The Complex Number System.

The set 'C' $= R \times R = \{(a,b) / a,b \in R\}$ is called the set of complex number if the following conditions are satisfied.

i) $(a,b) + (c,d) = (a+c,\ b+d)$ (Addition)

ii) $(a,b) \cdot (c,d) = (ac-bd,\ ad+bc)$ (Multiplication)

iii) $K(a,b) = (Ka, Kb)$ where $K \in R$ (Scalar Multiple)

iv) $(a,b) = (c,d)$ iff $a=c,\ b=d$ (Equality)

Note. $\boxed{(a,b) = a+bi}$ $a =$ Real Part, $b =$ Imag Part

where $(a,b) \in C$ & $a+bi \in C$

$\because (a,b) = a(1,0) + b(0,1)$

$(a,b) = a \cdot 1 + b\,i = a+bi$

$i = (0,1)$

$$i = (0,1)$$
$$i \cdot i = (0,1)(0,1)$$
$$i^2 = (0\cdot0 - 1\cdot1,\ 0\cdot1 + 1\cdot0)$$
$$i^2 = (-1, 0)$$
$$i^2 = -1 + 0i$$
$$i^2 = -1$$

$(a,b) = (a,0) + (0,b)$
$= a(1,0) + b(0,1)$
$= a(1) + b(i)$
$(a,b) = a+bi$

Modulus of $(a,b) \in C$

If $z = (a,b)$ then $|z| = \sqrt{a^2+b^2}$

Conjugate of $(a,b) \in C$

If $z = (a,b) = a+bi$
then $\bar{z} = \overline{a+bi} = a-bi$

Multiplicative Identity in C. $(1,0) = 1 = 1+0i$

Additive Identity in C. $(0,0) = 0 = 0+0i$

Additive Inverse of (a,b) is $(-a,-b)$

Multiplicative Inverse of $(a,b) \in C$

Let $z = (a,b) = a+ib$

$$z^{-1} = (a+ib)^{-1} = \frac{1}{a+ib}$$
$$= \frac{1}{a+ib} \times \frac{(a-ib)}{(a-ib)}$$
$$z^{-1} = \frac{a-ib}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{ib}{a^2+b^2}$$
$$z^{-1} = \left(\frac{a}{a^2+b^2},\ \frac{-b}{a^2+b^2}\right)$$

(To verify $zz^{-1} = (1,0) = 1$)

To Prove $z \cdot \bar{z} = |z|^2 = (\text{Re}\,z)^2 + (\text{Im}\,z)^2$

LHS $z \cdot \bar{z}$
$= (a+bi)(a-bi)$
$= a^2+b^2 = (\text{Re}\,z)^2 + (\text{Im}\,z)^2$

$$\boxed{z\bar{z} = |z|^2}$$

Also $z^2 = (a+bi)(a+bi)$

$$z^2 = a^2 - b^2 + 2abi$$
$$|z^2| = \sqrt{(a^2-b^2)^2 + (2ab)^2}$$
$$= \sqrt{a^4+b^4-2a^2b^2+4a^2b^2}$$
$$= \sqrt{a^4+b^4+2a^2b^2}$$
$$= \sqrt{(a^2+b^2)^2} = a^2+b^2 = |z|^2$$

$$\therefore \boxed{|z^2| = |z|^2}$$

Imp

Th Let z_1, z_2 be complex numbers.

Show that (i) $\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$

(ii) $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$

(iii) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$

Sol (i) Let $z_1 = a+bi \quad \bar{z}_1 = a-bi$

$z_2 = c+di, \quad \bar{z}_2 = c-di$

LHS

$z_1+z_2 = a+bi+c+di$

$z_1+z_2 = (a+c)+(b+d)i$

$\overline{z_1+z_2} = (a+c)-(b+d)i$ —— (I)

RHS

$\bar{z}_1+\bar{z}_2 = a-bi+c-di$

$= (a+c)-(b+d)i$ —— (II)

(I) = (II) $\Rightarrow \quad \overline{z_1+z_2} = \bar{z}_1+\bar{z}_2$

(ii)

LHS $z_1 z_2 = (a+bi)(c+di)$

$= ac+adi+bci+bd(i^2)$

$= ac+adi+bci+bd(-1)$

$z_1 z_2 = (ac-bd)+(ad+bc)i$

$\overline{z_1 z_2} = (ac-bd)-i(ad+bc)$ —— (I)

RHS $\bar{z}_1\bar{z}_2 = (a-bi)(c-di)$

$= ac-adi-bci+bd(i)^2$

$\bar{z}_1\bar{z}_2 = (ac-bd)-i(ad+bc)$ —— (II)

(I) = (2) $\Rightarrow \quad \overline{z_1 z_2} = \bar{z}_1\bar{z}_2$

(iii) LHS $\dfrac{z_1}{z_2} = \dfrac{a+bi}{c+di}$

$\dfrac{z_1}{z_2} = \dfrac{a+bi}{c+di} \times \dfrac{c-di}{c-di}$

$= \dfrac{ac-adi+bci-bd(i^2)}{c^2-(id)^2}$

$= \dfrac{ac-adi+bci-bd(-1)}{c^2+d^2}$

$\dfrac{z_1}{z_2} = \dfrac{(ac+bd)+i(bc-ad)}{c^2+d^2}$

$\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{(ac+bd)-i(bc-ad)i}{c^2+d^2}$ —— (I)

RHS $\dfrac{\bar{z}_1}{\bar{z}_2} = \dfrac{a-bi}{c-di}$

$= \dfrac{a-bi}{c-di} \times \dfrac{c+di}{c+di}$

$= \dfrac{ac+adi-bci-bd(i^2)}{c^2-(id)^2}$

$= \dfrac{ac+adi-bci+bd}{c^2+d^2}$

$\dfrac{\bar{z}_1}{\bar{z}_2} = \dfrac{(ac+bd)-i(bc-ad)}{c^2+d^2}$ —— (II)

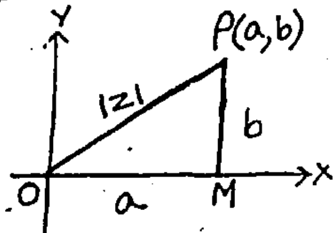
(I) = (II) $\Rightarrow \overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$

Imp

Th Show that the modulus $|z|$ of a complex number $z = a+bi$ is the distance of a point from origin, or length of $\overline{OP}$.

