Exercise 1.5

In each of Problem 1-5, evaluate The indicated sum.

1. (i) $\sin A + \sin 2A + \sin 3A + \cdots + \sin nA$

(ii) $\cos A + \cos 2A + \cos 3A + \cdots + \cos nA$

Sol: Let $S = \sin A + \cancel{\cos A} + \sin 2A + \sin 3A + \cdots + \sin nA$

$C = \cos A + \cos 2A + \cdots + \cos nA$

$C + iS = (\cos A + i\sin A) + (\cos 2A + i\sin 2A) + \cdots + (i\sin nA + \cos nA)$

$= e^{iA} + e^{i2A} + \cdots + e^{inA}$

This is geometric progression with $r = e^{iA}$

here $a = e^{iA}$, $r = e^{iA}$, $n = n$

$$C + iS = \frac{a[r^n - 1]}{r - 1}$$

$$= \frac{e^{iA}(e^{inA} - 1)}{e^{iA} - 1}$$

$$= \frac{e^{iA} \cdot e^{inA/2}(e^{inA/2} - e^{-iAn/2})}{e^{iA/2}(e^{iA/2} - e^{-iA/2})}$$

$$= e^{iA - iA/2 + inA/2} \frac{\left(\frac{e^{inA/2} - e^{-iAn/2}}{2i}\right)}{\left(\frac{e^{iA/2} - e^{-iA/2}}{2i}\right)}$$

$$= e^{iA/2 + inA/2} \frac{\sin(nA/2)}{\sin(A/2)}$$

$$= e^{iA/2(n+1)} \frac{\sin(nA/2)}{\sin(A/2)}$$

$$C + iS = \left(\cos\left(\frac{A(n+1)}{2}\right) + i\sin\left(\frac{A(n+1)}{2}\right)\right) \frac{\sin(nA/2)}{\sin(A/2)}$$

Equating real and imaginary part

(ii) $C = \cos\left(\frac{A(n+1)}{2}\right) \cdot \frac{\sin(nA/2)}{\sin(A/2)}$ Ans

(i) $S = \sin\left(\frac{A(n+1)}{2}\right) \frac{\sin(nA/2)}{\sin(A/2)}$ Ans

2. $\cos\theta + \cos 3\theta + \dots + \cos(2n-1)\theta = C$

Sol Let

$$C = \cos\theta + \cos 3\theta + \dots + \cos(2n-1)\theta$$

$$S = \sin\theta + \sin 3\theta + \dots + \sin(2n-1)\theta$$

$$C + iS = (\cos\theta + i\sin\theta) + (\cos 3\theta + i\sin 3\theta) + \dots + (\cos(2n-1)\theta - i\sin(2n-1)\theta)$$

$$C + iS = e^{i\theta} + e^{i3\theta} + e^{i5\theta} + \dots + e^{i(2n-1)\theta}$$

geometric progression

$$a = e^{i\theta}, \quad r = e^{i2\theta}, \quad n = n$$

$$C + iS = \frac{e^{i\theta}\left(e^{i2\theta n} - 1\right)}{e^{i2\theta} - 1} \qquad \because \text{Sum} = \frac{a(r^n - 1)}{(r-1)}$$

$$= \frac{\cancel{e^{i\theta}} \cdot e^{i\theta n}\left(e^{i\theta n} - e^{-i\theta n}\right)}{\cancel{e^{i\theta}}\left(e^{i\theta} - e^{-i\theta}\right)}$$

$$= e^{i\theta n}\,\frac{\left(\frac{e^{i\theta n} - e^{-i\theta n}}{2i}\right)}{\left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)}$$

