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Exercise 1.4

$\boxed{\log z = \ln|z| + i\theta}$

1. **Prove that**

(i) $\text{Log}\, i = \frac{\pi i}{2}$

Here
$z = i$
$|r| = \sqrt{1+0} = 1$
$\cos\theta = 0,\ \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$
$\text{Log}\, i = \log|z| + i\theta$
$= \ln(1) + i(\pi/2)$
$= 0 + i(\frac{\pi}{2})$
$\text{Log}\, i = i(\frac{\pi}{2})$

(ii) $\text{Log}(-5) = \ln 5 + \pi i$

Here $z = -5$, $|z| = 5$
$\cos\theta = -1$, $\sin\theta = 0$
quardrantal angle
$\theta = \pi$
$\text{Log}(-5) = \ln|z| + i(\theta)$
$\text{Log}(-5) = \ln(5) + i\pi$
Proved.

(iii) $\text{Log}(-1+i) = \frac{1}{2}\ln 2 + \frac{3\pi}{4}i$

Here $z = -1+i$
$|z| = \sqrt{1+1} = \sqrt{2}$
$\cos\theta = \frac{-1}{\sqrt{2}}$, $\sin\theta = \frac{1}{\sqrt{2}}$
refrence angle $= \alpha = \frac{\pi}{4}$
2nd quadrd
$\theta = \pi - \alpha = \pi - \frac{\pi}{4}$
$\theta = \frac{3\pi}{4}$
$\log(-1+i) = \ln(\sqrt{2}) + i(\frac{3\pi}{4})$
$= \frac{1}{2}\ln 2 + i(\frac{3\pi}{4})$
Proved

(iv) $\text{Log}(1+i) = \frac{1}{2}\ln 2 + \frac{\pi i}{4}$

Here $z = 1+i$, $|z| = \sqrt{2}$
$\cos\theta = \frac{1}{\sqrt{2}}$, $\sin\theta = \frac{1}{\sqrt{2}}$
$\theta = \frac{\pi}{4}$
$\log(1+i) = \ln(2^{1/2}) + i(\frac{\pi}{4})$
$\log(1+i) = \frac{1}{2}\ln 2 + \frac{\pi i}{4}$ proved.

(v). $\log\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = -\frac{2\pi}{3}i$

Here
$z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$, $|z| = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{\frac{4}{4}}$
$|z| = 1$
$\cos\theta = -\frac{1}{2}$, $\sin\theta = -\sqrt{3}/2$
refrence angle $= \alpha = \pi/3$
IIIrd quardrant
$\theta = \alpha - \pi = \frac{\pi}{3} - \alpha$
$\theta = -\frac{2\pi}{3}$
$\log\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = \ln(1) + i\left(-\frac{2\pi}{3}\right)$
$= 0 - \frac{2\pi i}{3}$
$\log\left(\frac{-1}{2} - \frac{\sqrt{3}}{2}i\right) = -\frac{2\pi}{3}i$

(vi) $\log(1-i) = \frac{1}{2}\ln 2 - \frac{\pi i}{4}$

$z = 1-i$, $|z| = \sqrt{2}$
$\cos\theta = \frac{1}{\sqrt{2}}$, $\sin\theta = \frac{-1}{\sqrt{2}}$
refrence angle $= \alpha = \pi/4$
4th quadrant,
$\theta = -\alpha = -\pi/4$
$\text{Log}(1-i) = \ln(2^{1/2}) + i(-\frac{\pi}{4})$
$= \frac{1}{2}\ln 2 - \frac{i\pi}{4}$
Proved

$\because\ 1-\tanh^2 z = \operatorname{sech}^2 z$
$1-\operatorname{sech}^2 z = \tanh^2 z$

2. Show that

(i) $\coth^{-1} z = \frac{1}{2}\log\left(\frac{z+1}{z-1}\right)$

Let $\coth^{-1} z = \omega$

$z = \coth\omega$

Now

$$\frac{z+1}{z-1} = \frac{\coth\omega+1}{\coth\omega-1}$$

$$= \frac{\frac{e^{\omega}+e^{-\omega}}{e^{\omega}-e^{-\omega}}+1}{\frac{e^{\omega}+e^{-\omega}}{e^{\omega}-e^{-\omega}}-1}$$

$$= \frac{e^{\omega}+\cancel{e^{-\omega}}+e^{\omega}-\cancel{e^{-\omega}}}{\cancel{e^{\omega}}+e^{-\omega}-\cancel{e^{\omega}}+e^{-\omega}}$$

$$= \frac{2e^{\omega}}{2e^{-\omega}}$$

$$\frac{z+1}{z-1} = e^{\omega+\omega} = e^{2\omega}$$

$$2\omega = \log\left(\frac{z+1}{z-1}\right)$$

$$\omega = \frac{1}{2}\log\left(\frac{z+1}{z-1}\right)$$

$$\coth^{-1} z = \frac{1}{2}\log\left(\frac{z+1}{z-1}\right)$$

(ii) $\operatorname{sech}^{-1} z = \log\left(\frac{1+\sqrt{1-z^2}}{z}\right)$

Let $\operatorname{sech}^{-1} z = \omega$

$\Rightarrow\ z = \operatorname{sech}\omega$

Now

$$\frac{1+\sqrt{1-z^2}}{z} = \frac{1+\sqrt{1-\operatorname{sech}^2\omega}}{\operatorname{sech}\omega}$$

