De Moivre's Theorem:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \qquad n \in Z$$

Proof:

For +ve integers, we use Mathematical Induction.

Case I: Put $n = 1$

$$(\cos\theta + i\sin\theta)^1 = \cos 1\theta + i\sin 1\theta$$
$$= \cos\theta + i\sin\theta.$$

This is true for $n = 1$.

Case 2: Suppose this is true for $n = k$ where $k \in Z^+$

$$(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$$

Multiply both sides by $(\cos\theta + i\sin\theta)$

$$(\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta) = (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$$

$$(\cos\theta + i\sin\theta)^{k+1} = \cos k\theta\cos\theta + i\cos k\theta\sin\theta + i\sin k\theta\cos\theta + i^2\sin k\theta\sin\theta$$

$$= (\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\cos k\theta\sin\theta + \sin k\theta\cos\theta)$$

$$= \cos(k\theta + \theta) + i\sin(k\theta + \theta)$$

$$(\cos\theta + i\sin\theta)^{k+1} = \cos(k+1)\theta + i\sin(k+1)\theta$$

$\Rightarrow$ This is true for $n = k+1$ where $k \in Z^+$

$\Rightarrow$ Theorem is valid for +ve integers.

For $n = 0$

$$(\cos\theta + i\sin\theta)^0 = \cos(0)\theta + i\sin(0)\theta$$
$$1 = 1 + i(0)$$
$$1 = 1 \qquad \text{True.}$$

For $n \in Z^-$

Let $n = -m$ where $m \in Z^+$

$$(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m} = \left[(\cos\theta + i\sin\theta)^m\right]^{-1}$$

$$= \left[\cos m\theta + i\sin m\theta\right]^{-1} \qquad \text{valid for } m \in Z^+$$

$$= \frac{1}{\cos m\theta + i\sin m\theta} \times \frac{\cos m\theta - i\sin m\theta}{\cos m\theta - i\sin m\theta}$$

$$= \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta = 1}$$

$$= \cos(-m\theta) + i\sin(-m\theta) \qquad \because \sin(-\theta) = -\sin\theta$$

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Hence Theorem is True for All integers

EXERCISE 1.2 (10)

Write each of the following expression in the form a+ib.

1. $(-\sqrt{3}+i)^2$

$z = -\sqrt{3}+i$

$x = -\sqrt{3}$, $y = 1$

$\therefore r = \sqrt{3+1} = 2$

$\cos\theta = \frac{-\sqrt{3}}{2}$, $\sin\theta = \frac{1}{2}$

θ is in IInd quardrant

refrence angle $= \pi/6$

$\theta = \pi - \pi/6 = 5\pi/6$

$z = r(\cos\theta + i\sin\theta)$

$z = 2\left[\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right]$

$(z)^2 = z^2\left[\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right]^2$

by de-Moivre's Theorem

$(-\sqrt{3}+i)^2 = 4\left[\cos\left(2\times\frac{5\pi}{6}\right) + i\sin\left(2\times\frac{5\pi}{6}\right)\right]$

$= 4\left[\cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)\right]$

$= 4\left[\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right]$

$= \frac{4}{2}(1-\sqrt{3}i)$

$= 2(1-\sqrt{3}i)$

$(-\sqrt{3}+i)^2 = 2 - 2\sqrt{3}i$

(ii) $(-3i)^4 = (-3)^4(i)^4$

$= 81(i^2)^2$

$= 81(-1)^2$

$(-3i)^4 = 81$

(iii) $\left(\frac{1-\sqrt{3}i}{1+\sqrt{3}i}\right)^6$

here $z = \frac{1-\sqrt{3}i}{1+\sqrt{3}i} \times \frac{1-\sqrt{3}i}{1-\sqrt{3}i}$

$z = \frac{(1-\sqrt{3}i)^2}{1+3} = \frac{(1-\sqrt{3}i)^2}{4}$

$z^6 = \left[\frac{(1-\sqrt{3}i)^2}{4}\right]^6$

$\left(\frac{1-\sqrt{3}i}{1+\sqrt{3}i}\right)^6 = \frac{(1-\sqrt{3}i)^{12}}{4096} = \frac{z_1}{4096}$

Let $z_1 = 1-\sqrt{3}i$

$|z_1| = \sqrt{1+3} = 2$

$\cos\theta = 1/2$, $\sin\theta = -\sqrt{3}/2$

IV IInd quardrant.

refrence Angle $= \alpha = \pi/3$

$\theta = -\frac{\pi}{3}$ $\because \theta = -\alpha$

$z_1 = r[\cos\theta + i\sin\theta]$

$z_1 = 2\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right]$

$(z_1)^{12} = (2)^{12}\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right]^{12}$

$(z_1)^{12} = 4096\left[\cos\left(-\frac{12\pi}{3}\right) + i\sin\left(-\frac{12\pi}{3}\right)\right]$

$\frac{(z_1)^{12}}{4096} = \cos(-4\pi) + i\sin(-4\pi)$

$\left(\frac{1-\sqrt{3}i}{1+\sqrt{3}i}\right)^6 = 1+0 = 1$

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2. Simplify.

(i) $$\frac{(\cos 2\theta + i\sin 2\theta)^5(\cos 3\theta - i\sin 3\theta)^6}{(\cos 4\theta - i\sin 4\theta)^7(\cos 5\theta + i\sin 5\theta)^8}$$

$$= \frac{(\cos 2\theta + i\sin 2\theta)^5(\cos(-3\theta) + i\sin(-3\theta))^6}{(\cos(-4\theta) + i\sin(-4\theta))^7(\cos 5\theta + i\sin 5\theta)^8}$$

$\because \cos(-\theta) = \cos\theta$, $\sin(-\theta) = -\sin\theta$

$$= \frac{(\cos 10\theta + i\sin 10\theta)(\cos(-18\theta) + i\sin(-18\theta))}{(\cos(-28\theta) + i\sin(-28\theta))(\cos 40\theta + i\sin 40\theta)}$$

$$= \frac{\text{Cis}(10\theta)\ \text{Cis}(-18\theta)}{\text{Cis}(-28\theta)\ \text{Cis}(40\theta)}$$

$\because \frac{\text{Cis}(\alpha)}{\text{Cis}(\beta)} = \text{Cis}(\alpha-\beta)$ & $\text{Cis}(\alpha).\text{Cis}(\beta) = \text{Cis}(\alpha+\beta)$

$$= \text{Cis}(10 - 18 + 28 \pm 40)\theta$$

$$= \text{Cis}(-20)\theta$$

$$= \cos(-20)\theta + i\sin(-20)\theta$$

$$= \cos 20\theta - i\sin 20\theta.$$

(ii) $$\frac{(\cos\alpha - i\sin\alpha)^{11}}{(\cos\beta + i\sin\beta)^9} = \frac{(\cos(-\alpha) + i\sin(-\alpha))^{11}}{(\cos\beta + i\sin\beta)^9}$$

$$= \frac{\cos(-11\alpha) + i\sin(-11\alpha)}{\cos(9\beta) + i\sin(9\beta)}$$

$$= \frac{\text{Cis}(-11\alpha)}{\text{Cis}(9\beta)}$$

$$= \text{Cis}(-11\alpha - 9\beta)$$

$$= \cos(-(11\alpha + 9\beta)) + i\sin(-(11\alpha + 9\beta))$$

$$= \cos(11\alpha + 9\beta) - i\sin(11\alpha + 9\beta)$$

(iii) $$\frac{(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)}{(\cos\gamma + i\sin\gamma)(\cos\delta + i\sin\delta)} = \frac{\text{Cis}\alpha \cdot \text{Cis}\beta}{\text{Cis}\gamma \cdot \text{Cis}\delta}$$

$$= \text{Cis}(\alpha + \beta - \gamma - \delta)$$

(iv) $$\frac{(3\,\text{Cis}\,\pi/6)^7}{(4\,\text{Cis}\,\pi/3)^5} = \frac{3^7}{4^5}\ \frac{\text{Cis}(7\pi/6)}{\text{Cis}(5\pi/3)}$$

$$= \frac{3^7}{4^5}\ \text{Cis}\left(\frac{7\pi}{6} - \frac{5\pi}{3}\right)$$

$$= \frac{3^7}{4^5}\ \text{Cis}\left(-\frac{\pi}{2}\right)$$

$$\frac{7\pi}{6} - \frac{5\pi}{3} = \frac{7\pi - 10\pi}{6} = \frac{-3\pi}{6} = -\frac{\pi}{2}$$

