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Exercise # 1.3

1. Show that

(i) e^z is never zero.

$e^z \cdot \frac{1}{e^z} = 1$

Multiplicative inverse of e^z is possible, so e^z can never be zero

OR

$e^z = e^{x+iy} = e^x e^{iy}$

$= e^x [\cos y + i \sin y]$

$\because e^x \neq 0$

there is no angle at which Cos & Sin become zero at same time

so $e^z \neq 0$

(ii) $|e^{iz}| = 1$

$|\cos z + i \sin z| = 1$

$\sqrt{\cos^2 z + \sin^2 z} = 1$

$\sqrt{1} = 1$

$1 = 1$

iii) $e^z = 1$ iff z is integral multiple of $2\pi i$.

Proof: Suppose $e^z = 1$

$e^{x+iy} = 1 = e^x e^{iy}$

$e^x [\cos y + i \sin y] = 1$

$e^x \cos y + i e^x \sin y = 1 + 0i$

Equating real and imaginary part.

$e^x \cos y = 1$ $\qquad$ $e^x \sin y = 0$

$e^x = 1 \; ; \; \cos y = 1$ $\qquad$ $e^x = 0$

$\Rightarrow x = 0 \; ; \;$ from (i) $\qquad$ $\sin y = 0$

$\cos(k\pi) = 1$ $\qquad$ $\Rightarrow y = k\pi \rightarrow$ (i) where $k \in Z$

$(-1)^k = 1$

$k = 2n \rightarrow$ put in (i)

$y = 2n\pi$

Now $z = x + iy = 0 + i2\pi n$

$\therefore z = (2\pi i)n$ So z is integral multiple of $2\pi i$.

Conversly; $z = 2\pi n i$

$e^z = e^{2n\pi i}$

$= \cos(2n\pi) + i \sin 2n\pi$

$e^z = 1 + 0 \Rightarrow e^z = 1$

Euler formula.

$e^{i\theta} = \cos\theta + i\sin\theta = \text{Cis}(\theta)$

(iv) $e^{z_1} = e^{z_2}$ iff $z_1 - z_2 = 2k\pi i$ where k is integer.

Proof: $e^{z_1} = e^{z_2} \Rightarrow \frac{e^{z_1}}{e^{z_2}} = 1$

$e^{z_1 - z_2} = 1$ $\qquad$ put $\because z_1 - z_2 = z$

$e^z = 1$

Now question is same as Q1 part (iii)

(v) $|e^z| = e^x$ where $x + iy = z$

$|e^z| = |e^{x+iy}|$

$= |e^x e^{iy}|$

$= e^x |e^{iy}|$

$= e^x \sqrt{\cos^2 y + \sin^2 y}$

$= e^x \sqrt{1}$

$|e^z| = e^x$

(vi) $e^{z_1} \cdot e^{z_2} \cdot \ldots \cdot e^{z_n} = e^{z_1+z_2+\cdots+z_n} \rightarrow (1)$

We'll prove this by Mathematical induction

Case I put $n=1$

$e^{z_1} = e^{z_1}$

put $n=2$

$e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}$

Condition is true for $n=1$ & 2.

Case II Suppose this is true for $n=k$ $k \in Z^+$

$e^{z_1} \cdot e^{z_2} \ldots e^{z_k} = e^{z_1+z_2+\cdots+z_k}$

Case III ~~Multiply both sides with~~ $e^{z_{k+1}}$

$e^{z_1} \cdot e^{z_2} \ldots e^{z_k} \cdot e^{z_{k+1}} = e^{z_1+z_2+\cdots+z_k} \cdot e^{z_{k+1}}$

$= e^{z_1+z_2+\cdots+z_k+z_{k+1}}$

This is true for two numbers C-I

Hence This is true for $n=k+1$

So by Induction (1) is true.

(vii) $(e^z)^n = e^{nz}$, n being an integer

$$
\begin{aligned}
L.H.S = (e^z)^n &= (e^{x+iy})^n \\
&= (e^x e^{iy})^n \\
&= e^{xn} (e^{iy})^n \\
&= e^{xn} (\cos y + i \sin y)^n \quad \text{Euler formula} \\
&= e^{xn} [\cos ny + i \sin ny] \\
&= e^{xn} e^{iny} \\
&= e^{xn+iny} = e^{n(iy+x)}
\end{aligned}
$$

$(e^z)^n = e^{nz}$

Formulae:

$\cosh z = \dfrac{e^z + e^{-z}}{2}$, $\sinh z = \dfrac{e^z - e^{-z}}{2}$, $\tanh z = \dfrac{e^z - e^{-z}}{e^z + e^{-z}}$

&

$\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$

$\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$

$\tan z = \dfrac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}$

Q2. Prove that $(\forall\ z_1, z_2, z \in C)$

(i) $\tan^2 z + 1 = \sec^2 z$

$$
\begin{aligned}
L.H.S &= 1 + \tan^2 z \\
&= 1 + \left(\frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}\right)^2 \\
&= 1 - \frac{(e^{iz} - e^{-iz})^2}{(e^{iz} + e^{-iz})^2}
\end{aligned}
$$

$$= \frac{(e^{iz}+e^{-iz})^2-(e^{iz}-e^{-iz})^2}{(e^{iz}+e^{-iz})^2}$$

$$= \frac{\cancel{e^{2iz}}+\cancel{e^{-2iz}}+2-(\cancel{e^{2iz}}+\cancel{e^{-2iz}}-2)}{(e^{iz}+e^{-iz})^2}$$

$$= \frac{2+2=4}{(e^{iz}+e^{-iz})^2}$$

$$= \left[\frac{2}{(e^{iz}+e^{-iz})}\right]^2$$

$$= \left[\frac{1}{\cos z}\right]^2$$

$$= (\sec z)^2$$

$$= \sec^2 z$$

$\because e^{iz}\cdot e^{-iz} = e^{iz-iz} = e^0 = 1$

$$1+\cot^2 z = \operatorname{cosec}^2 z$$

L.H.S $= 1+\cot^2 z$

$$= 1+\left[\frac{i(e^{iz}+e^{-iz})}{(e^{iz}-e^{-iz})}\right]^2 \quad \because i^2=-1$$

$$= 1-\frac{(e^{iz}+e^{-iz})^2}{(e^{iz}-e^{-iz})^2}$$

$$= \frac{(e^{iz}-e^{-iz})^2-(e^{iz}+e^{-iz})^2}{(e^{iz}-e^{-iz})^2}$$

$$= \frac{\cancel{e^{2iz}}+\cancel{e^{-2iz}}-2-(\cancel{e^{2iz}}+\cancel{e^{-2iz}}+2)}{(e^{iz}-e^{-iz})^2}$$