$$= \frac{1+\sqrt{\tanh^2\omega}}{\operatorname{sech}\omega}$$

$$= \frac{1+\tanh\omega}{\operatorname{sech}\omega}$$

$$= \frac{1+\frac{\sinh\omega}{\cosh\omega}}{\frac{1}{\cosh\omega}}$$

$$= \cosh\omega+\sinh\omega$$

$$= \frac{e^{\omega}+e^{-\omega}}{2}+\frac{e^{\omega}-e^{-\omega}}{2}$$

$$= \frac{1}{2}(e^{\omega}+\cancel{e^{-\omega}}+e^{\omega}-\cancel{e^{-\omega}})$$

$$= \frac{1}{2}(2e^{\omega})$$

$$\frac{1+\sqrt{1-z^2}}{z} = e^{\omega}$$

$$\omega = \log\left(\frac{1+\sqrt{1-z^2}}{z}\right)$$

(iii) $\operatorname{cosech}^{-1} z = \log\left[\frac{1+\sqrt{z^2+1}}{z}\right]$

Let

$\operatorname{cosech}^{-1} z = \omega \Rightarrow \operatorname{cosech}\omega = z$

$\because$ ~~$1-\coth^2\omega = 1-\operatorname{cosech}^2\omega$~~
$\because\ 1-\coth^2\omega = -\operatorname{cosech}^2\omega$
$1+\operatorname{cosech}^2\omega = \coth^2\omega$

$$\frac{1+\sqrt{z^2+1}}{z} = \frac{1+\sqrt{\operatorname{cosech}^2\omega+1}}{\operatorname{cosech}\omega}$$

$$= \frac{1+\sqrt{\coth^2\omega}}{\operatorname{cosech}\omega}$$

$$= \frac{1+\coth\omega}{\operatorname{cosech}\omega}$$

$$= \frac{1+\frac{\cosh\omega}{\sinh\omega}}{\frac{1}{\sinh\omega}}$$

$$= \sinh\omega+\cosh\omega$$

$$= \frac{e^{\omega}-e^{-\omega}}{2}+\frac{e^{\omega}+e^{-\omega}}{2}$$

$$= \frac{1}{2}(e^{\omega}-e^{-\omega}+e^{\omega}+e^{-\omega})$$

$$\frac{1+\sqrt{z^2+1}}{z} = \frac{1}{2}(2e^{\omega}) = e^{\omega}$$

$$\omega = \log\left[\frac{1+\sqrt{z^2+1}}{z}\right]$$

$\because 1+\tan^2 z = \sec^2 z$
$1-\sec^2 z = \tan^2 z$
$\Rightarrow \sec^2 z - 1 = \tan^2 z$

3. Prove that:

(i) $\cos^{-1} z = \frac{1}{i}\log\left(z+\sqrt{z^2-1}\right)$

Let $\cos^{-1} z = \omega$

$\Rightarrow \cos\omega = z$

$$z+\sqrt{z^2-1} = \cos\omega + \sqrt{\cos^2\omega - 1}$$
$$= \cos\omega + \sqrt{-(1-\cos^2\omega)}$$
$$= \cos\omega + \sqrt{-\sin^2\omega}$$
$$= \cos\omega + i\sin\omega$$
$$= \frac{e^{i\omega}+e^{-i\omega}}{2} + \cancel{i}\cdot\frac{e^{i\omega}-e^{-i\omega}}{2\cancel{i}}$$
$$= \frac{1}{2}\left(e^{i\omega}+e^{-i\omega}+e^{i\omega}-e^{-i\omega}\right)$$
$$= \frac{1}{2}\left(2e^{i\omega}\right)$$
$$z+\sqrt{z^2-1} = e^{i\omega}$$
$$i\omega = \log\left(z+\sqrt{z^2-1}\right)$$
$$\omega = \frac{1}{i}\log\left(z+\sqrt{z^2-1}\right)$$
$$\cos^{-1} z = \frac{1}{i}\log\left(z+\sqrt{z^2-1}\right)$$

(iii) $\cot^{-1} z = \frac{1}{2i}\log\left(\frac{z+i}{z-i}\right)$

Let $\cot^{-1} z = \omega \Rightarrow \cot\omega = z$

$$\frac{z+i}{z-i} = \frac{\cot\omega + i}{\cot\omega - i}$$
$$= \frac{\frac{\cos\omega}{\sin\omega} + i}{\frac{\cos\omega}{\sin\omega} - i}$$
$$= \frac{\cos\omega + i\sin\omega}{\cos\omega - i\sin\omega}$$
$$= \frac{\frac{e^{i\omega}+e^{-i\omega}}{2} + \cancel{i}\frac{e^{i\omega}-e^{-i\omega}}{2\cancel{i}}}{\frac{e^{i\omega}+e^{-i\omega}}{2} - \cancel{i}\frac{e^{i\omega}-e^{-i\omega}}{2\cancel{i}}}$$
$$= \frac{\frac{1}{2}\left(e^{i\omega}+\cancel{e^{-i\omega}}+e^{i\omega}-\cancel{e^{-i\omega}}\right)}{\frac{1}{2}\left(\cancel{e^{i\omega}}+e^{-i\omega}-\cancel{e^{i\omega}}+e^{-i\omega}\right)}$$
$$= \frac{\cancel{2}e^{i\omega}}{\cancel{2}e^{-i\omega}} = e^{i\omega+i\omega}$$
$$\frac{z+i}{z-i} = e^{2i\omega}$$
$$2i\omega = \log\left(\frac{z+i}{z-i}\right)$$
$$\omega = \cot^{-1} z = \frac{1}{2i}\log\left(\frac{z+i}{z-i}\right)$$

(ii) $\tan^{-1} z = \frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right)$