Sol We know to each complex number $z = a+bi$ there correspond a pt $P(a,b)$, in the cartesian plane and vice versa.

But the point (a,b) in the plane is represented as



$|\overline{OM}| = a$

$|\overline{PM}| = b$

$|\overline{OP}| = |z|$

By pythagoras Th.

$|\overline{OP}|^2 = |\overline{OM}|^2 + |\overline{PM}|^2$

$|OP|^2 = a^2+b^2$

$|OP| = \sqrt{a^2+b^2}$

$|z| = \sqrt{a^2+b^2}$ Distance of (a,b) from $(0,0)$

Note 1)

$z = a+ib$

$\bar{z} = a-ib$

$\bar{\bar{z}} = a+ib \quad \therefore \boxed{\bar{\bar{z}} = z}$

Note 2)

$z = a+ib \Rightarrow |z| = \sqrt{a^2+b^2}$

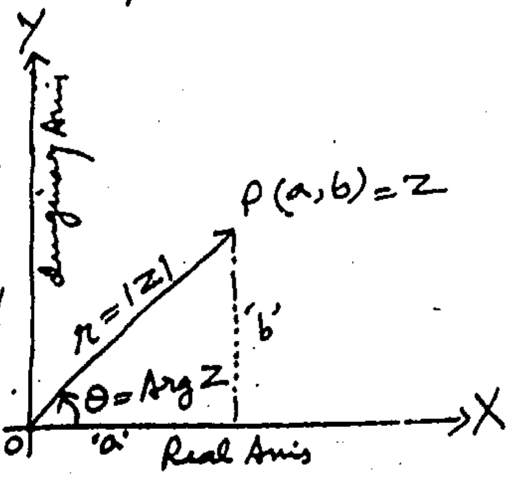
$\bar{z} = a-ib \Rightarrow |\bar{z}| = \sqrt{a^2+b^2}$

$-z = -a-ib \Rightarrow |-z| = \sqrt{a^2+b^2}$

$\therefore |z| = |-z| = |\bar{z}|$

Complex Plane

A complex number $z = a + bi$ corresponds to the pt (a,b) in the XY-plane and vice versa.



The XY-plane in which a complex number z is represented by a vector $\overline{OP}$ is called Complex Plane or Z-Plane, X-axis is called Real Axis and Y-axis is called Imaginary Axis & figure so obtained is called Argand Diagram.

The inclination 'θ' of a complex vector $\overline{OP}$ with positive direction of x-axis is called Argument or Amplitude of z, written as $\arg z$.

$$\arg z = \theta = \tan^{-1}\frac{b}{a}$$

$\sin\theta = \frac{b}{|z|}$

$\cos\theta = \frac{a}{|z|}$

If value of θ is such that $-\pi < \theta \leq \pi$ then θ is called Principal argument of z i.e $\text{Arg}\, z$

Arg of 0 is not defined.

Imp Note

i) When $z = (a,b)$ is in Ist Quad then angle is 'θ'

ii) When $z = (a,b)$ is in 2nd Quad then angle is $(\pi - \theta)$

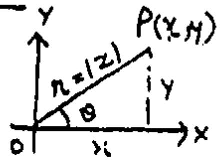
iii) When $z = (a,b)$ is in 3rd Quad then angle is $-(\pi - \theta)$

iv) When $z = (a,b)$ is in 4th Quad then angle is $-$'θ'

It is because $-\pi < \theta \leq \pi$ i.e value of θ is not greater than π.

Polar Form of Complex Number

From fig



$x = r\cos\theta$, $y = r\sin\theta$

$$z = x + iy \quad \text{——— (I)}$$

$$= r\cos\theta + i\, r\sin\theta$$

$$= r(\cos\theta + i\sin\theta)$$

$$z = r\, \text{Cis}\, \theta \quad \text{——— (II)}$$

(I) is Cartesian form of complex number z.

(II) is Polar form of complex number z.

Properties

i) $|z| = |-z| = |\bar{z}|$

ii) $|z|^2 = |z^2| = z\bar{z}$

iii) $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$

iv) $|z_1 z_2| = |z_1||z_2|$

v) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

vi) $|\text{Re}\, z| \leq |z|$

vii) $|\text{Im}\, z| \leq |z|$

viii) $z\bar{z} = (\text{Re}\, z)^2 + (\text{Im}\, z)^2$

ix) $|z_1 - z_2| = |z_2 - z_1|$

x) $|z_1 - z_2| \geq |z_1| - |z_2|$

xi) $|z_1 + z_2| \leq |z_1| + |z_2|$

Imp Th For all $z_1, z_2 \in C$

$$\boxed{|z_1 z_2| = |z_1||z_2|}$$

Proof Let $z_1 = a+ib$, $z_2 = c+id$, then

$z_1 z_2 = (a+ib)(c+id)$

$z_1 z_2 = ac + iad + ibc + i^2 bd$

$z_1 z_2 = (ac-bd) + i(ad+bc)$

$|z_1 z_2| = \sqrt{(ac-bd)^2 + (ad+bc)^2}$

$= \sqrt{a^2c^2 + b^2d^2 - \cancel{2acbd} + a^2d^2 + b^2c^2 + \cancel{2adbc}}$

$= \sqrt{a^2(c^2+d^2) + b^2(d^2+c^2)}$

$= \sqrt{(a^2+b^2)(c^2+d^2)}$

$= \sqrt{a^2+b^2}\,\sqrt{c^2+d^2}$

$|z_1 z_2| = |z_1||z_2|$ proved

Th Prove that for $z_1, z_2 \in C$

$$\boxed{|z_1| - |z_2| \le |z_1+z_2| \le |z_1| + |z_2|}$$

Proof Let $|z_1+z_2|^2$

$= (z_1+z_2)\overline{(z_1+z_2)}$ $\quad \because |z|^2 = z\bar{z}$

$= (z_1+z_2)(\bar{z}_1+\bar{z}_2)$ $\quad \because \overline{z_1+z_2} = \bar{z}_1+\bar{z}_2$

$= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2$ $\quad \because z_1\bar{z}_2 + z_2\bar{z}_1 = 2Re\, z_1\bar{z}_2$