$$= \frac{(-1)(2)^2}{(e^{iz}-e^{-iz})^2} \quad \because (-1)=i^2$$

$$= \left(\frac{2i}{e^{iz}-e^{-iz}}\right)^2$$

$$= \left(\frac{1}{\sin z}\right)^2$$

$$= \operatorname{cosec}^2 z = \text{R.H.S}$$

(iii) $\sin(z_1-z_2) = \sin z_1\cos z_2 - \cos z_1 \sin z_2$

R.H.S $= \sin z_1\cos z_2 - \cos z_1\sin z_2$

$$= \left(\frac{e^{iz_1}-e^{-iz_1}}{2i}\right)\left(\frac{e^{iz_2}+e^{-iz_2}}{2}\right) - \left(\frac{e^{iz_1}+e^{-iz_1}}{2}\right)\left(\frac{e^{iz_2}-e^{-iz_2}}{2i}\right)$$

$$= \frac{1}{4i}\left((e^{iz_1}-e^{-iz_1})(e^{iz_2}+e^{-iz_2}) - (e^{iz_1}+e^{-iz_1})(e^{iz_2}-e^{-iz_2})\right)$$

$$= \frac{1}{4i}\left[\cancel{e^{iz_1}e^{iz_2}} + e^{iz_1}e^{-iz_2} - e^{-iz_1}e^{iz_2} + (-)\cancel{e^{-iz_1}e^{-iz_2}} - \cancel{e^{iz_1}e^{iz_2}} + e^{iz_1}e^{-iz_2} - e^{-iz_1}e^{iz_2} + \cancel{e^{-iz_1}e^{-iz_2}}\right)$$

$$= \frac{1}{4i}\left(2e^{iz_1}e^{-iz_2} - 2e^{-iz_1}e^{iz_2}\right)$$

$$= \frac{1}{\underset{2}{\cancel{4}}i}\,\cancel{2}\left(e^{i(z_1-z_2)} - e^{-i(z_1-z_2)}\right)$$

$$= \frac{e^{i(z_1-z_2)}-e^{-i(z_1-z_2)}}{2i}$$

$$= \sin(z_1-z_2) = \text{L.H.S}$$

(iv) $\cos(z_1+z_2) = \cos z_1\cos z_2 - \sin z_1\sin z_2$

R.H.S $= \cos z_1\cos z_2 - \sin z_1 \sin z_2$

$$= \left(\frac{e^{iz_1}+e^{-iz_1}}{2}\right)\left(\frac{e^{iz_2}+e^{-iz_2}}{2}\right) - \left(\frac{e^{iz_1}-e^{-iz_1}}{2i}\right)\left(\frac{e^{iz_2}-e^{-iz_2}}{2i}\right)$$

$$= \frac{1}{4}\left((e^{iz_1}+e^{-iz_1})(e^{iz_2}+e^{-iz_2}) + (e^{iz_1}-e^{-iz_1})(e^{iz_2}-e^{-iz_2})\right)$$

$$= \frac{1}{4}\left[e^{iz_1}e^{iz_2} + \cancel{e^{iz_1}e^{-iz_2}} + \cancel{e^{-iz_1}e^{iz_2}} + e^{-iz_1}e^{-iz_2} + e^{iz_1}e^{iz_2} - \cancel{e^{iz_1}e^{-iz_2}} - \cancel{e^{-iz_1}e^{iz_2}} + e^{-iz_1}e^{-iz_2}\right]$$

$$= \frac{1}{4}\left[2e^{iz_1}e^{iz_2} + 2e^{-iz_1}e^{-iz_2}\right]$$

$$= \frac{1}{2}\left[e^{i(z_1+z_2)} + e^{-i(z_1+z_2)}\right]$$

$$= \cos(z_1+z_2)$$

(V) $\cos(z_1 - z_2) = \cos z_1 \cos z_2 + \sin z_1 \sin z_2$.

$$R.H.S = \cos z_1 \cos z_2 + \sin z_1 \sin z_2$$

$$= \left(\frac{e^{iz_1} + e^{-iz_1}}{2}\right)\left(\frac{e^{iz_2} + e^{-iz_2}}{2}\right) + \left(\frac{e^{iz_1} - e^{-iz_1}}{2i}\right)\left(\frac{e^{iz_2} - e^{-iz_2}}{2i}\right)$$

$$= \frac{1}{4}\left((e^{iz_1} + e^{-iz_1})(e^{iz_2} + e^{-iz_2}) + \frac{1}{i^2}(e^{iz_1} - e^{-iz_1})(e^{iz_2} - e^{-iz_2})\right)$$

$$= \frac{1}{4}\left(\cancel{e^{iz_1}e^{iz_2}} + e^{iz_1}e^{-iz_2} + e^{-iz_1}e^{iz_2} + \cancel{e^{-iz_1}e^{-iz_2}} - \cancel{e^{iz_1}e^{iz_2}} + e^{iz_1}e^{-iz_2} + e^{-iz_1}e^{iz_2} - \cancel{e^{-iz_1}e^{-iz_2}}\right)$$

$$= \frac{1}{4}\left(2e^{iz_1}e^{-iz_2} + 2e^{-iz_1}e^{iz_2}\right)$$

$$= \frac{1}{2}\left(e^{i(z_1 - z_2)} + e^{-i(z_1 - z_2)}\right)$$

$$= \cos(z_1 - z_2) = L.H.S.$$

(Vi) $\cos 2z = \cos^2 z - \sin^2 z = 2\cos^2 z - 1 = 1 - 2\sin^2 z$

$$\cos^2 z - \sin^2 z = \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 - \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2$$

$$= \frac{(e^{iz} + e^{-iz})^2}{4} - \frac{(e^{iz} - e^{-iz})^2}{4i^2} \qquad i^2 = -1$$

$$= \frac{1}{4}\left[(e^{iz} + e^{-iz})^2 + (e^{iz} - e^{-iz})^2\right]$$

$$= \frac{1}{4}\left[e^{2iz} + e^{-2iz} + \cancel{2} + e^{2iz} + e^{-2iz} - \cancel{2}\right]$$

$$= \frac{1}{4}\left[2e^{2iz} + 2e^{-2iz}\right]$$

$$= \frac{1}{2}\left[e^{2iz} + e^{-2iz}\right]$$

$$= \frac{e^{2iz} + e^{-2iz}}{2}$$

$$= \cos 2z \quad \text{Prove}$$

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$$2\cos^2 z - 1 = 2\left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 - 1$$

