Ex 1.2

1-2-1

① Write the following expression in the form of $a+ib$.

(i) $(-\sqrt{3}+i)^2$

Let $z = (-\sqrt{3}+i)$ $\Rightarrow x = -\sqrt{3}$, $y = 1$

$r = |z| = \sqrt{(-\sqrt{3})^2+1^2} = \sqrt{3+1} = 2$

$\cos\theta = \frac{x}{r} = \frac{-\sqrt{3}}{2}$ $\Rightarrow \theta = \cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)$

$\sin\theta = \frac{y}{r} = \frac{1}{2}$ $\Rightarrow \theta = \sin^{-1}\left(\frac{1}{2}\right)$

$\theta = \frac{5\pi}{6}$

$\because$ x is $-ve$, y is $+ve$ } So θ lies in 2nd Quad.

$\therefore$ Principal Arg of $z = \pi - \theta$
$= \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

Hence $z = r(\cos\theta + i\sin\theta)$

$-\sqrt{3}+i = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

squaring $(-\sqrt{3}+i)^2 = 2^2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)^2$

$= 4\left(\cos 2\left(\frac{5\pi}{6}\right) + i\sin 2\left(\frac{5\pi}{6}\right)\right)$

$= 4\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$

$= 4\left(\cos\left(\frac{-\pi}{3}\right) + i\sin\left(\frac{-\pi}{3}\right)\right)$

$= 4\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$

$= 4\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$

$= 2 - i2\sqrt{3}$

$\frac{5\pi}{3} - 2\pi = \frac{-\pi}{3}$

$\cos\frac{\pi}{3} = \frac{1}{2}$

$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

(ii) $(-3i)^4$

Let $z = -3i$
$= 0 - 3i$ $\Rightarrow x = 0$, $y = -3$

$r = |z| = \sqrt{0^2+(-3)^2} = 3.$

$\cos\theta = \frac{x}{r} = \frac{0}{3} = 0$ $\Rightarrow \theta = \cos^{-1}(0)$

$\sin\theta = \frac{y}{r} = \frac{-3}{3} = -1$ $\Rightarrow \theta = \cos^{-1}(-1)$

$\Rightarrow \theta = \frac{-\pi}{2}$

$\because$ x is $+ve$, y is $-ve$ } So θ lies in 4th Quad.

$\therefore$ Principal Arg of $z = -\theta$
$= -\frac{\pi}{2}$

Exercise 1.2 - Chapter 01
Mathematical Methods
by S.M. Yusuf, A. Majeed and M. Amin
ILMI KITAB KHANA, LAHORE.

$$z = r(\cos\theta + i\sin\theta)$$

$$-3i = 3\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$$

$$(-3i)^4 = 3^4\left[\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right]^4$$

$$= 81\left[\cos\left(4\cdot\frac{\pi}{2}\right) + i\sin\left(-4\cdot\frac{\pi}{2}\right)\right]$$

$$= 81(\cos(2\pi) + i\sin(-2\pi))$$

$$= 81(\cos 2\pi - i\sin 2\pi)$$

$$= 81\ (1 - 0)$$

$$(-3i)^4 = 81 \quad \text{Ans}$$

(iii) $\left(\dfrac{1-\sqrt{3}\,i}{1+\sqrt{3}\,i}\right)^6$

Let $z = \dfrac{1-\sqrt{3}\,i}{1+\sqrt{3}\,i}$

$$= \frac{1-\sqrt{3}\,i}{1+\sqrt{3}\,i} \times \frac{1-\sqrt{3}\,i}{1-\sqrt{3}\,i}$$

$$= \frac{(1-\sqrt{3}i)^2}{1+3}$$

$$= \frac{1^2 + (\sqrt{3}i)^2 - 2\cdot1\cdot\sqrt{3}\,i}{4}$$

$$= \frac{1-3-2\sqrt{3}\,i}{4}$$

$$= -\frac{2}{4} - \frac{2\sqrt{3}}{4}i$$

$$z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$\left.\begin{array}{l} x = -\frac{1}{2} \\ y = -\frac{\sqrt{3}}{2} \end{array}\right\} \Rightarrow r = |z| = \sqrt{\frac{1}{4} + \frac{3}{4}}$

$\boxed{r = 1}$

$\left.\begin{array}{ll} \cos\theta = \frac{x}{r} = -\frac{1/2}{1} & \Rightarrow \theta = \cos^{-1}\left(-\frac{1}{2}\right) \\ \sin\theta = \frac{y}{r} = -\frac{\sqrt{3}}{2} & \Rightarrow \theta = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) \end{array}\right\} \Rightarrow \theta = -\frac{2\pi}{3}$

$\because$ x is $-$ive, y is $-$ive $\}$ So θ lies in IIIrd Quad

So Principal Arg of $z = -(\pi - \theta)$

$= -\left(\pi - \frac{\pi}{3}\right) = \boxed{-\frac{2\pi}{3}}$

$$\therefore\ z = r(\cos\theta + i\sin\theta)$$

$$\therefore \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 1\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)$$

So $\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^6 = 1^6\left[\cos 6\left(-\frac{2\pi}{3}\right) + i\sin 6\left(-\frac{2\pi}{3}\right)\right]$

$= \cos(-4\pi) + i\sin(-4\pi)$ $\qquad \cos(-\theta)=\cos\theta$, $\sin(-\theta)=-\sin\theta$

$= \cos 4\pi - i\sin 4\pi$

$= 1-0 = 1$ ans.

Q. No. 2

Part-(i) Simplify $\dfrac{(\cos 2\theta + i\sin 2\theta)^5(\cos 3\theta - i\sin 3\theta)^6}{(\cos 4\theta - i\sin 4\theta)^7(\cos 5\theta + i\sin 5\theta)^8}$

Sol $= \dfrac{(\cos 2\theta + i\sin 2\theta)^5(\cos(-3\theta) + i\sin(-3\theta))^6}{[\cos(-4\theta) + i\sin(-4\theta)]^7[\cos 5\theta + i\sin 5\theta]^8}$

$= \dfrac{[(\cos\theta + i\sin\theta)^2]^5[(\cos\theta + i\sin\theta)^{-3}]^6}{[(\cos\theta + i\sin\theta)^{-4}]^7[(\cos\theta + i\sin\theta)^5]^8}$

$= \dfrac{(\cos\theta + i\sin\theta)^{10}(\cos\theta + i\sin\theta)^{-18}}{(\cos\theta + i\sin\theta)^{-28}(\cos\theta + i\sin\theta)^{40}}$

$= (\cos\theta + i\sin\theta)^{10-18+28-40}$

$= (\cos\theta + i\sin\theta)^{-20}$

$= \cos(-20\theta) + i\sin(-20\theta)$

$= \cos 20\theta - i\sin 20\theta$ ans.

Part-(ii) $\dfrac{(\cos\alpha - i\sin\alpha)^{11}}{(\cos\beta + i\sin\beta)^9}$

Sol $= \dfrac{\{\cos(-\alpha) + i\sin(-\alpha)\}^{11}}{(\cos\beta + i\sin\beta)^9}$ $\qquad \because$ To make $\cos\theta + i\sin\theta$ i.e. + ive sign.

$= [(\cos\alpha + i\sin\alpha)^{-1}]^{11}\{\cos\beta + i\sin\beta\}^{-9}$

$$= (\cos\alpha + i\sin\alpha)^{-11} (\cos\beta + i\sin\beta)^{-9}$$ [1-2-4]

$$= [\cos(-11\alpha) + i\sin(-11\alpha)][\cos(-9\beta) + i\sin(-9\beta)]$$

$$= (\cos(-11\alpha)\cos(-9\beta) - \sin(-11\alpha)\sin(-9\beta)) + i\begin{pmatrix}\cos(-11\alpha)\sin(-9\beta) \\ + \sin(-11\alpha)\cos(-9\beta)\end{pmatrix}$$

$$= \cos(-11\alpha - 9\beta) + i\sin(-11\alpha - 9\beta)$$

$$= \operatorname{Cis}(-11\alpha - 9\beta) = \operatorname{Cis}(-(11\alpha + 9\beta)) \quad \text{Ans.}$$

x ———— x

Part-(iii) $$\frac{(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)}{(\cos\gamma + i\sin\gamma)(\cos\delta + i\sin\delta)}$$

Sol $$= \frac{(\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\cos\alpha\sin\beta + \sin\alpha\cos\beta)}{(\cos\gamma\cos\delta - \sin\gamma\sin\delta) + i(\sin\gamma\cos\delta + \cos\gamma\sin\delta)}$$

$$= \frac{\cos(\alpha+\beta) + i\sin(\alpha+\beta)}{\cos(\gamma+\delta) + i\sin(\gamma+\delta)}$$

$$= [\cos(\alpha+\beta) + i\sin(\alpha+\beta)][\cos(\gamma+\delta) + i\sin(\gamma+\delta)]^{-1}$$

$$= [\cos(\alpha+\beta) + i\sin(\alpha+\beta)][\cos(-\gamma-\delta) + i\sin(-\gamma-\delta)]$$

$$= [\cos(\alpha+\beta)\cos(-\gamma-\delta) - \sin(\alpha+\beta)\sin(-\gamma-\delta)]$$
$$+ i[\sin(\alpha+\beta)\cos(-\gamma-\delta) + \sin(-\gamma-\delta)\cos(\alpha+\beta)]$$

$$= \cos(\alpha+\beta-\gamma-\delta) + i\sin(\alpha+\beta-\gamma-\delta)$$

$$= \operatorname{Cis}(\alpha+\beta-\gamma-\delta)$$

x ———— x

Part-(iv) $$\left(3\operatorname{cis}\frac{\pi}{6}\right)^7 \Big/ \left(4\operatorname{cis}\frac{\pi}{3}\right)^6$$

Sol: $$= 3^7\left(\operatorname{cis}\frac{\pi}{6}\right)^7 \Big/ 4^6\left(\operatorname{cis}\frac{\pi}{3}\right)^6$$

$$= \frac{3^7}{4^6}\cdot\frac{\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^7}{\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^6}$$

1.2-5

$$= \frac{3^7}{4^6} \cdot \left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{-6}$$

$$= \frac{3^7}{4^6} \cdot \left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)\left(\cos\left(-6\cdot\frac{\pi}{3}\right) + i\sin\left(-6\cdot\frac{\pi}{3}\right)\right)$$

$$= \frac{3^7}{4^6}\left[\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right]\left[\cos(-2\pi) + i\sin(-2\pi)\right]$$

$$= \frac{3^7}{4^6}\left[\cos\left(\frac{7\pi}{6} - 2\pi\right) + i\sin\left(\frac{7\pi}{6} - 2\pi\right)\right]$$

$\because \cos\frac{7\pi}{6}\cos(-2\pi) - \sin\frac{7\pi}{6}\sin(-2\pi) = \cos\left(\frac{7\pi}{6} + (-2\pi)\right)$

$$= \frac{3^7}{4^6}\, Cis\left(-\frac{5\pi}{6}\right) \quad \text{Ans.}$$

Q.3 (i) Prove that $\left[(\cos\theta - \cos\phi) + i(\sin\theta - \sin\phi)\right]^n + \left[(\cos\theta - \cos\phi) - i(\sin\theta - \sin\phi)\right]^n$

$$= 2^{n+1}\sin^n\left(\frac{\theta-\phi}{2}\right)\cos n\left(\frac{\theta+\phi+\pi}{2}\right)$$

Sol: L.H.S. $\left[(\cos\theta - \cos\phi) + i(\sin\theta - \sin\phi)\right]^n + \left[(\cos\theta - \cos\phi) - i(\sin\theta - \sin\phi)\right]^n$

using formulas

$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

and $\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$ we get

$$= \left[\left(-2\sin\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}\right) + i\left(2\cos\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}\right)\right]^n + \left[\left(-2\sin\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}\right) - i\left(2\cos\frac{\theta+\phi}{2}\cdot\sin\frac{\theta-\phi}{2}\right)\right]^n$$

$$= 2^n\sin^n\left(\frac{\theta-\phi}{2}\right)\left[-\sin\frac{\theta+\phi}{2} + i\cos\frac{\theta+\phi}{2}\right]^n + 2^n\sin^n\left(\frac{\theta-\phi}{2}\right)\left[-\sin\frac{\theta+\phi}{2} - i\cos\frac{\theta+\phi}{2}\right]^n$$

$\because \cos\left(\theta+\frac{\pi}{2}\right) = -\sin\theta$ and $\sin\left(\theta+\frac{\pi}{2}\right) = \cos\theta$

$$= 2^n\sin^n\left(\frac{\theta-\phi}{2}\right)\left[\left\{\cos\left(\frac{\pi}{2}+\frac{\theta+\phi}{2}\right) + i\sin\left(\frac{\pi}{2}+\frac{\theta+\phi}{2}\right)\right\}^n + \left\{\cos\left(\frac{\pi}{2}+\frac{\theta+\phi}{2}\right) - i\sin\left(\frac{\pi}{2}+\frac{\theta+\phi}{2}\right)\right\}^n\right]$$

$$= 2^n \sin^n\left(\frac{\theta-\phi}{2}\right)\left[\left\{\cos n\left(\frac{\pi+\theta+\phi}{2}\right)+i\sin n\left(\frac{\pi+\theta+\phi}{2}\right)\right\} + \left\{\cos n\left(\frac{\pi+\theta+\phi}{2}\right)-i\sin n\left(\frac{\pi+\theta+\phi}{2}\right)\right\}\right]$$

$$= 2^n \sin^n\left(\frac{\theta-\phi}{2}\right)\left[\cos n\left(\frac{\pi+\theta+\phi}{2}\right) \cancel{+ i \sin n\left(\frac{\pi+\theta+\phi}{2}\right)} + \cos n\left(\frac{\pi+\theta+\phi}{2}\right) \cancel{- i \sin n\left(\frac{\pi+\theta+\phi}{2}\right)}\right]$$

$$= 2^n \sin^n\left(\frac{\theta-\phi}{2}\right) 2\cos n\left(\frac{\pi+\theta+\phi}{2}\right)$$

$$= 2^{n+1} \sin^n\left(\frac{\theta-\phi}{2}\right)\cos n\left(\frac{\pi+\theta+\phi}{2}\right) = RHS$$

x ——————— x

P (ii) $$\left(\frac{1+\sin x+i\cos x}{1+\sin x-i\cos x}\right)^n = \cos n\left(\frac{\pi}{2}-x\right)+i\sin n\left(\frac{\pi}{2}-x\right)$$