$= |z_1|^2 + 2Re(z_1\bar{z}_2) + |z_2|^2$ $\quad \because |Re\, z| \le |z|$

$\le |z_1|^2 + 2|z_1\bar{z}_2| + |z_2|^2$ $\quad \because |z_1 z_2| = |z_1||z_2|$

$= |z_1|^2 + 2|z_1||\bar{z}_2| + |z_2|^2$ $\quad |\bar{z}| = |z|$

$= |z_1|^2 + 2|z_1||z_2| + |z_2|^2$

$|z_1+z_2|^2 \le (|z_1|+|z_2|)^2$

$\boxed{|z_1+z_2| \le |z_1| + |z_2|}$ — (I)

Now $|z_1| = |z_1+z_2-z_2|$ $\quad (+\,+\,-z_2)$

$\le |z_1+z_2| + |-z_2|$

$|z_1| = |z_1+z_2| + |z_2|$

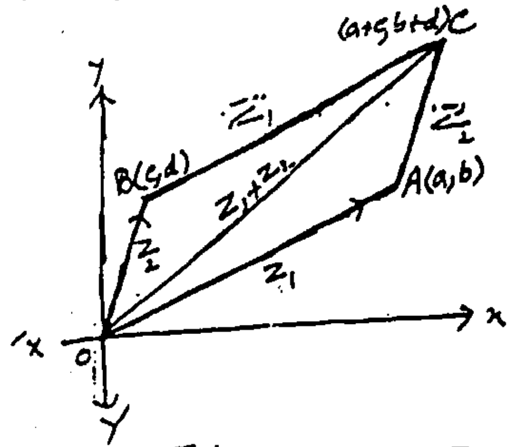
$\boxed{|z_1| - |z_2| \le |z_1+z_2|}$ — (II)

Combining (I) & (II) $|z_1| - |z_2| \le |z_1+z_2| \le |z_1| + |z_2|$

Imp 2nd Method (Also see on Page 10)

For any two complex numbers

$|z_1| - |z_2| \le |z_1+z_2| \le |z_1| + |z_2|$



Let $|z_1| = |\overline{OA}|$ $\quad |z_1+z_2| = |\overline{OC}|$

$|z_2| = |\overline{OB}|$

In ΔOAC $\quad |\overline{OA}| + |\overline{AC}| > |\overline{OC}|$ — (I)

$|z_1| + |z_2| > |z_1+z_2|$

For Collinear pts

$|\overline{OA}| + |\overline{AC}| = |\overline{OC}|$

$|z_1| + |z_2| = |z_1+z_2|$ — (II)

Combining (I) & (II)

$|z_1| + |z_2| \ge |z_1+z_2|$ proved

Now $|z_1| = |z_1+z_2-z_2|$

$|z_1| \le |z_1+z_2| + |-z_2|$

$|z_1| \le |z_1+z_2| + |z_2|$

$|z_1| - |z_2| \le |z_1+z_2|$ — (III)

Combining (II) + (III) we get the result.

To Prove $|z_1| - |z_2| \le |z_1 - z_2|$

Proof Since $|z_1+z_2| \le |z_1| + |z_2|$

Replace z_1 by $z_1 - z_2$

$\therefore |z_1 - z_2 + z_2| \le |z_1 - z_2| + |z_2|$

$\Rightarrow |z_1| \le |z_1 - z_2| + |z_2|$

$|z_1| - |z_2| \le |z_1 - z_2|$

To Prove $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$

Proof $\because z\bar{z} = |z|^2$

$\Rightarrow \frac{1}{z\bar{z}} = \frac{1}{|z|^2}$

$\Rightarrow \frac{1}{z} = \frac{\bar{z}}{|z|^2}$ proved.

EXERCISE 1.1

EXPRESS each of the following complex numbers in the polar form. (Problem 1—6):

Q.1 Let $z = x + iy = -\sqrt{3} + i \Rightarrow x = -\sqrt{3}$, $y = 1$, $r = |z| = \sqrt{(-\sqrt{3})^2 + 1^2} = \sqrt{3+1} = 2$

$\because \cos\theta = \frac{x}{r} = \frac{-\sqrt{3}}{2} \Rightarrow \theta = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

$\&\ \sin\theta = \frac{y}{r} = \frac{1}{2} \Rightarrow \theta = \sin^{-1}\left(\frac{1}{2}\right)$ $\Big\} \Rightarrow \theta = \frac{5\pi}{6}$

(x, is -ive & y, is +ive, So θ lies in 2nd Quad. Since 2nd Quadrant, $\therefore \theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$)

Hence $z = r(\cos\theta + i\sin\theta)$
$= 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$
$= 2\ \mathrm{cis}\,\frac{5\pi}{6}$ Ans

2nd Method

$z = x + iy = -\sqrt{3} + i \Rightarrow x = -\sqrt{3}$, $y = 1$

$\tan\theta = \frac{y}{x} = \frac{1}{-\sqrt{3}}$

$\theta = \tan^{-1}\left(\frac{1}{-\sqrt{3}}\right) = \frac{5\pi}{6}$

(x -ive, y +ive $\therefore$ θ lies in 2nd Quad. So Principal Arg $= \pi - \theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$)

Hence $z = r(\cos\theta + i\sin\theta)$
$= 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

Q.2 Let $z = x + iy = -i = 0 + (-i) \Rightarrow x = 0$, $y = -1$

$\Rightarrow r = |z| = \sqrt{0^2 + (-1)^2} = 1$

$\because \cos\theta = \frac{x}{r} = \frac{0}{1} = 0 \Rightarrow \theta = \cos^{-1}(0)$

$\&\ \sin\theta = \frac{y}{r} = -\frac{1}{1} = -1 \Rightarrow \theta = \sin^{-1}(-1)$ $\Big\} \theta = -\frac{\pi}{2}$

($\because$ x +ive & y -ive $\therefore$ 4th Quad. So Principal Arg$z = -\theta = -\frac{\pi}{2}$)

Hence $z = r(\cos\theta + i\sin\theta)$
$= 1\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$

or $z = \left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right)$

2nd Method

$z = x + iy = -i$
$\Rightarrow x = 0,\ y = -1$

$\tan\theta = \frac{y}{x} = \frac{-1}{0} = \infty$

$\theta = \tan^{-1}(\infty) = -\frac{\pi}{2}$

($\because$ x +ive, y -ive $\therefore$ 4th Quad. So Principal Arg $= -\theta = -\frac{\pi}{2}$)