$$= \frac{\cancel{2}}{\cancel{4}_2}(e^{iz} + e^{-iz})^2 - 1$$

$$= \frac{(e^{iz} + e^{-iz})^2 - 2}{2}$$

$$= \frac{e^{2iz} + e^{-2iz} + \cancel{2} - \cancel{2}}{2}$$

$$= \frac{e^{2iz} + e^{-2iz}}{2}$$

$$= \cos 2z$$

$$1 - 2\sin^2 z = 1 - \cancel{2}\frac{(e^{iz} - e^{-iz})^2}{{}_2\cancel{4}i^2}$$

$$= 1 + \frac{(e^{iz} - e^{-iz})^2}{2}$$

$$= \frac{2 + (e^{iz} - e^{-iz})^2}{2}$$

$$= \frac{2 + e^{2iz} + e^{-2iz} - 2}{2}$$

$$= \frac{e^{2iz} + e^{-2iz}}{2}$$

$$= \cos 2z$$

(vii) $\sin 2z = 2\sin z\cos z$

$$2\sin z\cos z = \cancel{2}\left(\frac{e^{iz}-e^{-iz}}{2i}\right)\left(\frac{e^{iz}+e^{-iz}}{\cancel{2}}\right)$$

$$= \frac{1}{2i}\left(e^{iz}-e^{-iz}\right)\left(e^{iz}+e^{-iz}\right)$$

$$= \frac{1}{2i}\left(e^{2iz}-e^{-2iz}\right)$$

$$= \sin 2z$$

viii) $\cos z_1 - \cos z_2 = 2\sin\frac{z_1+z_2}{2}\sin\frac{z_2-z_1}{2}$

$$R.H.S = 2\sin\left(\frac{z_1+z_2}{2}\right)\sin\left(\frac{z_2-z_1}{2}\right)$$

$$= \cancel{2}\left[\frac{e^{i\left(\frac{z_1+z_2}{2}\right)}-e^{-i\left(\frac{z_1+z_2}{2}\right)}}{\cancel{2}i}\right]\left[\frac{e^{i\left(\frac{z_2-z_1}{2}\right)}-e^{-i\left(\frac{z_2-z_1}{2}\right)}}{2i}\right]$$

$$= \frac{1}{2i^2}\left[\left(e^{i\left(\frac{z_1+z_2}{2}\right)}-e^{-i\left(\frac{z_1+z_2}{2}\right)}\right)\left(e^{i\left(\frac{z_2-z_1}{2}\right)}-e^{-i\left(\frac{z_2-z_1}{2}\right)}\right)\right]$$

$$= -\frac{1}{2}\left[e^{i\left(\frac{z_1+z_2}{2}\right)}e^{i\left(\frac{z_2-z_1}{2}\right)} - e^{i\left(\frac{z_1+z_2}{2}\right)}e^{-i\left(\frac{z_2-z_1}{2}\right)} - e^{-i\left(\frac{z_1+z_2}{2}\right)}e^{i\left(\frac{z_2-z_1}{2}\right)} + e^{-i\left(\frac{z_1+z_2}{2}\right)}e^{-i\left(\frac{z_2-z_1}{2}\right)}\right]$$

$$= -\frac{1}{2}\left[e^{i\left(\frac{z_1+z_2}{2}\right)+i\left(\frac{z_2-z_1}{2}\right)} - e^{i\left(\frac{z_1+z_2}{2}\right)+(-i)\left(\frac{z_2-z_1}{2}\right)} - e^{-i\left(\frac{z_1+z_2}{2}\right)+i\left(\frac{z_2-z_1}{2}\right)} + e^{-i\left(\frac{z_1+z_2}{2}\right)-i\left(\frac{z_2-z_1}{2}\right)}\right]$$

$$= -\frac{1}{2}\left[e^{i\left(\frac{\cancel{z_1}+z_2+z_2-\cancel{z_1}}{2}\right)} - e^{i\left(\frac{z_1+z_2-z_2+z_1}{2}\right)} - e^{-i\left(\frac{z_1+\cancel{z_2}-\cancel{z_2}+z_1}{2}\right)} + e^{-i\left(\frac{\cancel{z_1}+z_2+z_2-\cancel{z_1}}{2}\right)}\right]$$

$$= -\frac{1}{2}\left[\left(e^{iz_2} - e^{iz_1} - e^{-iz_1} + e^{-iz_2}\right)\right]$$

$$= -\frac{1}{2}\left[\left(e^{iz_2}+e^{-iz_2}\right) - \left(e^{iz_1}+e^{-iz_1}\right)\right]$$

$$= -\left(\frac{e^{iz_2}+e^{-iz_2}}{2}\right) + \left(\frac{e^{iz_1}+e^{-iz_1}}{2}\right)$$

$$= -\cos z_2 + \cos z_1 = L.H.S \quad \text{proved}$$

(ix) $\sin z_1 - \sin z_2 = 2\cos\left(\frac{z_1+z_2}{2}\right)\sin\left(\frac{z_1-z_2}{2}\right)$

$$R.H.S = 2\cos\left(\frac{z_1+z_2}{2}\right)\sin\left(\frac{z_1-z_2}{2}\right)$$

$$= \cancel{2}\left[\frac{e^{i\left(\frac{z_1+z_2}{2}\right)}+e^{-i\left(\frac{z_1+z_2}{2}\right)}}{\cancel{2}}\right]\left[\frac{e^{i\left(\frac{z_2-z_1}{2}\right)}-e^{-i\left(\frac{z_2-z_1}{2}\right)}}{2i}\right]$$

$$= \frac{1}{2i}\left[\left(e^{i\left(\frac{z_1+z_2}{2}\right)} + e^{-i\left(\frac{z_1+z_2}{2}\right)}\right)\left(e^{i\left(\frac{z_1-z_2}{2}\right)} - e^{-i\left(\frac{z_1-z_2}{2}\right)}\right)\right]$$

$$= \frac{1}{2i}\left[e^{i\left(\frac{z_1+z_2}{2}\right)} e^{i\left(\frac{z_1-z_2}{2}\right)} - e^{i\left(\frac{z_1+z_2}{2}\right)} e^{-i\left(\frac{z_1-z_2}{2}\right)} + e^{-i\left(\frac{z_1+z_2}{2}\right)} e^{i\left(\frac{z_1-z_2}{2}\right)} - e^{-i\left(\frac{z_1+z_2}{2}\right)} e^{-i\left(\frac{z_1-z_2}{2}\right)}\right]$$

$$= \frac{1}{2i}\left[e^{i\left(\frac{z_1+\cancel{z_2}+z_1-\cancel{z_2}}{2}\right)} - e^{i\left(\frac{\cancel{z_1}+z_2-\cancel{z_1}+z_2}{2}\right)} + e^{-i\left(\frac{\cancel{z_1}+z_2-\cancel{z_1}+z_2}{2}\right)} - e^{-i\left(\frac{z_1+\cancel{z_2}+z_1-\cancel{z_2}}{2}\right)}\right]$$

$$= \frac{1}{2i}\left[e^{iz_1} - e^{iz_2} + e^{-iz_2} - e^{-iz_1}\right]$$

$$= \frac{1}{2i}\left[(e^{iz_1} - e^{-iz_1}) - (e^{iz_2} - e^{-iz_2})\right]$$

$$= \frac{e^{iz_1} - e^{-iz_1}}{2i} - \frac{e^{iz_2} - e^{-iz_2}}{2i} = \sin z_1 - \sin z_2 = L.H.S.$$

