

Q.3(v) $\cos(-z) = \cos z$

L.H.S $\cos(-z) = \dfrac{e^{i(-z)} + e^{-i(-z)}}{2}$

$$= \frac{e^{-iz}+e^{+iz}}{2} = \frac{e^{iz}+e^{-iz}}{2} = \cos z = R.H.S$$

**Q.3(vi)** $\tan(-z) = -\tan z$

L.H.S. $\tan(-z) = \dfrac{e^{i(-z)} - e^{-i(-z)}}{i\left(e^{i(-z)} + e^{-i(-z)}\right)} = \dfrac{e^{-iz} - e^{iz}}{i\left(e^{-iz} + e^{iz}\right)}$



$$= -\left[\frac{e^{iz} - e^{-iz}}{i\left(e^{iz} + e^{-iz}\right)}\right] = -\tan z = R.H.S.$$

**Q.3(vii)** $\sinh(-z) = -\sinh z$

L.H.S. $\sinh(-z) = \dfrac{e^{-z} - e^{-(-z)}}{2} = \dfrac{e^{-z} - e^{+z}}{2}$

$$= -\left(\frac{e^{z} - e^{-z}}{2}\right) = -\sinh z = R.H.S.$$

**Q.3(viii)** $\cosh(-z) = \cosh z$

L.H.S. $= \cosh(-z) = \dfrac{e^{-z} + e^{-(-z)}}{2} = \dfrac{e^{-z} + e^{z}}{2} = \dfrac{e^{z} + e^{-z}}{2} = \cosh z$
$= R.H.S$

**Q.3(ix)** $\tanh(-z) = -\tanh z$

L.H.S $= \tanh(-z) = \dfrac{e^{-z} - e^{-(-z)}}{e^{-z} + e^{-(-z)}} = \dfrac{e^{-z} - e^{z}}{e^{-z} + e^{z}}$

$$= -\left(\frac{e^{z} - e^{-z}}{e^{z} + e^{-z}}\right) = -\tanh z = R.H.S.$$

**Q.3(x)** $\overline{\tanh z} = \tanh \bar{z}$

Sol We know that $i\tanh z = \tan(iz)$

Ex # $\boxed{1.3-16}$

Q4 (i) $\cosh^2 z - \sinh^2 z = 1$ To Prove

$\cos^2 z + \sin^2 z = 1$ by putting $z = iz$

$\cos^2 iz + \sin^2 iz = 1$ $\because \cos iz = \cosh z$

$\cosh^2 z + (i\sinh z)^2 = 1$ $\because \sin iz = i\sinh z$

$\cosh^2 z - \sinh^2 z = 1$ $\because i^2 = -1$

(ii) To Prove $\text{sech}^2 z = 1 - \tanh^2 z$

We know

$\sec^2 z = 1 + \tan^2 z$ by putting $z = iz$

$\sec^2 iz = 1 + \tan^2 iz$ $\because \sec iz = \text{sech} z$

$\text{sech}^2 z = 1 + (i\tanh z)^2$ $\because \tan iz = i\tanh z$

$\text{sech}^2 z = 1 - \tanh^2 z$ $\because i^2 = -1$

(iii) To Prove $\text{cosech}^2 z = \coth^2 z - 1$

We know

$1 + \cot^2 z = \text{cosec}^2 z$ by putting $z = iz$

$1 + \cot^2 iz = \text{cosec}^2 iz$

$1 + (-i\coth z)^2 = (-i\,\text{cosech} z)^2$ $\because \cot iz = -i\coth z$, $\because \text{cosec}\, iz = -i\,\text{cosech} z$