3. Prove that

(i) $\left[(\cos\theta-\cos\phi)+i(\sin\theta-\sin\phi)\right]^n+\left[(\cos\theta-\cos\phi)-i(\sin\theta-\sin\phi)\right]^n$

$$= 2^{n+1}\sin^n\left(\frac{\theta-\phi}{2}\right)\cos n\left(\frac{\theta+\phi+\pi}{2}\right) \quad (A)$$

Proof: Let $\cos\theta-\cos\phi = r\cos\alpha \to (i)$, $\sin\theta-\sin\phi = r\sin\alpha \to (ii)$

Squaring & adding (i) & (ii)

$$(\cos\theta-\cos\phi)^2+(\sin\theta-\sin\phi)^2 = r^2\cos^2\alpha+r^2\sin^2\alpha$$

$$\cos^2\theta+\cos^2\phi-2\cos\theta\cos\phi+\sin^2\theta+\sin^2\phi-2\sin\theta\sin\phi = r^2(\sin^2\alpha+\cos^2\alpha)$$

$$(\cos^2\theta+\sin^2\theta)+(\cos^2\phi+\sin^2\phi)-2(\cos\theta\cos\phi-\sin\theta\sin\phi) = r^2(1)$$

$$1+1-2\cos(\theta-\phi) = r^2$$

$$2-2\cos(\theta-\phi) = r^2$$

$$2(1-\cos(\theta-\phi)) = r^2 \qquad \because \sqrt{\frac{1-\cos\theta}{2}} = \sin\theta/2$$

$$2\left(2\sin^2\left(\frac{\theta-\phi}{2}\right)\right) = r^2 \qquad \Rightarrow \frac{1-\cos\theta}{2} = \sin^2\theta/2$$

$$r^2 = 4\sin^2\left(\frac{\theta-\phi}{2}\right) \qquad \Rightarrow 1-\cos\theta = 2\sin^2\theta/2$$

Square Root

$$\Rightarrow r = 2\sin\left(\frac{\theta-\phi}{2}\right) \longrightarrow (iii)$$

Dividing (ii) by (i)

$$\frac{\cancel{r}\sin\alpha}{\cancel{r}\cos\alpha} = \frac{\sin\theta-\sin\phi}{\cos\theta-\cos\phi}$$

$$\tan\alpha = \frac{2\cos\left(\frac{\theta+\phi}{2}\right)\cancel{\sin\left(\frac{\theta-\phi}{2}\right)}}{-\cancel{2}\sin\left(\frac{\theta+\phi}{2}\right)\cancel{\sin\left(\frac{\theta-\phi}{2}\right)}}$$

$$\tan\alpha = -\cot\left(\frac{\theta+\phi}{2}\right)$$

$$\tan\alpha = \tan\left(\frac{\pi}{2}+\frac{\theta+\phi}{2}\right) \qquad \because \tan\left(\frac{\pi}{2}+\alpha\right) = -\cot\alpha$$

$$\alpha = \frac{\pi}{2}+\frac{\theta+\phi}{2}$$

$$\alpha = \frac{\pi+\theta+\phi}{2} \rightarrow (iv)$$

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from (A)

$$\text{L.H.S} = (r\cos\alpha+ir\sin\alpha)^n+(r\cos\alpha-ir\sin\alpha)^n$$

$$= r^n(\cos\alpha+i\sin\alpha)^n+r^n(\cos\alpha-i\sin\alpha)^n \qquad \text{De-Moivre's Theorem}$$

$$= r^n\left[(\cos n\alpha+i\sin n\alpha)+(\cos(-\alpha)+i\sin(-\alpha))^n\right]$$

$$= r^n\left[\cos n\alpha+i\sin n\alpha+(\cos(-n\alpha)+i\sin(-n\alpha))\right]$$

$$= r^n\left[\cos n\alpha+\cancel{i\sin n\alpha}+\cos n\alpha-\cancel{i\sin n\alpha}\right]$$

$$= r^n\left[2\cos n\alpha\right]$$

$$= \left[2\sin\left(\frac{\theta-\phi}{2}\right)\right]^n 2\cos n\left(\frac{\pi+\theta+\phi}{2}\right) \qquad \text{from (iii) \& (iv)}$$

$$= 2^n \sin^n\left(\frac{\theta-\phi}{2}\right) 2 \cos n\left(\frac{\theta+\phi+\pi}{2}\right)$$

$$= 2^{n+1} \sin^n\left(\frac{\theta-\phi}{2}\right) \cos n\left(\frac{\theta+\phi+\pi}{2}\right) = \text{R.H.S proved}$$

(ii) $$\left(\frac{1+\sin x + i\cos x}{1+\sin x - i\cos x}\right)^n = \cos n\left(\frac{\pi}{2}-x\right) + i\sin n\left(\frac{\pi}{2}-x\right)$$

$$\frac{1+\sin x + i\cos x}{1+\sin x - i\cos x} = \frac{(1+\sin x)+i(\cos x)}{(1+\sin x)-i(\cos x)} \times \frac{(1+\sin x)+i(\cos x)}{(1+\sin x)+i(\cos x)}$$

$$= \frac{((1+\sin x)+i\cos x)^2}{(1+\sin x)^2-(i\cos x)^2}$$

$$= \frac{(1+\sin x)^2+(i\cos x)^2+2(1+\sin x)(i\cos x)}{1+\sin^2 x+2\sin x - i^2\cos^2 x}$$

$\because i^2=-1$

$$= \frac{(1+\sin x)^2-\cos^2 x+2(1+\sin x)(i\cos x)}{1+\sin^2 x+2\sin x+\cos^2 x}$$

$\because \cos^2 x+\sin^2 x=1$

$\therefore \cos^2 x = 1-\sin^2 x$

$$= \frac{(1+\sin x)^2-(1-\sin^2 x)+2(1+\sin x)(i\cos x)}{(1+\sin x)^2+(1-\sin^2 x)}$$

$\because a^2-b^2=(a+b)(a-b)$

$$= \frac{(1+\sin x)^2-(1-\sin x)(1+\sin x)+2(1+\sin x)(i\cos x)}{(1+\sin x)^2+(1-\sin x)(1+\sin x)}$$

$$= \frac{\cancel{(1+\sin x)}\left[1+\sin x-(1-\sin x)+2i\cos x\right]}{\cancel{(1+\sin x)}\left[(1+\cancel{\sin x})+(1-\cancel{\sin x})\right]}$$

$$= \frac{\cancel{1}+\sin x-\cancel{1}+\sin x+2i\cos x}{2}$$

$$= \frac{2\sin x+2i\cos x}{2}$$

$$= \frac{\cancel{2}(\sin x+i\cos x)}{\cancel{2}}$$

$$\frac{1+\sin x+i\cos x}{1+\sin x-i\cos x} = i\cos x+\sin x$$

Allide Angles

$\therefore \cos\left(\frac{\pi}{2}-x\right)=\sin x$

$\&\ \sin\left(\frac{\pi}{2}-x\right)=\cos x$

$$\left(\frac{1+\sin x+i\cos x}{1+\sin x-i\cos x}\right)^n = (\sin x+i\cos x)^n$$

$$= \left[\cos\left(\frac{\pi}{2}-x\right)+i\sin\left(\frac{\pi}{2}-x\right)\right]^n$$

De-Moivre's Theorem

$$= \cos n\left(\frac{\pi}{2}-x\right)+i\sin n\left(\frac{\pi}{2}-x\right) = \text{R.H.S}$$

PROVED

4. if $2\cos\theta = x+\frac{1}{x}$, $2\cos\phi = y+\frac{1}{y}$, $2\cos\psi = z+\frac{1}{z}$ then prove that

(i) $2\cos(\theta+\phi+\psi) = xyz + \frac{1}{x+y+z}$

Sol.

Let $x = \cos\theta + i\sin\theta = Cis(\theta)$

$\frac{1}{x} = \cos\theta - i\sin\theta = Cis(-\theta)$

$\Rightarrow x + \frac{1}{x} = 2\cos\theta$

$\because \frac{1}{x} = \frac{1}{\cos\theta + i\sin\theta}$

$= \frac{1}{\cos\theta + i\sin\theta} \times \frac{\cos\theta - i\sin\theta}{\cos\theta - i\sin\theta}$

$= \frac{\cos\theta - i\sin\theta}{\cos^2\theta + \sin^2\theta}$

$= \cos\theta - i\sin\theta$

Similarly,

$y = \cos\phi + i\sin\phi = Cis(\phi)$

$\frac{1}{y} = \cos\phi - i\sin\phi = Cis(-\phi)$

& also $z = \cos\psi + i\sin\psi = Cis(\psi)$

$1/z = \cos\psi - i\sin\psi = Cis(-\psi)$

$xyz = Cis(\theta)\ Cis(\phi)\ Cis(\psi)$

$xyz = Cis(\theta+\phi+\psi) \rightarrow ①$

and $\frac{1}{x}\cdot\frac{1}{y}\cdot\frac{1}{z} = Cis(-\theta)\ Cis(-\phi)\ Cis(-\psi)$

$\frac{1}{xyz} = Cis(-(\theta+\phi+\psi)) \rightarrow ②$

adding ① & ②

$xyz + \frac{1}{xyz} = Cis(\theta+\phi+\psi) + Cis(-(\theta+\phi+\psi))$

$= \cos(\theta+\phi+\psi) + i\sin(\theta+\phi+\psi) + \cos(-(\theta+\phi+\psi)) + i\sin(-(\theta+\phi+\psi))$

$= \cos(\theta+\phi+\psi) + i\sin(\cancel{\theta+\phi+\psi}) + \cos(\theta+\phi+\psi) - i\sin(\cancel{\theta+\phi+\psi})$

$xyz + \frac{1}{xyz} = 2\cos(\theta+\phi+\psi)$ Proved.