(X) $\sin 3z = 3\sin z - 4\sin^3 z$

$$3\sin z - 4\sin^3 z = 3\left(\frac{e^{iz} - e^{-iz}}{2i}\right) - 4\left(\frac{e^{iz} - e^{-iz}}{2i}\right)^3$$

$$= 3\left(\frac{e^{iz} - e^{-iz}}{2i}\right) - \frac{4}{8i^3}\left(e^{iz} - e^{-iz}\right)^3$$

$3e^{2iz-iz} = 3e^{iz}$ $\quad$ $3e^{iz-2iz} = 3e^{-iz}$

$$= \frac{3}{2i}\left(e^{iz} - e^{-iz}\right) - \frac{1}{-2i}\left(e^{3iz} - e^{-3iz} - 3e^{2iz}e^{-iz} + 3e^{iz}e^{-2iz}\right)$$

$$= \frac{1}{2i}\left[\cancel{3e^{iz}} - \cancel{3e^{-iz}} + e^{3iz} - e^{-3iz} - \cancel{3e^{iz}} + \cancel{3e^{-iz}}\right]$$

$$= \frac{e^{3iz} - e^{-3iz}}{2i}$$

$$= \sin 3z = R.H.S$$

(Xi) $\tan(z_1+z_2) = \dfrac{\tan z_1 + \tan z_2}{1 - \tan z_1 \tan z_2}$

$$\tan(z_1+z_2) = \frac{\sin(z_1+z_2)}{\cos(z_1+z_2)}$$

$$= \frac{\sin z_1 \cos z_2 + \cos z_1 \sin z_2}{\cos z_1 \cos z_2 - \sin z_1 \sin z_2}$$

$$= \frac{\frac{\sin z_1 \cancel{\cos z_2}}{\cos z_1 \cancel{\cos z_2}} + \frac{\cancel{\cos z_1} \sin z_2}{\cancel{\cos z_1} \cos z_2}}{\frac{\cancel{\cos z_1 \cos z_2}}{\cancel{\cos z_1 \cos z_2}} - \frac{\sin z_1 \sin z_2}{\cos z_1 \cos z_2}}$$

$$= \frac{\tan z_1 + \tan z_2}{1 - \tan z_1 \tan z_2}$$

(xii) $\tan(z_1-z_2) = \dfrac{\tan z_1 - \tan z_2}{1 + \tan z_1 \tan z_2}$

$$\tan(z_1-z_2) = \frac{\sin(z_1-z_2)}{\cos(z_1-z_2)}$$

$$= \frac{\sin z_1 \cos z_2 - \cos z_1 \sin z_2}{\cos z_1 \cos z_2 + \sin z_1 \sin z_2}$$

$$= \frac{\frac{\sin z_1 \cancel{\cos z_2}}{\cos z_1 \cancel{\cos z_2}} - \frac{\cancel{\cos z_1} \sin z_2}{\cancel{\cos z_1} \cos z_2}}{\frac{\cancel{\cos z_1 \cos z_2}}{\cancel{\cos z_1 \cos z_2}} + \frac{\sin z_1 \sin z_2}{\cos z_1 \cos z_2}}$$

$$\tan(z_1 - z_2) = \frac{\tan z_1 - \tan z_2}{1 + \tan z_1 \tan z_2}$$

Formulae:

- $\sin(iz) = i\sinh z$
- $\cos(iz) = \cosh z$
- $\tan(iz) = i\tanh z$

3. Show that.

(i) $\overline{\sin z} = \sin\bar{z}$

$$\sin z = \sin(x+iy)$$
$$= \sin x\cos(iy) + \cos x\sin(iy)$$
$$= \sin x\cosh y + \cos x\, i\sinh y$$
$$\overline{\sin z} = \overline{\sin x\cosh y + i\cos x\sinh y}$$
$$= \sin x\cosh y - i\cos x\sinh y \rightarrow (i)$$

$$\sin\bar{z} = \sin(\overline{x+iy})$$
$$= \sin(x-iy)$$
$$= \sin x\cos(iy) - \cos x\sin(iy)$$
$$= \sin x\cosh y - \cos x\, i\sinh y$$
$$\sin\bar{z} = \cosh y\sin x - i\cos x\sinh y \rightarrow (ii)$$

From (i) & (ii) $\overline{\sin z} = \sin\bar{z}$

(ii) $\overline{\cos z} = \cos\bar{z}$

$$\cos z = \cos(x+iy)$$
$$= \cos x\cos iy - \sin x\sin(iy)$$
$$= \cos x\cosh y - \sin x\, i\sinh y$$
$$\overline{\cos z} = \overline{\cos x\cosh y - i\sin x\sinh y}$$
$$= \cos x\cosh y + i\sin x\sinh y \rightarrow (i)$$

$$\cos\bar{z} = \cos(\overline{x+iy})$$
$$= \cos(x-iy)$$
$$= \cos x\cos iy + \sin x\sin(iy)$$
$$\cos\bar{z} = \cos x\cosh y + i\sin x\sinh y \quad (ii)$$

From (i) & (ii) $\overline{\cos z} = \cos\bar{z}$

(iii) $\overline{\tan z} = \tan\bar{z}$

$$\tan z = \tan(x+iy)$$
$$= \frac{\tan x + \tan(iy)}{1 - \tan(x)\tan(iy)}$$
$$= \frac{\tan x + i\tanh y}{1 - \tan x\, i\tanh y}$$
$$\overline{\tan z} = \overline{\left(\frac{\tan x + i\tanh y}{1 - i\tan x\tanh y}\right)} \quad \because \overline{\left(\frac{z}{z_1}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$$
$$= \frac{\overline{\tan x + i\tanh y}}{\overline{1 - i\tan x\tanh y}}$$
$$\overline{\tan z} = \frac{\tan x - i\tanh y}{1 + i\tan x\tanh y} \rightarrow (1)$$

$$\tan\bar{z} = \tan(\overline{x+iy})$$
$$= \tan(x-iy)$$
$$\tan\bar{z} = \frac{\tan x - \tan(iy)}{1 + \tan(x)\tan(iy)}$$
$$\tan\bar{z} = \frac{\tan x - i\tanh y}{1 + i\tan x\tanh y} \rightarrow (2)$$

From (1) & (2)

$\overline{\tan z} = \tan\bar{z}$

(iv) $\sin(-z) = -\sin z$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \rightarrow (1)$$
$$\sin(-z) = \frac{e^{-iz} - e^{iz}}{2i}$$
$$\sin(-z) = -\left(\frac{e^{iz} - e^{-iz}}{2i}\right) = -\sin z \text{ from } (1)$$

$\sin(-z) = -\sin z$

(v) $\cos(-z) = +\cos(z)$

$$\cos(z) = \frac{e^{iz}+e^{-iz}}{2} \rightarrow (i)$$

replace z by $-z$

$$\cos(-z) = \frac{e^{-iz}+e^{iz}}{2}$$

$\cos(-z) = \cos z$ from (i)