$1 - \coth^2 z = -\text{cosech}^2 z$

$\text{cosech}^2 z = \coth^2 z - 1$

(iv) $\cosh 2z = \cosh^2 z + \sinh^2 z = 2\cosh^2 z - 1 = 1 + 2\sinh^2 z$

We know

$\cos 2z = \cos^2 z - \sin^2 z$

$\cos 2iz = (\cos iz)^2 - (\sin iz)^2$

$= \cosh^2 z - (i\sinh z)^2$

$\cosh 2z = \cosh^2 z + \sinh^2 z$ proved

$= \cosh^2 z + (\cosh^2 z - 1)$

$= 2\cosh^2 z - 1$ proved

$= 2(1 + \sinh^2 z) - 1$

$= 2 + 2\sinh^2 z - 1$

$= 1 + 2\sinh^2 z$ proved

OR $\cosh 2z = 2\cosh^2 z - 1$

We know $\cos 2z = 2\cos^2 z - 1$

$\cos 2iz = 2(\cos iz)^2 - 1$

$\cosh 2z = 2\cosh^2 z - 1$ proved

OR $\cosh 2z = 1 + 2\sinh^2 z$

We know

$\cos 2z = 1 - 2\sin^2 z$

$\cos 2(iz) = 1 - 2(\sin iz)^2$

$\cosh 2z = 1 - 2(i\sinh z)^2$

$\cosh 2z = 1 + 2\sinh^2 z.$ proved

Q4(ix) $\tanh(z_1 \pm z_2) = \dfrac{\tanh z_1 \pm \tanh z_2}{1 \pm \tanh z_1 \tanh z_2}$

Sol We know

$\tan(z_1 \pm z_2) = \dfrac{\tan z_1 \pm \tan z_2}{1 \mp \tan z_1 \tan z_2}$

$\tan(iz_1 \pm iz_2) = \dfrac{\tan iz_1 \pm \tan iz_2}{1 \mp \tan iz_1 \tan iz_2}$ by putting $z_1 = iz_1$, $z_2 = iz_2$

$\tan i(z_1 + z_2) = \dfrac{i\tanh z_1 \pm i\tanh z_2}{1 \mp (i)^2 \tanh z_1 \tanh z_2}$ $(\because \tan i z_1 = i\tanh z_1)$

$i\tanh(z_1+z_2) = i\left(\dfrac{\tanh z_1 \pm \tanh z_2}{1 \pm \tanh z_1 \tanh z_2}\right)$

$\tanh(z_1+z_2) = \dfrac{\cancel{i}(\tanh z_1 \pm \tanh z_2)}{\cancel{i}(1 \pm \tanh z_1 \tanh z_2)}$ Proved

---

Q4(x) $\tanh 3z = \dfrac{3\tanh z + \tanh^3 z}{1 + 3\tanh^2 z}$

Sol We know

$\tan 3z = \dfrac{3\tan z - \tan^3 z}{1 - 3\tan^2 z}$

$\tan 3iz = \dfrac{3\tan iz - \tan^3 iz}{1 - 3\tan^2 iz}$

$i\tanh 3z = \dfrac{3i\tanh z - (i\tanh z)^3}{1 - 3ii\tanh^2 z}$

$i\tanh 3z = \dfrac{i(3\tanh z + \tanh^3 z)}{1 + 3\tanh^2 z}$

$\tanh 3z = \dfrac{\cancel{i}(3\tanh z + \tanh^3 z)}{\cancel{i}(1 + 3\tanh^2 z)}$ Proved