Sol: L.H.S $$\left(\frac{1+\sin x+i\cos x}{1+\sin x-i\cos x}\right)^n$$

$$= \left(\frac{(\sin^2 x+\cos^2 x)+(\sin x+i\cos x)}{1+\sin x-i\cos x}\right)^n$$

$$= \left[\frac{(\sin x+i\cos x)(\sin x-i\cos x)+(\sin x+i\cos x)}{1+\sin x-i\cos x}\right]^n$$

$$= \left\{\frac{(\sin x+i\cos x)(\cancel{\sin x-i\cos x+1})}{(\cancel{1+\sin x-i\cos x})}\right\}^n$$

$$= (\sin x+i\cos x)^n$$

$\because \cos\left(\frac{\pi}{2}-x\right) = \sin x$

and

$\sin\left(\frac{\pi}{2}-x\right) = \cos x$

$$= \left[\cos\left(\frac{\pi}{2}-x\right)+i\sin\left(\frac{\pi}{2}-x\right)\right]^n$$

applying Demoivre's th.

$$= \cos n\left(\frac{\pi}{2}-x\right)+i\sin n\left(\frac{\pi}{2}-x\right)$$

$= RHS$

Q.4 $2\cos\theta = x + \frac{1}{x}$; $2\cos\phi = y + \frac{1}{y}$, $2\cos\psi = z + \frac{1}{z}$,
then prove that 1.2-7

Part-(i) $2\cos(\theta+\phi+\psi) = xyz + \frac{1}{xyz}$

PROOF we have $2\cos\theta = x + \frac{1}{x}$, $\Rightarrow x = \cos\theta + i\sin\theta$

$2\cos\phi = y + \frac{1}{y} \Rightarrow y = \cos\phi + i\sin\phi$

and. $2\cos\psi = z + \frac{1}{z} \Rightarrow z = \cos\psi + i\sin\psi$

Then $x\cdot y\cdot z = (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)(\cos\psi + i\sin\psi)$

$= [(\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\sin\theta\cos\phi + \sin\phi\cdot\cos\theta)]$
$[\cos\psi + i\sin\psi]$

$= [\cos(\theta+\phi) + i\sin(\theta+\phi)][\cos\psi + i\sin\psi]$

$= (\cos(\theta+\phi)\cos\psi - \sin(\theta+\phi)\sin\psi)$
$+ i(\sin(\theta+\phi)\cos\psi + \cos(\theta+\phi)\sin\psi)$

$x\cdot y\cdot z = \cos(\theta+\phi+\psi) + i\sin(\theta+\phi+\psi) \longrightarrow (1)$

Similarly $\frac{1}{x\cdot y\cdot z} = \frac{1}{(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)(\cos\psi + i\sin\psi)}$

$= \frac{1}{\cos(\theta+\phi+\psi) + i\sin(\theta+\phi+\psi)}$

$= [\cos(\theta+\phi+\psi) + i\sin(\theta+\phi+\psi)]^{-1}$

$\frac{1}{xyz} = \cos(\theta+\phi+\psi) - i\sin(\theta+\phi+\psi) \longrightarrow (2)$

Adding Eqns (1) and (2), we get

$xyz + \frac{1}{xyz} = 2\cos(\theta+\phi+\psi)$ Proved.

x———x

Part-(ii) $2\cos(m\theta + n\phi) = x^m y^n + \frac{1}{x^m y^n}$

Sol: we are given that $2\cos\theta = x + \frac{1}{x}$
it is because if $x = \cos\theta + i\sin\theta$

$1.2-8$

and $2\cos\phi = y + \frac{1}{y} \Rightarrow y = \cos\phi + i\sin\phi$

So $x^m = (\cos\theta + i\sin\theta)^m = \cos m\theta + i\sin m\theta$

and $y^n = (\cos\phi + i\sin\phi)^n = \cos n\phi + i\sin n\phi$

then $x^m y^n = (\cos m\theta + i\sin m\theta)(\cos n\phi + i\sin n\phi)$

$= (\cos m\theta\cos n\phi - \sin m\theta\sin n\phi) + i(\sin m\theta\cos n\phi + \sin n\phi\cos m\theta)$

$$x^m y^n = \cos(m\theta + n\phi) + i\sin(m\theta + n\phi) \longrightarrow (1)$$

and $\frac{1}{x^m y^n} = \frac{1}{(\cos m\theta + i\sin m\theta)(\cos n\phi + i\sin \phi)}$

$= \frac{1}{\cos(m\theta + n\phi) + i\sin(m\theta + n\phi)}$

$= [\cos(m\theta + n\phi) + i\sin(m\theta + n\phi)]^{-1}$

$$\frac{1}{x^m y^n} = \cos(m\theta + n\phi) - i\sin(m\theta + n\phi) \longrightarrow (2)$$

Add equations (1) and (2), we get

$x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\theta + n\phi)$

Proved

Q.5(i) Find Three Cube roots of '$8i$'

Sol Let $Z^3 = 8i = 8(0 + i)$

or $Z^3 = 8\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$x = 0,\ y = 1,\ r = \sqrt{0+1} = 1$

$\left.\begin{array}{l}\sin\theta = \frac{1}{1} \\ \theta = \sin^{-1} 1\end{array}\right\}\ \theta = \frac{\pi}{2}$

means Ist and

or $Z^3 = 2^3\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$= 2^3\left(\cos\left(\frac{\pi}{2} + 2k\pi\right) + i\sin\left(\frac{\pi}{2} + 2k\pi\right)\right)$, where $k \in Z$

$$\Rightarrow Z_k = 2\left[\cos\left(2\pi k+\frac{\pi}{2}\right)+i\sin\left(2\pi k+\frac{\pi}{2}\right)\right]^{\frac{1}{3}}$$

where $k = 0, 1, 2.$

$$\therefore Z_k = 2\left[\cos\left(\frac{4k\pi+\pi}{6}\right)+i\sin\left(\frac{4k\pi+\pi}{6}\right)\right]$$

So put $k = 0, 1, 2,$ then required three roots are given by.

for $k=0$, $Z_0 = 2\left[\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right] = 2\left\{\frac{\sqrt{3}}{2}+\frac{i\cdot 1}{2}\right\} \Rightarrow \boxed{\sqrt{3}+i = Z_0}$

for $k=1$, $Z_1 = 2\left\{\cos\left(\frac{4\pi+\pi}{6}\right)+i\sin\left(\frac{4\pi+\pi}{6}\right)\right\} = 2\left\{\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right\}$

$$= 2\left\{-\frac{\sqrt{3}}{2}+\frac{i}{2}\right\} \Rightarrow \boxed{Z_1 = -\sqrt{3}+i}$$

and 3rd root is obtained by $k=2$, we get

$$Z_2 = 2\left\{\cos\left(\frac{8\pi+\pi}{6}\right)+i\sin\left(\frac{8\pi+\pi}{6}\right)\right\} = 2\left[\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right]$$

$$= 2\left[\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right] = 2(0-i) \Rightarrow \boxed{Z_2 = -2i}$$

x ———— x

Part-(ii) Find four fourth roots of each of the following complex number.

(a) $-16i$, (b) 64, (c) $-2\sqrt{3}+2i$

(a) Since we have find four fourth roots of $-16i$

So put $z^4 = -16i = 16(0-i)$

$$= 2^4\left[\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right]$$

$x=0, y=-1$ $r=\sqrt{0^2+(-1)^2}=1$

$\left.\begin{array}{l}\cos\theta = \frac{x}{r} = \frac{0}{1} = 0 \Rightarrow \theta = \cos^{-1}0 \\ \sin\theta = \frac{y}{r} = \frac{-1}{1} = -1 \Rightarrow \theta = \cos^{-1}(-1)\end{array}\right\} \Rightarrow \theta = \frac{-\pi}{2}$

(angle in IV th Q ∴ $-\theta$ Hence $-\frac{\pi}{2}$)

$$z^4 = 2^4\left[\cos\left(2\pi k-\frac{\pi}{2}\right)+i\sin\left(2k\pi-\frac{\pi}{2}\right)\right]$$

So fourth root of $-16i$ is

$$Z_k = 2\left[\cos\left(2\pi k-\frac{\pi}{2}\right)+i\sin\left(2k\pi-\frac{\pi}{2}\right)\right]^{\frac{1}{4}}$$

where $k = 0, 1, 2, 3$

$$= 2\left[\cos\frac{1}{4}\left(\frac{4k\pi-\pi}{2}\right)+i\sin\left(\frac{4k\pi-\pi}{2}\right)\frac{1}{4}\right]$$

$$Z_k = 2\left[\cos\left(\frac{4k\pi-\pi}{8}\right)+i\sin\left(\frac{4k\pi-\pi}{8}\right)\right], \quad k = 0, 1, 2, 3 \longrightarrow (1)$$

So required four roots can be obtained by putting $K = 0, 1, 2, 3$ in (1), we get 1.2-10

$$Z_0 = 2\left[\cos\left(\frac{0-\pi}{8}\right) + i\sin\left(\frac{0-\pi}{8}\right)\right] = 2\left(\cos\left(\frac{-\pi}{8}\right) + i\sin\left(\frac{-\pi}{8}\right)\right)$$

or $\boxed{Z_0 = {}_{2}\,\text{cis}\left(\frac{-\pi}{8}\right)}$

for $K=1$, $Z_1 = 2\left[\cos\left(\frac{4\pi-\pi}{8}\right) + i\sin\left(\frac{4\pi-\pi}{8}\right)\right] = 2\ \text{cis}\left(\frac{3\pi}{8}\right)$

for $K=2$, $Z_2 = 2\left[\cos\left(\frac{8\pi-\pi}{8}\right) + i\sin\left(\frac{8\pi-\pi}{8}\right)\right] = 2\ \text{cis}\left(\frac{7\pi}{8}\right)$

for $K=3$, $Z_3 = 2\left[\cos\left(\frac{12\pi-\pi}{8}\right) + i\sin\left(\frac{12\pi-\pi}{8}\right)\right] = 2\ \text{cis}\left(\frac{11\pi}{8}\right)$ $\because \frac{11\pi}{8} - 2\pi = -\frac{5\pi}{8}$

$= 2\,\text{cis}\left(-\frac{5\pi}{8}\right)$

x ——— x

(b) let $Z^4 = 64 = 64(1+0i) = 64(\cos 0 + i\sin 0)$

$= 64(\cos(2\pi K + 0) + i\sin(2\pi K + 0))$

$Z^4 = 64[\cos 2\pi K + i\sin 2\pi K]$

So four 4th root of 64 are

$Z_k = (64)^{\frac{1}{4}}[\cos 2\pi K + i\sin 2\pi K]^{\frac{1}{4}}$, where $k = 0,1,2,3$

$= (16\times 4)^{\frac{1}{4}}\left[\cos\frac{2\pi K}{4} + i\sin\frac{2\pi K}{4}\right]$

$= (2^4 \cdot 2^2)^{\frac{1}{4}}\left(\cos\frac{\pi K}{2} + i\sin\frac{\pi K}{2}\right)$

$Z_k = 2\sqrt{2}\left[\cos\frac{\pi K}{2} + i\sin\frac{\pi K}{2}\right]$

for first root, put $K=0$, $\Rightarrow Z_0 = 2\sqrt{2}[\cos 0 + i\sin 0] = 2\sqrt{2}(1+i0)$

$\boxed{Z_0 = 2\sqrt{2}}$

put $K=1$, $Z_1 = 2\sqrt{2}\left\{\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right\} = 2\sqrt{2}\{0+i\} \Rightarrow \boxed{2\sqrt{2}\,i = Z_1}$

put $K=2$, $Z_2 = 2\sqrt{2}\{\cos\pi + i\sin\pi\} = 2\sqrt{2}[-1+0i] \Rightarrow \boxed{Z_2 = -2\sqrt{2}}$

put $K=3$, $Z_3 = 2\sqrt{2}\left[\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right] = 2\sqrt{2}[0-i]$

$\Rightarrow \boxed{Z_3 = -2\sqrt{2}\,i}$

(C) Let $Z^4 = -2\sqrt{3} + 2i$ $r = |z| = \sqrt{(2\sqrt{3})^2 + 2^2} = \sqrt{16} = 4$