$$= (\cos n\theta + i\sin n\theta) \cdot \frac{\sin n\theta}{\sin\theta}$$

Equating Real part

$$C = \cos n\theta \cdot \frac{\sin n\theta}{\sin\theta} \quad \text{Ans}$$

3. $1 + x\cos\theta + x^2\cos 2\theta + \dots + x^n \cos n\theta$

Sol Let

$$C = 1 + x\cos\theta + x^2\cos 2\theta + \dots + x^n\cos n\theta$$

$$S = x\sin\theta + x^2\sin 2\theta + \dots + x^n \sin n\theta$$

$$C + iS = 1 + x(\cos\theta + i\sin\theta) + x^2(\cos 2\theta + i\sin 2\theta) + \dots + x^n(\cos n\theta + i\sin n\theta)$$

$$C + iS = 1 + xe^{i\theta} + x^2 e^{i2\theta} + \dots + e^{in\theta} \cdot x^n$$

$$a = 1, \quad r = xe^{i\theta}, \quad n = n+1$$

$$C + iS = \frac{1\left[x^{n+1} e^{i(n+1)\theta} - 1\right]}{xe^{i\theta} - 1} \qquad \because \frac{a(r^{n+1} - 1)}{r-1}$$

$$= \frac{x^{n+1}\left(\cos(n+1)\theta + i\sin(n+1)\theta\right) - 1}{x(\cos\theta + i\sin\theta) - 1}$$

$$= \frac{\left[x^{n+1}\cos(n+1)\theta - 1\right] + i\,x^{n+1}\sin(n+1)\theta}{(x\cos\theta - 1) + i\,x\sin\theta}$$

multiplying & dividing by $(x\cos\theta - 1) - ix\sin\theta$

$$C+iS = \frac{(x^{n+1}\cos(n+1)\theta - 1) + i x^{n+1}\sin(n+1)\theta}{(x\cos\theta - 1) + ix\sin\theta} \times \frac{(x\cos\theta - 1) - ix\sin\theta}{(x\cos\theta - 1) - ix\sin\theta}$$

Equaling real part

$$C = \frac{(x^{n+1}\cos(n+1)\theta - 1)(x\cos\theta - 1) + x^{n+1}\sin(n+1)\theta\sin\theta \, . \, x}{(x\cos\theta - 1)^2 + (x\sin\theta)^2}$$

$$C = \frac{x^{n+1}\cos(n+1)\theta\, x\cos\theta - x^{n+1}\cos(n+1)\theta - x\cos\theta + 1 + x^{n+2}\sin(n+1)\theta\sin\theta}{(x^2\cos^2\theta + 1 - 2x\cos\theta + x^2\sin^2\theta)}$$

$$C = \frac{x^{n+2}\left[\underbrace{\cos(n+1)\theta\cos\theta + \sin(n+1)\theta\sin\theta}_{\cos(\alpha-\beta)}\right] - x^{n+1}\cos(n+1)\theta - x\cos\theta + 1}{x^2(\cos^2\theta + \sin^2\theta) + 1 - 2x\cos\theta}$$

$$C = \frac{x^{n+2}(\cos(n+1-1)\theta) - x^{n+1}\cos(n+1)\theta - x\cos\theta + 1}{x^2 + 1 - 2x\cos\theta}$$

$$C = \frac{x^{n+2}\cos n\theta - x^{n+1}\cos(n+1)\theta - x\cos\theta + 1}{x^2 + 1 - 2x\cos\theta} \quad \text{Ans}$$

4. Let

Sol

$$S = 3\sin\alpha + 5\sin 2\alpha + 7\sin 3\alpha + \dots + (2n+1)\sin n\alpha$$

$$C = 3\cos\alpha + 5\cos 2\alpha + 7\cos 3\alpha + \dots + (2n+1)\cos n\alpha$$

$$C+iS = 3(\cos\alpha + i\sin\alpha) + 5(\cos 2\alpha + i\sin 2\alpha) + 7(\cos 3\alpha + i\sin 3\alpha) + \dots + (2n+1)(\cos n\alpha + i\sin n\alpha)$$

$$C+iS = 3e^{i\alpha} + 5e^{i2\alpha} + 7e^{i3\alpha} + \dots + (2n+1)e^{in\alpha} \rightarrow (i) \text{ multiply with } -e^{i\alpha}$$