Let $\tan^{-1} z = \omega \Rightarrow \tan\omega = z$

$$\frac{1+iz}{1-iz} = \frac{1+i\tan\omega}{1-i\tan\omega}$$
$$= \frac{1+i\frac{\sin\omega}{\cos\omega}}{1-i\frac{\sin\omega}{\cos\omega}}$$
$$= \frac{\cos\omega + i\sin\omega}{\cos\omega - i\sin\omega}$$
$$= \frac{\frac{e^{i\omega}+e^{-i\omega}}{2} + \cancel{i}\frac{e^{i\omega}-e^{-i\omega}}{2\cancel{i}}}{\frac{e^{i\omega}+e^{-i\omega}}{2} - \cancel{i}\frac{e^{i\omega}-e^{-i\omega}}{2\cancel{i}}}$$
$$= \frac{\frac{1}{2}\left(e^{i\omega}+\cancel{e^{-i\omega}}+e^{i\omega}-\cancel{e^{-i\omega}}\right)}{\frac{1}{2}\left(\cancel{e^{i\omega}}+e^{-i\omega}-\cancel{e^{i\omega}}+e^{-i\omega}\right)}$$
$$= \frac{\cancel{2}e^{i\omega}}{\cancel{2}e^{-i\omega}} = e^{i\omega+i\omega}$$
$$\frac{1+iz}{1-iz} = e^{2i\omega}$$
$$\log\left(\frac{1+iz}{1-iz}\right) = 2i\omega$$
$$\omega = \frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right)$$
$$\tan^{-1} z = \frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right)$$

(iv) $\sec^{-1} z = \frac{1}{i}\log\left(\frac{1+\sqrt{1-z^2}}{z}\right)$

Let $\sec^{-1} z = \omega \Rightarrow \sec\omega = z$

$$\frac{1+\sqrt{1-z^2}}{z} = \frac{1+\sqrt{1-\sec^2\omega}}{\sec\omega} = \frac{1+\sqrt{-(\sec^2 z-1)}}{\sec\omega}$$
$$= \frac{1+\sqrt{-\tan^2\omega}}{\sec\omega} = \frac{1+\sqrt{-1}\sqrt{\tan^2\omega}}{\sec\omega}$$
$$= \frac{1+i\tan\omega}{\sec\omega} = \frac{1+\frac{\sin\omega}{\cos\omega}}{1/\cos\omega}$$
$$= \frac{\frac{\cos\omega+i\sin\omega}{\cos\omega}}{1/\cos\omega}$$
$$= \cos\omega + i\sin\omega$$
$$= \frac{e^{i\omega}+e^{-i\omega}}{2} + \cancel{i}\frac{e^{i\omega}-e^{-i\omega}}{2\cancel{i}}$$
$$= \frac{1}{2}\left(e^{i\omega}+\cancel{e^{-i\omega}}+e^{i\omega}-\cancel{e^{-i\omega}}\right)$$
$$\frac{1+\sqrt{1-z^2}}{z} = \frac{1}{2}(2e^{i\omega}) = e^{i\omega}$$
$$i\omega = \log\left(\frac{1+\sqrt{1-z^2}}{z}\right)$$
$$\omega = \frac{1}{i}\log\left(\frac{1+\sqrt{1-z^2}}{z}\right)$$

(v) $\operatorname{Cosec}^{-1} z = \frac{1}{i} \log\left(\frac{i+\sqrt{z^2-1}}{z}\right)$

Let $\operatorname{Cosec}^{-1} z = w \Rightarrow \operatorname{Cosec} w = z$

$$\frac{i+\sqrt{z^2-1}}{z} = \frac{i+\sqrt{\operatorname{Cosec}^2 w - 1}}{\operatorname{Cosec} w}$$

$$= \frac{i+\sqrt{\operatorname{Cot}^2 w}}{\operatorname{Cosec} w} = \frac{i+\operatorname{Cot} w}{\operatorname{Cosec} w}$$

$$= \frac{i+\frac{\operatorname{Cos} w}{\operatorname{Sin} w}}{1/\operatorname{Sin} w} = \frac{i\operatorname{Sin} w + \operatorname{Cos} w}{\operatorname{Sin} w/\operatorname{Sin} w}$$

$$= \operatorname{Cos} w + i\operatorname{Sin} w$$

By Euler formul $e^{iw} = \operatorname{Cos} w + i\operatorname{Sin} w$.

$$\frac{i+\sqrt{z^2-1}}{z} = e^{iw}$$

$$iw = \log\left(\frac{i+\sqrt{z^2-1}}{z}\right)$$

$$\operatorname{Cosec}^{-1} w = w = \frac{1}{i} \log\left(\frac{i+\sqrt{z^2-1}}{z}\right)$$