$x = -2\sqrt{3}$, $y = 2$, $r = 4 = |z|$

$$= 4\left[-\frac{\sqrt{3}}{2} + \frac{1}{2} i\right]$$

1-2-11

$\left.\begin{array}{l}\cos\theta = \frac{x}{r} = \frac{-2\sqrt{3}}{4} = -\frac{\sqrt{3}}{2} \\ \sin\theta = \frac{y}{r} = \frac{2}{4} = \frac{1}{2} \\ 2nd\ Q. \therefore \pi - \theta \\ So\ \pi - \frac{\pi}{6} = \boxed{\frac{5\pi}{6}}\end{array}\right\} \Rightarrow \theta = \frac{5\pi}{6}$

$$= 4\left[\cos\left(\frac{5\pi}{6}\right) + i \sin\left(\frac{5\pi}{6}\right)\right]$$

$$z^4 = 4\left[\cos\left(\frac{5\pi}{6} + 2k\pi\right) + i \sin\left(\frac{5\pi}{6} + 2k\pi\right)\right]$$

$$\Rightarrow z^4 = (4)\left[\cos\left(2k\pi + \frac{5\pi}{6}\right) + i \sin\left(2k\pi + \frac{5\pi}{6}\right)\right]$$

$$\therefore Z_k = (4)^{\frac{1}{4}}\left[\cos\left(\frac{12k\pi + 5\pi}{6}\right) + i \sin\left(\frac{12k\pi + 5\pi}{6}\right)\right]^{\frac{1}{4}}$$

where $k = 0, 1, 2, 3$

$$\Rightarrow Z_k = (2^2)^{\frac{1}{4}}\left[\cos\left(\frac{12k\pi + 5\pi}{24}\right) + i \sin\left(\frac{12k\pi + 5\pi}{24}\right)\right]$$

$$= 2^{\frac{1}{2}} \text{cis}\left(\frac{k\pi}{2} + \frac{5\pi}{24}\right) \quad \text{where } k = 0, 1, 2, 3$$

put $k = 0$, we get

$$Z_0 = \sqrt{2}\left[\cos\frac{5\pi}{24} + i \sin\frac{5\pi}{24}\right] = \sqrt{2}\ \text{cis}\ \frac{5\pi}{24}$$

for $k = 1$, $Z_1 = \sqrt{2}\left[\cos\frac{17\pi}{24} + i \sin\frac{17\pi}{24}\right] = \sqrt{2}\ \text{cis}\ \frac{17\pi}{24}$

for $k = 2$, $Z_2 = \sqrt{2}\left[\cos\frac{29\pi}{24} + i \sin\frac{29\pi}{24}\right] = \sqrt{2}\ \text{cis}\left(-\frac{19\pi}{24}\right)$ $\frac{29\pi}{24} - 2\pi = -\frac{19\pi}{24}$

for $k = 3$, $Z_3 = \sqrt{2}\left[\cos\frac{41\pi}{24} + i \sin\frac{41\pi}{24}\right] = \sqrt{2}\ \text{cis}\left(-\frac{7\pi}{24}\right)$ $\frac{41\pi}{24} - 2\pi = -\frac{7\pi}{24}$

x ——— x

Q.6 Find six 6th roots of (a) -1, (b) $1+i$

Sol (a) Let $Z^6 = -1 = (-1 + 0i)$ $r = |z| = \sqrt{(-1)^2 + 0^2} = +1$

$\left.\begin{array}{l}\cos\theta = \frac{x}{r} = \frac{-1}{1} = -1 \Rightarrow \theta = \cos^{-1}(-1) \\ \sin\theta = \frac{y}{r} = \frac{0}{1} = 0 \Rightarrow \theta = \sin^{-1}(0) \\ So\ QII \quad \therefore \pi - \theta \text{ or } \pi - 0 = \boxed{\pi}\end{array}\right\} \theta = \pi$

$$= 1\left[\cos\pi + i \sin\pi\right]$$

$$Z^6 = \left[\cos(2k\pi + \pi) + i \sin(2k\pi + \pi)\right]$$

So six 6th roots of '-1' are given by

$$Z_k = \left[\cos(2k\pi + \pi) + i \sin(2k\pi + \pi)\right]^{\frac{1}{6}}$$

or $Z_k = \cos\left(\frac{2k\pi + \pi}{6}\right) + i \sin\left(\frac{2k\pi + \pi}{6}\right)$, where $k = 0, 1, 2, 3, 4, 5$

So for $k = 0$, $Z_0 = \cos\frac{\pi}{6} + i \sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{i}{2}$

for $k=1$, $Z_1 = \cos\left(\frac{2\pi+\pi}{6}\right) + i \sin\left(\frac{2\pi+\pi}{6}\right) = \text{Cis}\,\frac{\pi}{2}$

$\Rightarrow \boxed{Z_1 = 0 + i}$

1.2-12

for $k=2$, $Z_2 = \cos\left(\frac{4\pi+\pi}{6}\right) + i \sin\left(\frac{4\pi+\pi}{6}\right) = \cos\frac{5\pi}{6} + i \sin\frac{5\pi}{6}$

$\Rightarrow \boxed{Z_2 = -\frac{\sqrt{3}}{2} + \frac{i}{2}}$ $\left(\because \cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2},\ \sin\frac{5\pi}{6} = \frac{1}{2}\right)$

for $k=3$, $Z_3 = \cos\left(\frac{6\pi+\pi}{6}\right) + i \sin\left(\frac{6\pi+\pi}{6}\right) = \cos\frac{7\pi}{6} + i \sin\frac{7\pi}{6}$

$\frac{7\pi}{6} - 2\pi = -\frac{5\pi}{6}$

$= \cos\left(-\frac{5\pi}{6}\right) + i \sin\left(-\frac{5\pi}{6}\right)$

$= \cos\frac{5\pi}{6} - i \sin\frac{5\pi}{6} = \boxed{-\frac{\sqrt{3}}{2} - \frac{i}{2}}$

for $k=4$, $Z_4 = \cos\left(\frac{8\pi+\pi}{6}\right) + i \sin\left(\frac{8\pi+\pi}{6}\right)$

$= \cos\frac{3\pi}{2} + i \sin\frac{3\pi}{2} = \text{Cis}\left(-\frac{\pi}{2}\right)$ $\quad \frac{3\pi}{2} - 2\pi = -\frac{\pi}{2}$

or $\boxed{Z_4 = 0 - i}$

for $k=5$, $Z_5 = \cos\left(\frac{10\pi+\pi}{6}\right) + i \sin\left(\frac{10\pi+\pi}{6}\right) = \cos\frac{11\pi}{6} + i \sin\frac{11\pi}{6}$

$Z_5 = \cos\left(\frac{-\pi}{6}\right) + i \sin\left(\frac{-\pi}{6}\right)$

or $\frac{11\pi}{6} - 2\pi = -\frac{\pi}{6}$ $\Rightarrow \boxed{Z_5 = \frac{\sqrt{3}}{2} - \frac{i}{2}}$

(b) Let $Z^6 = 1 + i$ $\quad \therefore |Z| = |1+i| = \sqrt{1+1} = \sqrt{2} = r$

$\theta = \tan^{-1}\frac{y}{x} = \frac{1}{1} = 1 \Rightarrow \boxed{\theta = \frac{\pi}{4}}$

OR $\cos\theta = \frac{x}{r} = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$

$\sin\theta = \frac{y}{r} = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$

x+ive y+ive so Ist Q. $\therefore \theta$

$= \sqrt{2}\left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i\right] = \sqrt{2}\left[\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}\right]$

$Z^6 = \sqrt{2}\left[\cos\left(2k\pi + \frac{\pi}{4}\right) + i \sin\left(2k\pi + \frac{\pi}{4}\right)\right]$

So six 6th roots of $1+i$ are

$Z_k = \left[(2)^{\frac{1}{2}}\right]^{\frac{1}{6}}\left[\cos\left(\frac{8\pi k+\pi}{4}\right) + i \sin\left(\frac{8k\pi+\pi}{4}\right)\right]^{\frac{1}{6}}$

or $Z_k = (2)^{\frac{1}{12}}\left[\cos\left(\frac{8\pi k+\pi}{24}\right) + i \sin\left(\frac{8\pi k+\pi}{24}\right)\right]$

where $k = 0, 1, 2, 3, 4, 5$

for $k=0$, $Z_0 = (2)^{\frac{1}{12}}\left[\cos\frac{\pi}{24} + i\sin\frac{\pi}{24}\right] = 2^{\frac{1}{12}} cis \frac{\pi}{24}$

for $k=1$, $Z_1 = (2)^{\frac{1}{12}}\left[\cos\frac{9\pi}{24} + i\sin\frac{9\pi}{24}\right] = 2^{\frac{1}{12}} cis \frac{3\pi}{8}$

for $k=2$, $Z_2 = (2)^{\frac{1}{12}}\left[\cos\frac{17\pi}{24} + i\sin\frac{17\pi}{24}\right] = 2^{\frac{1}{12}} cis \frac{17\pi}{24}$

for $k=3$, $Z_3 = (2)^{\frac{1}{12}}\left[\cos\frac{25\pi}{24} + i\sin\frac{25\pi}{24}\right]$

$= (2)^{\frac{1}{12}}\left[\cos\left(\pi+\frac{\pi}{24}\right) + i\sin\left(\pi+\frac{\pi}{24}\right)\right]$

$= (2)^{\frac{1}{12}}\left[-\cos\frac{\pi}{24} - i\sin\frac{\pi}{24}\right]$

$= -(2)^{\frac{1}{12}}\left[\cos\frac{\pi}{24} + i\sin\frac{\pi}{24}\right] = -2^{\frac{1}{12}} cis \frac{\pi}{24}$

(Also $\frac{25\pi}{24} - 2\pi = -\frac{23\pi}{24}$, $Z_3 = 2^{\frac{1}{12}} cis\left(-\frac{23\pi}{24}\right)$)

for $k=4$, $Z_4 = (2)^{\frac{1}{12}}\left[\cos\frac{33\pi}{24} + i\sin\frac{33\pi}{24}\right]$

$= (2)^{\frac{1}{12}}\left[\cos\left(\pi+\frac{9\pi}{24}\right) + i\sin\left(\pi+\frac{9\pi}{24}\right)\right]$

$= (2)^{\frac{1}{12}}\left[\cos\left(\pi+\frac{3\pi}{8}\right) + i\sin\left(\pi+\frac{3\pi}{8}\right)\right]$

$= (2)^{\frac{1}{12}}\left[-\cos\frac{3\pi}{8} - i\sin\frac{3\pi}{8}\right]$

$= -(2)^{\frac{1}{12}}\left[\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right]$

(Also $\frac{33\pi}{24} - 2\pi = -\frac{15\pi}{24}$, $Z_4 = 2^{\frac{1}{12}} cis\left(-\frac{15\pi}{24}\right)$)

for $k=5$, $Z_5 = (2)^{\frac{1}{12}}\left[\cos\frac{41\pi}{24} + i\sin\frac{41\pi}{24}\right]$

$= (2)^{\frac{1}{2}}\left[\cos\left(\pi+\frac{17\pi}{24}\right) + i\sin\left(\pi+\frac{17\pi}{24}\right)\right]$

$= +(2)^{\frac{1}{2}}\left[-\cos\frac{17\pi}{24} - i\sin\frac{17\pi}{24}\right]$

$= -(2)^{\frac{1}{2}}\left[\cos\frac{17\pi}{24} + i\sin\frac{17\pi}{24}\right]$

(Also $\frac{41\pi}{24} - 2\pi = -\frac{7\pi}{24}$, $Z_5 = 2^{\frac{1}{12}} cis\left(-\frac{7\pi}{24}\right)$)

Q:7 Find the squares of all the fifth roots of $\frac{1}{2}+\frac{\sqrt{3}}{2}i$

Sol

$r=\sqrt{(\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}=\sqrt{\frac{1}{4}+\frac{3}{4}}=\boxed{1}$

$\cos\theta=\frac{x}{r}=\frac{1}{2} \Rightarrow \theta=\frac{\pi}{3}$

$\sin\theta=\frac{y}{r}=\frac{\sqrt{3}}{2} \Rightarrow \theta=\frac{\pi}{3}$

so Quad I

let $z^5=\frac{1}{2}+\frac{\sqrt{3}}{2}i = 1\left[\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right]$

or $z^5=\left[\cos\left(2k\pi+\frac{\pi}{3}\right)+i\sin\left(2k\pi+\frac{\pi}{3}\right)\right]$

$\boxed{1\cdot2-14}$

$$z_k=\left[\cos\left(\frac{6k\pi+\pi}{3}\right)+i\sin\left(\frac{6k\pi+\pi}{3}\right)\right]^{1/5}$$

$$z_k=\left[\cos\left(\frac{6k\pi+\pi}{15}\right)+i\sin\frac{6k\pi+\pi}{15}\right]$$ where $k=0,1,2,3,4$

Now square of all the fifth roots are

$$z_k=\left[\cos\left(\frac{6k\pi+\pi}{15}\right)+i\sin\left(\frac{6k\pi+\pi}{15}\right)\right]^2$$ where $k=0,1,2,3,4$

or $$z_k=\cos\left(\frac{12k\pi+2\pi}{15}\right)+i\sin\left(\frac{12k\pi+2\pi}{15}\right)$$ where $k=0,1,2,3,4$

for $k=0$, $z_0=\cos\frac{2\pi}{15}+i\sin\frac{2\pi}{15}=\text{cis}\,\frac{2\pi}{15}$

for $k=1$, $z_1=\cos\frac{14\pi}{15}+i\sin\frac{14\pi}{15}=\text{cis}\,\frac{14\pi}{15}$

for $k=2$, $z_2=\cos\frac{26\pi}{15}+i\sin\frac{26\pi}{15}$

$\frac{26\pi}{15}-2\pi=\frac{-4\pi}{15}$

$$=\cos\left(\frac{-4\pi}{15}\right)+i\sin\left(\frac{-4\pi}{15}\right)=\text{cis}\left(\frac{-4\pi}{15}\right)$$

for $k=3$, $z_3=\cos\frac{38\pi}{15}+i\sin\frac{38\pi}{15}$

$\frac{38\pi}{15}-2\pi=\frac{8\pi}{15}$

$$z_3=\cos\frac{8\pi}{15}+i\sin\frac{8\pi}{15}=\text{cis}\left(\frac{8\pi}{15}\right)$$ $\left(3+\frac{5}{15}\right)\pi$

$$z_4=\cos\frac{50\pi}{15}+i\sin\frac{50\pi}{15}$$

for $k=4$, $z_4=\cos\frac{20\pi}{15}+i\sin\frac{20\pi}{15}$

$\frac{50\pi}{15}-2\pi=\frac{20\pi}{15}$

$\frac{20\pi}{15}-2\pi=-\frac{10\pi}{15}=\frac{-2\pi}{3}$

$$=\cos\left(\frac{-2\pi}{3}\right)+i\sin\left(\frac{-2\pi}{3}\right)=\text{cis}\left(-\frac{2\pi}{3}\right)$$