$$-e^{i\alpha}(C+iS) = -3e^{i2\alpha} - 5e^{i3\alpha} + \dots - (2n-1)e^{in\alpha} - (2n+1)e^{i(n+1)\alpha}$$

adding

$$(C+iS) - e^{i\alpha}(C+iS) = 3e^{i\alpha} + 2e^{i2\alpha} + 2e^{i3\alpha} + \dots + 2e^{in\alpha} - (2n+1)e^{i(n+1)\alpha}$$

$$(C+iS)(1-e^{i\alpha}) = 3e^{i\alpha} + 2\left[e^{i2\alpha} + e^{i3\alpha} + \dots + e^{in\alpha}\right] - (2n+1)e^{i(n+1)\alpha}.$$

$\rightarrow$ g.p $a = e^{i2\alpha}$, $r = e^{i\alpha}$, $n = n-1$

$$(C+iS)(1-e^{i\alpha}) = 3e^{i\alpha} + 2\left[\frac{e^{i2\alpha}\left[e^{i(n-1)\alpha} - 1\right]}{e^{i\alpha} - 1}\right] - (2n+1)e^{i(n+1)\alpha}$$

$$= \frac{3e^{i\alpha}(e^{i\alpha} - 1) + 2e^{i2\alpha}(e^{i(n-1)\alpha} - 1) - (2n+1)e^{i(n+1)\alpha}(e^{i\alpha} - 1)}{e^{i\alpha} - 1}$$

$$= \frac{3e^{i\alpha}(e^{i\alpha} - 1) + 2\left[e^{i2\alpha + in\alpha - i\alpha} - e^{i2\alpha}\right] - (2n+1)e^{i(n+1)\alpha}(e^{i\alpha} - 1)}{e^{i\alpha} - 1}$$

$$= \frac{3e^{i2\alpha} - 3e^{i\alpha} + 2e^{i(n+1)\alpha} - 2e^{i2\alpha} - (2n+1)e^{i(n+2)\alpha} + (2n+1)e^{i(n+1)\alpha}}{e^{i\alpha} - 1}$$

$$= \frac{2e^{i2\alpha} - 3e^{i\alpha} + (2+2n+1)\, e^{i(n+1)\alpha} - (2n+1)\, e^{i(n+2)\alpha}}{e^{i\alpha} - 1}$$

$$= \frac{2e^{i2\alpha} - 3e^{i\alpha} + (2n+3)\, e^{i(n+1)\alpha} - (2n+1)e^{i(n+2)\alpha}}{e^{i\alpha} - 1}$$

$$(C+iS)(1-e^{i\alpha}) = \frac{e^{i\alpha}\left(e^{i\alpha} - 3 + (2n+3)e^{in\alpha} - (2n+1)\, e^{i(n+1)\alpha}\right)}{e^{i\alpha} - 1}$$

$$(C+iS) = \frac{e^{i\alpha}\left(e^{i\alpha} - 3 + (2n+3)e^{in\alpha} - (2n+1)e^{i(n+1)\alpha}\right)}{-(e^{i\alpha}-1)(e^{i\alpha}-1) = -(e^{i\alpha}-1)^2}$$

$$= \frac{e^{i\alpha}\left(e^{i\alpha} - 3 + (2n+3)\, e^{in\alpha} - (2n+1)e^{i(n+1)\alpha}\right)}{-\left(e^{i\alpha/2}\left(\frac{e^{i\alpha/2} - e^{-\alpha i/2}}{2i}\right)\right)^2 \cdot (2i)^2}$$