1·3−18

Q4 (v) To Prove $\sinh 2z = 2\sinh z \cosh z$

We know

$$\sin 2z = 2\sin z \cos z$$

$$\sin 2iz = 2\sin iz \cos iz$$

$$i\sinh 2z = 2(i\sinh z)(\cosh z)$$

$$\sinh 2z = 2\sinh z \cosh z \quad \text{proved}$$

(vi) To Prove $\sinh 3z = 3\sinh z + 4\sinh^3 z$

We know

$$\sin 3z = 3\sin z - 4\sin^3 z$$

$$\sin 3iz = 3\sin iz - 4\sin^3 iz$$

$$i\sinh 3z = 3i\sinh z - 4(i\sinh z)^3$$

$$i\sinh 3z = 3i\sinh z + 4i\sinh^3 z$$

$$\sinh 3z = \frac{\cancel{i}}{\cancel{i}}(3\sinh z + 4\sinh^3 z)$$

proved

vii) To Prove $\cosh 3z = 4\cosh^3 z - 3\cosh z$

We know

$$\cos 3z = 4\cos^3 z - 3\cos z$$

$$\cos 3iz = 4\cos^3 iz - 3\cos iz$$

$$\cosh 3z = 4\cosh^3 z - 3\cosh z \quad \text{proved}$$

viii) To Prove $\sinh(z_1 - z_2) = \sinh z_1 \cosh z_2 - \cosh z_1 \sinh z_2$

We know

$$\sin(z_1 - z_2) = \sin z_1 \cos z_2 - \cos z_1 \sin z_2$$

$$\sin(iz_1 - iz_2) = \sin iz_1 \cos iz_2 - \cos iz_1 \sin iz_2$$

$$\sin i(z_1 - z_2) = i\sinh z_1 \cosh z_2 - \cosh z_1 (i\sinh z_2)$$

$$i\sinh(z_1 - z_2) = i(\sinh z_1 \cosh z_2 - \cosh z_1 \sinh z_2)$$

$$\sinh(z_1 - z_2) = \sinh z_1 \cosh z_2 - \cosh z_1 \sinh z_2$$

proved

$\boxed{1.3 - 19}$ 85

**Q5** (i) If $z = x + iy$, Prove that
$\sin z = \sin x \cosh y + i \cos x \sinh y$

**Sol** $\sin z = \sin(x+iy)$
$= \sin x \cos iy + \cos x \sin iy$
$= \sin x \cosh y + i \cos x \sinh y$ Proved

ii) Prove that $\tan z = \dfrac{\sin 2x + i \sinh 2y}{\cos 2x + \cosh 2y}$

$\tan z = \tan(x+iy)$

$= \dfrac{\sin(x+iy)}{\cos(x+iy)}$

x & ÷ by $2\cos(x-iy)$

$= \dfrac{2\sin(x+iy)\cos(x-iy)}{2\cos(x+iy)\cos(x-iy)}$

$= \dfrac{\sin(x+\cancel{iy}+x-\cancel{iy}) + \sin(\cancel{x}+iy-\cancel{x}+iy)}{\cos(x+\cancel{iy}+x-\cancel{iy}) + \cos(\cancel{x}+iy-\cancel{x}+iy)}$

$= \dfrac{\sin 2x + \sin 2iy}{\cos 2x + \cos 2iy}$

$= \dfrac{\sin 2x + i \sinh 2y}{\cos 2x + \cosh 2y}$ Proved

**2nd Method**

$\tan z = \tan(x+iy) = \dfrac{\sin(x+iy)}{\cos(x+iy)} = \dfrac{\sin x \cos iy + \cos x \sin iy}{\cos x \cos iy - \sin x \sin iy}$

$= \dfrac{\sin x \cosh y + i \cos x \sinh y}{\cos x \cosh y - i \sin x \sinh y}$

$= \dfrac{(\sin x \cosh y + i \cos x \sinh y)}{(\cos x \cosh y - i \sin x \sinh y)} \times \dfrac{(\cos x \cosh y + i \sin x \sinh y)}{(\cos x \cosh y + i \sin x \sinh y)}$

$= \dfrac{\sin x \cos x \cosh^2 y - \cos x \sin x \sinh^2 y + i(\sin^2 x \cosh y \sinh y + \cos^2 x \sinh y \cosh y)}{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y}$

$= \dfrac{\sin x \cos x (\cosh^2 y - \sinh^2 y) + i \cosh y \sinh y (\sin^2 x + \cos^2 x)}{\cos^2 x (1 + \sinh^2 y) + \sin^2 x \sinh^2 y}$

(x & ÷ by 2)