(ii) $x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\theta + n\phi)$

$x = \cos\theta + i\sin\theta$

$x^m = (\cos\theta + i\sin\theta)^m$

$x^m = \cos m\theta + i\sin m\theta$

$x^m = Cis(m\theta)$

$\frac{1}{x} = \cos\theta - i\sin\theta = \cos(-\theta) + i\sin(-\theta)$

$\frac{1}{x^m} = (\cos(-\theta) + i\sin(-\theta))^m$

$\frac{1}{x^m} = \cos(-m\theta) + i\sin(-m\theta)$

$\frac{1}{x^m} = Cis(-m\theta)$

&

$y = \cos\phi + i\sin\phi$

$y^n = (\cos\phi + i\sin\phi)^n$

$= \cos n\phi + i\sin n\phi$

$y^n = Cis(n\phi)$

$1/y = \cos\phi - i\sin\phi = \cos(-\phi) + i\sin(-\phi)$

$1/y^n = (\cos(-\phi) + i\sin(-\phi))^n$

$= \cos(-n\phi) + i\sin(-n\phi)$

$1/y^n = Cis(-n\phi)$

$x^m y^n = Cis(m\theta)Cis(n\phi)$

$x^m y^n = Cis(m\theta + n\phi) \rightarrow (1)$

$\frac{1}{x^m}\cdot\frac{1}{y^n} = Cis(-m\theta)\,Cis(-n\phi)$

$= Cis(-(m\theta+n\phi)) \rightarrow (2)$

adding (1) & (2)

$$x^m y^n + \frac{1}{x^m y^n} = Cis(m\theta+n\phi) + Cis(-(m\theta+n\phi))$$

$$= Cos(m\theta+n\phi) + iSin(m\theta+n\phi) + Cos(-(m\theta+n\phi)) + iSin(-(m\theta+n\phi))$$

$$= Cos(m\theta+n\phi) + \cancel{iSin(m\theta+n\phi)} + Cos(m\theta+n\phi) - \cancel{iSin(m\theta+n\phi)}$$

$$x^m y^n + \frac{1}{x^m y^n} = 2Cos(m\theta+n\phi)$$

5. Find the three cube roots of 8i.

$\because z = 8i$ here $x=0$, $y=8$, $|z| = r = \sqrt{0+8^2} = 8$

$Cos\theta = 0$, $Sin\theta = 1$ $\quad \because Cos\theta = x/r$, $Sin\theta = y/r$

Here θ is quardrantal Angle
& $\theta = 90° = \pi/2$

formula for finding n-Roots

$$z_k = |z|^{1/n}\left[Cos\left(\frac{\theta+2k\pi}{n}\right) + iSin\left(\frac{\theta+2k\pi}{n}\right)\right], \quad k = 0,1,2,\dots,n-1$$

for 3-roots.

$$z_k = |z|^{1/3}\left[Cos\left(\frac{\theta+2k\pi}{3}\right) + i\,Sin\left(\frac{\theta+2k\pi}{3}\right)\right], \quad k = 0,1,2$$

for $8i$, $\theta = \pi/2$ & $|z| = 8$

$$z_k = [8]^{1/3}\left[Cos\left(\frac{\pi/2 + 2k\pi}{3}\right) + i\,Sin\left(\frac{\pi/2+2k\pi}{3}\right)\right], \quad k = 0,1,2$$

$$z_k = 2\left[Cos\left(\frac{\pi+4k\pi}{6}\right) + i\,Sin\left(\frac{\pi+4k\pi}{6}\right)\right], \quad k = 0,1,2$$

For $k=0$

$$z_0 = 2\left[Cos\left(\frac{\pi+0}{6}\right) + i\,Sin\left(\frac{\pi+0}{6}\right)\right]$$

$$= 2\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$$

$$z_0 = \sqrt{3} + i$$

For $k=1$

$$z_1 = 2\left[Cos\left(\frac{\pi+4\pi}{6}\right) + i\,Sin\left(\frac{\pi+4\pi}{6}\right)\right]$$

$$z_1 = 2\left[Cos\left(\frac{5\pi}{6}\right) + i\,Sin\left(\frac{5\pi}{6}\right)\right]$$

$$z_1 = 2\left[-\frac{\sqrt{3}}{2} + \frac{i}{2}\right]$$

$$z_1 = -\sqrt{3} + i$$

For $k=2$

$$z_2 = 2\left[\cos\left(\frac{\pi+8\pi}{6}\right) + i\sin\left(\frac{\pi+8\pi}{6}\right)\right]$$

$$= 2\left[\cos\left(\frac{9\pi}{6}\right) + i\sin\left(\frac{9\pi}{6}\right)\right]$$

$$= 2\left[\cos\left(3\cdot\frac{\pi}{2}\right) + i\sin\left(3\cdot\frac{\pi}{2}\right)\right]$$

$$= 2\left(0 + i(-1)\right)$$

$$= -2i$$

Three roots of $8i$ are $\sqrt{3}+i$, $-\sqrt{3}+i$, $-2i$

Find four fourth roots of each of the complex numbers.

(i) $-16i$ here $x=0$, $y=-16$, $r=|z|=16$

$\cos\theta = 0$, $\sin\theta = -1$

θ is quardrantal angle & $\theta = -\pi/2$

$$z_k = |z|^{1/4}\left[\cos\left(\frac{\theta+2k\pi}{4}\right) + i\sin\left(\frac{\theta+2k\pi}{4}\right)\right]; \quad k=0,1,2,3$$

$$z_k = (16)^{1/4}\left[\cos\left(\frac{-\pi/2+2k\pi}{4}\right) + i\sin\left(-\frac{\pi/2+2k\pi}{4}\right)\right]; \quad k=0,1,2,3$$

$$z_k = 2\left[\cos\left(\frac{-\pi+4k\pi}{8}\right) + i\sin\left(\frac{-\pi+4k\pi}{8}\right)\right]; \quad k=0,1,2,3.$$

for $k=0$

$$z_0 = 2\left[\cos\left(\frac{-\pi}{8}\right) + i\sin\left(-\frac{\pi}{8}\right)\right]$$

for $k=1$

$$z_1 = 2\left[\cos\left(\frac{-\pi+4\pi}{8}\right) + i\sin\left(\frac{-\pi+4\pi}{8}\right)\right]$$

$$z_1 = 2\left[\cos\left(\frac{3\pi}{8}\right) + i\sin\left(\frac{3\pi}{8}\right)\right]$$

for $k=2$

$$z_2 = 2\left[\cos\left(\frac{-\pi+8\pi}{8}\right) + i\sin\left(\frac{-\pi+8\pi}{8}\right)\right]$$

$$z_2 = 2\left[\cos\left(\frac{7\pi}{8}\right) + i\sin\left(\frac{7\pi}{8}\right)\right]$$

for $k=3$

$$z_3 = 2\left[\cos\left(\frac{-\pi+12\pi}{8}\right) + i\sin\left(\frac{-\pi+12\pi}{8}\right)\right]$$

$$z_3 = 2\left[\cos\left(\frac{11\pi}{8}\right) + i\sin\left(\frac{11\pi}{8}\right)\right]$$

(ii) 64.