4. Prove that

(i) $i^i = e^{-\pi/2}$

$$i^i = e^{\log i^i}$$

$$= e^{i\log i}$$

$$= e^{i[\ln|i| + i\theta]}$$

here $z = i$, $|z| = 1$

$\operatorname{Cos}\theta = 0$, $\sin\theta = 1$

$\theta = \pi/2$

$$i^i = e^{i[\ln(1) + i\pi/2]}$$

$$= e^{i(0 + i\pi/2)}$$

$$= e^{i^2\pi/2}$$

$$i^i = e^{-\pi/2}$$

(ii) $(-1)^i = e^{-\pi}$

$$(-1)^i = e^{\log(-1)^i}$$

$$= e^{i\log(-1)}$$

$$= e^{i[\ln|-1| + i\theta]}$$

here $z = -1$, $|z| = 1$

$\log|z| = \ln(1) = 0$

$\operatorname{Cos}\theta = -1$, $\sin\theta = 0$

$\theta = \pi$

$$(-1)^i = e^{i[0 + i\pi]}$$

$$= e^{i^2\pi}$$

$$(-1)^i = e^{-\pi}$$

(iii) $(-i)^{-i} = e^{-\pi/2}$

$$(-i)^{-i} = e^{\log(-i)^{-i}}$$

$$= e^{-i\log(-i)}$$

$$= e^{-i[\ln|-i| + i\theta]}$$

$$= e^{-i[0 - i\pi/2]}$$

$$= e^{i^2\pi/2}$$

$$(-i)^{-i} = e^{-\pi/2}$$

here $z = -i$ $|z| = 1$

$\cos\theta = 0$, $\sin\theta = -1$ $\log|z| = \ln|z| = 0$

$\theta = -\pi/2$

(IV) $a^i = \cos(\ln a) + i\sin(\ln a)$, $a>0$

$$a^i = e^{\log a^i} = e^{i\log a}$$

here $z = a$ $|z| = a$

$\cos\theta = \frac{a}{a} = 1$, $\sin\theta = 0$

$\theta = 0^\circ$

$$a^i = e^{i[\ln|a| + i\theta]}$$

$$a^i = e^{i[\ln a + 0]}$$

$$a^i = e^{i(\ln a)}$$ by euler formula

$$= \cos(\ln a) + i\sin(\ln a) \quad , a>0$$

5. Prove that $\tanh^{-1} z = \sinh^{-1}\left(\frac{z}{\sqrt{1-z^2}}\right)$

Let $\tanh^{-1} z = w \Rightarrow z = \tanh w$

$$\frac{z}{\sqrt{1-z^2}} = \frac{\tanh w}{\sqrt{1-\tanh^2 w}} = \frac{\tanh w}{\sqrt{\operatorname{sech}^2 w}}$$

$$= \frac{\tanh w}{\operatorname{sech} w} = \frac{\sinh w}{\cosh w}.\cosh w$$

$$= \sinh w$$

$$w = \sinh^{-1}\left(\frac{z}{\sqrt{1-z^2}}\right)$$

$$\tanh^{-1} z = \sinh^{-1}\left(\frac{z}{\sqrt{1-z^2}}\right)$$

6. Show that if $z = x+iy$, then $\log\left(\frac{z}{\bar{z}}\right) = 2i\tan^{-1}\left(\frac{y}{x}\right)$

$$\log\left(\frac{z}{\bar{z}}\right) = \log z - \log\bar{z}$$

$$= \log(x+iy) - \log(x-iy)$$

$\because \log z = \ln|z| + i\theta$

for $z = x+iy$, $|z| = \sqrt{x^2+y^2}$ & $\theta = \tan^{-1}(y/x)$

& For $\bar{z} = x-iy$, $|\bar{z}| = \sqrt{x^2+y^2}$ & $\theta = \tan^{-1}(-y/x) = -\tan^{-1}(y/x)$

$$\log\left(\frac{z}{\bar{z}}\right) = \cancel{\ln(\sqrt{x^2+y^2})} + i\tan^{-1}(y/x) - \cancel{\ln(\sqrt{x^2+y^2})} + i\tan^{-1}(y/x)$$

$$\log\left(\frac{z}{\bar{z}}\right) = 2i\tan^{-1}(y/x)$$

7. if $a^{\alpha+i\beta} = (x+iy)^{p+iq}$, $a>0$ prove that

(i) $\alpha = \frac{1}{2}p\log_a(x^2+y^2) - q\tan^{-1}(y/x)\log_a e$

(ii) $\log_a(x^2+y^2) = \dfrac{2(\alpha p + \beta q)}{p^2+q^2}$

Sol:

for $z = a \Rightarrow |z| = a$
$\cos\theta = 1$, $\sin\theta = 0$
$\Rightarrow \theta = 0°$
& for $z = x+iy$, $|z| = \sqrt{x^2+y^2}$
& $\theta = \tan^{-1}(y/x)$

$$a^{\alpha+i\beta} = (x+iy)^{p+iq}$$
$$\log a^{\alpha+i\beta} = \log(x+iy)^{p+iq}$$
$$(\alpha+i\beta)\log a = (p+iq)\log(x+iy)$$
$$(\alpha+i\beta)\left[\ln|a| + i(0)\right] = (p+iq)\left[\ln\left[\sqrt{x^2+y^2}\right] + i\tan^{-1}(y/x)\right]$$
$$(\alpha+i\beta)(\ln a) = (p+iq)\left[\tfrac{1}{2}\ln(x^2+y^2) + i\tan^{-1}(y/x)\right]$$
$$\alpha\ln a + i\beta\ln a = \frac{p}{2}\ln(x^2+y^2) + ip\tan^{-1}(y/x) + i\frac{q}{2}\ln(x^2+y^2) + i^2 q\tan^{-1}(y/x)$$
$$\alpha\ln a + \beta\ln a = \frac{p}{2}\ln(x^2+y^2) - q\tan^{-1}(y/x) + i\left(p\tan^{-1}(y/x) + \frac{q}{2}\ln(x^2+y^2)\right)$$

(i) Equating real part.
$$\alpha\ln a = \frac{p}{2}\ln(x^2+y^2) - q\tan^{-1}(y/x)\cdot 1$$
$$= \frac{p}{2}\ln(x^2+y^2) - q\tan^{-1}(y/x)\ln e.$$
$$\alpha = \frac{p}{2}\frac{\ln(x^2+y^2)}{\ln a} - q\tan^{-1}(y/x)\frac{\ln e}{\ln a}$$
$$\alpha = \frac{p}{2}\log_a(x^2+y^2) - q\tan^{-1}(y/x)\log_a e \rightarrow (1)$$ **Proved**