$= \dfrac{\sin x \cos x \cdot 1 + i \cosh y \sinh y}{\cos^2 x + \sinh^2 y(\cos^2 x + \sin^2 x)} = \dfrac{2\sin x \cos x + i\, 2\cosh y \sinh y}{2\cos^2 x + 2\sinh^2 y}$

$\because \cos 2x = 2\cos^2 x - 1$
$\because \cosh 2y = 2\sinh^2 y + 1$
$\sinh 2y = 2\cosh y \sinh y$

$= \dfrac{\sin 2x + i \sinh 2y}{(2\cos^2 x - 1) + (2\sinh^2 y + 1)} = \dfrac{\sin 2x + i \sinh 2y}{\cos 2x + \cosh 2y}$ proved

(2+) Ex 1.4 Ex # 1.4 (i)

**Q11** Prove that
$\sec(x+iy) = \dfrac{2(\cos x \cosh y + i \sin x \sinh y)}{\cos 2x + \cosh 2y}$

**Sol** $\sec(x+iy)$

x & ÷ by $2\cos(x-iy)$

$= \dfrac{1}{\cos(x+iy)} \times \dfrac{2\cos(x-iy)}{2\cos(x-iy)}$

$= \dfrac{2(\cos x \cos iy + \sin x \sin iy)}{2\cos(x+iy)\cos(x-iy)}$

$= \dfrac{2(\cos x \cosh y + i \sin x \sinh y)}{\cos(x+\cancel{iy}+x-\cancel{iy}) + \cos(\cancel{x}+iy-\cancel{x}+iy)}$

$= \dfrac{2(\cos x \cosh y + i \sin x \sinh y)}{\cos 2x + \cos 2iy}$

$= \dfrac{2(\cos x \cosh y + i \sin x \sinh y)}{\cos 2x + \cosh 2y}$ proved

Q6 If $\sin(A+iB) = x+iy$ then show that,

(i) $\dfrac{x^2}{\sin^2 A} - \dfrac{y^2}{\cos^2 A} = 1$

(ii) $\dfrac{x^2}{\cosh^2 B} + \dfrac{y^2}{\sinh^2 B} = 1$

Sol

$$\sin(A+iB) = x+iy$$

$$\sin A\cos iB + \cos A\sin iB = x+iy$$

$$\Rightarrow \sin A\cosh B + i\cos A\sinh B = x+iy$$

Equating Real & Imaginary Parts

$\sin A\cosh B = x$ —(i) & $\cos A\sinh B = y$ —(ii)

$\dfrac{x}{\sin A} = \cosh B$ $\qquad$ $\dfrac{y}{\cos A} = \sinh B$

Subtracting $\dfrac{x^2}{\sin^2 A} - \dfrac{y^2}{\cos^2 A} = \cosh^2 B - \sinh^2 B$

$$\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = 1 \quad \text{proved}$$

from (i) $\dfrac{x}{\cosh B} = \sin A$

from (ii) $\dfrac{y}{\sinh B} = \cos A$

Squaring + adding

$$\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = \sin^2 A + \cos^2 A$$

$$\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = 1$$

proved

Q7 (i) If $\tan(\alpha+i\beta) = x+iy$, then Show that $x^2+y^2+2x\cot 2\alpha = 1$

Sol $\tan(\alpha+i\beta) = x+iy$

$$\alpha+i\beta = \tan^{-1}(x+iy) \text{ —(i)}$$

Conjugate $\alpha - i\beta = \tan^{-1}(x-iy)$ —(ii)

Add (i) & (ii) To eliminate $\beta$ $\quad \alpha+\cancel{i\beta}+\alpha-\cancel{i\beta} = \tan^{-1}(x+iy)+\tan^{-1}(x-iy)$

$$2\alpha = \tan^{-1}\left[\frac{(x+iy)+(x-iy)}{1-(x+iy)(x-iy)}\right]$$

$$2\alpha = \tan^{-1}\left[\frac{2x}{1-(x^2+y^2)}\right]$$

$$\tan 2\alpha = \frac{2x}{1-x^2-y^2}$$

Reciprocal

$$\cot 2\alpha = \frac{1-x^2-y^2}{2x}$$

$$2x\cot 2\alpha = 1-x^2-y^2$$

$$x^2+y^2+2x\cot 2\alpha = 1$$

proved.