Hence $z = r(\cos\theta + i\sin\theta)$
$= 1\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$
$z = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2}$

Q.3 Let $z = x + iy = -1 - \sqrt{3}\,i \Rightarrow x = -1$, $y = -\sqrt{3}$

$\Rightarrow r = |z| = \sqrt{(-1)^2 + (-\sqrt{3})^2}$

$\Rightarrow r = \sqrt{1+3} = \sqrt{4} = 2$

$\because \cos\theta = \frac{-1}{2} \Rightarrow \theta = \cos^{-1}\left(-\frac{1}{2}\right)$

$\&\ \sin\theta = \frac{-\sqrt{3}}{2} \Rightarrow \theta = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ $\Big\} \theta = -\frac{2\pi}{3}$

(3) $z = x + iy = -1 - \sqrt{3}i \Rightarrow x = -1$, $y = -\sqrt{3}$

$\tan\theta = \frac{y}{x} = \frac{-\sqrt{3}}{-1}$, $r = \sqrt{1+3} = \sqrt{4}$, $r = 2$

$\theta = \tan^{-1}\left(\frac{-\sqrt{3}}{-1}\right)$

$\theta = -\frac{2\pi}{3}$

$\because$ x -ive, y -ive III Quad

$\therefore -\left(\pi - \frac{\pi}{3}\right) = -\frac{2\pi}{3}$

$\because$ x is $-$ve $\quad$ $\therefore$ θ is in 3rd Quad.
y is $-$ve
So Principal Arg $z = -(\pi - \theta)$
Principal Arg $z = -(\pi - \frac{\pi}{3})$
$= -\frac{2\pi}{3}$

Hence $z = r(\cos\theta + i\sin\theta)$

$z = 2\left(\cos\left(\frac{-2\pi}{3}\right) + i\sin\left(\frac{-2\pi}{3}\right)\right)$

$= 2\left(\cos\left(\frac{2\pi}{3}\right) - i\sin\left(\frac{2\pi}{3}\right)\right)$

Q.4 Let $z = x + iy = -1 + i \Rightarrow x = -1,\ y = 1$

$\Rightarrow r = |z| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

$\cos\theta = \frac{x}{r} = \frac{-1}{\sqrt{2}} \Rightarrow \theta = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

$\sin\theta = \frac{y}{r} = \frac{1}{\sqrt{2}} \Rightarrow \theta = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$\theta = \frac{3\pi}{4}$ $\therefore \theta = \frac{3\pi}{4}$.

$\because$ x is $-$ve
y is $+$ve $\therefore$ θ is in 2nd Quad
Principal Arg $z = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

Hence $z = r(\cos\theta + i\sin\theta)$
$= \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$
$= \sqrt{2}\ \text{Cis}\,\frac{3\pi}{4}$ Ans

Q4 2nd Method

$z = x + iy = -1 + i$ $\quad$ $x = -1$, $y = 1$, $r = \sqrt{1+1} = \sqrt{2}$

$\tan\theta = \frac{y}{x} = \frac{1}{-1} = -1$

$\theta = \tan^{-1}(-1) = \frac{3\pi}{4}$

x $-$ve, y $+$ve II Quad
$\therefore \pi - \theta$
$\pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$\therefore z = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\,S\,\frac{3\pi}{4}\right)$

Q5 Let $z = (-2 + 2i)(1 - i)$
$= -2 + 2i + 2i - 2i^2$
$= \cancel{-2} + 4i + \cancel{2}$

$z = 4i$

$z = 0 + 4i \Rightarrow x = 0,\ y = 4$

$r = |z| = \sqrt{0^2 + 4^2} = 4$

$\cos\theta = \frac{x}{r} = \frac{0}{4} = 0 \Rightarrow \theta = \frac{\pi}{2}$

$\sin\theta = \frac{y}{r} = \frac{4}{4} = 1 \Rightarrow \theta = \frac{\pi}{2}$

(x is $+$ve, y is $+$ve θ is in Ist Quad So Principal Arg $z = \frac{\pi}{2}$)

Hence $z = r(\cos\theta + i\sin\theta) = 4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 4\,\text{Cis}\,\frac{\pi}{2}$ Ans.

Q5 $\quad z = (-2 + 2i)(1 - i)$
$= 4i$

$z = x + iy = 4i$ $\quad$ $x = 0$, $y = 4$

$\tan\theta = \frac{y}{x} = \frac{4}{0} = \infty$ $\quad$ $r = \sqrt{0^2 + 4^2}$, $r = 4$

$\theta = \tan^{-1}(\infty)$

$\theta = \frac{\pi}{2}$ $\quad$ $\because$ x $+$ve, y $+$ve I Quad

$\therefore z = 4\left(\cos\frac{\pi}{2} + i\,S\,\frac{\pi}{2}\right)$

Q6 Let $z = -\dfrac{34i}{5-3i}$

$$= \frac{-34i}{5-3i} \times \frac{5+3i}{5+3i}$$

$$= \frac{-34i(5+3i)}{25+9}$$

$$= -i(5+3i)$$

$$= -5i+3 \quad \Rightarrow x = 3, \; y = -5$$

$$r = \sqrt{3^2+(-5)^2} = \sqrt{34}$$

$$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\left(\frac{-5}{3}\right)$$

$\because x$ +ive, y −ive $\therefore \theta$ is in IVth Q $\therefore$ Principal Arg $= -\theta$

So Principal Arg $z = -\tan^{-1}\left(\frac{5}{3}\right) = \tan^{-1}\left(-\frac{5}{3}\right)$

$$z = r(\cos\theta + i\sin\theta)$$

$$= \sqrt{34}\left(\cos\left(\tan^{-1}\left(-\tfrac{5}{3}\right)\right) + i\sin\left(\tan^{-1}\left(-\tfrac{5}{3}\right)\right)\right)$$

$$= \sqrt{34}\, \text{cis}\left(\tan^{-1}\left(-\tfrac{5}{3}\right)\right)$$

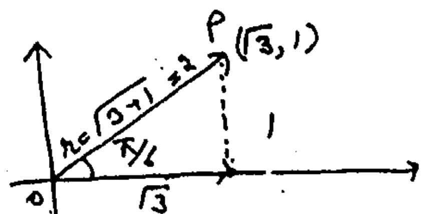
Q7 Express the given Complex number in Cartesian form and in Argand Diagram.

$$z = 2\,\text{cis}\left(\frac{\pi}{6}\right)$$

$$= 2\left[\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right]$$

$$= 2\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$$

$$= \sqrt{3} + i = (\sqrt{3}, 1)$$



$\because x$ is +ive & y is +ive so Ist Quad.