(ii) Equating imaginary part.
$$\beta\ln a = p\tan^{-1}(y/x) + \frac{q}{2}\ln(x^2+y^2).$$
$$\beta\ln a = p\tan^{-1}(y/x)\cdot 1 + \frac{q}{2}\ln(x^2+y^2)$$
$$= \frac{q}{2}\ln(x^2+y^2) + p\tan^{-1}(y/x)\ln e$$
$$\beta = \frac{q}{2}\frac{\ln(x^2+y^2)}{\ln a} + p\tan^{-1}(y/x)\frac{\ln e}{\ln a}$$
$$= \frac{q}{2}\log_a(x^2+y^2) + p\tan^{-1}(y/x)\log_a e \rightarrow (ii)$$

multiplying (i) with p and (ii) with q & add
$$p\alpha = \frac{p^2}{2}\log_a(x^2+y^2) - \cancel{pq\tan^{-1}(y/x)\log_a e}$$
$$q\beta = \frac{q^2}{2}\log_a(x^2+y^2) + \cancel{pq\tan^{-1}(y/x)\log_a e}$$

$$p\alpha + q\beta = \frac{(p^2+q^2)\log_a(x^2+y^2)}{2}$$
$$\log_a(x^2+y^2) = \frac{2(\alpha p + \beta q)}{p^2+q^2}$$ **Proved**

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8. if $\operatorname{Log}\sin(x+iy) = u+iv$, show that

(i) $\cosh 2y = \cos 2x + 2e^{2u}$ (ii) $e^{2y} = \dfrac{\cos(x-v)}{\cos(x+v)}$

Sol.

$$\operatorname{Log}\sin(x+iy) = u+iv$$
$$\sin(x+iy) = e^{u+iv} = e^{u}e^{iv}$$
$$\therefore \sin x\cosh y + \cos x\sin(iy) = e^{u}(\cos v + i\sin v)$$
$$\sin x\cosh y + i\cos x\sinh y = e^{u}\cos v + ie^{u}\sin v$$

Equating Real & imaginary parts;

$$\sin x\cosh y = e^{u}\cos v \rightarrow \text{(i)}\ ,\quad \cos x\sinh y = e^{u}\sin v \rightarrow \text{(ii)}$$

Squaring & adding (i) & (ii)

$$\sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y = e^{2u}\cos^2 v + e^{2u}\sin^2 v$$
$$(1-\cos^2 x)\cosh^2 y + \cos^2 x(\cosh^2 y - 1) = e^{2u}(\cos^2 v + \sin^2 v)$$
$$\cosh^2 y - \cancel{\cos^2 x\cosh^2 y} + \cancel{\cos^2 x\cosh^2 y} - \cos^2 x = e^{2u}(1)$$
$$\cosh^2 y - \cos^2 x = e^{2u}$$
$$\cosh^2 y = \cos^2 x + e^{2u}$$
$$\frac{1+\cosh 2y}{2} = \frac{1+\cos 2x}{2} + e^{2u}$$
$$\cancel{1} + \cosh 2y = \cancel{1} + \cos 2x + 2e^{2u}$$
$$\cosh 2y = \cos 2x + 2e^{2u}$$

Proved

dividing (i) & (ii)

$$\frac{\sin x\cosh y}{\cos x\sinh y} = \frac{e^{u}\cos v}{e^{u}\sin v}$$
$$\frac{\sin x\cosh y}{\cos x\sinh y} = \frac{\cos v}{\sin v}$$
$$\frac{\cos x\cos v}{\sin x\sin v} = \frac{\cosh y}{\sinh y}$$

componendo-dividendo Theorem

$$\frac{\cos x\cos v + \sin x\sin v}{\cos x\cos v - \sin x\sin v} = \frac{\cosh y + \sinh y}{\cosh y - \sinh y}$$
$$\frac{\cos(x-v)}{\cos(x+v)} = \frac{\frac{e^{y}+e^{-y}}{2} + \frac{e^{y}-e^{-y}}{2}}{\frac{e^{y}+e^{-y}}{2} - \frac{e^{y}-e^{-y}}{2}}$$
$$= \frac{\cancel{1/2}}{\cancel{1/2}}\left(\frac{e^{y}+\cancel{e^{-y}}+e^{y}-\cancel{e^{-y}}}{\cancel{e^{y}}+e^{-y}-\cancel{e^{y}}+e^{-y}}\right)$$
$$= \frac{\cancel{2}e^{y}}{\cancel{2}e^{-y}} = e^{y+y} = e^{2y}$$
$$\frac{\cos(x-v)}{\cos(x+v)} = e^{2y}$$

Proved

9. Show that

$$\operatorname{Log}(1+\cos\theta + i\sin\theta) = \ln(2\cos(\theta/2)) + i\,\theta/2$$

Sol.

$$\operatorname{Log}((1+\cos\theta) + i\sin\theta)$$
$$= \ln(\sqrt{2+2\cos\theta}) + i\tan^{-1}\left(\frac{\sin\theta}{1+\cos\theta}\right)$$
$$= \frac{1}{2}\ln(2(1+\cos\theta)) + i\tan^{-1}\left(\frac{\sin\theta}{1+\cos\theta}\right)$$

put (i) & (ii)

$$= \frac{1}{2}\ln\left(2\cos\frac{\theta}{2}\right)^2 + i\tan^{-1}\left(\frac{\cancel{2}\sin\theta/2\,\cancel{\cos\theta/2}}{\cancel{2}\cos^{\cancel{2}}\theta/2}\right)$$
$$= \frac{\cancel{2}}{\cancel{2}}\ln(2\cos\theta/2) + i\,\cancel{\tan^{-1}}(\cancel{\tan}(\theta/2))$$
$$= \ln(2\cos(\theta/2)) + i(\theta/2)$$