$(2i)^2 = 4i^2 = -4$

$$= \frac{e^{i\alpha}\left(e^{i\alpha} - 3 + (2n+3)e^{in\alpha} \mp (2n+1)e^{i(n+1)\alpha}\right)}{-(e^{i\alpha/2})^2 \left(\frac{e^{i\alpha/2} - e^{-\alpha i/2}}{2i}\right)^2 (-4)}$$

$$= \frac{\cancel{e^{i\alpha}}\left(e^{i\alpha} - 3 + (2n+3)\, e^{in\alpha} \mp (2n+1)e^{i(n+1)\alpha}\right)}{4\, \cancel{e^{i\alpha}}\, (\sin\alpha/2)^2}$$

$$C+iS = \frac{\cos\alpha + i\sin\alpha - 3 + (2n+3)(i\sin n\alpha + \cos n\alpha) - (2n+1)(\cos(n+1)\alpha + i\sin(n+1)\alpha)}{4\sin^2\alpha/2 = \left(\frac{1-\cos\alpha}{2}\right)}$$

Equaling imaginary part

$$S = \frac{\sin\alpha + (2n+3)\sin n\alpha - (2n+1)\sin(n+1)\alpha}{2\cancel{4}\left(\frac{1-\cos\alpha}{\cancel{2}}\right)}$$

$$S = \frac{\sin\alpha + (2n+3)\sin n\alpha - (2n+1)\sin(n+1)\alpha}{1-\cos\alpha}$$

5. $\cos^2\theta + \cos^2 2\theta + \cos^2 3\theta + \dots + \cos^2 n\theta = C_1$

Sol $C_1 = \frac{1+\cos 2\theta}{2} + \frac{1+\cos 4\theta}{2} + \frac{1+\cos 6\theta}{2} + \dots + \frac{1+\cos 2n\theta}{2}$

$= \frac{1}{2}\left[1+\cos 2\theta + 1+\cos 4\theta + 1+\cos 6\theta + \dots + 1+\cos 2n\theta\right]$

$= \frac{1}{2}\left[\underbrace{(1+1+1+\dots+1)}_{\text{up to } n \text{ terms}} + (\cos 2\theta + \cos 4\theta + \dots + \cos 2n\theta)\right]$

$$C_1 = \frac{1}{2}\left[n + (\cos 2\theta + \cos 4\theta + \cos 6\theta + \cdots + \cos 2n\theta)\right] \rightarrow (1)$$

Let

$$C = \cos 2\theta + \cos 4\theta + \cdots + \cos 2n\theta$$
$$S = \sin 2\theta + \sin 4\theta + \cdots + \sin 2n\theta$$

$$C + iS = (\cos 2\theta + i\sin 2\theta) + (\cos 4\theta + \sin 4\theta) + \cdots + (\cos 2n\theta + i\sin 2n\theta)$$

$$C + iS = e^{i2\theta} + e^{i4\theta} + e^{i6\theta} + \cdots + e^{i2n\theta}$$

$a = e^{i2\theta}$, $r = e^{i2\theta}$, $n = n$ $\qquad \frac{a(r^n - 1)}{r-1} = \text{sum}$

$$C + iS = \frac{e^{i2\theta}\left[e^{i2n\theta} - 1\right]}{e^{i2\theta} - 1}$$

$$= \frac{e^{i2\theta} \cdot e^{in\theta}\left[e^{in\theta} - e^{-in\theta}\right]}{e^{i\theta}\left[e^{i\theta} - e^{-i\theta}\right]}$$

$$= e^{(i2\theta - i\theta + in\theta)}\left[\frac{\frac{e^{in\theta} - e^{-in\theta}}{2i}}{\frac{e^{i\theta} - e^{-i\theta}}{2i}}\right]$$

$$= e^{i\theta + in\theta}\left(\frac{\sin n\theta}{\sin\theta}\right) = e^{i(n+1)\theta}\,\frac{\sin n\theta}{\sin\theta}$$

$$C + iS = \frac{\sin n\theta}{\sin\theta}\left(\cos(n+1)\theta + i\sin(n+1)\theta\right)$$

Equating Real part

$$C = \frac{\sin n\theta}{\sin\theta} \cdot \cos(n+1)\theta \qquad \text{put in } (1)$$

$$C_1 = \frac{1}{2}\left[n + \frac{\sin n\theta}{\sin\theta}\cos(n+1)\theta\right]$$

$$C_1 = \frac{n}{2} + \frac{\sin n\theta}{2\sin\theta}\cos(n+1)\theta \qquad \text{Proved.}$$

In Problems 6-15. find sum of each infinite series.