Q.8 (i) Solve the equation $x^7+1=0$ [1.2-15]

Sol We have $x^7+1=0$

$$\Rightarrow x^7 = -1 = -1+0i = 1\left(\cos\pi + i\sin\pi\right)$$

or $x^7 = \cos(\pi+2k\pi)+i\sin(\pi+2k\pi)$

$$\Rightarrow x = \left[\cos(\pi+2k\pi)+i\sin(\pi+2k\pi)\right]^{\frac{1}{7}}$$

So Seven 7th roots of -1 are

$$x_k = \cos\left(\frac{\pi+2k\pi}{7}\right)+i\sin\left(\frac{\pi+2k\pi}{7}\right)$$

Where $k=0,1,2,3,4,5,6$

$\frac{9\pi}{7} = \left(1+\frac{2}{7}\right)\pi = \pi+\frac{2\pi}{7}$

for $k=0$, $x_0 = \cos\frac{\pi}{7}+i\sin\frac{\pi}{7} = \operatorname{cis}\frac{\pi}{7}$

for $k=1$, $x_1 = \cos\frac{3\pi}{7}+i\sin\frac{3\pi}{7} = \operatorname{cis}\frac{3\pi}{7}$

for $k=2$, $x_2 = \cos\frac{5\pi}{7}+i\sin\frac{5\pi}{7} = \operatorname{cis}\frac{5\pi}{7}$

for $k=3$, $x_3 = \cos\pi+i\sin\pi = -1+0i = -1$

for $k=4$, $x_4 = \cos\frac{9\pi}{7}+i\sin\frac{9\pi}{7}$

$\frac{9\pi}{7}-2\pi = -\frac{5\pi}{7}$

$$= \cos\left(-\frac{5\pi}{7}\right)+i\sin\left(-\frac{5\pi}{7}\right)$$

$$x_4 = \operatorname{cis}\left(-\frac{5\pi}{7}\right)$$

for $k=5$ $\quad x_5 = \cos\left(\frac{11\pi}{7}\right)+i\sin\left(\frac{11\pi}{7}\right)$

$\frac{11\pi}{7}-2\pi = -\frac{3\pi}{7}$

$$= \cos\left(-\frac{3\pi}{7}\right)+i\sin\left(-\frac{3\pi}{7}\right)$$

$$= \operatorname{cis}\left(-\frac{3\pi}{7}\right)$$

for $k=6$, $x_6 = \cos\frac{13\pi}{7}+i\sin\frac{13\pi}{7}$

$\frac{13\pi}{7}-2\pi = -\frac{\pi}{7}$

$$= \cos\left(-\frac{\pi}{7}\right)+i\sin\left(-\frac{\pi}{7}\right) = \operatorname{cis}\left(-\frac{\pi}{7}\right)$$

Note we can also take values of $k=0,\pm1,\pm2,\pm3$ instead of $k=0,1,2,3,4,5,6$

1.2-16

P-(ii) $x^7+x^4+x^3+1=0$

Sol $x^4[x^3+1]+1[x^3+1]=0$

$\Rightarrow (x^4+1)(x^3+1)=0$

$\Rightarrow x^4+1=0$ and $x^3+1=0$

In case of $x^4+1=0 \Rightarrow x^4=-1=-1+0i$

or $x^4=\cos\pi+i\sin\pi = \cos(2k\pi+\pi)+i\sin(2k\pi+\pi)$

$\Rightarrow x_k=\cos\left(\frac{2k\pi+\pi}{4}\right)+i\sin\left(\frac{2k\pi+\pi}{4}\right)$, where $k=0,1,2,3$

for $k=0$, $x_0=\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}=\frac{1}{\sqrt2}+\frac{i}{\sqrt2}$

for $k=1$, $x_1=\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}=-\frac{1}{\sqrt2}+\frac{i}{\sqrt2}$

for $k=2$, $x_2=\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}=\cos\frac{-3\pi}{4}+i\sin\left(\frac{-3\pi}{4}\right)=-\frac{1}{\sqrt2}-\frac{i}{\sqrt2}$

$\frac{5\pi}{4}-2\pi=-\frac{3\pi}{4}$

for $k=3$, $x_3=\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}=\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)=\frac{1}{\sqrt2}-\frac{i}{\sqrt2}$

$\frac{7\pi}{4}-2\pi=-\frac{\pi}{4}$

In case of $x^3+1=0 \Rightarrow x^3=-1=-1+0i$

or $x^3=\cos\pi+i\sin\pi = \cos(2k\pi+\pi)+i\sin(2k\pi+\pi)$

$\Rightarrow x_k=\cos\left(\frac{2k\pi+\pi}{3}\right)+i\sin\left(\frac{2k\pi+\pi}{3}\right)$, $k=0,1,-1$

for $k=-1$, $x_{-1}=\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)=\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}=\frac12-\frac{\sqrt3}{2}i$

for $k=0$, $x_0=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}=\frac12+\frac{\sqrt3}{2}i$

for $k=1$, $x_1=\cos\pi+i\sin\pi$

$=-1+0=-1$

1.2-17

$r = |z| = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$

Q.8(iii) $x^6 + 1 = \sqrt{3}\,i$

$\Rightarrow x^6 = -1 + \sqrt{3}\,i$

or $x^6 = 2\left[-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right]$ (x and ÷ by 2, we get)

2nd quad

or $x^6 = 2\left[\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right]$

For finding θ, find θ = cos⁻¹(−0.5)

$= 2\left[\cos\left(2k\pi + \frac{2\pi}{3}\right) + i\sin\left(2k\pi + \frac{2\pi}{3}\right)\right]$

$\Rightarrow x = (2)^{\frac{1}{6}}\left[\cos\left(2k\pi + \frac{2\pi}{3}\right) + i\sin\left(2k\pi + \frac{2\pi}{3}\right)\right]^{1/6}$

or $x = (2)^{\frac{1}{6}}\left[\cos\left(\frac{6k\pi + 2\pi}{18}\right) + i\sin\left(\frac{6k\pi + 2\pi}{18}\right)\right]$

where $k = 0, 1, 2, 3, 4, 5$

for $k = 0$, $x_0 = (2)^{\frac{1}{6}}\left[\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}\right]$

for $k = 1$, $x_1 = (2)^{\frac{1}{6}}\left[\cos\frac{8\pi}{18} + i\sin\frac{8\pi}{18}\right] = (2)^{\frac{1}{6}}\operatorname{cis}\left(\frac{4\pi}{9}\right)$

for $k = 2$, $x_2 = (2)^{\frac{1}{6}}\left[\cos\frac{7\pi}{9} + i\sin\frac{7\pi}{9}\right] = (2)^{\frac{1}{6}}\operatorname{cis}\left(\frac{7\pi}{9}\right)$

for $k = 3$, $x_3 = (2)^{\frac{1}{6}}\left[\cos\frac{20\pi}{18} + i\sin\frac{20\pi}{18}\right]$

$\frac{20\pi}{18} - 2\pi = \frac{-16\pi}{18}$

$= (2)^{\frac{1}{6}}\left[\cos\left(\frac{-16\pi}{18}\right) + i\sin\left(\frac{-16\pi}{18}\right)\right] = (2)^{\frac{1}{6}}\operatorname{cis}\left(\frac{-8\pi}{9}\right)$

for $k = 4$, $x_4 = (2)^{\frac{1}{6}}\left[\cos\frac{13\pi}{9} + i\sin\frac{13\pi}{9}\right] = (2)^{\frac{1}{6}}\operatorname{cis}\left(\frac{-5\pi}{9}\right)$

$\frac{13\pi}{9} - 2\pi = \frac{-5\pi}{9}$

for $k = 4$, $x_5 = (2)^{\frac{1}{6}}\left[\cos\frac{16\pi}{9} + i\sin\frac{16\pi}{9}\right]$

$\frac{16\pi}{9} - 2\pi = \frac{-2\pi}{9}$

$= (2)^{\frac{1}{6}}\left[\cos\left(\frac{-2\pi}{9}\right) + i\sin\left(\frac{-2\pi}{9}\right)\right] = (2)^{\frac{1}{6}}\operatorname{cis}\left(\frac{-2\pi}{9}\right)$

x———x

Q.9 Solve the equation $x^{12} - 1 = 0$ and find which of its roots satisfy the equation $x^4 + x^2 + 1 = 0$

Sol: $x^{12} - 1 = 0$ gives

$x^{12} = 1 = 1 + 0i = \cos 0 + i\sin 0$

or $x^{12} = \cos(0 + 2k\pi) + i\sin(0 + 2k\pi)$

Easy

(21)

12-18

Now $x^4+x^2+1=0$

2nd Method:

$x^4+x^2+1=0$

$x^4+x^2+x^2-x^2+1=0$

$x^4+2x^2+1-x^2=0$

$(x^2+1)^2-x^2=0$

$(x^2+1+x)(x^2+1-x)=0$

Either

$x^2+1+x=0$

$x=\dfrac{-1\pm\sqrt{1-4}}{2}$

$x=\dfrac{-1\pm i\sqrt{3}}{2}$

OR

$x^2+1-x=0$

$x=\dfrac{1\pm\sqrt{1-4}}{2}$

$=\dfrac{1\pm i\sqrt{3}}{2}$

Put $x^2=y$

$y^2+y+1=0$

$y=\dfrac{-1\pm\sqrt{1-4}}{2}=\dfrac{-1\pm i\sqrt{3}}{2}$

x & ÷ by 2 to complete square

$x^2=\dfrac{-2\pm2\sqrt{-3}}{4}$

$=\dfrac{-2\pm2\cdot1\cdot\sqrt{-3}}{4}$

$=\dfrac{(1-3)\pm2\cdot1\cdot\sqrt{-3}}{4}$

$x^2=\dfrac{1^2+(\sqrt{-3})^2\pm2\cdot1\cdot\sqrt{-3}}{4}$

$x^2=\left(\dfrac{1\pm\sqrt{-3}}{2}\right)^2$

$x=\pm\left(\dfrac{1\pm\sqrt{-3}}{2}\right)$

$\Rightarrow x=\dfrac{1\pm\sqrt{3}i}{2},\ \dfrac{-1\pm\sqrt{3}i}{2}$

$x=\dfrac{1+\sqrt{3}i}{2},\ \dfrac{1-\sqrt{3}i}{2},\ \dfrac{-1+\sqrt{3}i}{2},\ \dfrac{-1-\sqrt{3}i}{2}$

We see that in twelve 12th roots of $x^{12}=-1$

x_2, x_4, x_8 and x_{10} are satisfy the roots of equation $x^4+x^2+1=0$

x——————x

Q.10 Expand the following in series of Sines or Cosines of multiples of θ

$(a+b)^n=a^n+na^{n-1}\cdot b+\frac{n(n-1)}{\lfloor 2}a^{n-2}b^2+\ldots\ldots b^n$

(i) $\cos^4\theta$

Sol. let $x=\cos\theta+i\sin\theta$, then $\dfrac{1}{x}=\dfrac{1}{\cos\theta+i\sin\theta}$

$=(\cos\theta+i\sin\theta)^{-1}$

$\dfrac{1}{x}=\cos\theta-i\sin\theta$

$\Rightarrow x+\dfrac{1}{x}=2\cos\theta$

So $(2\cos\theta)^4=\left(x+\dfrac{1}{x}\right)^4$

$2^4\cos^4\theta=x^4+4x^3\left(\dfrac{1}{x}\right)+\dfrac{4\cdot3}{2\cdot1}x^2\cdot\dfrac{1}{x^2}+\dfrac{4\cdot3\cdot2}{3\cdot2\cdot1}x\cdot\dfrac{1}{x^3}+\dfrac{1}{x^4}$

$=x^4+4x^2+6+\dfrac{4}{x^2}+\dfrac{1}{x^4}$

or $x^{12} = \cos 2\pi k + i \sin 2\pi k$

$\Rightarrow x_k = (\cos 2\pi k + i \sin 2\pi k)^{1/12}$

$\Rightarrow x_k = \cos \frac{\pi k}{6} + i \sin \frac{\pi k}{6}$

where $k = 0, 1, 2, \ldots \ldots 11$

For $k=0$, $x_0 = \cos 0 + i \sin 0 = 1$

For $k=1$, $x_1 = \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} = \frac{\sqrt{3}}{2} + i \frac{1}{2} = \frac{\sqrt{3}+i}{2}$

For $k=2$, $x_2 = \cos \frac{2\pi}{6} + i \sin \frac{2\pi}{6} = \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2} i = \boxed{\frac{1+\sqrt{3}i}{2}}$

For $k=3$, $x_3 = \cos \frac{3\pi}{6} + i \sin \frac{3\pi}{6} = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = 0 + i = i$

For $k=4$, $x_4 = \cos \frac{4\pi}{6} + i \sin \frac{4\pi}{6} = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} = \cos\left(\frac{\pi}{3} + \frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3} + \frac{\pi}{3}\right)$

$= \cos \frac{\pi}{3} \cdot \cos \frac{\pi}{3} - \sin \frac{\pi}{3} \sin \frac{\pi}{3} + i\left(2 \sin \frac{\pi}{3} \cos \frac{\pi}{3}\right)$

$= \frac{1}{2} \cdot \frac{1}{2} - \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} + 2i \frac{\sqrt{3}}{2} \cdot \frac{1}{2}$

$= \frac{1}{4} - \frac{(\sqrt{3})^2}{4} + i \frac{\sqrt{3}}{2} = -\frac{2}{4} + \frac{\sqrt{3}i}{2} = -\frac{1}{2} + \frac{\sqrt{3}i}{2} = \boxed{\frac{-1+\sqrt{3}i}{2}}$