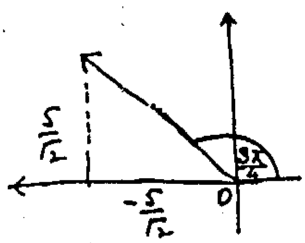
Q8 $z = 5\,\text{cis}\left(\frac{3\pi}{4}\right)$

$$= 5\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right]$$

$$= 5\left[-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right]$$

$$= \frac{-5}{\sqrt{2}} + i\frac{5}{\sqrt{2}}$$

$$= \left(-\frac{5}{\sqrt{2}}, \frac{5}{\sqrt{2}}\right)$$



$\frac{3\pi}{4} = 135^\circ$

$\because x$ is −ive, y is +ive so IInd Q

Q9 $z = \sqrt{3}\,\text{Cis}\,\frac{7\pi}{6}$

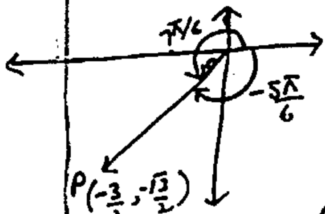
$= \sqrt{3}\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$

$= \sqrt{3}\left(\cos\left(2\pi - \frac{5\pi}{6}\right) + i\sin\left(2\pi - \frac{5\pi}{6}\right)\right)$

$= \sqrt{3}\left(\cos\frac{5\pi}{6} - i\sin\frac{\pi}{6}\right)$

$= \sqrt{3}\left(-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)$

$= -\frac{3}{2} - i\frac{\sqrt{3}}{2} \qquad = \left(-\frac{3}{2}, -\frac{\sqrt{3}}{2}\right)$



Note: both are equal
$\left(\pi + \frac{\pi}{6}\right) = \frac{7\pi}{6} = \left(2\pi - \frac{5\pi}{6}\right)$

$\frac{7\pi}{6} - 2\pi = -\frac{5\pi}{6}$

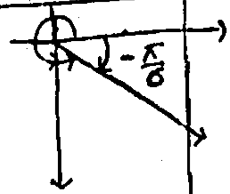
$\frac{5\pi}{6} = 150°$

$\cos\left(2\pi - \frac{5\pi}{6}\right) = \cos\frac{5\pi}{6}$

$\sin\left(2\pi - \frac{5\pi}{6}\right) = -\sin\frac{5\pi}{6}$

x is $-$ve, y is $-$ve so III rd Quad.
$\cos^{-1}\frac{\sqrt{3}}{2} = \boxed{\frac{\pi}{6}}$, $\sin^{-1}\frac{1}{2} = \boxed{\frac{\pi}{6}}$ So $-(\pi - \theta)$
$-\left(\pi - \frac{\pi}{6}\right) = -\frac{5\pi}{6}$

Q10 $z = \dfrac{5\,\text{Cis}\left(\frac{\pi}{3}\right)}{2\,\text{Cis}\left(\frac{\pi}{2}\right)}$



$= \dfrac{5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)}{2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)}$

$= \dfrac{5\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)}{2(0 + i\cdot 1)}$

$\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$

$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

$x \,\&\, \div$ by i $\quad = \dfrac{5\cdot\left(\frac{i}{2} - \frac{\sqrt{3}}{2}\right)}{-2}$

$= \frac{5}{2}\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right) \quad = \frac{5\sqrt{3}}{4} - \frac{5}{4}i \quad = \left(\frac{5\sqrt{3}}{4}, -\frac{5}{4}\right)$

2nd Method

$z = \dfrac{5\,\text{Cis}\left(\frac{\pi}{3}\right)}{2\,\text{Cis}\left(\frac{\pi}{2}\right)}$

$= \frac{5}{2}\,\text{Cis}\left(\frac{\pi}{3} - \frac{\pi}{2}\right)$

$= \frac{5}{2}\,\text{Cis}\left(-\frac{\pi}{6}\right)$

$= \frac{5}{2}\left[\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right]$

$= \frac{5}{2}\left[\frac{\sqrt{3}}{2} - i\frac{1}{2}\right]$

$= \frac{5}{4}\left(\sqrt{3} - i\right)$

x +ve, y $-$ve So IV th Q.
So $-\theta$
So $-\frac{\pi}{6}$

Q11 (i)

Find $|z|$ where $z = -2i(1+i)(2+4i)(3+i)$

$|z| = |-2i(1+i)(2+4i)(3+i)|$

$= |-2i|\,|1+i|\cdot|2+4i|\cdot|3+i| \qquad \because |z_1 z_2| = |z_1|\cdot|z_2|$

$= \sqrt{(-2)^2}\sqrt{1^2+1^2}\sqrt{2^2+4^2}\sqrt{3^2+1^2}$

$= 2\sqrt{2}\sqrt{20}\sqrt{10}$

$= 2\sqrt{2}\,2\sqrt{5}\,\sqrt{5}\sqrt{2}$

$= 4(2)(5) \qquad = 40$

(ii) $z = \dfrac{(3+4i)(-1+2i)}{(-1-i)(3-i)}$

$|z| = \left|\dfrac{(3+4i)(-1+2i)}{(-1-i)(3-i)}\right|$

$= \dfrac{|(3+4i)(-1+2i)|}{|(-1-i)(3-i)|} \qquad \because \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

$|z| = \dfrac{|3+4i|\cdot|-1+2i|}{|-1-i|\,|3-i|}$

$= \dfrac{\sqrt{9+16}\,\sqrt{1+4}}{\sqrt{1+1}\,\sqrt{9+1}}$

$= \dfrac{5\,\sqrt{5}}{\sqrt{2}\,\sqrt{10}}$

$= \dfrac{5\sqrt{5}}{2\sqrt{5}}$

$= \frac{5}{2}$ Ans.