(vi) $\tan(-z) = -\tan z$

$$\tan(z) = \frac{e^{iz}-e^{-iz}}{i(e^{iz}+e^{-iz})} \rightarrow ①$$

replace z by $(-z)$

$$\tan(-z) = \frac{e^{-iz}-e^{iz}}{i(e^{-iz}+e^{iz})}$$

$$\tan(-z) = -\frac{(e^{iz}-e^{-iz})}{i(e^{iz}+e^{-iz})}$$

$\tan(-z) = -\tan(z)$ from (i)

(vii) $\sinh(-z) = -\sinh z$

$$\sinh z = \frac{e^{z}-e^{-z}}{2} \rightarrow ①$$

replace z by $-z$

$$\sinh(-z) = \frac{e^{-z}-e^{z}}{2}$$

$$= -\left(\frac{e^{z}-e^{-z}}{2}\right)$$

$\sinh(-z) = -\sinh z$ from (i)

(viii) $\cosh(-z) = \cosh z$

$$\cosh z = \frac{e^{z}+e^{-z}}{2} \rightarrow ①$$

replace z by $-z$

$$\cosh(-z) = \frac{e^{-z}+e^{z}}{2}$$

$\cosh(-z) = \cosh z$ from ①

(ix) $\tanh(-z) = -\tanh z$

$$\tanh z = \frac{e^{z}-e^{-z}}{e^{z}+e^{-z}} \rightarrow (i)$$

replace z by $-z$

$$\tanh(-z) = \frac{e^{-z}-e^{z}}{e^{-z}+e^{z}}$$

$$\tanh(-z) = -\left(\frac{e^{z}-e^{-z}}{e^{z}+e^{-z}}\right)$$

$\tanh(-z) = -\tanh z$ from (i)

(x) $\overline{\tanh z} = \tanh \bar{z}$

By formula

$$i\tanh z = \tan(iz)$$

$$\tanh z = \frac{\tan(iz)}{i}$$

$$\overline{\tanh z} = \overline{\left(\frac{\tan(iz)}{i}\right)} = \frac{\overline{\tan(iz)}}{\bar{i}}$$

$$= \frac{\tan(\overline{iz})}{-i} \quad \because \overline{\tan z} = \tan\bar{z}$$

$$= \frac{\tan(-i\bar{z})}{-i}$$

$$= -\frac{\tan(i\bar{z})}{-i}$$

$$= \frac{\tan(i\bar{z})}{i}$$

$$= \frac{i\tanh(\bar{z})}{i}$$

$\overline{\tanh z} = \tanh \bar{z}$ proved

4. Prove the following identities

(i) $\cosh^2 z - \sinh^2 z = 1$

$\because \cos^2 z + \sin^2 z = 1$ replace z by iz

$$[\cos(iz)]^2 + [\sin(iz)]^2 = 1$$

$$(\cosh z)^2 + (i\sinh z)^2 = 1$$

$$\cosh^2 z + i^2 \sinh^2 z = 1 \quad \because i^2 = -1$$

$$\cosh^2 z - \sinh^2 z = 1$$

proved.

(ii) $\operatorname{Sech}^2 z = 1 - \tanh^2 z$

$\because \sec^2 z = 1 + \tan^2 z$

replace z by iz

$[\sec(iz)]^2 = 1 + [\tan(iz)]^2$

$(\operatorname{Sech} z)^2 = 1 + (i \tanh z)^2$

$\operatorname{Sech}^2 z = 1 - \tanh^2 z$

(iii) $\operatorname{Cosech}^2 z = \coth^2 z - 1$

$\because \operatorname{Cosec}^2 z = 1 + \cot^2(z)$

replace z by iz

$[\operatorname{Cosec}(iz)]^2 = 1 + [\cot(iz)]^2$

$[i \operatorname{Cosech} z]^2 = 1 + [i \coth z]^2$

$i^2 \operatorname{Cosech}^2 z = 1 + i^2 \coth^2 z$

$-\operatorname{Cosech}^2 z = 1 - \coth^2 z$

$\operatorname{Cosech}^2 z = \coth^2 z - 1$ proved.

(iv) $\cosh 2z = \cosh^2 z + \sinh^2 z = 2\cosh^2 z - 1 = 1 + 2\sinh^2 z$

$\because \cos 2z = \cos^2 z - \sin^2 z$

replace z by iz

$\cos(2iz) = [\cos(iz)]^2 - [\sin(iz)]^2$

$\cosh(2z) = (\cosh z)^2 - (i \sinh z)^2 \quad \because i^2 = -1$

$\cosh 2z = \cosh^2 z + \sinh^2 z \rightarrow (i)$ $\boxed{\text{Proved}}$

$\because \cosh^2 z = 1 + \sinh^2 z$ (put in (i))

$\cosh 2z = 1 + \sinh^2 z + \sinh z$

$= 1 + 2\sinh^2 z$ $\boxed{\text{Proved}}$

$\because \sinh^2 z = -1 + \cosh^2 z$ (put in (i))

$\cosh 2z = \cosh^2 z + \cosh^2 z - 1$

$= 2\cosh^2 z - 1$ $\boxed{\text{Proved}}$

(v) $\sinh 2z = 2\sinh z \cosh z$

$\because \sin 2z = 2 \sin z \cos z$

replace z by iz

$\sin(2iz) = 2 \sin(iz) \cos iz$

$i \sinh 2z = 2 i \sinh z \cosh z$

$\sinh 2z = 2 \sinh z \cosh z$

(vi) $\sinh 3z = 3\sinh z + 4\sinh^3 z$

$\because \sin 3z = 3\sin z - 4\sin^3 z$

replace z by iz

$\sin(3iz) = 3\sin(iz) - 4(\sin(iz))^3$

$i \sinh 3z = i3 \sinh z - 4(i \sinh z)^3$

$i \sinh 3z = i3 \sinh z - 4i^2 i \sinh^3 z$

$\sinh 3z = 3\sinh z + 4\sinh^3 z$

Proved

(vii) $\cosh 3z = 4\cosh^3 z - 3\cosh z$

$\because \cos 3z = 4\cos^3 z - 3\cos z$

replace z by iz

$\cos(3iz) = 4(\cos(iz))^3 - 3\cos(iz)$

$\cosh 3z = 4(\cosh z)^3 - 3\cosh z$

$= 4\cosh^3 z - 3\cosh z$

5. if $z = x+iy$, prove that

(i) $\operatorname{Sin} z = \operatorname{Sin}x \operatorname{Cosh}y + i\operatorname{Cos}x \operatorname{Sinh}y$

$\operatorname{Sin}z = \operatorname{Sin}(x+iy) = \operatorname{Sin}x \operatorname{Cos}(iy) + \operatorname{Cos}x \operatorname{Sin}(iy)$