$x_5 = \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} = -\frac{\sqrt{3}}{2} + \frac{i}{2} = \frac{-\sqrt{3}+i}{2}$

$x_6 = \cos \pi + i \sin \pi = -1 + 0i = -1$

$\because \frac{7\pi}{6} - 2\pi = -\frac{5\pi}{6}$ $x_7 = \cos \frac{7\pi}{6} + i \sin \frac{7\pi}{6} = \cos\left(-\frac{5\pi}{6}\right) + i \sin\left(-\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2} - \frac{i}{2}$

$\frac{8\pi}{6} - 2\pi = -\frac{4\pi}{6}$ $x_8 = \cos \frac{8\pi}{6} + i \sin \frac{8\pi}{6} = \cos\left(-\frac{4\pi}{6}\right) + i \sin\left(-\frac{4\pi}{6}\right) = \operatorname{cis}\left(-\frac{2\pi}{3}\right) = \boxed{\frac{-1-i\sqrt{3}}{2}}$

$\frac{9\pi}{6} - 2\pi = -\frac{3\pi}{6}$ $x_9 = \cos \frac{9\pi}{6} + i \sin \frac{9\pi}{6} = \cos\left(-\frac{3\pi}{2}\right) + i \sin\left(-\frac{3\pi}{2}\right) = 0 - i(1) = -i$

$\frac{5\pi}{3} - 2\pi = -\frac{\pi}{3}$ $x_{10} = \cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3} = \cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right)$

$= \cos \frac{\pi}{3} - i \sin \frac{\pi}{3} = \boxed{\frac{1-\sqrt{3}i}{2}}$

$\frac{11\pi}{6} - 2\pi = -\frac{\pi}{6}$ $x_{11} = \cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6} = \cos\left(-\frac{\pi}{6}\right) + i \sin\left(-\frac{\pi}{6}\right)$

$= \cos \frac{\pi}{6} - i \sin \frac{\pi}{6} = \frac{\sqrt{3}}{2} - \frac{i}{2} = \frac{\sqrt{3}-i}{2}$

$$2^4\cos^4\theta = \left(x^4+\frac{1}{x^4}\right)+4\left(x^2+\frac{1}{x^2}\right)+6$$

If $x = \cos\theta + i\sin\theta$ then $x^4 = \cos 4\theta + i\sin 4\theta$

$\frac{1}{x^4} = \frac{1}{\cos 4\theta + i\sin 4\theta}$

$\frac{1}{x^4} = \cos 4\theta - i\sin 4\theta$

$x^4 + \frac{1}{x^4} = 2\cos 4\theta$

$$= 2\cos 4\theta + 4\cdot 2\cos 2\theta + 6$$

$$2^4\cos^4\theta = 2\left[\cos 4\theta + 4\cos 2\theta + 3\right]$$

$$\cos^4\theta = \frac{1}{2^3}\left[\cos 4\theta + 4\cos 2\theta + 3\right] = \frac{1}{8}\left[\cos 4\theta + 4\cos 2\theta + 3\right]$$

x———x

(ii) $\sin^4\theta$

Sol if $x = \cos\theta + i\sin\theta$, then $\frac{1}{x} = \cos\theta - i\sin\theta$

so $x - \frac{1}{x} = 2i\sin\theta$, thus

$$(2i\sin\theta)^4 = \left(x-\frac{1}{x}\right)^4 = x^4 - 4x^2 + 6 - \frac{4}{x^2} + \frac{1}{x^4}$$

(∵ similar to part (i))

$$2^4 i^4 \sin^4\theta = \left(x^4+\frac{1}{x^4}\right) - 4\left(x^2+\frac{1}{x^2}\right) + 6 \quad \boxed{1\cdot 2-2\theta}$$

$$= 2\cos 4\theta - 4(2\cos 2\theta) + 6$$

$$16 i^4\sin^4\theta = 2(\cos 4\theta - 4\cos 2\theta + 3)$$

$$\sin^4\theta = \frac{2}{16}\left[\cos 4\theta - 4\cos 2\theta + 3\right]$$

$$\Rightarrow \sin^4\theta = \frac{1}{8}\left[\cos 4\theta - 4\cos 2\theta + 3\right]$$

x———x

(iii) $\sin^6\theta$

Sol, Let $x = \cos\theta + i\sin\theta$, then $\frac{1}{x} = \cos\theta - i\sin\theta$

$$x - \frac{1}{x} = 2i\sin\theta$$

$$\Rightarrow (2i\sin\theta)^6 = \left(x - \frac{1}{x}\right)^6$$

$$(2i\sin\theta)^6 = x^6 - 6x^5\cdot\frac{1}{x} + \frac{6\cdot 5}{2\cdot 1}x^4\cdot\frac{1}{x^2} - \frac{6\cdot 5\cdot 4}{3\cdot 2\cdot 1}x^3\cdot\frac{1}{x^3} + \frac{6\cdot 5\cdot 4\cdot 3}{4\cdot 3\cdot 2\cdot 1}x^2\cdot\frac{1}{x^4}$$

$$- \frac{6\cdot 5\cdot 4\cdot 3\cdot 2}{5\cdot 4\cdot 3\cdot 2\cdot 1}\cdot x\cdot\frac{1}{x^5} + \frac{1}{x^6}$$

$$= x^6 - 6x^4 + 15x^2 - 20 + \frac{15}{x^2} - \frac{6}{x^4} + \frac{1}{x^6}$$

$$2^6 i^6 \sin^6\theta = \left(x^6 + \frac{1}{x^6}\right) - 6\left(x^4 + \frac{1}{x^4}\right) + 15\left(x^2 + \frac{1}{x^2}\right) - 20$$

$$-2^6 \sin^6\theta = (2\cos 6\theta) - 6(2\cos 4\theta) + 15(2\cos 2\theta) - 20$$

$$-2^6 \sin^6\theta = 2\left[\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10\right]$$

$$\Rightarrow \boxed{\sin^6\theta = -\frac{1}{2^5}\left[\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10\right]}$$

ans

x ———— x

1-2-21

(iv) 10 $\cos^7\theta = ?$

Sol: let $x = \cos\theta + i\sin\theta$, then $\frac{1}{x} = \cos\theta - i\sin\theta$

$$\Rightarrow x + \frac{1}{x} = 2\cos\theta$$

So $[2\cos\theta]^7 = \left(x + \frac{1}{x}\right)^7$

$$2^7\cos^7\theta = x^7 + 7x^6\cdot\frac{1}{x} + \frac{7\cdot 6}{2\cdot 1}x^5\cdot\frac{1}{x^2} + \frac{7\cdot 6\cdot 5}{3\cdot 2\cdot 1}x^4\cdot\frac{1}{x^3} + \frac{7\cdot 6\cdot 5\cdot 4}{4\cdot 3\cdot 2\cdot 1}x^3\cdot\frac{1}{x^4} + \frac{7\cdot 6\cdot 5\cdot 4\cdot 3}{5\cdot 4\cdot 3\cdot 2\cdot 1}x^2\cdot\frac{1}{x^5}$$

$$+ \frac{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}x\cdot\frac{1}{x^6} + \frac{1}{x^7}$$

or $$2^7\cos^7\theta = x^7 + 7x^5 + 21x^3 + 35x + \frac{35}{x} + \frac{21}{x^3} + \frac{7}{x^5} + \frac{1}{x^7}$$

$$2^7\cos^7\theta = \left(x^7 + \frac{1}{x^7}\right) + 7\left(x^5 + \frac{1}{x^5}\right) + 21\left(x^3 + \frac{1}{x^3}\right) + 35\left(x + \frac{1}{x}\right)$$

$$= 2\cos 7\theta + 7(2\cos 5\theta) + 21(2\cos 3\theta) + 35(2\cos\theta)$$

$$\Rightarrow 2^7\cos^7\theta = 2\left[\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta\right]$$

$$\Rightarrow \boxed{\cos^7\theta = \frac{1}{2^6}\left[\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta\right]}$$

ans.

(v) Q-10 $\sin^9\theta = ?$

1-2-22

Sol:- let $x = \cos\theta + i\sin\theta$, then

$\frac{1}{x} = \cos\theta - i\sin\theta$

$\Rightarrow x - \frac{1}{x} = 2i\sin\theta$

$\Rightarrow (2i\sin\theta)^9 = \left(x - \frac{1}{x}\right)^9$

$$\text{or } 2^9 i^9 \sin^9\theta = x^9 - 9x^8\frac{1}{x} + \frac{9\cdot 8\, x^7}{2\cdot 1\cdot x^2} - \frac{9\cdot 8\cdot 7}{3\cdot 2\cdot 1}\frac{x^6}{x^3} + \frac{9\cdot 8\cdot 7\cdot 6}{4\cdot 3\cdot 2\cdot 1}\frac{x^5}{x^4}$$

$$- \frac{9\cdot 8\cdot 7\cdot 6\cdot 5}{5\cdot 4\cdot 3\cdot 2\cdot 1}\cdot\frac{x^4}{x^5} + \frac{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}\cdot\frac{x^3}{x^6} - \frac{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}\frac{x^2}{x^7}$$

$$+ \frac{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}{8!}\frac{x}{x^8} - \frac{1}{x^9}$$

$$2^9 i^{8+1}\sin^9\theta = x^9 - 9x^7 + 36x^5 - 84x^3 + 126x - \frac{126}{x} + \frac{84}{x^3} - \frac{36}{x^5} + \frac{9}{x^7} - \frac{1}{x^9}$$

$$2^9 i \sin^9\theta = x^9 - 9x^7 + 36x^5 - 84x^3 + 126x - \frac{126}{x} + \frac{84}{x^3} - \frac{36}{x^5} + \frac{9}{x^7} - \frac{1}{x^9}$$

$$= \left(x^9 - \frac{1}{x^9}\right) - 9\left(x^7 - \frac{1}{x^7}\right) + 36\left(x^5 - \frac{1}{x^5}\right) - 84\left(x^3 - \frac{1}{x^3}\right) + 126\left(x - \frac{1}{x}\right)$$

$$= 2i\sin 9\theta - 9(2i\sin 7\theta) + 36(2i\sin 5\theta) - 84(2i\sin 3\theta) + 126(2i\sin\theta)$$

$$2^9 i \sin^9\theta = 2i\left[\sin 9\theta - 9\sin 7\theta + 36\sin 5\theta - 84\sin 3\theta + 126\sin\theta\right]$$

$$\boxed{\sin^9\theta = \frac{1}{2^8}\left[\sin 9\theta - 9\sin 7\theta + 36\sin 5\theta - 84\sin 3\theta + 126\sin\theta\right]}$$

Ans.

(vi) Q-10 $\sin^6\theta\cos^2\theta = ?$

Sol:- let $x = \cos\theta + i\sin\theta$, then $\frac{1}{x} = \cos\theta - i\sin\theta$ $\Big\}$ $\therefore x + \frac{1}{x} = 2\cos\theta$ and $x - \frac{1}{x} = 2i\sin\theta$

$$2^6 i^6 2^2 \sin^6\theta \cos^2\theta = \left(x-\frac{1}{x}\right)^6\left(x+\frac{1}{x}\right)^2$$

$$2^8 i^6 \sin^6\theta \cos^2\theta = \left[x-\frac{1}{x}\right]^4\left[x-\frac{1}{x}\right]^2\left[x+\frac{1}{x}\right]^2 = \left[x-\frac{1}{x}\right]^4\left[\left(x-\frac{1}{x}\right)\left(x+\frac{1}{x}\right)\right]^2$$

$$= \left[x-\frac{1}{x}\right]^4\left[x^2-\frac{1}{x^2}\right]^2$$

$$= \left[x^4 - 4x^3\cdot\frac{1}{x} + \frac{4\cdot3}{2\cdot1}x^2\cdot\frac{1}{x^2} - \frac{4\cdot3\cdot2}{3\cdot2\cdot1}\frac{x}{x^3} + \frac{1}{x^4}\right]\left[x^4+\frac{1}{x^4}-2\right]$$

$$= \left[x^4 - 4x^2 + 6 - \frac{4}{x^2} + \frac{1}{x^4}\right]\left[x^4+\frac{1}{x^4}-2\right]$$

$$= x^8 + 1 - 2x^4 - 4x^6 - \frac{4}{x^2} + 8x^2 + 6x^4 + \frac{6}{x^4} - 12 - 4x^2$$
$$- \frac{4}{x^6} + \frac{8}{x^2} + 1 + \frac{1}{x^8} - \frac{2}{x^4}$$

$\boxed{i^6 = -1}$

$$= \left(x^8+\frac{1}{x^8}\right) - 4\left(x^6+\frac{1}{x^6}\right) + \left(-2x^4+6x^4+\frac{6}{x^4}-\frac{2}{x^4}\right)$$
$$+ \left(8x^2-4x^2+\frac{8}{x^2}-\frac{4}{x^2}\right) - 10$$

$$-2^8 \sin^6\theta\cos^2\theta = \left(x^8+\frac{1}{x^2}\right) - 4\left(x^6+\frac{1}{x^6}\right) + 4\left(x^4+\frac{1}{x^4}\right) + 4\left(x^2+\frac{1}{x^2}\right) - 10$$

$$= 2\cos 8\theta - 4(2\cos 6\theta) + 4(2\cos 4\theta) + 4(2\cos 2\theta) - 10$$

$$-2^8 \cdot \sin^6\theta\cos^2\theta = 2\left[\cos 8\theta - 4\cos 6\theta + 4\cos 4\theta + 4\cos 2\theta - 5\right]$$

$$\Rightarrow \sin^6\theta\cos^2\theta = -\frac{1}{2^7}\left(\cos 8\theta - 4\cos 6\theta + 4\cos 4\theta + 4\cos 2\theta - 5\right) \quad \text{Ans}$$

or

$$\boxed{\sin^6\theta\cos^2\theta = \frac{1}{2^7}\left(-\cos 8\theta + 4\cos 6\theta - 4\cos 4\theta - 4\cos 2\theta + 5\right)}$$