Imp Q12(i) Show that $z = a + ib$ is real iff $\boxed{z = \bar{z}}$

Let $z = a + ib$ is real $\Rightarrow b = 0$ (Imaginary part $b = 0$)

$\therefore z = a$ —— (i)

$\bar{z} = a$ —— (ii)

from (i) & (ii) $\therefore \boxed{z = \bar{z}}$

Conversely Suppose $z = \bar{z}$

$a + ib = \overline{a + ib}$

$a + ib = a - ib$

$\cancel{a} - \cancel{a} + ib + ib = 0$

$2ib = 0$

$b = 0$ $\because \left(\begin{matrix} i = \sqrt{-1} \neq 0 \\ 2 \neq 0 \end{matrix} \right.$

Hence $z = a + 0i$

$z = a$ which is real

Imp Q12 (ii) Show that $z = a + ib$ is pure imaginary iff $\boxed{z = -\bar{z}}$

(Real part $a = 0$)

Let $z = a + ib$ is pure imaginary $\Rightarrow a = 0$

So $z = ib$ —— (1)

$\bar{z} = -ib$ —— (2)

$\bar{z} = -(z)$ using (1) in (2)

$\boxed{z = -\bar{z}}$

Conversely Let $z = -\bar{z}$

$a + ib = -(a - ib)$

$a + \cancel{ib} + a - \cancel{ib} = 0$

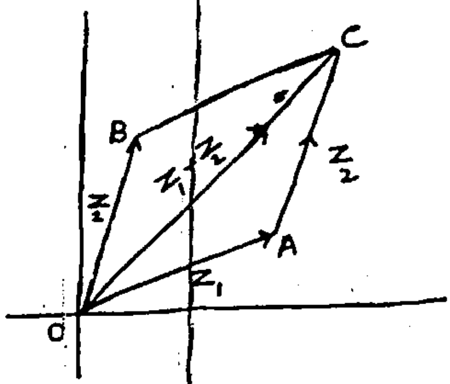
$2a = 0$

$a = 0$ (Real part of z is zero)

So $z = 0 + ib$

$z = ib$ which is pure imaginary.

Example 3 Let z_1, z_2 be two complex numbers. Determine the greatest and least values of $|z_1+z_2|$



Sol Let $z_1 = \overline{OA}$

& $z_2 = \overline{OB}$

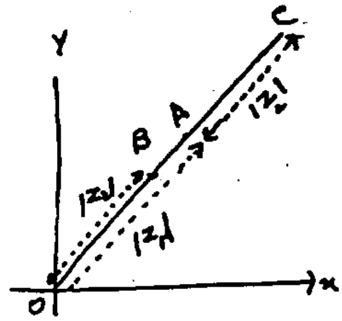
then $z_1+z_2 = \overline{OC}$

Now $|z_1| = |\overline{OA}|$

$|z_2| = |\overline{OB}| = |\overline{AC}|$

& $|z_1+z_2| = |\overline{OC}|$

In ΔOAC



$|\overline{OA}| + |\overline{AC}| > |\overline{OC}|$ ($\because$ Sum of lengths of two sides is greater than the length of third side in any Δ)

$|z_1| + |z_2| > |z_1+z_2|$ —— (I)

when $\operatorname{Arg} z_1 = \operatorname{Arg} z_2$

then OA is // to OB

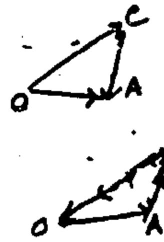
hence //gm OABC becomes straight line

$\therefore\ OA + AC = OC$

$|z_1| + |z_2| = |z_1+z_2|$ —— (II)

from (I) & (II) $|z_1| + |z_2| \geq |z_1+z_2|$ —— (III)

Thus greatest possible value of $|z_1+z_2|$ is $|z_1|+|z_2|$



Case I

Case II

In ΔOAC $OC + CA > OA$ and $CO + OA > CA$

$|z_1+z_2| + |z_2| > |z_1|$ $\qquad$ $|z_1+z_2| + |z_1| > |z_2|$

$|z_1+z_2| > |z_1| - |z_2|$ —— (IV) $\qquad$ $|z_1+z_2| > |z_2| - |z_1|$

$|z_1+z_2| > -(|z_1| - |z_2|)$

$-|z_1+z_2| < |z_1| - |z_2|$ —— (V)

from (IV) & (V) $-|z_1+z_2| < |z_1| - |z_2| < |z_1+z_2|$ —— (VI)

& (VI) together with the extreme case when O, A, B, C are collinear gives

$$||z_1| - |z_2|| \leq |z_1+z_2| \quad \text{—— (VII)} \qquad \because \begin{pmatrix} -a \leq x \leq a \\ \Rightarrow |x| \leq a \end{pmatrix}$$

Thus least possible value of $|z_1+z_2|$ is $||z_1| - |z_2||$

Combining (III) & (VII)

$$||z_1| - |z_2|| \leq |z_1+z_2| \leq |z_1| + |z_2|$$

x —————— x

Imp S2001

Q13 Prove analytically for complex No z_1, z_2

$$||z_1|-|z_2|| \leq |z_1-z_2| \leq |z_1+z_2| \leq |z_1|+|z_2|$$

Sol Let $|z_1+z_2|^2 = (z_1+z_2)(\overline{z_1+z_2})$ $\quad \because |z|^2 = z\bar{z}$

$= (z_1+z_2)(\bar{z}_1+\bar{z}_2)$ $\quad \because \overline{z_1+z_2} = \bar{z}_1+\bar{z}_2$

$= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2$

$= |z_1|^2 + 2Re(z_1\bar{z}_2) + |z_2|^2$ $\quad \because z_1\bar{z}_2 + z_2\bar{z}_1 = 2Re z_1\bar{z}_2$ proved earlier

$\leq |z_1|^2 + 2|z_1\bar{z}_2| + |z_2|^2$ $\quad \because |Re z| \leq |z|$

$= |z_1|^2 + 2|z_1||z_2| + |z_2|^2$ $\quad \because |z_1\bar{z}_2| = |z_1||\bar{z}_2|$ & $|\bar{z}_1| = |z_1|$