here $x = 64$, $y=0$, $|z| = 64$

$\cos\theta = \frac{x}{r} = 1$, $\sin\theta = y/r = 0$

θ is quardrantal angle

& $\theta = 0^\circ$

$$z_k = (64)^{1/4}\left[\cos\left(\frac{0+2k\pi}{4}\right) + i\sin\left(\frac{0+2k\pi}{4}\right)\right]; \quad k=0,1,2,3$$

$$(4^2)^{1/4}\, 4^{1/4}$$
$$= (2^4)^{1/4}\,(2^2)^{1/4}$$
$$= 2.\sqrt{2}$$

$$Z_K = (4^2.4)^{1/4}\left[\cos\left(\frac{k\pi}{2}\right) + i\sin\left(\frac{k\pi}{2}\right)\right] \quad ; k=0,1,2,3$$
$$= 2\sqrt{2}\left[\cos\left(\frac{k\pi}{2}\right) + i\sin\left(\frac{k\pi}{2}\right)\right] \quad ; k=0,1,2,3$$

for $k=0$

$$Z_0 = 2\sqrt{2}\,[\underbrace{\cos(0) + i\sin(0)}_{(1+0i)}]$$
$$Z_0 = 2\sqrt{2}$$

for $k=1$

$$Z_1 = 2\sqrt{2}\,[\underbrace{\cos(\pi/2) + i\sin(\pi/2)}_{(0+i)}]$$
$$Z_1 = 2\sqrt{2}(i)$$

for $k=2$

$$Z_2 = 2\sqrt{2}\,[\underbrace{\cos\pi + i\sin\pi}_{(-1+0i)}]$$
$$Z_2 = -2\sqrt{2}$$

for $k=3$

$$Z_3 = 2\sqrt{2}\,[\underbrace{\cos\tfrac{3\pi}{2} + i\sin\tfrac{3\pi}{2}}_{(0-i)}]$$
$$Z = -2\sqrt{2}i$$

Forth roots of 64 are $\pm 2\sqrt{2}$, $\pm 2\sqrt{2}i$

(iii) $-2\sqrt{3} + 2i$

here $x = -2\sqrt{3}$, $y = 2$

$$|Z| = \sqrt{x^2+y^2} = \sqrt{4(3)+4} = \sqrt{12+4} = \sqrt{16} = 4 \Rightarrow \boxed{|Z| = 4}$$

$$\cos\theta = \frac{x}{r} = \frac{-2\sqrt{3}}{4} = -\frac{\sqrt{3}}{2}$$
$$\sin\theta = \frac{y}{r} = \frac{2}{4} = \frac{1}{2}$$

θ lies in IInd quardrant.
refrence Angle $= \alpha = \pi/6$
$\theta = \pi - \alpha = \pi - \frac{\pi}{6} \Rightarrow \theta = \frac{5\pi}{6}$

formula

$$Z_K = (4)^{1/4}\left[\cos\left(\frac{5\pi/6 + 2k\pi}{4}\right) + i\sin\left(\frac{5\pi/6 + 2k\pi}{4}\right)\right], k=0,1,2,3$$
$$= (2)^{2\times 1/4}\left[\cos\left(\frac{5\pi + 12k\pi}{24}\right) + i\sin\left(\frac{5\pi+12k\pi}{24}\right)\right], k=0,1,2,3$$

For $k=0$

$$Z_0 = \sqrt{2}\left[\cos\left(\frac{5\pi}{24}\right) + i\sin\left(\frac{5\pi}{24}\right)\right]$$

$$Z_1 = \sqrt{2}\left[\cos\left(\frac{5\pi+12\pi}{24}\right) + i\sin\left(\frac{5\pi+12\pi}{24}\right)\right] = \sqrt{2}\left[\cos\left(\frac{17\pi}{24}\right) + i\sin\left(\frac{17\pi}{24}\right)\right]$$

$$Z_2 = \sqrt{2}\left[\cos\left(\frac{5\pi+24\pi}{24}\right) + i\sin\left(\frac{5\pi+24\pi}{24}\right)\right] = \sqrt{2}\left[\cos\left(\frac{29\pi}{24}\right) + i\sin\left(\frac{29\pi}{24}\right)\right]$$

$$Z_3 = \sqrt{2}\left[\cos\left(\frac{5\pi+36\pi}{24}\right) + i\sin\left(\frac{5\pi+36\pi}{24}\right)\right] = \sqrt{2}\left[\cos\left(\frac{41\pi}{24}\right) + i\sin\left(\frac{41\pi}{24}\right)\right]$$

6. Find six 6th roots of.

(i) -1 $\qquad z_1^6 = -1$

$z = -1 + 0i \qquad r = \sqrt{1+0} = 1$

$x = -1 \quad , y = 0$

$\cos\theta = -1 \quad , \quad \sin\theta = 0$

quardrantal angle $\theta = \pi$

$$z_k = (1)^{1/6}\left[\cos\left(\frac{\pi+2k\pi}{6}\right) + i\sin\left(\frac{\pi+2k\pi}{6}\right)\right], \; k = 0,1,2,3,4,5,6$$

For $k=0$

$$z_0 = \cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + i\frac{1}{2}$$

$$z_1 = \cos\left(\frac{\pi+2\pi}{6}\right) + i\sin\left(\frac{\pi+2\pi}{6}\right) = \cos\left(\overset{\pi/2}{\frac{3\pi}{6}}\right) + i\sin\left(\frac{3\pi}{6}\right) = 0 + i$$

$$z_2 = \cos\left(\frac{\pi+4\pi}{6}\right) + i\sin\left(\frac{\pi+4\pi}{6}\right) = \cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2} + i\frac{1}{2}$$

$$z_3 = \cos\left(\frac{\pi+6\pi}{6}\right) + i\sin\left(\frac{\pi+6\pi}{6}\right) = \cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2} - i\frac{1}{2}$$

$$z_4 = \cos\left(\frac{\pi+8\pi}{6}\right) + i\sin\left(\frac{\pi+8\pi}{6}\right) = \cos\left(\overset{3\pi/2}{\frac{9\pi}{6}}\right) + i\sin\left(\frac{9\pi}{6}\right) = 0 - 1i$$

$$z_5 = \cos\left(\frac{\pi+10\pi}{6}\right) + i\sin\left(\frac{\pi+10\pi}{6}\right) = \cos\left(\frac{11\pi}{6}\right) + i\sin\left(\frac{11\pi}{6}\right) = \frac{\sqrt{3}}{2} - i\frac{1}{2}$$

(ii) $1 + i = z$

Here $x = 1 \quad , y = 1 \quad , |z| = \sqrt{1+1} = \sqrt{2}$

$\cos\theta = \frac{1}{\sqrt{2}} \quad , \quad \sin\theta = \frac{1}{\sqrt{2}}$

θ Lies in 1st quardrant

$\alpha = \theta = \pi/4$

$$z_k = (\sqrt{2})^{1/6}\left[\cos\left(\frac{\pi/4 + 2k\pi}{6}\right) + i\sin\left(\frac{\pi/4 + 2k\pi}{6}\right)\right], k = 0,1,2,3,4,5$$

$$= (2)^{1/12}\left[\cos\left(\frac{\pi + 8k\pi}{24}\right) + i\sin\left(\frac{\pi+8k\pi}{24}\right)\right], k = 0,1,2,3,4,5$$

$$z_0 = (2)^{1/12}\left[\cos\left(\frac{\pi}{24}\right) + i\sin\left(\frac{\pi}{24}\right)\right] = (2)^{1/12}\,\mathrm{Cis}(\pi/24)$$

$$z_1 = (2)^{1/12}\left[\cos\left(\frac{\pi+8\pi}{24}\right) + i\sin\left(\frac{\pi+8\pi}{24}\right)\right] = (2)^{1/12}\,\mathrm{Cis}\left(\frac{9\pi}{24}\right)$$

$$z_2 = (2)^{1/12}\left[\cos\left(\frac{\pi+16\pi}{24}\right) + i\sin\left(\frac{\pi+16\pi}{24}\right)\right] = (2)^{1/12}\,\mathrm{Cis}\left(\frac{17\pi}{24}\right)$$

$$z_3 = (2)^{1/12}\left[\cos\left(\frac{\pi+24\pi}{24}\right) + i\sin\left(\frac{\pi+24\pi}{24}\right)\right] = (2)^{1/12}\,\mathrm{Cis}\left(\frac{25\pi}{24}\right)$$

$$z_4 = (2)^{1/12}\left[\cos\left(\frac{\pi+32\pi}{24}\right) + i\sin\left(\frac{\pi+32\pi}{24}\right)\right] = (2)^{1/12}\,\mathrm{Cis}\left(\frac{33\pi}{24}\right)$$

$$z_5 = (2)^{1/12}\left[\cos\left(\frac{\pi+40\pi}{24}\right) + i\sin\left(\frac{\pi+40\pi}{24}\right)\right] = (2)^{1/12}\,\mathrm{Cis}\left(\frac{41\pi}{24}\right)$$

7. Find the squares of all 5th roots of $\frac{1}{2}+\frac{\sqrt{3}}{2}i$

Sol. $z = \frac{1}{2} + \frac{\sqrt{3}}{2}$, $|z| = \sqrt{\frac{1}{4}+\frac{3}{4}} = \sqrt{\frac{4}{4}} \Rightarrow \boxed{|z| =}$