(ii) If $\tan(\alpha+i\beta) = x+iy$ then show that $x^2+y^2-2y\coth 2\beta = -1$

Sol $\tan(\alpha+i\beta) = x+iy$

$$\alpha+i\beta = \tan^{-1}(x+iy) \text{ —(i)}$$

$$\alpha-i\beta = \tan^{-1}(x-iy) \text{ —(ii)}$$

(To eliminate $\alpha$) Subtracting

$$\alpha+i\beta-(\alpha-i\beta) = \tan^{-1}(x+iy)-\tan^{-1}(x-iy)$$

$$2i\beta = \tan^{-1}\left[\frac{(x+iy)-(x-iy)}{1+(x+iy)(x-iy)}\right]$$

$$\tan 2i\beta = \frac{2iy}{1+x^2+y^2}$$

$$i\tanh 2\beta = \frac{2iy}{1+x^2+y^2}$$

$$\tanh 2\beta = \frac{2y}{1+x^2+y^2}$$

$$\coth 2\beta = \frac{1+x^2+y^2}{2y}$$

$$2y\coth 2\beta = 1+x^2+y^2$$

$$-1 = x^2+y^2-2y\coth 2\beta$$

proved

Note $\tan^{-1}x+\tan^{-1}y = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$

1·3–21

Q8 If $\sin(\theta+\phi i) = \cos\alpha + i\sin\alpha$, prove that $\cos^2\theta = \pm\sin\alpha$

Sol $\sin(\theta+\phi i) = \cos\alpha + i\sin\alpha$

$\sin\theta\cos(i\phi)+\cos\theta\sin(i\phi) = \cos\alpha+i\sin\alpha$

$\sin\theta\cosh\phi + i\cos\theta\sinh\phi = \cos\alpha + i\sin\alpha$

Equating real & imaginary parts

$\sin\theta\cosh\phi = \cos\alpha$ & $\cos\theta\sinh\phi = \sin\alpha$

$\cosh\phi = \dfrac{\cos\alpha}{\sin\theta}$ $\qquad$ $\sinh\phi = \dfrac{\sin\alpha}{\cos\theta}$

squaring & subtracting

$$\cosh^2\phi - \sinh^2\phi = \frac{\cos^2\alpha}{\sin^2\theta} - \frac{\sin^2\alpha}{\cos^2\theta}$$

$$1 = \frac{\cos^2\alpha\cos^2\theta - \sin^2\alpha\sin^2\theta}{\sin^2\theta\cos^2\theta}$$

$$\sin^2\theta\cos^2\theta = \cos^2\alpha\cos^2\theta - \sin^2\alpha\sin^2\theta$$

$$(1-\cos^2\theta)\cos^2\theta = (1-\sin^2\alpha)\cos^2\theta - \sin^2\alpha(1-\cos^2\theta)$$

$$\cos^2\theta - \cos^4\theta = \cos^2\theta - \cancel{\sin^2\alpha\cos^2\theta} - \sin^2\alpha + \cancel{\sin^2\alpha\cos^2\theta}$$

$$\cancel{\cos^2\theta} - \cancel{\cos^2\theta} + \sin^2\alpha = \cos^4\theta$$

$$\pm\sin\alpha = \cos^2\theta \quad \text{Proved.}$$

Q10 Prove that $\sinh\left(\frac{x}{2}\right) = \sqrt{\dfrac{\cosh x-1}{2}}$ if $x \geq 0$