6. $\sin\theta + \frac{1}{2}\sin 3\theta + \frac{1.3}{2.4}\sin 5\theta + \cdots = S$

Sol

$$C = \cos\theta + \frac{1}{2}\cos 3\theta + \frac{1.3}{2.4}\cos 5\theta + \cdots$$

$$C + iS = (\cos\theta + i\sin\theta) + \frac{1}{2}(\cos 3\theta + i\sin 3\theta) + \frac{1.3}{2.4}(\cos 5\theta + i\sin 5\theta) + \cdots$$

$$= e^{i\theta} + \frac{1}{2} e^{i3\theta} + \frac{1.3}{2.4} e^{i5\theta} + \cdots$$

$$C+iS = e^{i\theta}\left(1+\frac{1}{2}e^{i2\theta}+\frac{1.3}{2.4}e^{i4\theta}+\cdots\right)$$

$$= e^{i\theta}\left(1-e^{i2\theta}\right)^{-1/2}$$

$$\because \quad (1-x)^{-1/2} = 1+\frac{1}{2}x+\frac{1.3}{2.4}x^2+\frac{1.3.5}{2.4.6}x^3+\cdots$$

$$C+iS = e^{i\theta}\left(1-(\cos 2\theta+i\sin 2\theta)\right)^{-1/2}$$

$$= e^{i\theta}\left((1-\cos 2\theta)-i\sin 2\theta\right)^{-1/2} \qquad \frac{1-\cos 2\theta}{2}=\sin^2\theta$$

$$= e^{i\theta}\left(2\sin^2\theta - i\,2\sin\theta\cos\theta\right)^{-1/2} \qquad 1-\cos 2\theta = 2\sin^2\theta$$

$$= e^{i\theta}\left[2\sin\theta\left(\sin\theta - i\cos\theta\right)\right]^{-1/2}$$

$$= e^{i\theta}(2\sin\theta)^{-1/2}\left[\sin\theta-i\cos\theta\right]^{-1/2}$$

$$= e^{i\theta}(2\sin\theta)^{-1/2}\left[\cos\left(\frac{\pi}{2}-\theta\right)-i\sin\left(\frac{\pi}{2}-\theta\right)\right]^{-1/2} \qquad \text{De-Moivres Theorem}$$

$$= e^{i\theta}(2\sin\theta)^{-1/2}\left[\cos\left(-\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right)-i\sin\left(-\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right)\right]$$

$$= e^{i\theta}(2\sin\theta)^{-1/2}\left[\cos\left(\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right)+i\sin\left(\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right)\right]$$

$$= \underset{(\cos\theta+i\sin\theta)}{\underline{e^{i\theta}}}(2\sin\theta)^{-1/2}\left[\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)+i\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]$$

Equating imaginary part

$$S = (2\sin\theta)^{-1/2}\left[\cos\theta\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)+\sin\theta\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]$$

$$= (2\sin\theta)^{-1/2}\left[\sin\left(\theta+\frac{\pi}{4}-\frac{\theta}{2}\right)\right]$$

$$S = (2\sin\theta)^{-1/2}\sin\left(\frac{\theta}{2}+\frac{\pi}{4}\right) \qquad \text{Ans.}$$

7. $$\sinh\theta + \frac{\sinh 2\theta}{2!}+\frac{\sinh 3\theta}{3!}+\cdots \qquad \sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2}$$

Sol.