$\operatorname{Sin}z = \operatorname{Sin}x \operatorname{Cosh}y + i\operatorname{Cos}x \operatorname{Sinh}y$ proved

(ii) $\tan z = \dfrac{\operatorname{Sin}2x + i\operatorname{Sinh}2y}{\operatorname{Cos}2x + \operatorname{Cosh}2y}$

Formula used

$2\operatorname{Sin}A\operatorname{Cos}B = \operatorname{Sin}(A+B) + \operatorname{Sin}(A-B)$

&

$2\operatorname{Cos}A\operatorname{Cos}B = \operatorname{Cos}(A+B) + \operatorname{Cos}(A-B)$

$\tan z = \tan(x+iy)$

$= \dfrac{\operatorname{Sin}(x+iy)}{\operatorname{Cos}(x+iy)} \times \dfrac{\operatorname{Cos}(x-iy)}{\operatorname{Cos}(x-iy)}$

$= \dfrac{2\operatorname{Sin}(x+iy)\operatorname{Cos}(x-iy)}{2\operatorname{Cos}(x+iy)\operatorname{Cos}(x-iy)}$

$= \dfrac{\operatorname{Sin}(x+\cancel{iy}+x-\cancel{iy}) + \operatorname{Sin}(\cancel{x}+iy-\cancel{x}+iy)}{\operatorname{Cos}(x+\cancel{iy}+x-\cancel{iy}) + \operatorname{Cos}(\cancel{x}+iy-\cancel{x}+iy)}$

$= \dfrac{\operatorname{Sin}(2x) + \operatorname{Sin}(2iy)}{\operatorname{Cos}(2x) + \operatorname{Cos}(2iy)}$

$\therefore = \dfrac{\operatorname{Sin}2x + i\operatorname{Sinh}2y}{\operatorname{Cos}2x + i\operatorname{Cosh}2y}$ proved

6. if $\operatorname{Sin}(A+iB) = x+iy$, show that

$\dfrac{x^2}{\operatorname{Sin}^2A} - \dfrac{y^2}{\operatorname{Cos}^2A} = 1$ & $\dfrac{x^2}{\operatorname{Cosh}^2B} + \dfrac{y^2}{\operatorname{Sinh}^2B} = 1$

$\operatorname{Sin}(A+iB) = x+iy$

$\operatorname{Sin}A\operatorname{Cos}(iB) + \operatorname{Cos}A\operatorname{Sin}(iB) = x+iy$

$\operatorname{Sin}A\operatorname{Cosh}B + i\operatorname{Cos}A\operatorname{Sinh}B = x+iy$

equating real and imaginary parts

$\operatorname{Sin}A\operatorname{Cosh}B = x$; $\operatorname{Cos}A\operatorname{Sinh}B = y$

$\operatorname{Cosh}B = \dfrac{x}{\operatorname{Sin}A} \rightarrow$ (i) ; $\operatorname{Sinh}B = \dfrac{y}{\operatorname{Cos}A} \rightarrow$ (i)

& $\operatorname{Sin}A = \dfrac{x}{\operatorname{Cosh}B} \rightarrow$ (iii) ; $\operatorname{Cos}A = \dfrac{y}{\operatorname{Sinh}B} \rightarrow$ (iv)

Squaring & Subtracting (i) & (iii)

$\dfrac{x^2}{\operatorname{Sin}^2A} - \dfrac{y^2}{\operatorname{Cos}^2A} = \operatorname{Cosh}^2B - \operatorname{Sinh}^2B$

$\dfrac{x^2}{\operatorname{Sin}^2A} - \dfrac{y^2}{\operatorname{Cos}^2A} = 1$

Proved

Squaring & adding (iii) & (iv)

$\dfrac{x^2}{\operatorname{Cosh}^2B} + \dfrac{y^2}{\operatorname{Sinh}^2B} = \operatorname{Cos}^2A + \operatorname{Sin}^2A$

$\dfrac{x^2}{\operatorname{Cosh}^2B} + \dfrac{y^2}{\operatorname{Sinh}^2B} = 1$

Proved

(viii) $$\sinh(z_1 - z_2) = \sinh z_1 \cosh z_2 - \cosh z_2 \sinh z_2$$

$$\because \sin(z_1 - z_2) = \sin z_1 \cos z_2 - \cos z_1 \sin z_2$$

replace z_1 by iz_1 and z_2 by iz_2

$$\sin(iz_1 - iz_2) = \sin(iz_1)\cos(iz_2) - \cos(iz_1)\sin(iz_2)$$

$$\sin i(z_1 - z_2) = i\sinh z_1 \cosh z_2 - \cosh z_1 \, i\sinh z_2$$

$$\cancel{i}\sinh(z_1 - z_2) = \cancel{i}\left[\sinh z_1 \cosh z_2 - \cosh z_1 \sinh z_2\right]$$

$$\sinh(z_1 - z_2) = \sinh z_1 \cosh z_2 - \cosh z_1 \sinh z_2$$ proved

(ix) $$\tanh(z_1 \pm z_2) = \frac{\tanh z_1 \pm \tanh z_2}{1 \mp \tanh z_1 \tanh z_2}$$

$$\because \tan(z_1 \pm z_2) = \frac{\tan z_1 \pm \tan z_2}{1 \mp \tan z_1 \tan z_2}$$

replace z_1 by iz_1 and z_2 by iz_2

$$\tan(iz_1 \pm iz_2) = \frac{\tan(iz_1) \pm \tan(iz_2)}{1 \mp \tan(iz_1)\tan(iz_2)}$$

$$\tan i(z_1 \pm z_2) = \frac{i\tanh z_1 + i\tanh z_2}{1 \mp i\tanh z_1 \, i\tanh z_2} = \frac{i\tanh z_1 + i\tanh z_2}{1 \mp i^2 \tanh z_1 \tanh z_2}$$

$$\cancel{i}\tanh(z_1 \pm z_2) = \cancel{i}\,\frac{(\tanh z_1 + \tanh z_2)}{1 \pm \tanh z_1 \tanh z_2}$$

$$\tanh(z_1 \pm z_2) = \frac{\tanh z_1 \pm \tanh z_2}{1 \pm \tanh z_1 \tanh z_2}$$ proved