Proved

here $z = 1+\cos\theta + i\sin\theta$

$$|z|^2 = (1+\cos\theta)^2 + \sin^2\theta$$
$$= 1+\cos^2\theta + 2\cos\theta + \sin^2\theta$$
$$= 1+1+2\cos\theta$$
$$|z| = \sqrt{2+2\cos\theta}$$
$$\theta' = \tan^{-1}\left(\frac{\sin\theta}{1+\cos\theta}\right)$$

① $\because \dfrac{1+\cos\theta}{2} = \cos^2\theta/2$

$$1+\cos\theta = 2\cos^2\theta/2$$

& $\cancel{4}\dfrac{(1+\cos\theta)}{\cancel{2}} = 4\cos^2\theta/2$

② $2(1+\cos\theta) = 4\cos^2\theta/2 = (2\cos\theta/2)^2$

10. Prove that.

$$\tan^{-1}\left(\frac{x+iy}{x-iy}\right) = \frac{\pi}{4} + \frac{i}{2}\ln\left(\frac{x+y}{x-y}\right) \quad \text{, if } x>y>0$$

Sol:

Let $\alpha + i\beta = \tan^{-1}\left(\frac{x+iy}{x-iy}\right) \rightarrow (i)$ $\quad \because \overline{\tan^{-1} z} = \tan^{-1}\bar{z}$

$$\overline{\alpha + i\beta} = \tan^{-1}\overline{\left(\frac{x+iy}{x-iy}\right)} = \tan^{-1}\left(\frac{\overline{x+iy}}{\overline{x-iy}}\right) \quad \because \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$$

$$\alpha - i\beta = \tan^{-1}\left(\frac{x-iy}{x+iy}\right) \longrightarrow (ii)$$

adding (i) & (ii)

$$2\alpha = \tan^{-1}\left(\frac{x+iy}{x-iy}\right) + \tan^{-1}\left(\frac{x-iy}{x+iy}\right) \quad \because \tan^{-1}(A+B) = \tan^{-1}\left(\frac{A+B}{1-AB}\right)$$

$$= \tan^{-1}\left(\frac{\frac{x+iy}{x-iy} + \frac{x-iy}{x+iy}}{1 - \left(\frac{\cancel{x+iy}}{\cancel{x-iy}}\right)\left(\frac{\cancel{x-iy}}{\cancel{x+iy}}\right)}\right)$$

$$= \tan^{-1}\left(\frac{\frac{(x+iy)^2 + (x-iy)^2}{(x+iy)(x-iy)}}{0}\right) \qquad \frac{\text{any no}}{0} = \infty$$

$$= \tan^{-1}(\infty)$$

$$2\alpha = \frac{\pi}{2} \Rightarrow \boxed{\alpha = \frac{\pi}{4}}$$

Subtracting (i) & (ii)

$$i\beta + i\beta = \tan^{-1}\left(\frac{x+iy}{x-iy}\right) - \tan^{-1}\left(\frac{x-iy}{x+iy}\right) \quad \because$$

$$2i\beta = \tan^{-1}\left[\frac{\frac{x+iy}{x-iy} - \frac{x-iy}{x+iy}}{1 + \frac{\cancel{x+iy}}{\cancel{x-iy}} \cdot \frac{\cancel{x-iy}}{\cancel{x+iy}}}\right] \qquad \tan^{-1}(A-B) = \tan^{-1}\left(\frac{A-B}{1+AB}\right)$$

$$2i\beta = \tan^{-1}\left[\frac{\frac{(x+iy)^2 - (x-iy)^2}{(x+iy)(x-iy)}}{1+1}\right]$$

$$2i\beta = \tan^{-1}\left[\frac{\cancel{x^2} - \cancel{y^2} + 2xyi - \cancel{x^2} + \cancel{y^2} + 2xyi}{2(x^2+y^2)}\right]$$

$$= \tan^{-1}\left(\frac{i\,{}^2\cancel{4}xy}{\cancel{2}(x^2+y^2)}\right)$$

$$2i\beta = \tan^{-1}\left(i\frac{2xy}{(x^2+y^2)}\right)$$

$$\tan(2i\beta) = i\left(\frac{2xy}{x^2+y^2}\right)$$

$$\cancel{i}\tanh 2\beta = \cancel{i}\left(\frac{2xy}{x^2+y^2}\right)$$

$$\frac{e^{2\beta} - e^{-2\beta}}{e^{2\beta} + e^{-2\beta}} = \frac{2xy}{x^2+y^2} \Rightarrow \frac{e^{2\beta} + e^{-2\beta}}{e^{2\beta} - e^{-2\beta}} = \frac{x^2+y^2}{2xy}$$

Componendo-dividendo Theorem.

$$\frac{e^{2\beta}+\cancel{e^{-2\beta}}+e^{2\beta}-\cancel{e^{-2\beta}}}{\cancel{e^{2\beta}}+e^{-2\beta}-\cancel{e^{2\beta}}+e^{-2\beta}} = \frac{x^2+y^2+2xy}{x^2+y^2-2xy} = \frac{(x+y)^2}{(x-y)^2}$$

$$\frac{\cancel{2}e^{2\beta}}{\cancel{2}e^{-2\beta}} = \frac{(x+y)^2}{(x-y)^2}$$

$$e^{4\beta} = \left(\frac{x+y}{x-y}\right)^2$$

$$\cancel{2}\,\cancel{4}\beta = \ln\left(\frac{x+y}{x-y}\right)^2 = \cancel{2}\ln\left(\frac{x+y}{x-y}\right)$$

$$2\beta = \ln\left(\frac{x+y}{x-y}\right)$$

$$\beta = \frac{1}{2}\ln\left(\frac{x+y}{x-y}\right)$$

$$\because \tan^{-1}\left(\frac{x+iy}{x-iy}\right) = \alpha+i\beta$$

$$\tan^{-1}\left(\frac{x+iy}{x-iy}\right) = \frac{\pi}{4}+\frac{i}{2}\ln\left(\frac{x+y}{x-y}\right)$$ Proved