$$= \frac{e^{\theta}-e^{-\theta}}{2}+\frac{e^{2\theta}-e^{-2\theta}}{2.2!}+\frac{e^{3\theta}-e^{-3\theta}}{3!.2}+\cdots$$

$$= \frac{1}{2}\left[e^{\theta}-e^{-\theta}+\frac{e^{2\theta}}{2}-\frac{e^{-2\theta}}{2}+\frac{e^{3\theta}}{3!}-\frac{e^{-3\theta}}{3!}+\cdots\right]$$

$$= \frac{1}{2}\left[1-1+e^{\theta}-e^{-\theta}+\frac{e^{2\theta}}{2}-\frac{e^{-2\theta}}{2}+\frac{e^{3\theta}}{3!}-\frac{e^{-3\theta}}{3!}+\cdots\right]$$

$$= \frac{1}{2}\left[\left(1+e^{\theta}+\frac{e^{2\theta}}{2!}+\frac{e^{3\theta}}{3!}+\cdots\right)-\left(1+\frac{e^{-\theta}}{1!}+\frac{e^{2\theta}}{2!}+\frac{e^{3\theta}}{3!}+\cdots\right)\right]$$

9. $C = 1 - \frac{1}{2}\cos\theta + \frac{1.3}{2.4}\cos 2\theta - \frac{1.3.5}{2.4.6}\cos 4\theta + \dots$

$S = -\frac{1}{2}\sin\theta + \frac{1.3}{2.4}\sin 2\theta - \frac{1.3.5}{2.4.6}\sin 4\theta + \dots$

$C + iS = 1 - \frac{1}{2}(\cos\theta + i\sin\theta) + \frac{1.3}{2.4}(\cos 2\theta + i\sin\theta) - \frac{1.3.5}{2.4.6}(\cos 4\theta - \sin 4\theta) + \dots$

$= 1 - \frac{1}{2}e^{i\theta} + \frac{1.3}{2.4}e^{i2\theta} - \frac{1.3.5}{2.4.6}e^{i4\theta} \pm \dots$

$C + iS = (1 + e^{i\theta})^{-1/2}$

$= (1 + \cos\theta + i\sin\theta)^{-1/2}$

$= (2\cos^2\theta/_2 + i2\cos\theta/_2 \sin\theta/_2)^{-1/2}$

$= (2\cos\theta/_2(\cos\theta/_2 + i\sin\theta/_2))^{-1/2}$

$= (2\cos\theta/_2)^{-1/2}\left[\cos\left(-\frac{1}{2}\cdot\frac{\theta}{2}\right) + i\sin\left(-\frac{1}{2}\,\frac{\theta}{2}\right)\right]$

$= (2\cos\theta/_2)^{-1/2}\left[\cos\left(\frac{\theta}{4}\right) - i\sin(\theta/4)\right]$

Equating real part.

$C = (2\cos\theta/_2)^{-1/2}\cos(\theta/4)$

10. $n\sin\theta + \frac{n(n+1)}{2!}\sin 2\theta + \frac{n(n+1)(n+2)}{3!}\sin 3\theta + \dots = S$

$C = 1 + n\cos\theta + \frac{n(n+1)}{2!}\cos 2\theta + \frac{n(n+1)(n+2)}{3!}\cos 3\theta + \dots$

$C + iS = 1 + n(\cos\theta + i\sin\theta) + \frac{n(n+1)}{2!}(\cos 2\theta + i\sin 2\theta) + \frac{n(n+1)(n+2)}{3!}(\cos 3\theta + i\sin 3\theta) + \dots$

$C + iS = 1 + ne^{i\theta} + \frac{n(n+1)}{2!}e^{i2\theta} + \frac{n(n+1)(n+2)}{3!}e^{i3\theta} + \dots$

$= (1 - e^{i\theta})^{-n}$

$\frac{1 - \cos\theta}{2} = \sin^2\theta/_2$

$= (1 - \cos\theta - i\sin\theta)^{-n}$

$1 - \cos\theta = 2\sin^2\theta/_2$

$= \left(2\sin^2\frac{\theta}{2} - i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^{-n}$

$= [2\sin\theta/_2(\sin\theta/_2 - i\cos\theta/_2)]^{-n}$

$= (2\sin\theta/_2)^{-n}\left(\cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right) - i\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right)^{-n}$