$\cos\theta = \frac{1}{2}$, $\sin\theta = \frac{\sqrt{3}}{2}$

θ is in 1st quardrant, $\theta = \alpha = \pi/3$

for five roots.

$$z_k = (1)^{1/5}\left[\cos\left(\frac{\pi/3+2k\pi}{5}\right)+i\sin\left(\frac{\pi/3+2k\pi}{5}\right)\right], \; k=0,1,2,3,4$$

$$z_k = \left[\cos\left(\frac{\pi+6k\pi}{15}\right)+i\sin\left(\frac{\pi+6k\pi}{15}\right)\right], \; k=0,1,2,3,4$$

Taking square on both side ∵(we find squares of fifth roots)

$$z_k^2 = \left[\cos\left(\frac{\pi+6k\pi}{15}\right)+i\sin\left(\frac{\pi+6k\pi}{15}\right)\right]^2$$

by De-Moivre's Theorem

$$z_k^2 = \cos 2\left(\frac{\pi+6k\pi}{15}\right)+i\sin 2\left(\frac{\pi+6k\pi}{16}\right)$$

$$z_k^2 = \cos\left(\frac{2\pi+12k\pi}{15}\right)+i\sin\left(\frac{2\pi+12k\pi}{15}\right)$$

for $k=0$.

$$z_0^2 = \cos\left(\frac{2\pi}{15}\right)+i\sin\left(\frac{2\pi}{15}\right) = \text{Cis}\left(\frac{2\pi}{15}\right)$$

for $k=1$.

$$z_1^2 = \cos\left(\frac{2\pi+12\pi}{15}\right)+i\sin\left(\frac{2\pi+12\pi}{15}\right) = \text{Cis}\left(\frac{14\pi}{15}\right)$$

for $k=2$.

$$z_2^2 = \cos\left(\frac{2\pi+24\pi}{15}\right)+i\sin\left(\frac{2\pi+24\pi}{15}\right) = \text{Cis}\left(\frac{26\pi}{15}\right)$$

for $k=3$.

$$z_3^2 = \cos\left(\frac{2\pi+36\pi}{15}\right)+i\sin\left(\frac{2\pi+36\pi}{15}\right) = \text{Cis}\left(\frac{38\pi}{15}\right)$$

for $k=4$.

$$z_4^2 = \cos\left(\frac{2\pi+48\pi}{15}\right)+i\sin\left(\frac{2\pi+48\pi}{15}\right) = \text{Cis}\left(\frac{50\pi}{15}\right)$$

8. Solve the following equations.

(i) $x^7+1=0$

$$x^7 = (-1)$$

$$x = (-1)^{1/7}$$

Let $z = -1+0i$

∴ $x = -1$, $y = 0$ $\quad |z| = 1$

$$\left.\begin{aligned}\cos\theta &= \frac{x}{r} = -1\\ \sin\theta &= y/r = 0\end{aligned}\right\}$$ quardrantal Angle. $\theta = \pi$

$$Z_k = (1)^{1/7}\left[\cos\left(\frac{\pi+2k\pi}{7}\right) + i\sin\left(\frac{\pi+2k\pi}{7}\right)\right], k = 0,1,2,3,4,5,6$$

for $k=0$

$$Z_0 = \left[\cos\left(\frac{\pi}{7}\right) + i\sin\left(\frac{\pi}{7}\right)\right] = Cis\left(\frac{\pi}{7}\right)$$

for $k=1$.

$$Z_1 = \left[\cos\left(\frac{\pi+2\pi}{7}\right) + i\sin\left(\frac{\pi+2\pi}{7}\right)\right] = Cis\left(\frac{3\pi}{7}\right)$$

for $k=2$

$$Z_2 = \left[\cos\left(\frac{\pi+4\pi}{7}\right) + i\sin\left(\frac{\pi+4\pi}{7}\right)\right] = Cis\left(\frac{5\pi}{7}\right)$$

for $k=3$

$$Z_3 = \left[\cos\left(\frac{\pi+6\pi}{7}\right) + i\sin\left(\frac{\pi+6\pi}{7}\right)\right] = \cos(\pi) + i\sin(\pi)$$

$\frac{7\pi}{7} = \pi$

$$Z_3 = -1 + 0i = -1$$

for $k=4$

$$Z_4 = \left[\cos\left(\frac{\pi+8\pi}{7}\right) + i\sin\left(\frac{\pi+8\pi}{7}\right)\right] = Cis\left(\frac{9\pi}{7}\right)$$

for $k=5$

$$Z_5 = \left[\cos\left(\frac{\pi+10\pi}{7}\right) + i\sin\left(\frac{\pi+10\pi}{7}\right)\right] = Cis\left(\frac{11\pi}{7}\right)$$

for $k=6$

$$Z_6 = \left[\cos\left(\frac{\pi+12\pi}{7}\right) + i\sin\left(\frac{\pi+12\pi}{7}\right)\right] = Cis\left(\frac{13}{7}\right)$$

(ii) $\boldsymbol{x^7 + x^4 + x^3 + 1 = 0}$

$$x^4(x^3+1) + 1(x^3+1) = 0$$

$$(x^3+1)(x^4+1) = 0$$

Either $x^3+1 = 0$; $x^4+1 = 0$

$x^3 = (-1)$; $x^4 = (-1)$

$x = (-1)^{1/3}$; $x = (-1)^{1/4}$

Let $z = (-1)$

$x = -1$, $y = 0$, $|z| = 1$

$$\left.\begin{aligned}\cos\theta &= -1\\ \sin\theta &= 0\end{aligned}\right\} \quad \theta = \pi$$

$$Z_K = (1)^{1/3}\left[\cos\left(\frac{\pi+2k\pi}{3}\right)+i\sin\left(\frac{\pi+2k\pi}{3}\right)\right],\ k=0,1,2$$

$$Z_K = \cos\left(\frac{\pi+2k\pi}{3}\right)+i\sin\left(\frac{\pi+2k\pi}{3}\right)$$

for $k=0$

$$Z_0 = \cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right) = \frac{1}{2}+i\frac{\sqrt{3}}{2}$$

for $k=1$

$$Z_1 = \cos\left(\frac{\pi+2\pi}{3}\right)+i\sin\left(\frac{\pi+2\pi}{3}\right) = \cos(\pi)+i\sin(\pi) = -1$$

for $k=2$

$$Z_2 = \cos\left(\frac{\pi+4\pi}{3}\right)+i\sin\left(\frac{\pi+4\pi}{3}\right) = \cos\left(\frac{5\pi}{3}\right)+i\sin\left(\frac{5\pi}{3}\right)$$

$$Z_2 = \frac{1}{2}-i\frac{\sqrt{3}}{2}$$

When:

$$x = (-1)^{1/4},\quad \text{Let } z=-1,\quad |z| = 1$$

$$\left.\begin{array}{l}\cos\theta = -1\\ \sin\theta = 0\end{array}\right\}\ \theta=\pi$$

$$Z_K = (1)^{1/4}\left[\cos\left(\frac{\pi+2k\pi}{4}\right)+i\sin\left(\frac{\pi+2k\pi}{4}\right)\right];\ k= 0,1,2,3$$

for $k=0$

$$Z_0 = \cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}$$

for $k=1$

$$Z_1 = \cos\left(\frac{\pi+2}{4}\right)+i\sin\left(\frac{\pi+2}{4}\right) = \cos\left(\frac{3\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}\right)$$

$$Z_1 = -\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}$$

for $k=2$

$$Z_2 = \cos\left(\frac{\pi+4\pi}{4}\right)+i\sin\left(\frac{\pi+4\pi}{4}\right) = \cos\left(\frac{5\pi}{4}\right)+i\sin\left(\frac{5\pi}{4}\right)$$

$$Z_2 = -\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i$$

for $k=3$

$$Z_3 = \cos\left(\frac{\pi+6\pi}{4}\right)+i\sin\left(\frac{\pi+6\pi}{4}\right) = \cos\left(\frac{7\pi}{4}\right)+i\sin\left(\frac{7\pi}{4}\right)$$

$$Z_3 = \frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}$$

(iii) $x^6 + 1 = \sqrt{3}\,i$

$$x^6 = -1 + \sqrt{3}\,i$$
$$x = (-1 + \sqrt{3}i)^{1/6}$$

Let
$$z = -1 + \sqrt{3}i$$
$$x = 1 \quad , y = \sqrt{3} \quad , |z| = \sqrt{1+3} = \sqrt{4} \ni \boxed{2 = r}$$