11. Prove that

(i) $\operatorname{Sec}(x+iy) = 2\left[\dfrac{\operatorname{Cos}x\operatorname{Cosh}y + i\operatorname{Sin}x\operatorname{Sinh}y}{\operatorname{Cos}2x+\operatorname{Cosh}2y}\right]$

Sol:

$$\operatorname{Sec}(x+iy) = \frac{1}{\operatorname{Cos}(x+iy)}\cdot\frac{\operatorname{Cos}(x-iy)}{\operatorname{Cos}(x-iy)}$$

$$= \frac{\operatorname{Cos}(x-iy)}{\operatorname{Cos}(x+iy)\operatorname{Cos}(x-iy)}$$

$$= \frac{2\operatorname{Cos}(x-iy)}{2\operatorname{Cos}(x+iy)\operatorname{Cos}(x-iy)}$$

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$$= \frac{2\operatorname{Cos}(x-iy)}{\operatorname{Cos}(x+\cancel{iy}+x-\cancel{iy})+\operatorname{Cos}(\cancel{x}+iy-\cancel{x}+iy)}$$

$$= 2\left[\frac{\operatorname{Cos}x\operatorname{Cos}(iy) + \operatorname{Sin}x\operatorname{Sin}(iy)}{\operatorname{Cos}(2x)+\operatorname{Cos}(2iy)}\right]$$

$$= 2\left[\frac{\operatorname{Cos}x\operatorname{Cosh}y + i\operatorname{Sin}x\operatorname{Sinh}y}{\operatorname{Cos}2x+\operatorname{Cosh}2y}\right]$$ proved

(ii) $\cos^{-1}(\cos\theta + i\sin\theta) = \sin^{-1}\sqrt{\sin\theta} + i\ln(\sqrt{1+\sin\theta} - \sqrt{\sin\theta})$

Sol:

$$\alpha + i\beta = \cos^{-1}(\cos\theta + i\sin\theta)$$

$$\cos(\alpha + i\beta) = \cos\theta + i\sin\theta$$

$$\cos\alpha\cos(i\beta) - \sin\alpha\sin(i\beta) = \cos\theta + i\sin\theta$$

$$\cos\alpha\cosh\beta - i\sin\alpha\sinh\beta = \cos\theta + i\sin\theta$$

Equating real and imaginary parts

$$\cos\alpha\cosh\beta = \cos\theta \quad ; \quad -\sin\alpha\sinh\beta = \sin\theta$$

$$\Rightarrow \cosh\beta = \frac{\cos\theta}{\cos\alpha} \quad ; \quad \sinh\beta = -\frac{\sin\theta}{\sin\alpha}$$

Squaring & Subtracting

$$\cosh^2\beta - \sinh^2\beta = \frac{\cos^2\theta}{\cos^2\alpha} - \frac{\sin^2\theta}{\sin^2\alpha}$$

$$= \frac{\cos^2\theta\sin^2\alpha - \sin^2\theta\cos^2\alpha}{\cos^2\alpha\sin^2\alpha}$$

$$= \frac{(1-\sin^2\theta)\sin^2\alpha - \sin^2\theta(1-\sin^2\alpha)}{\cos^2\alpha\sin^2\alpha}$$

$$= \frac{\sin^2\alpha - \cancel{\sin^2\alpha\sin^2\theta} - \sin^2\theta + \cancel{\sin^2\alpha\sin^2\theta}}{\cos^2\alpha\sin^2\alpha}$$

$$\cos^2\alpha\sin^2\alpha = \sin^2\alpha - \sin^2\theta$$

$$(1-\sin^2\alpha)\sin^2\alpha = \cancel{\sin^2\alpha} - \sin^4\alpha = \cancel{\sin^2\alpha} - \sin^2\theta$$

$$\Rightarrow \sin^4\alpha = \sin^2\theta$$

$$\Rightarrow \sin^2\alpha = \sin\theta$$

$$\Rightarrow \sin\alpha = \sqrt{\sin\theta} \quad \Rightarrow \quad \boxed{\alpha = \sin^{-1}\sqrt{\sin\theta}}$$

Now

$$\cos\alpha = \frac{\cos\theta}{\cosh\beta} \quad , \quad \sin\alpha = -\frac{\sin\theta}{\sinh\beta}$$

Squaring & adding.

$$\cos^2\alpha + \sin^2\alpha = \frac{\cos^2\theta}{\cosh^2\beta} + \frac{\sin^2\theta}{\sinh^2\beta}$$

$$1 = \frac{\cos^2\theta\sinh^2\beta + \sin^2\theta\cosh^2\beta}{\cosh^2\beta\sinh^2\beta}$$

$$\cosh^2\beta\sinh^2\beta = \cos^2\theta\sinh^2\beta + \sin^2\theta\cosh^2\beta$$

$$(1+\sinh^2\beta)\sinh^2\beta = (1-\sin^2\theta)\sinh^2\beta + \sin^2\theta(1+\sinh^2\beta)$$

$$\cancel{\sinh^2\beta} + \sinh^4\beta = \cancel{\sinh^2\beta} - \cancel{\sin^2\theta\sinh^2\beta} + \sin^2\theta + \cancel{\sin^2\theta\sinh^2\beta}$$