—x— ans —x—

(vii) Q-10 $\cos^4\theta \sin^3\theta = ?$

Sol: Let $x = \cos\theta + i\sin\theta$
then $\frac{1}{x} = \cos\theta - i\sin\theta$ $\Big\}$ $\Rightarrow x + \frac{1}{x} = 2\cos\theta$ and $x - \frac{1}{x} = 2i\sin\theta$

So $(2\cos\theta)^4(2i\sin\theta)^3 = \left(x+\frac{1}{x}\right)^4\left(x-\frac{1}{x}\right)^3$

$$2^4 \cdot 2^3\, i^3 \cos^4\theta \sin^3\theta = \left(x+\frac{1}{x}\right)\left(x+\frac{1}{x}\right)^3\left(x-\frac{1}{x}\right)^3$$

$$-2^7 i \cos^4\theta \sin^3\theta = \left(x+\frac{1}{x}\right)\left[\left(x+\frac{1}{x}\right)\left(x-\frac{1}{x}\right)\right]^3$$ 1.2-24

$$= \left(x+\frac{1}{x}\right)\left[x^2-\frac{1}{x^2}\right]^3$$

$$= \left(x+\frac{1}{x}\right)\left[(x^2)^3 - 3(x^2)^2\left(\frac{1}{x^2}\right) + 3x^2\cdot\frac{1}{(x^2)^2} - \frac{1}{(x^2)^3}\right]$$

$$= \left(x+\frac{1}{x}\right)\left[x^6 - 3x^2 + \frac{3}{x^2} - \frac{1}{x^6}\right]$$

$$= x^7 - 3x^3 + \frac{3}{x} - \frac{1}{x^5} + x^5 - 3x + \frac{3}{x^3} - \frac{1}{x^7}$$

$$= \left(x^7 - \frac{1}{x^7}\right) - 3\left(x^3 - \frac{1}{x^3}\right) - 3\left(x - \frac{1}{x}\right) + \left(x^5 - \frac{1}{x^5}\right)$$

$$= 2i\sin 7\theta - 6i\sin 3\theta - 6i\sin\theta + 2i\sin 5\theta$$

$$-2^7 i\cos^4\theta\sin^3\theta = 2i\left[\sin 7\theta + \sin 5\theta - 3\sin 3\theta - 3\sin\theta\right]$$

$$\boxed{\Rightarrow \cos^4\theta \sin^3\theta = -\frac{1}{2^6}\left[\sin 7\theta + \sin 5\theta - 3\sin 3\theta - 3\sin\theta\right]}$$

—x—— Ans ——x—

VIII Q-10 $\cos^5\theta \sin^7\theta = ?$

Sol:- Let $x = \cos\theta + i\sin\theta$, $\frac{1}{x} = \cos\theta - i\sin\theta$ $\Rightarrow$ $x + \frac{1}{x} = 2\cos\theta$ and $x - \frac{1}{x} = 2i\sin\theta$

So: $(2\cos\theta)^5(2i\sin\theta)^7 = \left(x+\frac{1}{x}\right)^5\left(x-\frac{1}{x}\right)^7$

$$-2^{12} i\cos^5\theta\sin^7\theta = \left(x+\frac{1}{x}\right)^5\left(x-\frac{1}{x}\right)^5\left(x-\frac{1}{x}\right)^2$$

$$= \left[\left(x+\frac{1}{x}\right)\left(x-\frac{1}{x}\right)\right]^5\left[x-\frac{1}{x}\right]^2$$

$$= \left[x^2-\frac{1}{x^2}\right]^5\left[x-\frac{1}{x}\right]^2$$

$$= \left[(x^2)^5 - 5(x^2)^4\cdot\frac{1}{x^2} + \frac{5\cdot 4}{2\cdot 1}(x^2)^3\cdot\frac{1}{(x^2)^2} - \frac{5\cdot 4\cdot 3}{3\cdot 2\cdot 1}(x^2)^2\cdot\frac{1}{(x^2)^3} + \frac{5\cdot 4\cdot 3\cdot 2}{4\cdot 3\cdot 2}x^2\cdot\frac{1}{(x^2)^4} - \frac{1}{(x^2)^5}\right]\left[x^2+\frac{1}{x^2}-2\right]$$

$$-2^{12} i \cos^5\theta \sin^7\theta = \left[x^{10} - 5x^6 + 10x^2 - \frac{10}{x^2} + \frac{5}{x^6} - \frac{1}{x^{10}}\right]\left[x^2 + \frac{1}{x^2} - 2\right]$$

$$-2^{12} i \cos^5\theta \sin^7\theta = x^{12} - 5x^8 + 10x^4 - 10 + \frac{5}{x^4} - \frac{1}{x^8} + x^8 - 5x^4 + 10 - \frac{10}{x^4} + \frac{5}{x^8} - \frac{1}{x^{12}} - 2x^{10} + 10x^6 - 20x^2 + \frac{20}{x^2} - \frac{10}{x^6} + \frac{2}{x^{10}}$$

12-25

$$= \left(x^{12} - \frac{1}{x^{12}}\right) - 2\left(x^{10} - \frac{1}{x^{10}}\right) - 4\left(x^8 - \frac{1}{x^8}\right) + 10\left(x^6 - \frac{1}{x^6}\right) + 5\left(x^4 - \frac{1}{x^4}\right) - 20\left(x^2 - \frac{1}{x^2}\right)$$

$$= 2i\sin 12\theta - 2(2i\sin 10\theta) - 4(2i\sin 8\theta) + 10(2i\sin 6\theta) + 5(2i\sin 4\theta) - 20(2i\sin 2\theta)$$

$$-2^{12} i \cos^5\theta \sin^7\theta = 2i\left(\sin 12\theta - 2\sin 10\theta - 4\sin 8\theta + 10\sin 6\theta + 5\sin 4\theta - 20\sin 2\theta\right)$$

$$\Rightarrow \boxed{\cos^5\theta \sin^7\theta = -\frac{1}{2^{11}}\left(\sin 12\theta - 2\sin 10\theta - 4\sin 8\theta + 10\sin 6\theta + 5\sin 4\theta - 20\sin 2\theta\right)}$$

— x — Ans — x —

Q.11 Show that $\cos^4\theta + \sin^4\theta = \frac{1}{4}(\cos 4\theta + 3)$

Sol:- let $x = \cos\theta + i\sin\theta$, then $\frac{1}{x} = \cos\theta - i\sin\theta$ $\Rightarrow$ $2\cos\theta = x + \frac{1}{x}$, $2i\sin\theta = x - \frac{1}{x}$

So $(2\cos\theta)^4 = \left(x + \frac{1}{x}\right)^4$

$$2^4\cos^4\theta = x^4 + 4x^3\cdot\frac{1}{x} + \frac{4\cdot 3}{2\cdot 1}x^2\cdot\frac{1}{x^2} + \frac{4\cdot 3\cdot 2}{3\cdot 2\cdot 1}x\cdot\frac{1}{x^3} + \frac{4\cdot 3\cdot 2\cdot 1}{4\cdot 3\cdot 2\cdot 1}\cdot\frac{1}{x^4} \to (1)$$

$$(2i\sin\theta)^4 = \left(x - \frac{1}{x}\right)^4 = x^4 - 4x^3\cdot\frac{1}{x} + \frac{4\cdot 3}{2\cdot 1}x^2\cdot\frac{1}{x^2} - \frac{4\cdot 3\cdot 2}{3\cdot 2\cdot 1}x\cdot\frac{1}{x^3} + \frac{1}{x^4} \to (2)$$

Add (1) and (2), we get

$$2^4\cos^4\theta + 2^4 i^4\sin^4\theta = 2\left\{x^4 + \frac{4\cdot 3}{2\cdot 1} + \frac{1}{x^4}\right\}$$

$$2^4\left[\cos^4\theta + \sin^4\theta\right] = 2\left[x^4 + 6 + \frac{1}{x^4}\right] \qquad (i^4 = 1)$$

$$\cos^4\theta + \sin^4\theta = \frac{1}{2^3}\left[\left(x^4 + \frac{1}{x^4}\right) + 6\right] = \frac{1}{8}\left[2\cos 4\theta + 6\right]$$

$$\boxed{\cos^4\theta + \sin^4\theta = \frac{1}{4}\left[\cos 4\theta + 3\right]}$$

—x— Ans —x—

P.2-26

Q.12 Prove that $64(\cos^8\theta + \sin^8\theta) = \cos 8\theta + 28\cos 4\theta + 35$

Sol: Let $x = \cos\theta + i\sin\theta$, $\frac{1}{x} = \cos\theta - i\sin\theta$ $\Rightarrow$ $x + \frac{1}{x} = 2\cos\theta$ and $x - \frac{1}{x} = 2i\sin\theta$

So $(2\cos\theta)^8 + (2i\sin\theta)^8 = \left[\left(x+\frac{1}{x}\right)^8 + \left(x-\frac{1}{x}\right)^8\right]$

$$2^8\left[\cos^8\theta + \sin^8\theta\right] = \begin{cases} x^8 + 8x^7\frac{1}{x} + \frac{8\cdot 7}{2}x^6\cdot\frac{1}{x^2} + \frac{8\cdot 7\cdot 6}{3\cdot 2}x^5\cdot\frac{1}{x^3} + \frac{8\cdot 7\cdot 6\cdot 5}{4\cdot 3\cdot 2}x^4\cdot\frac{1}{x^4} \\ + \frac{8\cdot 7\cdot 6\cdot 5\cdot 4}{5\cdot 4\cdot 3\cdot 2}x^3\cdot\frac{1}{x^5} + \frac{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3}{6\cdot 5\cdot 4\cdot 3\cdot 2}x^2\cdot\frac{1}{x^6} + \frac{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}x\cdot\frac{1}{x^7} + \frac{1}{x^8} \\ x^8 - 8x^7\frac{1}{x} + \frac{8\cdot 7}{2}x^6\cdot\frac{1}{x^2} - \frac{8\cdot 7\cdot 6}{3\cdot 2}x^5\cdot\frac{1}{x^3} + \frac{8\cdot 7\cdot 6\cdot 5}{4\cdot 3\cdot 2}x^4\cdot\frac{1}{x^4} \\ - \frac{8\cdot 7\cdot 6\cdot 5\cdot 4}{5\cdot 4\cdot 3\cdot 2}x^3\cdot\frac{1}{x^5} + \frac{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3}{6\cdot 5\cdot 4\cdot 3\cdot 2}x^2\cdot\frac{1}{x^6} - \frac{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}x\cdot\frac{1}{x^7} + \frac{1}{x^8} \end{cases}$$

$$2^8\left[\cos^8\theta + \sin^8\theta\right] = 2\left[x^8 + 28x^6\cdot\frac{1}{x^2} + 70 + 28\cdot x^2\cdot\frac{1}{x^6} + \frac{1}{x^8}\right]$$

$$= 2\left[\left(x^8 + \frac{1}{x^8}\right) + 28\left(x^4 + \frac{1}{x^4}\right) + 70\right]$$

$$= 2\left[2\cos 8\theta + 28(2\cos 4\theta) + 70\right]$$

$$2^6(\cos^8\theta + \sin^8\theta) = \left[\cos 8\theta + 28\cos 4\theta + 35\right]$$

$$\boxed{\Rightarrow 64(\cos^8\theta + \sin^8\theta) = \cos 8\theta + 28\cos 4\theta + 35}$$

Proved.

Q-13 PROVE THAT:-

P-(i) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

PROOF let $x = \cos\theta + i\sin\theta$, then $\frac{1}{x} = \cos\theta - i\sin\theta$

$\Rightarrow x - \frac{1}{x} = 2i\sin\theta$

Thus $(2i\sin\theta)^3 = \left(x - \frac{1}{x}\right)^3 = x^3 - 3x + \frac{3}{x} - \frac{1}{x^3}$

$2^3 i^3 \sin^3\theta = \left(x^3 - \frac{1}{x^3}\right) - 3\left(x - \frac{1}{x}\right)$

$-2^3 i \sin^3\theta = 2i\sin 3\theta - 3(2i\sin\theta)$

$-8i\sin^3\theta = 2i\sin 3\theta - 6i\sin\theta$

$\Rightarrow -4\sin^3\theta = \sin 3\theta - 3\sin\theta$

$\Rightarrow \boxed{\sin 3\theta = 3\sin\theta - 4\sin^3\theta}$

Proved

Part-(ii) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

PROOF:- let $x = \cos\theta + i\sin\theta$, then $\frac{1}{x} = \cos\theta - i\sin\theta$

$\Rightarrow x + \frac{1}{x} = 2\cos\theta$

Thus $(2\cos\theta)^3 = \left(x + \frac{1}{x}\right)^3 = x^3 + 3x^2\cdot\frac{1}{x} + 3x\cdot\frac{1}{x^2} + \frac{1}{x^3}$

$2^3\cos^3\theta = x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}$

$2^3\cos^3\theta = \left(x^3 + \frac{1}{x^3}\right) + 3\left(x + \frac{1}{x}\right)$

$\cancel{2}^3 \cos^3\theta = 2\cos 3\theta + 3(2\cos\theta)$

$2^2\cos^3\theta = \cos 3\theta + 3\cos\theta$

$\Rightarrow \boxed{\cos 3\theta = 4\cos^3\theta - 3\cos\theta}$

—x— Proved —x—

ALTERNATE OF PART-(i) AND PART-(ii)

$\because (\cos 3\theta + i\sin 3\theta) = (\cos\theta + i\sin\theta)^3$

(By using De-moivre's Th.)