$= -\sqrt{\dfrac{\cosh x-1}{2}}$ if $x < 0$

Sol RHS $\sqrt{\dfrac{\cosh x - 1}{2}}$

$$= \sqrt{\frac{\left(\frac{e^x+e^{-x}}{2}\right)-1}{2}} = \sqrt{\frac{e^x+e^{-x}-2}{4}}$$

$$= \sqrt{\frac{\left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)^2}{4}} = \pm\left(\frac{e^{\frac{x}{2}} - e^{-\frac{x}{2}}}{2}\right) = \pm\sinh\left(\frac{x}{2}\right)$$

$\Rightarrow \sqrt{\dfrac{\cosh x-1}{2}} = \sinh\left(\frac{x}{2}\right)$ for $x \geq 0$

& $\sqrt{\dfrac{\cosh x-1}{2}} = -\sinh\left(\frac{x}{2}\right)$ for $x<0$

$\Rightarrow \sinh\left(\frac{x}{2}\right) = \sqrt{\dfrac{\cosh x-1}{2}}$ for $x \geq 0$

$\Rightarrow \sinh\left(\frac{x}{2}\right) = -\sqrt{\dfrac{\cosh x-1}{2}}$ for $x<0$

proved

2nd Method

$\cosh x = 1 + 2\sinh^2\frac{x}{2}$

$\cosh x - 1 = 2\sinh^2\frac{x}{2}$

$\dfrac{\cosh x-1}{2} = \sinh^2\frac{x}{2}$

$\pm\sqrt{\dfrac{\cosh x-1}{2}} = \sinh\frac{x}{2}$

**Q9. (i)** If $\tan(\theta+\phi i) = \tan\alpha + i\sec\alpha$,
Prove that $e^{2\phi} = \pm\cot\frac{\alpha}{2}$

**Sol** $\tan(\theta+\phi i) = \tan\alpha + i\sec\alpha$

$$\theta+\phi i = \tan^{-1}(\tan\alpha + i\sec\alpha) \quad \text{—(i)}$$

$$\theta-\phi i = \tan^{-1}(\tan\alpha - i\sec\alpha) \quad \text{—(ii)}$$

$$\tan^{-1}(\tan\alpha+i\sec\alpha) - \tan^{-1}(\tan\alpha - i\sec\alpha)$$

$$2i\phi = \tan^{-1}\left[\frac{(\tan\alpha+i\sec\alpha)-(\tan\alpha-i\sec\alpha)}{1+(\tan\alpha+i\sec\alpha)(\tan\alpha-i\sec\alpha)}\right]$$

$$\tan 2i\phi = \frac{2i\sec\alpha}{1+\tan^2\alpha+\sec^2\alpha}$$

$$i\tanh 2\phi = \frac{2i\sec\alpha}{\sec^2\alpha+\sec^2\alpha}$$

$$\tanh 2\phi = \frac{\cancel{2}\sec\alpha}{\cancel{2}\sec^2\alpha}$$

$$= \frac{1}{\sec\alpha}$$

$$\tanh 2\phi = \cos\alpha$$

$$\frac{e^{2\phi}-e^{-2\phi}}{e^{2\phi}+e^{-2\phi}} = \frac{\cos\alpha}{1}$$

Componendo Dividendo $\left(\because \frac{a}{b} = \frac{a+b}{a-b}\right)$

$$\frac{(e^{2\phi}-e^{-2\phi})+(e^{2\phi}+e^{-2\phi})}{(e^{2\phi}-e^{-2\phi})-(e^{2\phi}+e^{-2\phi})} = \frac{\cos\alpha+1}{\cos\alpha-1}$$

$$\frac{\cancel{2}e^{2\phi}}{-\cancel{2}e^{-2\phi}} = \frac{\cos\alpha+1}{-(1-\cos\alpha)}$$