$\left.\begin{array}{l}\cos\theta = -1/2 \\ \sin\theta = \sqrt{3}/2\end{array}\right]$ $\theta \in$ II quardrant, refrence angle $= \alpha = \pi/3$

$$\therefore \theta = \pi - \alpha = \pi - \pi/3 = \frac{2\pi}{3}$$

$$x_k = (2)^{1/6}\left[\cos\left(\frac{2\pi/3 + 2k\pi}{6}\right) + i\sin\left(\frac{2\pi/3 + 2k\pi}{6}\right)\right]; k = 0,1,2,3,4,5$$

$$x_k = (2)^{1/6}\left[\cos\left(\frac{2\pi + 6k\pi}{18}\right) + i\sin\left(\frac{2\pi + 6k\pi}{18}\right)\right]$$

$$x_k = (2)^{1/6}\left[\cos\left(\frac{\pi + 3k\pi}{9}\right) + i\sin\left(\frac{\pi + 3k\pi}{9}\right)\right]; k = 0,1,2,3,4,5$$

for $k = 0$
$$x_0 = (2)^{1/6}\left[\cos\left(\frac{\pi}{9}\right) + i\sin\left(\frac{\pi}{9}\right)\right] = (2)^{1/6}\,\text{Cis}\left(\frac{\pi}{9}\right)$$

for $k = 1$
$$x_1 = (2)^{1/6}\left[\cos\left(\frac{\pi + 3\pi}{9}\right) + i\sin\left(\frac{\pi + 3\pi}{9}\right)\right] = (2)^{1/6}\,\text{Cis}\left(\frac{4\pi}{9}\right)$$

for $k = 2$
$$x_2 = (2)^{1/6}\left[\cos\left(\frac{\pi + 6\pi}{9}\right) + i\sin\left(\frac{\pi + 6\pi}{9}\right)\right] = (2)^{1/6}\,\text{Cis}\left(\frac{7\pi}{9}\right)$$

for $k = 3$
$$x_3 = (2)^{1/6}\left[\cos\left(\frac{\pi + 9\pi}{9}\right) + i\sin\left(\frac{\pi + 9\pi}{9}\right)\right] = (2)^{1/6}\,\text{Cis}\left(\frac{10\pi}{9}\right)$$

for $k = 4$
$$x_4 = (2)^{1/6}\left[\cos\left(\frac{\pi + 12\pi}{9}\right) + i\sin\left(\frac{\pi + 12\pi}{9}\right)\right] = (2)^{1/6}\,\text{Cis}\left(\frac{13\pi}{9}\right)$$

for $k = 5$
$$x_5 = (2)^{1/6}\left[\cos\left(\frac{\pi + 15\pi}{9}\right) + i\sin\left(\frac{\pi + 15\pi}{9}\right)\right] = (2)^{1/6}\,\text{Cis}\left(\frac{16\pi}{9}\right)$$

9. Solve the equation $x^{12}-1$ and find which of its roots satisfy the equation $x^4+x^2+1=0$

Sol. $x^{12}-1=0 \Rightarrow (x^6)^2-(1)^2=0$

$(x^6-1)(x^6+1)=0$

Either $x^6-1=0$ or $x^6+1=0$

$\Rightarrow x^6=1$ & $x^6=-1$

$\Rightarrow x=(1)^{1/6}$

here $z=1$

$x=1, y=0 \quad |z|=1$

$\left.\begin{array}{l}\cos\theta=1\\ \sin\theta=0\end{array}\right]$ quardrantal angle, $\theta=0^\circ$

$$x_k=(1)^{1/6}\left[\cos\left(\frac{2k\pi}{6}\right)+i\sin\left(\frac{2k\pi}{6}\right)\right]$$

$$x_k=\cos\left(\frac{k\pi}{3}\right)+i\sin\left(\frac{k\pi}{3}\right)$$

$k=0,1,2,3,4,5$

for $k=0$

$x_0=\cos(0)+i\sin(0)=1$

for $k=1$

$x_1=\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)=\frac{1}{2}+i\frac{\sqrt{3}}{2}$

for $k=2$

$x_2=\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)=-\frac{1}{2}+\frac{i\sqrt{3}}{2}$

for $k=3$

$x_3=\cos\left(\frac{3\pi}{3}\right)+i\sin\left(\frac{3\pi}{3}\right)=-1$

for $k=4$

$x_4=\cos\left(\frac{4\pi}{3}\right)+i\sin\left(\frac{4\pi}{3}\right)=-\frac{1}{2}-i\frac{\sqrt{3}}{2}$

for $k=5$

$x_5=\cos\left(\frac{5\pi}{3}\right)+i\sin\left(\frac{5\pi}{3}\right)=\frac{1}{2}-\frac{i\sqrt{3}}{2}$

$x=(-1)^{1/6}$

Same Solution as in Q6 part (i)

$z_0=\frac{\sqrt{3}}{2}+\frac{i}{2}$

$z_1=i$

$z_2=-\frac{\sqrt{3}}{2}+\frac{i}{2}$

$z_3=-\frac{\sqrt{3}}{2}-\frac{i}{2}$

$z_4=-i$

$z_5=\frac{\sqrt{3}}{2}-\frac{i}{2}$

So roots of $x^{12}-1=0$ are

$\pm1, \pm i, \frac{1}{2}\pm\frac{\sqrt{3}}{2}i, -\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$

$\frac{\sqrt{3}}{2}\pm\frac{i}{2}$ and $-\frac{\sqrt{3}}{2}\pm\frac{i}{2}$

Now $x^6-1=0$

$\Rightarrow (x^2-1)(x^4+x^2-1)=0$

$x^2-1=0$, $x^4+x^2-1=0$

$\Rightarrow x=\pm1$

Hence roots of $x^4+x^2-1=0$ are

$\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$ and $-\frac{1}{2}\pm\frac{\sqrt{3}i}{2}$

10. Express the following in series of sines or cosines of moltiples of θ.

(i) $\cos^4\theta$

$$x = \cos\theta + i\sin\theta$$
$$2\cos\theta = x + \frac{1}{x}$$
$$(2\cos\theta)^4 = \left(x + \frac{1}{x}\right)^4$$
$$2^4\cos^4\theta = \binom{4}{0}x^4\left(\frac{1}{x}\right)^0 + \binom{4}{1}x^3\frac{1}{x} + \binom{4}{2}x^2\frac{1}{x^2} + \binom{4}{1}x\frac{1}{x^3} + \binom{4}{0}x^0\frac{1}{x^4}$$
$$2^4\cos^4\theta = x^4 + 4x^2 + 6 + 4\left(\frac{1}{x^2}\right) + \frac{1}{x^4}$$
$$= \left(x^4 + \frac{1}{x^4}\right) + 4\left(x^2 + \frac{1}{x^2}\right) + 6$$
$$2^4\cos^4\theta = 2\cos 4\theta + 4(2\cos 2\theta) + 6$$
$$2^4\cos^4\theta = 2(\cos 4\theta + 4\cos 2\theta + 3)$$
$$2^3\cos^4\theta = \cos 4\theta + 4\cos 2\theta + 3$$
$$\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$$

(ii) $\sin^4\theta$

$$x = \cos\theta + i\sin\theta$$
$$x - \frac{1}{x} = 2i\sin\theta$$

$\because i^4 = (i^2)^2 = (-1)^2 = 1$

$$(2i\sin\theta)^4 = \left(x - \frac{1}{x}\right)^4$$
$$2^4 i^4 \sin^4\theta = \binom{4}{0}x^4\left(\frac{1}{x}\right)^0 - \binom{4}{1}x^3\frac{1}{x} + \binom{4}{2}x^2\frac{1}{x^2} - \binom{4}{3}x\frac{1}{x^3} + \binom{4}{4}x^0\frac{1}{x^4}$$
$$\sin^4\theta = \frac{1}{2^4}\left[x^4 - 4x^2 + 6 - 4\left(\frac{1}{x^2}\right) + \left(\frac{1}{x^4}\right)\right]$$
$$= \frac{1}{2^4}\left[\left(x^4 + \frac{1}{x^4}\right) - 4\left(x^2 + \frac{1}{x^2}\right) + 6\right]$$
$$= \frac{1}{2^4}\left[2\cos 4\theta - 4(2\cos 2\theta) + 6\right]$$
$$= \frac{2}{2^4}\left[\cos 4\theta - 4\cos 2\theta + 6\right] = \frac{1}{2^3}\left[\cos 4\theta - 4\cos 2\theta + 3\right]$$
$$\sin^4\theta = \frac{1}{8}\left[\cos 4\theta - 4\cos 2\theta + 3\right]$$