$$\sinh^4\beta = \sin^2\theta$$

$$\sinh^2\beta = \sin\theta$$

$$\sinh\beta = \sqrt{\sin\theta}$$

$$\sinh\beta = \sqrt{\sin\theta}$$

$$\frac{e^{\beta}-e^{-\beta}}{2} = \sqrt{\sin\theta} \Rightarrow e^{\beta}-e^{-\beta} = 2\sqrt{\sin\theta}$$

$$e^{\beta}-\frac{1}{e^{\beta}} = 2\sqrt{\sin\theta}$$

$$\frac{e^{2\beta}-1}{e^{\beta}} = 2\sqrt{\sin\beta} \Rightarrow e^{2\beta}-1 = 2\sqrt{\sin\theta}\, e^{\beta}$$

$$e^{2\beta}-2e^{\beta}\sqrt{\sin\theta}-1 = 0$$

quardratic in e^{β}

$$a=1, \quad b=-2\sqrt{\sin\theta}, \quad c=-1$$

$$e^{\beta} = \frac{2\sqrt{\sin\theta} \pm \sqrt{4\sin\theta+4}}{2}$$

$$e^{\beta} = \frac{2\sqrt{\sin\theta} \pm 2\sqrt{\sin\theta+1}}{2} = \frac{\cancel{2}(\sqrt{\sin\theta} \pm \sqrt{\sin\theta+1})}{\cancel{2}}$$

$$e^{\beta} = \sqrt{\sin\theta} \pm \sqrt{\sin\theta+1}$$

$$\beta = \ln\left|\pm\sqrt{\sin\theta+1} \pm \sqrt{\sin\theta}\right| \Rightarrow \beta = \ln\left|\sqrt{\sin\theta} - \sqrt{\sin\theta-1}\right| = \ln\left|\sqrt{\sin\theta-1} - \sqrt{\sin\theta}\right|$$

$$\cos^{-1}(\cos\theta+i\sin\theta) = \alpha+i\beta = \sin^{-1}\sqrt{\sin\theta} + i\ln\left(\sqrt{\sin\theta+1} - \sqrt{\sin\theta}\right)$$ Proved.

(iii) $\tan^{-1}(\cos\theta+i\sin\theta) = \pm\frac{\pi}{4} + \frac{i}{4}\ln\frac{1+\sin\theta}{1-\sin\theta}$

Sol

$$\alpha+i\beta = \tan^{-1}(\cos\theta+i\sin\theta) \rightarrow (i)$$

$$\alpha-i\beta = \tan^{-1}(\cos\theta-i\sin\theta) \rightarrow (ii)$$

adding (i) & (ii)

$$2\alpha = \tan^{-1}(\cos\theta+i\sin\theta) + \tan^{-1}(\cos\theta-i\sin\theta)$$

$$= \tan^{-1}\left[\frac{\cos\theta+\cancel{i\sin\theta}+\cos\theta-\cancel{i\sin\theta}}{1-(\cos\theta+i\sin\theta)(\cos\theta-i\sin\theta)}\right]$$

$$= \tan^{-1}\left[\frac{2\cos\theta}{1-(\cos^2\theta+\sin^2\theta)}\right]$$

$$= \tan^{-1}\left[\frac{2\cos\theta}{1-1}\right]$$

$$= \tan^{-1}\left(\frac{2\cos\theta}{0}\right)$$

$$= \tan^{-1}(\infty)$$

$$2\alpha = \pm\pi/2 \Rightarrow \alpha = \pm\pi/4$$

Subtracting (i) & (ii)

$$2i\beta = \tan^{-1}(\cos\theta+i\sin\theta) - \tan^{-1}(\cos\theta-i\sin\theta)$$

$$2i\beta = \tan^{-1}\left[\frac{\cancel{\cos\theta} + i\sin\theta - \cancel{\cos\theta} + i\sin\theta}{1 + (\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)}\right]$$

$$= \tan^{-1}\left(\frac{2i\sin\theta}{1 + \cos^2\theta + \sin^2\theta}\right)$$

$$\tan(2i\beta) = \frac{2i\sin\theta}{1+1}$$

$$i\tanh 2\beta = \frac{\cancel{2}\cancel{i}\sin\theta}{\cancel{2}}$$

$$\tanh 2\beta = \sin\theta$$

$$\frac{1}{\tanh 2\beta} = \frac{1}{\sin\theta}$$

$$\frac{e^{2\beta} + e^{-2\beta}}{e^{2\beta} - e^{-2\beta}} = \frac{1}{\sin\theta}$$

$$\frac{e^{2\beta} + \cancel{e^{-2\beta}} + e^{+2\beta} - \cancel{e^{-2\beta}}}{\cancel{e^{2\beta}} + e^{-2\beta} - \cancel{e^{2\beta}} + e^{-2\beta}} = \frac{1 + \sin\theta}{1 - \sin\theta}$$

$$\frac{\cancel{2}e^{2\beta}}{\cancel{2}e^{-2\beta}} = \frac{1+\sin\theta}{1-\sin\theta} \qquad = e^{2\beta + 2\beta}$$

$$e^{4\beta} = \frac{1+\sin\theta}{1-\sin\theta}$$

$$4\beta = \ln\left(\frac{1+\sin\theta}{1-\sin\theta}\right)$$

$$\beta = \frac{1}{4}\ln\left(\frac{1+\sin\theta}{1-\sin\theta}\right)$$

Putting value of α and β.

$$\tan^{-1}(\cos\theta + i\sin\theta) = \alpha + i\beta = \pm\frac{\pi}{4} + \frac{i}{4}\ln\left(\frac{1+\sin\theta}{1-\sin\theta}\right) \quad \text{proved}$$