But $(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3i\cos^2\theta\sin\theta + 3i^2\cos\theta\sin^2\theta + i^3\sin^3\theta$

$$(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$$

or $\cos 3\theta + i\sin 3\theta = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$

$$= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)$$

$$= [\cos^3\theta - 3\cos\theta(1-\cos^2\theta)] + i[3\sin\theta(1-\sin^2\theta) - \sin^3\theta]$$

$$= [\cos^3\theta - 3\cos\theta + 3\cos^3\theta] + i[3\sin\theta - 3\sin^3\theta - \sin^3\theta]$$

$$\cos 3\theta + i\sin 3\theta = [4\cos^3\theta - 3\cos\theta] + i[3\sin\theta - 4\sin^3\theta]$$

Equating real and imaginary parts we get

$$\boxed{\begin{aligned}\cos 3\theta &= 4\cos^3\theta - 3\cos\theta \\ \text{and}\quad \sin 3\theta &= 3\sin\theta - 4\sin^3\theta\end{aligned}}$$

Proved

$\boxed{1\cdot 2\text{-}28}$

PART-(iii)&(iv) $\sin 4\theta = 4(\cos^3\theta\sin\theta - \cos\theta\sin^3\theta)$

and $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$

PROOF:-

$\because (\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta \longrightarrow (1)$

(De-Moivre's Th.)

but $(\cos\theta + i\sin\theta)^4 = \cos^4\theta + 4i\cos^3\theta\sin\theta + \frac{4\cdot 3}{2}i^2\cos^2\theta\sin^2\theta$

$+ \frac{4\cdot 3\cdot 2}{3\cdot 2}i^3\cos\theta\sin^3\theta + i^4\sin^4\theta$

$$= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta$$

(Binomial Th.)

or $(\cos\theta + i\sin\theta)^4 = (\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta) + i(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$

using (1)

$$\cos 4\theta + i\sin 4\theta = [\cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + \sin^4\theta] + i(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$$

$$\cos 4\theta + i\sin 4\theta = [\cos^4\theta + 6\cos^4\theta - 6\cos^2\theta + (1-\cos^2\theta)^2] + i[4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta]$$

$$= [7\cos^4\theta - 6\cos^2\theta + 1 + \cos^4\theta - 2\cos^2\theta] + i[4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta]$$

$$\cos 4\theta + i\sin 4\theta = [8\cos^4\theta - 8\cos^2\theta + 1] + i[4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta]$$

Equating real and imaginary parts, we get

$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1 \longrightarrow$ Part (iv)

$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta \longrightarrow$ Part (iii)

$\boxed{12-29}$

(v) $\frac{\sin 5\theta}{\sin\theta} = 16\cos^4\theta - 12\cos^2\theta \pm 1$

1st Method

$x^5 = \cos 5\theta + i\sin 5\theta =$

$\frac{1}{x^5} = \cos 5\theta - i\sin 5\theta$

$x^5 - \frac{1}{x^5} = 2i\sin 5\theta$

$\left(x - \frac{1}{x}\right)^5 = 2^5 i^5 \sin^5\theta$

$32 i \sin^5\theta = x^5 - 5x^4\frac{1}{x} + \frac{5\cdot4}{2}x^3\frac{1}{x^2} - \frac{5\cdot4\cdot3}{1\cdot2\cdot3}x^2\frac{1}{x^3} + \frac{5\cdot4\cdot3\cdot2}{1\cdot2\cdot3\cdot4}x\frac{1}{x^4} - \frac{1}{x^5}$

$= \left(x^5 - \frac{1}{x^5}\right) - 5\left(x^3 - \frac{1}{x^3}\right) + 10\left(x - \frac{1}{x}\right)$

$= (2i\sin 5\theta) - 5(2i\sin 3\theta) + 10(2i\sin\theta)$

$32 i \sin^5\theta = 2i(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$

$\sin\theta(16\sin^4\theta) + 5(3\sin\theta - 4\sin^3\theta) - 10\sin\theta = \sin 5\theta$

$\frac{\sin\theta}{\sin\theta}\left[16\sin^4\theta + 15 - 20\sin^2\theta - 10\right) = \frac{\sin 5\theta}{\sin\theta}$

$16(1-\cos^2\theta)^2 + 15 - 20(1-\cos^2\theta) - 10$

$16(1+\cos^4\theta - 2\cos^2\theta) + 15 - 20 + 20\cos^2\theta - 10$

$16 + 16\cos^4\theta - 32\cos^2\theta - 20 + 20\cos^2\theta - 10$

$1 + 16\cos^4\theta - 12\cos^2\theta \quad = \frac{\sin 5\theta}{\sin\theta}$

proved

2nd Method.

Part-(V) $\frac{\sin 5\theta}{\sin\theta} = 16\cos^4\theta - 12\cos^2\theta + 1$

[1.2-30]

PROOF According to De moivres Th.

$$[\cos\theta + i\sin\theta]^5 = \cos 5\theta + i\sin 5\theta \longrightarrow ①$$

but $[\cos\theta + i\sin\theta]^5 = \cos^5\theta + 5i\cos^4\theta\cdot\sin\theta + \frac{5\cdot4}{2\cdot1}\cos^3\theta\sin^2\theta - \frac{5\cdot4\cdot3}{3\cdot2\cdot1}i\cos^2\theta\sin^3\theta$

by Binomial Th

$$+\frac{5\cdot4\cdot3\cdot2}{4\cdot3\cdot2\cdot1}\cos\theta\sin^4\theta + i\sin^5\theta$$

$$[\cos\theta + i\sin\theta]^5 = \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$$

using ①, we get

$$\cos 5\theta + i\sin 5\theta = (\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta) + i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)$$

Equating imaginary parts, we get

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$

$$\Rightarrow \frac{\sin 5\theta}{\sin\theta} = \frac{\cancel{\sin\theta}(5\cos^4\theta - 10\cos^2\theta\sin^2\theta + \sin^4\theta)}{\sin\theta}$$

by $\sin\theta$

$$= 5\cos^4\theta - 10\cos^2\theta\sin^2\theta + \sin^4\theta$$

$$= 5\cos^4\theta - 10\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$$

$$= 5\cos^4\theta - 10\cos^2\theta + 10\cos^4\theta + \cos^4\theta + 1 - 2\cos^2\theta$$

or $$\frac{\sin 5\theta}{\sin\theta} = 16\cos^4\theta - 12\cos^2\theta + 1$$

x———— Proved ————x

Q.14 Prove that $\tan 6\theta = 2t\left(\frac{3-10t^2+3t^4}{1-15t^2+15t^4-t^6}\right)$ Where $t=\tan\theta$

PROOF According to Demoivre's Th.

$$(\cos\theta + i\sin\theta)^6 = \cos 6\theta + i\sin 6\theta \longrightarrow ①$$

but $(\cos\theta + i\sin\theta)^6 = \left\{\cos^6\theta + 6i\cos^5\theta\sin\theta + \frac{6\cdot5}{2}\cos^4\theta\sin^2\theta - \frac{6\cdot5\cdot4}{3\cdot2}i\cos^3\theta\sin^3\theta\right.$

by Binomial Th

$$\left. + \frac{6\cdot5\cdot4\cdot3}{4\cdot3\cdot2}\cos^2\theta\sin^4\theta + \frac{6\cdot5\cdot4\cdot3\cdot2}{5\cdot4\cdot3\cdot2}i\cos\theta\sin^5\theta - \sin^6\theta\right\}$$

Q:

$$\cos 6\theta + i\sin 6\theta = \left\{\begin{array}{l}\cos^6\theta + 6i\cos^5\theta\sin\theta - 15\cos^4\theta\sin^2\theta - 20i\cos^3\theta\sin^3\theta \\ + 15\cos^2\theta\sin^4\theta + 6i\cos\theta\sin^5\theta - \sin^6\theta\end{array}\right\}$$

1.2-31

Equating real and imaginary parts.

$$\cos 6\theta = \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta \rightarrow (ii)$$

and $\sin 6\theta = 6\cos^5\theta\sin\theta - 20\sin^3\theta\cos^3\theta + 6\cos\theta\sin^5\theta$

$$\frac{\sin 6\theta}{\cos 6\theta} = \left(\frac{6\cos^5\theta\sin\theta - 20\sin^3\theta\cos^3\theta + 6\cos\theta\sin^5\theta}{\cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta}\right)$$

$$\tan 6\theta = \frac{\frac{6\cos^5\theta\sin\theta - 20\sin^3\theta\cos^3\theta + 6\cos\theta\sin^5\theta}{\cos^6\theta}}{\frac{\cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta}{\cos^6\theta}}$$

$$= \frac{6\tan\theta - 20\tan^3\theta + 6\tan^5\theta}{1 - 15\tan^2\theta + 15\tan^4\theta - \tan^6\theta}$$

R.H.S.

$$= \frac{2\tan\theta(3 - 10\tan^2\theta + 3\tan^4\theta)}{1 - 15\tan^2\theta + 15\tan^4\theta - \tan^6\theta}$$

$$= \frac{2t(3 - 10t^2 + 3t^4)}{1 - 15t^2 + 15t^4 - t^6}$$

Q.15 Prove that $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$ and hence solve the equation $1-3t^2 = 3t - t^3$

Sol: Since $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta \rightarrow ①$

but $(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$ by B.Th.

using ①, we get

$$\cos 3\theta + i\sin 3\theta = (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)$$

$$= [\cos^3\theta - 3\cos\theta(1-\cos^2\theta)] + i[3(1-\sin^2\theta)\sin\theta - \sin^3\theta]$$

$$= [\cos^3\theta - 3\cos\theta + 3\cos^3\theta] + i[3\sin\theta - 3\sin^3\theta - \sin^3\theta]$$

$$\cos 3\theta + i\sin 3\theta = [4\cos^3\theta - 3\cos\theta] + i[3\sin\theta - 4\sin^3\theta]$$

Equating real and imaginary parts, we get

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta, \quad \sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

$$\Rightarrow \frac{\sin 3\theta}{\cos 3\theta} = \frac{3\sin\theta - 4\sin^3\theta}{4\cos^3\theta - 3\cos\theta}$$

$\boxed{1.2\text{-}32}$

$$= \frac{\sin\theta(3-4\sin^2\theta)}{\cos\theta(4\cos^2\theta - 3)} = \frac{\tan\theta(3-4\sin^2\theta)}{4\cos^2\theta - 3}$$

$$= \frac{\tan\theta[3\cdot 1 - 4\sin^2\theta]}{4\cos^2\theta - 3\cdot 1} = \frac{\tan\theta[3(\cos^2\theta + \sin^2\theta) - 4\sin^2\theta]}{4\cos^2\theta - 3(\sin^2\theta + \cos^2\theta)}$$

$$= \frac{\tan\theta[3\cos^2\theta - \sin^2\theta]}{\cos^2\theta - 3\sin^2\theta}$$

$\div$ N & D by $\cos^2\theta$

$$\tan 3\theta = \frac{\tan\theta(3-\tan^2\theta)}{1-3\tan^2\theta}$$ proved

Now Put $\tan\theta = t$

$$\Rightarrow \tan 3\theta = \frac{t(3-t^2)}{1-3t^2} = \frac{3t - t^3}{1-3t^2} \rightarrow ②$$

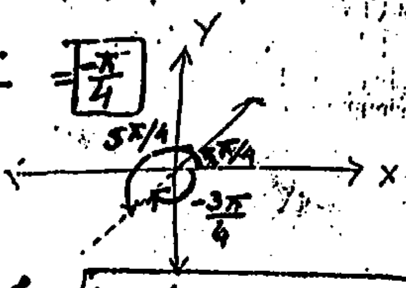
Since we are asked to solve $1-3t^2 = 3t - t^3$

$$\Rightarrow 1 = \frac{3t-t^3}{1-3t^2} \rightarrow ③ \Rightarrow \tan 3\theta = 1 \quad (\text{from ② and ③})$$

$3\theta = \tan^{-1}(1) = \pi/4, \frac{5\pi}{4}, -\frac{3\pi}{4}$

$\Rightarrow 3\theta = \frac{\pi}{4}, \quad 3\theta = \frac{5\pi}{4}, \quad$ and $3\theta = -\frac{3\pi}{4}$

$\Rightarrow \boxed{\theta = \frac{\pi}{12}}, \quad \boxed{\theta = \frac{5\pi}{12}}, \quad \theta = \frac{-3\pi}{(3)4} = \boxed{\frac{-\pi}{4}}$



Since $t = \tan\theta$

So $t = \tan\frac{\pi}{12}$, $\quad t = \tan\frac{5\pi}{12}$ & $\boxed{t = \tan(\frac{-\pi}{4}) = -1}$

$\because \tan\frac{\pi}{6} = \tan\left(\frac{\pi}{12} + \frac{\pi}{12}\right)$ $\quad \left(\because \tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}\right)$

$\Rightarrow \frac{1}{\sqrt{3}} = \frac{2\tan\frac{\pi}{12}}{1-\tan^2\frac{\pi}{12}}$ $\quad$ $\boxed{\because \sin\frac{\pi}{6} = \frac{1}{2} \;\&\; \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}}$

$\Rightarrow 1 - \tan^2\frac{\pi}{12} = 2\sqrt{3}\tan\frac{\pi}{12}$

or $\tan^2\frac{\pi}{12} + 2\sqrt{3}\tan\frac{\pi}{12} = 1$

$\Rightarrow \tan^2\pi/12 + 2\sqrt{3}\tan\pi/12 + 3 = 1+3$ $\quad$ (completing square)

$\Rightarrow \left(\tan\frac{\pi}{12} + \sqrt{3}\right)^2 = 2^2$

$\Rightarrow \tan\frac{\pi}{12} + \sqrt{3} = 2 \quad \Rightarrow \quad \boxed{\tan\frac{\pi}{12} = 2 - \sqrt{3} = t}$