(x) $$\tanh 3z = \frac{3\tanh z + \tanh^3 z}{1 + 3\tanh^2 z}$$

$$\because \tan 3z = \frac{3\tan z - \tan^3 z}{1 - 3\tan^2 z}$$

replace z by iz

$$\tan(3iz) = \frac{3\tan(iz) - (\tan(iz))^3}{1 - 3(\tan(iz))^2}$$

$$= \frac{i3\tanh z - i^3 \tanh^3 z}{1 - 3i^2 \tanh^2 z}$$ $i^3 = i^2 . i = -i$

$$= \frac{i3\tanh z + i\tanh^3 z}{1 + 3\tanh^2 z}$$

$$\cancel{i}\tanh 3z = \cancel{i}\,\frac{(3\tanh z + \tanh^3 z)}{1 + 3\tanh^2 z}$$

$$\tanh 3z = \frac{3\tanh z + \tanh^3 z}{1 + 3\tanh^2 z}$$ Proved

7. if $\tan(\alpha+i\beta) = x+iy$, show that

$x^2+y^2+2x\cot 2\alpha = 1$ and $x^2+y^2-2y\coth 2\beta = -1$

Sol. $\tan(\alpha+i\beta) = x+iy \Rightarrow \alpha+i\beta = \tan^{-1}(x+iy) \rightarrow$ (i)

$\overline{\tan(\alpha+i\beta)} = \overline{x+iy}$

$\tan(\overline{\alpha+i\beta}) = x-iy = \tan(\alpha-i\beta)$

$\alpha-i\beta = \tan^{-1}(x-iy) \rightarrow$ (ii)

Adding (i) & (ii)

$\alpha+\cancel{i\beta}+\alpha-\cancel{i\beta} = \tan^{-1}(x+iy)+\tan^{-1}(x-iy)$

$2\alpha = \tan^{-1}(x+iy)+\tan^{-1}(x-iy)$ $\quad\tan^{-1}(A)+\tan^{-1}B = \tan^{-1}\left(\frac{A+B}{1-AB}\right)$

$2\alpha = \tan^{-1}\left[\frac{x+\cancel{iy}+x-\cancel{iy}}{1-(x+iy)(x-iy)}\right]$

$\tan(2\alpha) = \frac{2x}{1-(x^2+y^2)}$ $\quad\because (x+iy)(x-iy) = x^2-i^2y^2 = x^2+y^2$

$\frac{1}{\cot 2\alpha} = \frac{2x}{1-(x^2+y^2)}$

$1-(x^2+y^2) = 2x\cot 2\alpha$

$\Rightarrow (x^2+y^2+2x\cot 2\alpha = 1$ proved.

Subtracting (i) & (ii)

$\cancel{\alpha}+i\beta-\cancel{\alpha}+i\beta = \tan^{-1}(x+iy)-\tan^{-1}(x-iy)$

$2i\beta = \tan^{-1}\left(\frac{x+iy-x+iy}{1+(x+iy)(x-iy)}\right)$

$\tan(2i\beta) = \frac{2iy}{1+x^2+y^2}$

$\cancel{i}\tanh(2\beta) = \cancel{i}\frac{2y}{1+x^2+y^2}$

$\frac{1}{\coth(2\beta)} = \frac{2y}{1+x^2+y^2}$

$1+x^2+y^2 = 2y\coth(2\beta)$

$x^2+y^2-2y\coth(2\beta) = -1$ proved

8. if $\sin(\theta+i\phi) = \cos\alpha+i\sin\alpha$ ~~x+iy~~, show that $\cos^2\theta = \pm\sin\alpha$.

Sol. $\sin(\theta+i\phi) = \cancel{x+iy}\ \cos\alpha+i\sin\alpha$

$\sin\theta\cos(i\phi)+\cos\theta\sin(i\phi) = \cancel{x+iy}\ \cos\alpha+i\sin\alpha$

$\sin\theta\cosh\phi+i\cos\theta\sinh\phi = \cancel{x+iy}\ \cos\alpha+i\sin\alpha$

Equating real & imaginary parts

$\sin\theta\cosh\phi = \cos\alpha \qquad \cos\theta\sinh\phi = \sin\alpha$

$\cosh\phi = \frac{\cos\alpha}{\sin\theta} \rightarrow (i) \qquad \sinh\phi = \frac{\sin\alpha}{\cos\theta} \rightarrow (ii)$

Squaring & subtracting (i) & (ii)

$$\cosh^2\phi - \sinh^2\phi = \frac{\cos^2\alpha}{\sin^2\theta} - \frac{\sin^2\alpha}{\cos^2\theta}$$

$$1 = \frac{\cos^2\alpha\cos^2\theta - \sin^2\alpha\sin^2\theta}{\sin^2\theta\cos^2\theta}$$

$$1 = \frac{(1-\sin^2\alpha)\cos^2\theta - \sin^2\alpha(1-\cos^2\theta)}{\sin^2\theta\cos^2\theta}$$

$$\sin^2\theta\cos^2\theta = \cos^2\theta - \cancel{\cos^2\theta\sin^2\alpha} - \sin^2\alpha + \cancel{\sin^2\alpha\cos^2\theta}$$

$$(1-\cos^2\theta)\cos^2\theta = \cancel{\cos^2\theta} - \cos^4\theta = \cancel{\cos^2\theta} - \sin^2\alpha$$

$$-\cos^4\theta = -\sin^2\alpha$$

$$\therefore \cos^4\theta = \sin^2\alpha$$

$$\cos^2\theta = \pm\sin\alpha \quad \text{Proved}$$

Cos

9. if $\tan(\theta+i\phi) = \tan\alpha + i\sec\alpha$, prove that $e^{2\phi} = \pm\cot\frac{\alpha}{2}$ & $2\theta = n\pi + \frac{\pi}{2} + \alpha$

Sol

$$\tan(\theta+i\phi) = \tan\alpha + i\sec\alpha$$

$$\theta + i\phi = \tan^{-1}(\tan\alpha + i\sec\alpha) \rightarrow (i)$$

$$\tan(\theta - i\phi) = \tan\alpha - i\sec\alpha \qquad \because \overline{\tan z} = \tan\bar{z}$$

$$\theta - i\phi = \tan^{-1}(\tan\alpha - i\sec\alpha) \rightarrow (ii)$$

adding (i) and (ii)

$$2\theta = \tan^{-1}(\tan\alpha + i\sec\alpha) + \tan^{-1}(\tan\alpha - i\sec\alpha)$$

$$2\theta = \tan^{-1}\left[\frac{\tan\alpha + i\cancel{\sec\alpha} + \tan\alpha - i\cancel{\sec\alpha}}{1-(\tan\alpha+i\sec\alpha)(\tan\alpha - i\sec\alpha)}\right] \qquad \because \tan^{-1}A + \tan^{-1}B = \tan^{-1}\left[\frac{A+B}{1-AB}\right]$$

$$2\theta = \tan^{-1}\left[\frac{2\tan\alpha}{1-(\tan^2\alpha - i^2\sec^2\alpha)}\right] \qquad \because i^2 = -1$$

$$2\theta = \tan^{-1}\left[\frac{2\tan\alpha}{1-(\tan^2\alpha + \sec^2\alpha)}\right]$$

$$2\theta = \tan^{-1}\left[\frac{2\tan\alpha}{1-\tan^2\alpha - \sec^2\alpha}\right] \qquad \because \sec^2\alpha = 1+\tan^2\alpha$$