$= (2\sin\theta/_2)^{-n}\left[\cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2} - \frac{\pi}{2}\right)\right]^{-n}$

De-Moivre's Theorem

$$\because e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$= \frac{1}{2}\left(e^{e^{\theta}} - e^{e^{-\theta}}\right) \qquad \because e^{i\theta} = \cos\theta + i\sin\theta, \quad e^{\theta} = \cosh\theta + \sinh\theta$$

$$= \frac{1}{2}\left[e^{\cosh\theta + \sinh\theta} - e^{\cosh(-\theta) + \sinh(-\theta)}\right]$$

$$= \frac{1}{2}\left[e^{\cosh\theta} \cdot e^{\sinh\theta} - e^{\cosh\theta - \sinh\theta}\right]$$

$$= \frac{1}{2}\left[e^{\cosh\theta} \cdot e^{\sinh\theta} - e^{\cosh\theta}\, e^{-\sinh\theta}\right]$$

$$= \frac{1}{2} e^{\cosh\theta}\left[e^{\sinh\theta} - e^{-\sinh\theta}\right]$$

$$= e^{\cosh\theta}\left[\frac{e^{\sinh\theta} - e^{-\sinh\theta}}{2}\right]$$

$$= e^{\cosh\theta} \sinh(\sinh\theta) \qquad \text{Ans}$$

8. $\sin\alpha \sin\alpha + \sin^2\alpha \sin 2\alpha + \sin^3\alpha \sin 3\alpha + \cdots = S$

Sol. $C = \sin\alpha\cos\alpha + \sin^2\alpha\cos 2\alpha + \sin^3\alpha\cos 3\alpha + \cdots$

$$C + iS = \sin\alpha(\cos\alpha + i\sin\alpha) + \sin^2\alpha(\cos 2\alpha + i\sin 2\alpha) + \sin^3\alpha(\cos 3\alpha + i\sin 3\alpha) + \cdots$$

$$C + iS = \sin\alpha\, e^{i\alpha} + \sin^2\alpha\, e^{i2\alpha} + \sin^3\alpha\, e^{i3\alpha} + \cdots$$

$a = \sin\alpha\, e^{i\alpha}$, $r = \sin\alpha\, e^{i\alpha}$ geometric Series. Sum $= \frac{a}{1-r}$

$$C + iS = \frac{\sin\alpha \cdot e^{i\alpha}}{1 - e^{i\alpha}\sin\alpha}$$

$$= \frac{\sin\alpha(\cos\alpha + i\sin\alpha)}{1 - \sin\alpha(\cos\alpha + i\sin\alpha)}$$

$$= \frac{\sin\alpha\cos\alpha + i\sin^2\alpha}{(1 - \sin\alpha\cos\alpha) - i\sin^2\alpha} \times \frac{(1 - \sin\alpha\cos\alpha) + i\sin^2\alpha}{(1 - \sin\alpha\cos\alpha) + i\sin^2\alpha}$$

$$C + iS = \frac{(\cos\alpha\sin\alpha + i\sin^2\alpha)((1 - \sin\alpha\cos\alpha) + i\sin^2\alpha)}{(1 - \sin\alpha\cos\alpha)^2 + (\sin^2\alpha)^2}$$

Equating imaginary parts.

$$S = \frac{\cos\alpha\sin\alpha\sin^2\alpha + \sin^2\alpha(1 - \sin\alpha\cos\alpha)}{1 + \sin^2\alpha\cos^2\alpha - 2\sin\alpha\cos\alpha + \sin^4\alpha}$$

$$= \frac{\cos\alpha\sin\alpha\sin^2\alpha + \sin^2\alpha - \cos\alpha\sin\alpha\sin^2\alpha}{1 + \sin^2\alpha(\cos^2\alpha + \sin^2\alpha) - 2\cos\alpha\sin\alpha}$$