$|z_1+z_2|^2 \leq (|z_1|+|z_2|)^2$

Taking Square Root $\boxed{|z_1+z_2| \leq |z_1|+|z_2|}$ ——— ①

Now $|z_1| = |z_1+z_2-z_2|$ $\quad (+\ \&\ -z_2)$

$\leq |z_1+z_2| + |-z_2|$ $\quad \because |z_1+z_2| \leq |z_1|+|z_2|$

$= |z_1+z_2| + |z_2|$ $\quad \because |z_2| = |-z_2|$

$\boxed{|z_1|-|z_2| \leq |z_1+z_2|}$ ——— (II)

Put $z_2 = -z_2$ in (II)

$|z_1|-|-z_2| \leq |z_1-z_2|$

$\boxed{|z_1|-|z_2| \leq |z_1-z_2|}$ ——— (III)

Also $|z_2| = |z_2-z_1+z_1|$ $\quad (+\ \&\ -z_1)$

$\leq |z_2-z_1|+|z_1|$

$|z_2|-|z_1| \leq |z_2-z_1|$

$|z_2|-|z_1| \leq |z_1-z_2|$ $\quad \because |z_1-z_2| = |z_2-z_1|$

$-|z_1-z_2| \leq |z_1|-|z_2|$ ——— (IV)

from (III) & (IV) $\therefore -|z_1-z_2| \leq |z_1|-|z_2| \leq |z_1-z_2|$

$\therefore ||z_1|-|z_2|| \leq |z_1-z_2|$ ——— (V) $\quad$ as if $-a \leq x \leq a$ then $|x| \leq a$

Now Obviously $|z_1-z_2| \leq |z_1+z_2|$ ——— (VI)

from ① & (VI) $|z_1-z_2| \leq |z_1+z_2| \leq |z_1|+|z_2|$ ——— (VII)

from (V) & (VII) $||z_1|-|z_2|| \leq |z_1-z_2| \leq |z_1+z_2| \leq |z_1|+|z_2|$ Proved.

$z_1 = 24+7i$, $|z_2| = 6$

$|z_1| = \sqrt{(24)^2+7^2}$

$= \sqrt{576+49}$

$= \sqrt{625} = 25$

We know $\big||z_1| - |z_2|\big| \leq |z_1 + z_2| \leq |z_1| + |z_2|$

So greatest value of $|z_1 + z_2|$ is

$= |z_1| + |z_2| = 25+6 = 31$

also since $\big||z_1| - |z_2|\big| \leq |z_1 + z_2|$

so least value of $|z_1 + z_2|$ is

$= \big||z_1| - |z_2|\big|$

$= |25 - 6| = |19| = 19$

Q.15 If z_1, z_2 are complex numbers, show that

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right)$$

PROOF L.H.S. $|z_1 + z_2|^2 + |z_1 - z_2|^2$

$= (z_1 + z_2)(\overline{z_1 + z_2}) + (z_1 - z_2)(\overline{z_1 - z_2})$ $\left(\because z\bar{z} = |z|^2\right)$

$= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) + (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$

$= z_1\bar{z}_1 + z_2\bar{z}_2 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_1\bar{z}_1 + z_2\bar{z}_2 - z_1\bar{z}_2 - \cancel{z_2\bar{z}_1}$

$= 2z_1\bar{z}_1 + 2z_2\bar{z}_2 - 2(z_1\bar{z}_1 + z_2\bar{z}_2)$

$= 2\left(|z_1|^2 + |z_2|^2\right) = $ R.H.S.

599 Imp

Q.16 Prove that $\left|\dfrac{az+b}{\bar{b}z+\bar{a}}\right| = 1$ for $|z| = 1$

Sol We have L.H.S. $= \left|\dfrac{az+b}{\bar{b}z+\bar{a}}\right| = \dfrac{|az+b|}{|\bar{b}z+\bar{a}|}$ $\left(\because \left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}\right)$

$\left(\because |z| = |\bar{z}|\right)$

$= \dfrac{|az+b|}{|\overline{\bar{b}z+\bar{a}}|} = \dfrac{|az+b|}{|\bar{\bar{b}}\bar{z}+\bar{\bar{a}}|}$

$= \dfrac{|az+b|}{|\bar{\bar{b}}\bar{z}+\bar{\bar{a}}|} = \dfrac{|az+b|}{|b\bar{z}+a|}$ $\left(\because \bar{\bar{z}} = z\right)$

$= \dfrac{|z|\,|az+b|}{|z|\,|b\bar{z}+a|} = \dfrac{|z|\,|az+b|}{|bz\bar{z}+az|}$ $\left(\because |z_1z_2| = |z_1||z_2|\right)$

$$= \frac{|z||az+b|}{|az+b|} \quad \left(\because z\bar{z} = |z|^2 = 1 \text{ as } |z|=1 \text{ given}\right)$$

$$= |z| = 1 = R.H.S$$

Q.17 Find locus of the points in the plane where satisfying each of the following conditions:

Part-(i) $|z-5| = 6$

Sol $|z-5| = 6 \rightarrow (i)$ let $z = a+ib$

Then (i) will become

$|a+ib-5| = 6$ or $|(a-5)+ib| = 6$

or $\sqrt{(a-5)^2+b^2} = 6$ or $(a-5)^2+(b-0)^2 = 36$

which shows that the locus is a circle having centre at point $(5,0)$ and radius $= 6$ unit

Part-(ii) $|z-2i| \geq 1$

Sol we have $|z-2i| \geq 1 \rightarrow (i)$

let $z = x+iy$. So (i) will be

$|x+iy-2i| \geq 1$ or $|x+i(y-2)| \geq 1$

or $\sqrt{x^2+(y-2)^2} \geq 1$ or $x^2+(y-2)^2 \geq 1$

or $(x-0)^2+(y-2)^2 \geq 1 \longrightarrow (ii)$

inequality give that required locus is a set of those points that lies on the circle or outside the circle having centre at $(0,2)$ and radius $= 1$

Part-(iii) $Re(z+2) = -1$

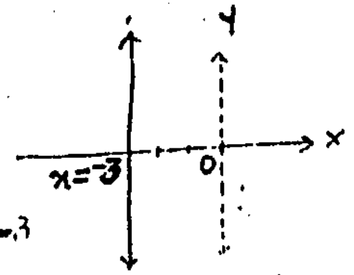
Sol We are given that

$Re(z+2) = -1 \longrightarrow (i)$

let $z = x+iy$, put in (i)

$\Rightarrow \quad Re\,(x+iy+2) = -1$

or $\quad Re\,((x+2)+iy) = -1$



$\Rightarrow \quad x+2 = -1 \quad$ or $\quad x = -3$

The locus is the line $\parallel$ to Y-axis on left side of y-axis.