$$2\theta = \tan^{-1}\left[\frac{2\tan\alpha}{\cancel{1}-\tan^2\alpha - \cancel{1}-\tan^2\alpha}\right]$$

$$2\theta = \tan^{-1}\left(\frac{2\tan\alpha}{-2\tan^2\alpha}\right)$$

$$= \tan^{-1}\left(-\frac{1}{\tan\alpha}\right) \qquad \because \tan\left(\frac{\pi}{2}+\alpha\right) = -\cot\alpha$$

$$= \tan^{-1}(-\cot\alpha)$$

$$2\theta = \tan^{-1}\left[\tan\left(\frac{\pi}{2}+\alpha+\pi n\right)\right]$$

$$2\theta = \frac{\pi}{2}+\alpha+n\pi$$

(ii) Subtracting (i) & (ii)

$$2i\phi = \tan^{-1}(\tan\alpha + i\sec\alpha) - \tan^{-1}(\tan\alpha - i\sec\alpha)$$

$$2i\phi = \tan^{-1}\left[\frac{\tan\alpha + i\sec\alpha - (\tan\alpha - i\sec\alpha)}{1+(\tan\alpha+i\sec\alpha)(\tan\alpha - i\sec\alpha)}\right]$$

$$= \tan^{-1}\left[\frac{\cancel{\tan\alpha} + i\sec\alpha - \cancel{\tan\alpha} + i\sec\alpha}{1+\tan^2\alpha+\sec^2\alpha}\right]$$

$$= \tan^{-1}\left[\frac{2i\sec\alpha}{\sec^2\alpha+\sec^2\alpha}\right]$$

$$= \tan^{-1}\left[\frac{\cancel{2}i\cancel{\sec\alpha}}{\cancel{2}\sec^{\cancel{2}}\alpha}\right]$$

$$\tan(2i\phi) = \frac{i}{\sec\alpha}$$

$$\cancel{i}\tanh 2\phi = \cancel{i}\cos\alpha$$

$$\tanh 2\phi = \cos\alpha$$

$$\frac{e^{2\phi}-e^{-2\phi}}{e^{2\phi}+e^{-2\phi}} = \cos\alpha$$

$$\frac{e^{2\phi}+e^{-2\phi}}{e^{2\phi}-e^{-2\phi}} = \frac{1}{\cos\alpha}$$

Using componendo-dividendo Theorem.

$$\frac{e^{2\phi}+\cancel{e^{-2\phi}}+e^{2\phi}-\cancel{e^{-2\phi}}}{\cancel{e^{2\phi}}+e^{-2\phi}-\cancel{e^{2\phi}}+e^{-2\phi}} = \frac{1+\cos\alpha}{1-\cos\alpha}$$

$$\frac{\cancel{2}e^{2\phi}}{\cancel{2}e^{-2\phi}} = \frac{\cancel{2}\cos^2(\alpha/2)}{\cancel{2}\sin^2(\alpha/2)}$$

$$e^{2\phi+2\phi} = \cot^2\alpha/2$$

$$e^{4\phi} = \cot^2\alpha/2$$

$$e^{2\phi} = \pm\cot(\alpha/2) \qquad \text{Proved}$$

10. Prove that.

$$\sinh\left(\frac{x}{2}\right) = \sqrt{\frac{\cosh x - 1}{2}} \quad \text{if } x \geq 0$$

$$= -\sqrt{\frac{\cosh x - 1}{2}} \quad \text{if } x < 0$$

Sol:

$$\sqrt{\frac{\cosh x - 1}{2}} = \sqrt{\frac{\frac{e^x + e^{-x}}{2} - 1}{2}} = \sqrt{\frac{e^x + e^{-x} - 2}{4}}$$

$$= \sqrt{\frac{(e^{x/2} - e^{-x/2})^2}{(2)^2}}$$

$$= \sqrt{\left(\frac{e^{x/2} - e^{-x/2}}{2}\right)^2}$$

$\because (e^{x/2} - e^{-x/2})^2 = e^{2x/2} + e^{-2x/2} - 2e^{x/2}e^{-x/2}$

$= e^x + e^{-x} - 2$

$$\therefore \quad = \sqrt{\left(\sinh\left(\frac{x}{2}\right)\right)^2}$$

$$= \pm \sinh\left(\frac{x}{2}\right)$$

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$$\sinh\left(\frac{x}{2}\right) = \sqrt{\frac{\cosh x - 1}{2}} \quad \text{when } x \geq 0$$

$$\sinh\frac{x}{2} = -\sqrt{\frac{\cosh x - 1}{2}} \quad \text{when } x < 0$$

11. Show that multiplication of a vector z by $e^{i\alpha}$, where α is real no. rotates the vector z counterclockwise through an angle of measure α.

Sol. Let $Z = r[\cos\theta + i\sin\theta]$

$$Z = re^{i\theta}$$

$$Z(e^{i\alpha}) = re^{i\theta} \cdot r^{i\alpha}$$

$$Z(e^{i\alpha}) = re^{i(\theta+\alpha)}$$

$$= r\left[\cos(\theta+\alpha) + i\sin(\theta+\alpha)\right] \quad \text{proved.}$$

12. Show that

(i) $2+i = \sqrt{5}\, e^{i\tan^{-1}(1/2)}$

$Z = 2+i \quad , r = |Z| = \sqrt{4+1}$

$r = 5$

$\cos\theta = \frac{2}{\sqrt{5}} \quad , \sin\theta = \frac{1}{\sqrt{5}}$

Ist quardrant.

$\tan\theta = \frac{1/\sqrt{5}}{2/\sqrt{5}} = 1/2$

$\theta = \tan^{-1}(1/2)$

$Z = r(\cos\theta + i\sin\theta)$

$= \sqrt{5}\left[\cos\left(\tan^{-1}\left(\frac{1}{2}\right)\right) + i\sin\left(\tan^{-1}\left(\frac{1}{2}\right)\right)\right]$

$Z = \sqrt{5}\, e^{i\tan^{-1}(1/2)}$

(ii) $-3-4i = 5e^{i(\pi+\tan^{-1}(\frac{4}{3}))}$

$Z = -3-4i \quad , r = \sqrt{9+16} = \sqrt{25}$

$\boxed{r=5}$

$\cos\theta = -\frac{3}{5} \quad , \sin\theta = -\frac{4}{5}$

$\cancel{\tan}\theta = \pi+\alpha \qquad \tan\alpha = 4/3$

$\theta = \pi + \tan^{-1}\left(\frac{4}{3}\right) \qquad \alpha = \tan^{-1}\left(\frac{4}{3}\right)$

$Z = r(\cos\theta + i\sin\theta)$

$Z = 5\left[\cos\left(\pi+\tan^{-1}\left(\frac{4}{3}\right)\right) + i\sin\left(\pi+\tan^{-1}\left(\frac{4}{3}\right)\right)\right]$

$Z = 5e^{i(\pi+\tan^{-1}(4/3))}$