$$S = \frac{\sin^2\alpha}{1 + \sin^2\alpha - 2\cos\alpha\sin\alpha} \qquad \text{Ans}$$

$$C+iS = \left(2\sin\theta/_2\right)^{-n}\left[\cos(-n)\left(\frac{\pi}{2}-\frac{\theta}{2}\right)+i\sin\left(-n\left(\frac{\theta}{2}-\frac{\pi}{2}\right)\right)\right]$$

$$= \left(2\sin\theta/_2\right)^{-n}\left[\cos n\left(\frac{\pi}{2}-\frac{\theta}{2}\right)+i\sin n\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right]$$

Equating imaginary part.

$$S = \left(2\sin\theta/2\right)^{-n}\sin n\left(\frac{\pi}{2}-\frac{\theta}{2}\right) \quad \text{proved.}$$

11. $$C = 1+\frac{1}{2}\cos\theta+\frac{1.3}{2.4}\cos 2\theta+\frac{1.3.5}{2.4.6}\cos 3\theta+\cdots$$

Let $$S = \frac{1}{2}\sin\theta+\frac{1.3}{2.4}\sin 2\theta+\frac{1.3.5}{2.4.6}\sin 3\theta+\cdots$$

$$C+iS = 1+\frac{1}{2}(\cos\theta+i\sin\theta)+\frac{1.3}{2.4}(\cos 2\theta+i\sin 2\theta)+\frac{1.35}{2.4.6}(\cos 3\theta+i\sin 3\theta) + \cdots$$

$$= 1+\frac{1}{2}e^{i\theta}+\frac{1.3}{2.4}e^{i2\theta}+\frac{1.3.5}{2.4.6}e^{i3\theta}+\cdots$$

$$= (1-e^{i\theta})^{-1/2} \qquad 1-\cos\theta = 2\sin^2\theta/_2$$

$$= (1-\cos\theta-i\sin\theta)^{-1/2}$$

$$= \left(2\sin^2\theta/_2 - i2\sin\theta/_2\cos\theta/_2\right)^{-1/2}$$

$$= \left(2\sin\theta/_2\right)^{-1/2}\left(\sin\theta/_2 - i\cos\theta/_2\right)^{-1/2}$$

$$= \left(2\sin\theta/_2\right)^{-1/2}\left(\cos\left(\frac{\pi}{2}-\frac{\theta}{2}\right)-i\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right)^{-1/2}$$

$$= \left(2\sin\theta/_2\right)^{-1/2}\left(\cos\left(\frac{\pi}{2}-\frac{\theta}{2}\right)+i\sin\left(-\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right)\right)^{-1/2}$$

De-Moivre's Theorem

$$= \left(2\sin\theta/_2\right)^{-1/2}\left(\cos\left(-\frac{1}{2}\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right)+i\sin\left(-\frac{1}{2}\left(-\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right)\right)\right)$$

$$= \left(2\sin\theta/_2\right)^{-1/2}\left[\cos\left(\frac{\pi}{4}-\frac{\theta}{4}\right)+i\sin\left(\frac{\pi}{4}-\frac{\theta}{4}\right)\right]$$

equating real part.

$$C = \left(2\sin\theta/_2\right)^{-1/2}\cos\left(\frac{\pi}{4}-\frac{\theta}{4}\right) \quad \text{Ans}$$

12. $$C = \cos\alpha - \frac{\cos(\alpha+2\beta)}{3!}+\frac{\cos(\alpha+4\beta)}{5!}+\cdots$$

$$S = \sin\alpha - \frac{\sin(\alpha+2\beta)}{3!}+\frac{\sin(\alpha+4\beta)}{5!}+\cdots$$

$$C+iS = (\cos\alpha+i\sin\alpha)-\frac{1}{3!}\left(\cos(\alpha+2\beta)+i\sin(\alpha+2\beta)\right)+\frac{1}{5!}\left(\cos(\alpha+4\beta)+i\sin(\alpha+4\beta)\right) + \cdots$$