Also $\tan\frac{5\pi}{6} = \tan\left(\frac{5\pi}{12} + \frac{5\pi}{12}\right)$ $\quad$ $\boxed{\begin{array}{l}\sin\frac{5\pi}{6} = \frac{1}{2} \\ \Rightarrow \cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} \\ \Rightarrow \tan\frac{5\pi}{6} = -\frac{1}{\sqrt{3}}\end{array}}$

or $\frac{-1}{\sqrt{3}} = \frac{2\tan\frac{5\pi}{12}}{1-\tan^2\frac{5\pi}{12}}$

$-1 + \tan^2\frac{5\pi}{12} = 2\sqrt{3}\tan\frac{5\pi}{12}$

or $\tan^2\frac{5\pi}{12} - 2\sqrt{3}\tan\frac{5\pi}{12} = 1$

Completing sq. we get

$\tan^2\frac{5\pi}{12} - 2\sqrt{3}\tan\frac{5\pi}{12} + (\sqrt{3})^2 = 1+3$

$\left(\tan\frac{5\pi}{12} - \sqrt{3}\right)^2 = 2^2$

$\Rightarrow \tan\frac{5\pi}{12} - \sqrt{3} = 2$

$\Rightarrow \boxed{t = \tan\frac{5\pi}{12} = 2 + \sqrt{3}}$

Hence the required roots of cubic equation $1 - 3t^2 = 3t - t^3$ are

$-1, \; 2+\sqrt{3}, \; 2-\sqrt{3}$

ans

Q.16 Prove that $\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} = \frac{1}{2}$ $\boxed{1.2-34}$

Sol Consider the seventh roots of unity

i.e let $x^7 = 1 \Rightarrow x^7 = 1 + 0i$

$\Rightarrow x^7 = \cos 0 + i\sin 0 = \cos(0 + 2k\pi) + i\sin(0 + 2\pi k)$

or $x^7 = \cos 2\pi k + i\sin 2k\pi$

So seven 7th roots of unity are

$$x_k = (\cos 2\pi k + i\sin 2k\pi)^{\frac{1}{7}}$$

where $k = 0, \pm1, \pm2, \pm3$

$\Rightarrow x_k = \cos\frac{2\pi k}{7} + i\sin\frac{2k\pi}{7} \rightarrow (1)$, where $k = 0, \pm1, \pm2, \pm3$

after putting values of k in (1), we get its seven roots

i.e $1, \ \cos\frac{2\pi}{7} \pm i\sin\frac{2\pi}{7}, \ \cos\frac{4\pi}{7} \pm i\sin\frac{4\pi}{7}, \ \cos\frac{6\pi}{7} \pm i\sin\frac{6\pi}{7}$

(Since $\sin(-\alpha) = -\sin\alpha$ and $\cos(-\alpha) = \cos\alpha$)

Now from theory of equations, the sum of root of $x^7 - 1 = 0$ is zero

$$\Rightarrow 1 + \left(\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right) + \left(\cos\frac{4\pi}{7} + i\sin\frac{4\pi}{7}\right) + \left(\cos\frac{6\pi}{7} + i\sin\frac{6\pi}{7}\right.$$
$$\left.\cos\frac{2\pi}{7} - i\sin\frac{2\pi}{7} + \cos\frac{4\pi}{7} - i\sin\frac{4\pi}{7} + \cos\frac{6\pi}{7} - i\sin\frac{6\pi}{7}\right) = 0$$

$\Rightarrow 1 + 2\cos\frac{2\pi}{7} + 2\cos\frac{4\pi}{7} + 2\cos\frac{6\pi}{7} = 0$

$\Rightarrow 2\left(\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}\right) = -1$

$\boxed{\cos(\pi - \alpha) = -\cos\alpha}$

$\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$

$\cos\frac{2\pi}{7} + \cos\left(\pi - \frac{3\pi}{7}\right) + \cos\left(\pi - \frac{\pi}{7}\right) = -\frac{1}{2}$

$\pi - \frac{3\pi}{7} = \frac{7\pi - 3\pi}{7} = \frac{4\pi}{7}$

$\pi - \frac{\pi}{7} = \frac{7\pi - \pi}{7} = \frac{6\pi}{7}$

$\cos\frac{2\pi}{7} - \cos\frac{3\pi}{7} - \cos\frac{\pi}{7} = -\frac{1}{2}$

$\Rightarrow \boxed{\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} = \frac{1}{2}}$ Proved

Q.17 Prove the following relations ($\forall$ m, n belongs to $\mathbb{Z}$ mean m & n are integer)

(i) $Z^m Z^n = Z^{m+n}$

$\boxed{1\cdot2-35}$

Proof Let $Z = r(\cos\theta + i\sin\theta)$

$\Rightarrow Z^m = r^m(\cos\theta + i\sin\theta)^m$

$Z^m = r^m(\cos m\theta + i\sin m\theta)$

Similarly $Z^n = r^n(\cos n\theta + i\sin n\theta)$

Then $L.H.S. = Z^m \cdot Z^n$

$= r^m[\cos m\theta + i\sin m\theta]\, r^n[\cos n\theta + i\sin n\theta]$

$= r^m r^n[\cos m\theta + i\sin m\theta][\cos n\theta + i\sin n\theta]$

$= r^{m+n}\left[(\cos m\theta\cos n\theta - \sin m\theta\sin n\theta) + i(\sin m\theta\cos n\theta + \sin n\theta\cos m\theta)\right]$

$= r^{m+n}[\cos(m\theta + n\theta) + i\sin(m\theta + n\theta)]$

$= r^{m+n}[\cos(m+n)\theta + i\sin(m+n)\theta]$

$= r^{m+n}(\cos\theta + i\sin\theta)^{m+n}$ by Demoivre's Th.

$= Z^{m+n} = R.H.S.$

(ii) $(Z^m)^n = Z^{mn}$

Proof:- Let $Z = r(\cos\theta + i\sin\theta)$

$\Rightarrow Z^m = r^m(\cos\theta + i\sin\theta)^m$ (De-moivre's Th.)

$Z^m = r^m(\cos m\theta + i\sin m\theta)$

$\Rightarrow (Z^m)^n = r^{mn}(\cos m\theta + i\sin m\theta)^n$ (De-moivre's Th.)

$$L.H.S = \left(Z^m\right)^n = r^{mn}\left[\cos mn\alpha + i\sin mn\alpha\right] = r^{mn}\left[\cos\alpha + i\sin\alpha\right]^{mn}$$
$$= Z^{mn} = R.H.S.$$

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(iii) $(Z_1 Z_2)^n = Z_1^n Z_2^n$

Proof let $Z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $Z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Hence $Z_1 \cdot Z_2 = r_1(\cos\theta_1 + i\sin\theta_1)\cdot r_2(\cos\theta_2 + i\sin\theta_2)$
$$= r_1 r_2\left[\cos\theta_1 + i\sin\theta_1\right]\left[\cos\theta_2 + i\sin\theta_2\right]$$
$$= r_1 r_2\left[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\right]$$
$$Z_1 Z_2 = r_1 r_2\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]$$
$$L.H.S = (Z_1 Z_2)^n = r_1^n r_2^n\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]^n$$
by Demoivre th.

$\boxed{\begin{aligned} &\text{if } Z = r(\cos\alpha + i\sin\alpha)\\ &\Rightarrow Z^n = \left[r(\cos\alpha + i\sin\alpha)\right]^n\\ &= r^n(\cos\alpha + i\sin\alpha)^n\\ &= r^n\left[\cos n\alpha + i\sin n\alpha\right]\end{aligned}}$

$$= r_1^n r_2^n\left[\cos n(\theta_1+\theta_2) + i\sin n(\theta_1+\theta_2)\right]$$
$$= r_1^n r_2^n\left[\cos(n\theta_1 + n\theta_2) + i\sin(n\theta_1 + n\theta_2)\right]$$
$$= r_1^n r_2^n\left[(\cos n\theta_1\cos n\theta_2 - \sin n\theta_1\sin n\theta_2) + i(\sin n\theta_1\cos n\theta_2 + \cos n\theta_1\sin n\theta_2)\right]$$
$$= r_1^n r_2^n\left[(\cos n\theta_1\cos n\theta_2 + i^2\sin n\theta_1\sin n\theta_2) + (i\sin n\theta_1\cos n\theta_2 + i\cos n\theta_1\sin n\theta_2)\right]$$
$$= r_1^n r_2^n\left[(\cos n\theta_1\cos n\theta_2 + i\cos n\theta_1\sin n\theta_2) + (i\sin n\theta_1\cos n\theta_2 + i^2\sin n\theta_1\sin n\theta_2)\right]$$
$$= r_1^n r_2^n\left[\cos n\theta_1(\cos n\theta_2 + i\sin n\theta_2) + i\sin n\theta_1(\cos n\theta_2 + i\sin n\theta_2)\right]$$
$$= r_1^n r_2^n\left[(\cos n\theta_1 + i\sin n\theta_1)(\cos n\theta_2 + i\sin n\theta_2)\right]$$
$$= r_1^n(\cos n\theta_1 + i\sin n\theta_1)\cdot r_2^n(\cos n\theta_2 + i\sin n\theta_2)$$
$$= r_1^n(\cos\theta_1 + i\sin\theta_1)^n \cdot r_2^n(\cos\theta_2 + i\sin\theta_2)^n$$
$$= Z_1^n Z_2^n = R.H.S$$

(iv) $\frac{z^m}{z^n} = z^{m-n}, \quad z \neq 0$

Proof let $z = r(\cos\theta + i\sin\theta)$

$\Rightarrow z^m = r^m(\cos\theta + i\sin\theta)^m$ & $z^n = r^n(\cos\theta + i\sin\theta)^n$

$$L.H.S = \frac{z^m}{z^n} = \frac{r^m(\cos\theta + i\sin\theta)^m}{r^n(\cos\theta + i\sin\theta)^n}$$

$$= r^{m-n} \cdot \frac{(\cos m\theta + i\sin m\theta)}{(\cos n\theta + i\sin n\theta)}$$

$$= r^{m-n} \cdot (\cos m\theta + i\sin m\theta)(\cos n\theta + i\sin n\theta)^{-1}$$

$$= r^{m-n}[\cos m\theta + i\sin m\theta][\cos(-n\theta) + i\sin(-n\theta)]$$

$$= r^{m-n}[\cos m\theta + i\sin m\theta][\cos n\theta - i\sin n\theta].$$

$$= r^{m-n}\left[(\cos m\theta\cos n\theta + \sin m\theta\sin n\theta) + i(\sin m\theta\cos n\theta - \cos m\theta\sin n\theta)\right]$$

$$= r^{m-n}[\cos(m\theta - n\theta) + i\sin(m\theta - n\theta)]$$

$$= r^{m-n}[\cos(m-n)\theta + i\sin(m-n)\theta]$$

$$= r^{m-n}[\cos\theta + i\sin\theta]^{m-n}$$ using Demoivre Th.

$$= z^{m-n} = R.H.S.$$

(v) $\left(\frac{z_1}{z_2}\right)^n = \frac{z_1^n}{z_2^n}, \quad z_2 \neq 0$

Proof let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$$\Rightarrow \frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

$\boxed{1.2\text{-}38}$

$$L.H.S = \left(\frac{Z_1}{Z_2}\right)^n = \left(\frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}\right)^n$$

$$\Rightarrow \left(\frac{Z_1}{Z_2}\right)^n = \frac{r_1^n}{r_2^n}\left(\frac{(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)}\right)^n$$

$$= \frac{r_1^n}{r_2^n}\left(\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1)}{\cos^2\theta_2 + \sin^2\theta_2}\right)^n$$

$$= \frac{r_1^n}{r_2^n}\left(\frac{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)}{1}\right)^n$$

$$= \frac{r_1^n}{r_2^n}\left(\cos n(\theta_1 - \theta_2) + i\sin n(\theta_1 - \theta_2)\right)$$

$$= \frac{r_1^n}{r_2^n}\left[\cos(n\theta_1 - n\theta_2) + i\sin(n\theta_1 - n\theta_2)\right]$$

$$= \frac{r_1^n}{r_2^n}\left((\cos n\theta_1\cos n\theta_2 + \sin n\theta_1\sin n\theta_2) + i(\sin n\theta_1\cos n\theta_2 - \sin n\theta_2\cos n\theta_1)\right)$$

$$= \frac{r_1^n}{r_2^n}\left((\cos n\theta_1\cos n\theta_2 - i^2\sin n\theta_1\sin n\theta_2) + i(\sin n\theta_1\cos n\theta_2 - \sin n\theta_2\cos n\theta_1)\right)$$

$$= \frac{r_1^n}{r_2^n}\left((\cos n\theta_1\cos n\theta_2 - i\sin n\theta_2\cos n\theta_1) + (i\sin n\theta_1\cos n\theta_2 - i^2\sin n\theta_1\sin n\theta_2)\right)$$

$$= \frac{r_1^n}{r_2^n}\left(\cos n\theta_1(\cos n\theta_2 - i\sin n\theta_2) + i\sin n\theta_1(\cos n\theta_2 - i\sin n\theta_2)\right)$$

$$= \frac{r_1^n}{r_2^n}\left[(\cos n\theta_1 + i\sin n\theta_1)(\cos \theta_2 n - i\sin n\theta_2)\right]$$

$$= \frac{r_1^n}{r_2^n} \times (\cos\theta_1 + i\sin\theta_1)^n(\cos\theta_2 n + i\sin n\theta_2)^{-1}$$

$$= \frac{r_1^n}{r_2^n}\frac{(\cos\theta_1 + i\sin\theta_1)^n}{(\cos n\theta_2 + i\sin n\theta_2)} = \frac{r_1^n}{r_2^n} \times \frac{(\cos\theta_1 + i\sin\theta_1)^n}{(\cos\theta_2 + i\sin\theta_2)^n} = \frac{Z_1^n}{Z_2^n}$$

$$= R.H.S$$