(iii) $\sin^6\theta$

$$x = \cos\theta + i\sin\theta$$
$$2i\sin\theta = x - \frac{1}{x}$$

$$(2i\sin\theta)^6 = \left(x - \frac{1}{x}\right)^6$$
$$2^6 i^6 \sin^6\theta = \binom{6}{0}x^6\left(\frac{1}{x}\right)^0 - \binom{6}{1}x^5\frac{1}{x} + \binom{6}{2}x^4\frac{1}{x^2} - \binom{6}{3}x^3\frac{1}{x^3}$$
$$+ \binom{6}{4}x^2\frac{1}{x^4} - \binom{6}{5}x\frac{1}{x^5} + \binom{6}{6}x^0\frac{1}{x^6}$$

$$i^6 = (i^2)^3 = (-1)^3 = -1$$

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$$-\sin^6\theta = \frac{1}{2^6}\left[x^6 - 6x^4 + 15x^2 - 20 + \frac{15}{x^2} - \frac{6}{x^4} + \frac{1}{x^6}\right]$$

$$= \frac{1}{2^6}\left[\left(x^6 + \frac{1}{x^6}\right) - 6\left(x^4 + \frac{1}{x^4}\right) + 15\left(x^2 + \frac{1}{x^2}\right) - 20\right]$$

$$= \frac{1}{64}\left[2\cos 6\theta - 6(2\cos 4\theta) + 15(2\cos 2\theta) - 20\right]$$

$$= \frac{2}{64}\left[\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10\right]$$

$$\sin^6\theta = \frac{1}{32}\left[\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10\right] \quad \text{Ans.}$$

(iv) $\cos^7\theta$.

$$x = \cos\theta + i\sin\theta$$
$$x + \frac{1}{x} = 2\cos\theta$$

$$(2\cos\theta)^7 = \left(x + \frac{1}{x}\right)^7$$

$$2^7\cos^7\theta = \binom{7}{0}x^7\left(\frac{1}{x}\right)^0 + \binom{7}{1}x^6\frac{1}{x} + \binom{7}{2}x^5\left(\frac{1}{x^2}\right) + \binom{7}{3}x^4\frac{1}{x^3}$$
$$+ \binom{7}{4}x^3\frac{1}{x^4} + \binom{7}{5}x^2\frac{1}{x^5} + \binom{7}{6}x\frac{1}{x^6} + \binom{7}{7}x^0\frac{1}{x^7}$$

$$2^7\cos^7\theta = x^7 + 7x^5 + 21x^3 + 35x + \frac{35}{x} + \frac{21}{x^3} + \frac{7}{x^5} + \frac{1}{x^7}$$
$$= \left(x^7 + \frac{1}{x^7}\right) + 7\left(x^5 + \frac{1}{x^5}\right) + 21\left(x^3 + \frac{1}{x^3}\right) + 35\left(x + \frac{1}{x}\right)$$
$$= 2\cos 7\theta + 7(2\cos 5\theta) + 21(2\cos 3\theta) + 35(2\cos\theta)$$

$$\cos^7\theta = \frac{1}{2^7}2\left(\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta\right)$$

$$\cos^7\theta = \frac{1}{2^6}\left(\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta\right) \quad \text{Ans.}$$

(V) $\sin^9\theta$.

$$x - \frac{1}{x} = 2i\sin\theta$$

$$\because i^9 = (i^2)^4 i = (-1)^4 i = i$$

$$(2i\sin\theta)^9 = \left(x - \frac{1}{x}\right)^9$$

$$2^9 i^9 \sin^9\theta = \binom{9}{0}x^9\left(\frac{1}{x}\right)^0 - \binom{9}{1}x^8\frac{1}{x} + \binom{9}{2}x^7\frac{1}{x^2} - \binom{9}{3}x^6\frac{1}{x^3} + \binom{9}{4}x^5\frac{1}{x^4}$$
$$- \binom{9}{5}x^4\frac{1}{x^5} + \binom{9}{6}x^3\frac{1}{x^6} - \binom{9}{7}x^2\frac{1}{x^7} + \binom{9}{8}x\frac{1}{x^8} - \binom{9}{9}x^0\frac{1}{x^9}$$

$$2^9 i\sin^9\theta = x^9 - 9x^7 + 36x^5 - 84x^3 + 126x - \frac{126}{x} + \frac{84}{x^3} - \frac{36}{x^5} + \frac{9}{x^7} - \frac{1}{x^9}$$
$$= \left(x^9 - \frac{1}{x^9}\right) - 9\left(x^7 - \frac{1}{x^7}\right) + 36\left(x^5 - \frac{1}{x^5}\right) - 84\left(x^3 - \frac{1}{x^3}\right) + 126\left(x - \frac{1}{x}\right)$$

$$= 2i\sin 9\theta - 9(2i\sin 7\theta) + 36(2i\sin 5\theta) - 84(2i\sin 3\theta) + 126(2i\sin\theta)$$

$$2^{\cancel{9}\,8}\cancel{i}\sin^9\theta = \cancel{2}\cancel{i}\left(\sin 9\theta - 9\sin 7\theta + 36\sin 5\theta - 84\sin 3\theta + 126\sin\theta\right)$$

$$\sin^9\theta = \frac{1}{2^8}\left[\sin 9\theta - 9\sin 7\theta + 36\sin 5\theta - 84\sin 3\theta + 126\sin\theta\right]$$

$i^6 = (i^2)^3 = (-1)^3 = -$

(vi) $\sin^6\theta\cos^2\theta$

$x + \frac{1}{x} = 2\cos\theta \quad , x - \frac{1}{x} = 2i\sin\theta$

$(2\cos\theta)^2(2i\sin\theta)^6 = \left(x+\frac{1}{x}\right)^2\left(x-\frac{1}{x}\right)^6$

$2^2\cos^2\theta \; 2^6 i^6 \sin^6\theta = \left(x+\frac{1}{x}\right)^2\left(x-\frac{1}{x}\right)^2\left(x-\frac{1}{x}\right)^4$

$-2^8 \sin^6\theta\cos^2\theta = \left(\left(x+\frac{1}{x}\right)\left(x-\frac{1}{x}\right)\right)^2\left(x-\frac{1}{x}\right)^4$

$= \left(x^2-\frac{1}{x^2}\right)^2\left(x-\frac{1}{x}\right)^4$

$-2^8\sin^6\theta\cos^2\theta = \left(x^4+\frac{1}{x^4}-2\right)\left[\binom{4}{0}x^4\cdot\frac{1}{x^0} - \binom{4}{1}x^3\frac{1}{x} + \binom{4}{2}x^2\frac{1}{x^2} - \binom{4}{3}x\frac{1}{x^3} + \binom{4}{4}x^0\cdot\frac{1}{x^4}\right)$

$= \left(x^4+\frac{1}{x^4}-2\right)\left(x^4-4x^2+6-\frac{4}{x^2}+\frac{1}{x^4}\right)$

$= x^8 - 4x^6 + 6x^4 - 4x^2 + 1 + 1 - \frac{4}{x^2} + \frac{6}{x^4} - \frac{4}{x^6} + \frac{1}{x^8}$
$-2x^4 + 8x^2 - 12 + \frac{8}{x^2} - \frac{2}{x^4}$

$-2^8\sin^6\theta\cos^2\theta = \left(x^8+\frac{1}{x^8}\right) - 4\left(x^6+\frac{1}{x^6}\right) + 4\left(x^4+\frac{1}{x^4}\right) + 4\left(x^2+\frac{1}{x^2}\right) - 10$

$\sin^6\theta\cos^2\theta = -\frac{1}{2^8}\left[2\cos8\theta - 4(2\cos6\theta) + 4(2\cos4\theta) + 4(2\cos2\theta) - 10\right]$

$= -\frac{2}{2^8}\left[\cos8\theta - 4\cos6\theta + 4\cos4\theta + 4\cos2\theta - 5\right]$

$\sin^6\theta\cos^2\theta = \frac{1}{2^7}\left[\cos8\theta - 4\cos6\theta + 4\cos4\theta + 4\cos2\theta - 5\right]$

(vii) $\cos^4\theta\sin^3\theta$

$\because i^3 = i\,i^2 = -i$

$(2\cos\theta)^4(2i\sin\theta)^3 = \left(x+\frac{1}{x}\right)^4\left(x-\frac{1}{x}\right)^3$

$2^4\cos^4\theta \; 2^3 i^3\sin^3\theta = \left(x+\frac{1}{x}\right)^3\left(x-\frac{1}{x}\right)^3\left(x+\frac{1}{x}\right)$

$-2^7 i\cos^4\theta\sin^3\theta = \left(x^2-\frac{1}{x^2}\right)^3\left(x+\frac{1}{x}\right)$