×————×

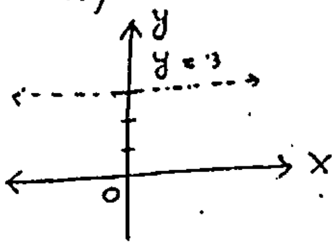
Part - (iv) $\quad Re(i\bar{z}) = 3$

Sol We have $\quad Re\,(i\,\bar{z}) = 3 \longrightarrow$ (i)

let $\quad z = x+iy \quad$ then (i) will be

$Re\,(i\,\overline{x+iy}) = 3 \quad$ or $\quad Re\,(i\,(x-iy)) = 3$

or $\quad Re\,(y+ix) = 3 \quad \Rightarrow \quad y = 3$



$\Rightarrow$ loucus is the horizontal line $y=3$.

×————×

Part - (v) $\quad |z+i| = |z-i| \longrightarrow$ (i)

Sol Put $\quad z = x+iy$ in (i) we get

$|x+iy+i| = |x+iy-i|$

or $\quad |x+i(y+1)| = |x+i(y-1)|$

$\Rightarrow \quad \sqrt{x^2+(y+1)^2} = \sqrt{x^2+(y-1)^2}$

Sq. we get

$x^2+(y+1)^2 = x^2+(y-1)^2$

or $\quad (y+1)^2 = (y-1)^2$

$\Rightarrow \quad y^2+2y+1 = y^2-2y+1 \quad \Rightarrow \quad 2y = -2y$

or $\quad 4y = 0 \quad \Rightarrow \quad y = 0 \longrightarrow$ (ii)

Eq. (ii) gives that required locus is set of all those points that lies on x- axis.

Part-(vi) $|z+3| + |z+1| = 4 \rightarrow (i)$

Sol Put $z = x+iy$ in (i) we get

$|x+yi+3| + |x+iy+1| = 4$

or $|(x+3)+iy| + |(x+1)+iy| = 4$

or $\sqrt{(x+3)^2+y^2} + \sqrt{(x+1)^2+y^2} = 4$

$\Rightarrow \sqrt{(x+3)^2+y^2} = 4 - \sqrt{(x+1)^2+y^2}$

Sq. we get

$(x+3)^2+y^2 = 16 + (x+1)^2 + y^2 - 8\sqrt{(x+1)^2+y^2}$

$\cancel{x^2}+\cancel{y^2}+6x+9-\cancel{x^2}-2x-17-\cancel{y^2} = -8\sqrt{(x+1)^2+y^2}$

$4x-8 = -8\sqrt{(x+1)^2+y^2}$

or $x-2 = -2\sqrt{(x+1)^2+y^2}$

Sq. we get

$x^2-4x+4 = 4(x^2+2x+1+y^2)$

or $3x^2+12x+4y^2 = 0$ which is required locus

x ——— x

Part-(vii) $-1 \leq Re\, z < 1$

Sol Put $x+iy = z$ in (i)

$\Rightarrow -1 \leq Re(x+iy) < 1$

or $-1 \leq x < 1$

$\Rightarrow$ The value of x lies in the interval $[-1, 1[$.

x ——— x

Part-(viii) $Im\, z < 0 \rightarrow (i)$

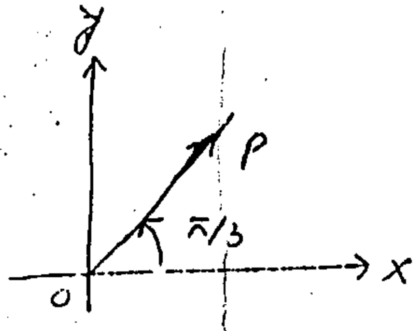
Sol Put $z = x+iy$ in (i)

$\Rightarrow Im(x+iy) < 0$ or $y < 0$ which is required locus i.e. value of y is -ive.

Part-(ix) $\quad Arg\, z = \frac{\pi}{3}$

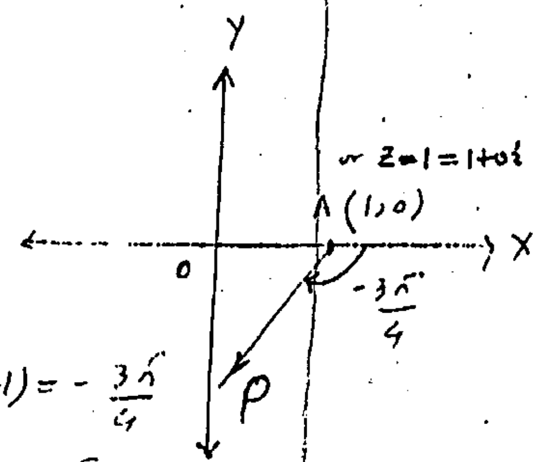
$\underline{Sol}$ Let $z = \vec{OP}$. then $Arg\, z = Arg\, \vec{OP} = \frac{\pi}{3}$

So, the required locus is a line $\vec{OP}$ that makes $\theta = \frac{\pi}{3}$ with +ive x-axis, as shown in fig



× ——— ×

Part-(x) $\quad Arg(z-1) = -\frac{3\pi}{4}$



$\underline{Sol}$ put $z = x+iy$

$\Rightarrow Arg(z-1) = Arg(x+iy-1) = -\frac{3\pi}{4}$

or $\quad Arg\left((x-1)+iy\right) = -\frac{3\pi}{4}$

or $\quad Arg\left((x-1)+i(y-0)\right) = -\frac{3\pi}{4}$

The required locus is represented by the line $\vec{AP}$ which makes an angle of measure $= -\frac{3\pi}{4}$ with +ive x-axis at pt $A(1,0)$ on x-axis

× ——— ×

De Moivre's Theorem

STATEMENT:- If n is any integer, then

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \rightarrow (1)$$

PROOF:- CASE-1 Put $n=0$ in (1), then

$L.H.S = (\cos\theta + i\sin\theta)^0 = 1$

$R.H.S = \cos(0\cdot\theta) + i\sin(0\cdot\theta) = \cos 0 + i\sin 0 = 1 + 0i = 1$

$\Rightarrow L.H.S. = R.H.S.$