$$e^{4\phi} = \frac{2\cos^2\alpha/2}{2\sin^2\frac{\alpha}{2}}$$

$$e^{4\phi} = \cot^2\frac{\alpha}{2}$$

$$\boxed{e^{2\phi} = \pm\cot\frac{\alpha}{2}}$$

**(ii)** If $\tan(\theta+\phi i) = \tan\alpha + i\sec\alpha$
Prove that $2\theta = n\pi + \frac{\pi}{2} + \alpha$

**Sol** Add (i)+(ii)

$$2\theta = \tan^{-1}(\tan\alpha+i\sec\alpha)+\tan^{-1}(\tan\alpha-i\sec\alpha)$$

$$= \tan^{-1}\left[\frac{(\tan\alpha+i\sec\alpha)+(\tan\alpha-i\sec\alpha)}{1-(\tan\alpha+i\sec\alpha)(\tan\alpha-i\sec\alpha)}\right]$$

$$\tan 2\theta = \frac{2\tan\alpha}{1-(\tan^2\alpha+\sec^2\alpha)}$$

$$= \frac{2\tan\alpha}{1-\tan^2\alpha-\sec^2\alpha}$$

$$= \frac{2\tan\alpha}{1-\tan^2\alpha-(1+\tan^2\alpha)}$$

$$= \frac{2\tan\alpha}{-2\tan^2\alpha}$$

$$= -\frac{1}{\tan\alpha}$$

$$\tan 2\theta = -\cot\alpha$$

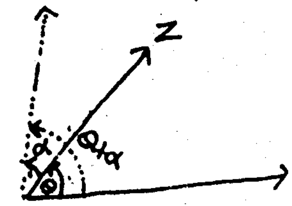
$$= \tan\left(\frac{\pi}{2}+\alpha\right)$$

$$\tan 2\theta = \tan\left(n\pi+\frac{\pi}{2}+\alpha\right)$$

$$2\theta = n\pi+\frac{\pi}{2}+\alpha$$

proved.

Q11 Show that multiplication of a vector $z$ by $e^{i\alpha}$, where $\alpha$ is a real number rotates the vector $z$ counterclockwise through an angle '$\alpha$'.




Sol $z = r(\cos\theta + i\sin\theta)$

$= r\, e^{i\theta}$

$z\, e^{i\alpha} = r\, e^{i\theta} e^{i\alpha}$

$= r\, e^{i(\theta+\alpha)}$

$z\, e^{i\alpha} = r\left(\cos(\theta+\alpha) + i\sin(\theta+\alpha)\right)$

Thus vector $z$ is rotated through an angle $\alpha$ counterclockwise.

Q12 Show that $2+i = \sqrt{5}\, e^{i\tan^{-1}\left(\frac{1}{2}\right)}$

$z = 2+i \quad \Rightarrow r = |z| = \sqrt{2^2+1^2} = \sqrt{5}$

$\cos\theta = \frac{x}{r} = \frac{2}{\sqrt{5}}$

$\sin\theta = \frac{y}{r} = \frac{1}{\sqrt{5}}$

$\tan\theta = \frac{1}{2} \quad \Rightarrow \quad \theta = \tan^{-1}\frac{1}{2}$

In polar form $z = r(\cos\theta + i\sin\theta) = r\, e^{i\theta}$

$2+i = \sqrt{5}\, e^{i\tan^{-1}\frac{1}{2}}$

(ii) $z = -3-4i \quad \Rightarrow \quad r = |z| = \sqrt{9+16} = \sqrt{25} = 5$

$\cos\theta = \frac{x}{r} = \frac{-3}{5}$

$\sin\theta = \frac{y}{r} = \frac{-4}{5}$

$\tan\theta = \frac{-4}{-3} \quad \Rightarrow \quad \theta = \tan^{-1}\left(\frac{4}{3}\right)$

$\theta = \pi + \tan^{-1}\left(\frac{4}{3}\right) \quad \because$ 3rd Quad

In polar form $z = r\, e^{i\theta}$

$= 5\, e^{i\left(\pi + \tan^{-1}\frac{4}{3}\right)}$ Ans.