$= \left[\binom{3}{0}(x^2)^3\frac{1}{(x^2)^0} - \binom{3}{1}(x^2)^2\frac{1}{x^2} + \binom{3}{2}(x^2)\frac{1}{(x^2)^2} - \binom{3}{3}(x^2)^0\frac{1}{(x^2)^3}\right]\left(x+\frac{1}{x}\right)$

$= \left[x^6 - 3x^2 + \frac{3}{x^2} - \frac{1}{x^6}\right]\left(x+\frac{1}{x}\right)$

$= x^7 - 3x^3 + \frac{3}{x} - \frac{1}{x^5} + x^5 - 3x + \frac{3}{x^3} - \frac{1}{x^7}$

$-2^7 i\cos^4\theta\sin^3\theta = \left(x^7-\frac{1}{x^7}\right) - 3\left(x^3-\frac{1}{x^3}\right) + \left(x^5-\frac{1}{x^5}\right) - 3\left(x-\frac{1}{x}\right)$

$\cos^4\theta\sin^3\theta = -\frac{1}{2^7 i}\left[2i\sin7\theta - 3(2i\sin3\theta) + 2i\sin5\theta - 3(2i\sin\theta)\right]$

$$= -\frac{\cancel{2}\cancel{i}}{2^{\cancel{7}6}\cancel{i}}\left[\sin 7\theta + \sin 5\theta - 3\sin 3\theta - 3\sin\theta\right]$$

$$\cos^4\theta\sin^3\theta = -\frac{1}{2^6}\left[\sin 7\theta + \sin 5\theta - 3\sin 3\theta - 3\sin\theta\right]$$

$i^7 = (i^2)^3 \cdot i = -i$

(viii) $\cos^5\theta\sin^7\theta$

$$(2\cos\theta)^5(2i\sin\theta)^7 = \left(x+\frac{1}{x}\right)^5\left(x-\frac{1}{x}\right)^7$$

$$2^5\cos^5\theta \cdot 2^7 i^7\sin^7\theta = \left(x+\frac{1}{x}\right)^5\left(x-\frac{1}{x}\right)^5\left(x-\frac{1}{x}\right)^2$$

$$-2^{12} i\cos^5\theta\sin^7\theta = \left(x^2-\frac{1}{x^2}\right)^5\left(x^2+\frac{1}{x^2}-2\right)$$

$$=\left[\binom{5}{0}(x^2)^5\left(\frac{1}{x^2}\right)^0 - \binom{5}{1}(x^2)^4\left(\frac{1}{x^2}\right) + \binom{5}{2}(x^2)^3\frac{1}{(x^2)^2} - \binom{5}{3}(x^2)^2\frac{1}{(x^2)^3}\right.$$
$$\left.+\binom{5}{4}(x^2)\cdot\frac{1}{(x^2)^4} - \binom{5}{5}(x^2)^0\left(\frac{1}{x^2}\right)^5\right]\left(x^2+\frac{1}{x^2}-2\right)$$

$$= \left(x^{10} - 5x^6 + 10x^2 - \frac{10}{x^2} + \frac{5}{x^6} - \frac{1}{x^{10}}\right)\left(x^2+\frac{1}{x^2}-2\right)$$

$$= x^{12} - 5x^8 + \underline{10x^4} - \cancel{10} + \frac{5}{x^4} - \frac{1}{x^8} - \underline{5x^4} + \cancel{10} - \frac{10}{x^4} + \frac{5}{x^8}$$
$$-\frac{1}{x^{12}} - 2x^{10} + 10x^6 - 20x^2 + \frac{20}{x^2} - \frac{10}{x^6} - \frac{2}{x^{10}}$$

$$= \left(x^{12}-\frac{1}{x^{12}}\right) - 2\left(x^{10}-\frac{1}{x^{10}}\right) - 4\left(x^8-\frac{1}{x^8}\right) + 10\left(x^6-\frac{1}{x^6}\right) + 5\left(x^4-\frac{1}{x^4}\right)$$
$$-20\left(x^2-\frac{1}{x^2}\right)$$

$$-2^{12} i\cos^5\theta\sin^7\theta = 2i\sin 12\theta - 2(2i\sin 10\theta) - 4(2i\sin 8\theta) + 10(2i\sin 6\theta) + 5(2i\sin 4\theta)$$
$$-20(2i\sin 4\theta)$$

$$-2^{\cancel{12}\,11}\cancel{i}\cos^5\theta\sin^7\theta = \cancel{2}\cancel{i}(\sin 12\theta - 2\sin 10\theta - 4\sin 8\theta + 10\sin 6\theta + 5\sin 4\theta - 20\sin 4\theta)$$

$$\cos^5\theta\sin^7\theta = -\frac{1}{2^{11}}\left(\sin 12\theta - 2\sin 10\theta - 4\sin 8\theta + 10\sin 6\theta + 5\sin 4\theta - 20\sin 2\theta\right]$$

11. Show that $\cos^4\theta + \sin^4\theta = \frac{1}{4}(3+\cos 4\theta)$

From question 10 part (i) & (ii)

$$\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3) \rightarrow (A)$$

$$\sin^4\theta = \frac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3) \rightarrow (B)$$

Adding (A) and (B)

$$\cos^4\theta + \sin^4\theta = \frac{1}{8}(\cos 4\theta + \cancel{4\cos 2\theta} + 3 + \cos 4\theta - \cancel{4\cos 2\theta} + 3)$$

$$= \frac{1}{8}(2\cos 4\theta + 6)$$

$$= \frac{\cancel{2}}{\cancel{8}_4}(\cos 4\theta + 3)$$

$$\cos^4\theta + \sin^4\theta = \frac{1}{4}(3 + \cos 4\theta) \quad \text{Proved}$$

12. Prove that.

$$64(\cos^8\theta + \sin^8\theta) = \cos^8\theta + 28\cos 4\theta + 35$$

Proof:

$$(2\cos\theta)^8 = \left(x + \frac{1}{x}\right)^8$$

$$2^8\cos^8\theta = \binom{8}{0}x^8 \cdot \frac{1}{x^0} + \binom{8}{1}x^7 \cdot \frac{1}{x} + \binom{8}{2}x^6 \cdot \frac{1}{x^2} + \binom{8}{3}x^5 \cdot \frac{1}{x^3} + \binom{8}{4}x^4 \cdot \frac{1}{x^4}$$

$$+ \binom{8}{5}x^3 \frac{1}{x^5} + \binom{8}{6}x^2 \cdot \frac{1}{x^6} + \binom{8}{7}x \cdot \frac{1}{x^7} + \binom{8}{8}x^0 \cdot \frac{1}{x^8}$$

$$2^8\cos^8\theta = x^8 + 8x^6 + 28x^4 + 56x^2 + 70 + \frac{56}{x^2} + \frac{28}{x^4} + \frac{8}{x^6} + \frac{1}{x^8}$$

$$= \left(x^8 + \frac{1}{x^8}\right) + 8\left(x^6 + \frac{1}{x^6}\right) + 28\left(x^4 + \frac{1}{x^4}\right) + 56\left(x^2 + \frac{1}{x^2}\right) + 70$$

$$2^2 \cdot 2^6\cos^8\theta = 2\cos 8\theta + 8(2\cos 6\theta) + 28(2\cos 4\theta) + 56(2\cos 2\theta) + 70 \rightarrow (1)$$

$$(2i\sin\theta)^8 = \left(x - \frac{1}{x}\right)^8 \qquad i^8 = (i^2)^4 = 1$$

$$2^8 i^8 \sin^8\theta = \binom{8}{0}x^8\frac{1}{x^0} - \binom{8}{1}x^7\frac{1}{x} + \binom{8}{2}x^6 \cdot \frac{1}{x^2} - \binom{8}{3}x^5 \cdot \frac{1}{x^3} + \binom{8}{4}x^4 \cdot \frac{1}{x^4}$$

$$- \binom{8}{5}x^3 \cdot \frac{1}{x^5} + \binom{8}{6}x^2\frac{1}{x^6} - \binom{8}{7}x \cdot \frac{1}{x^7} + \binom{8}{8}x^0 \cdot \frac{1}{x^8}$$

$$= x^8 - 8x^6 + 28x^4 - 56x^2 + 70 - \frac{56}{x^2} + \frac{28}{x^4} - \frac{8}{x^6} + \frac{1}{x^8}$$

$$= \left(x^8 + \frac{1}{x^8}\right) - 8\left(x^6 + \frac{1}{x^6}\right) + 28\left(x^4 + \frac{1}{x^4}\right) - 56\left(x^2 + \frac{1}{x^7}\right) + 70$$

$$2^2 2^6 \sin^8\theta = 2\cos 8\theta - 8(2\cos 6\theta) + 28(2\cos 4\theta) - 56(2\cos 2\theta) + 70 \rightarrow (2)$$

adding (1) & (2)

$$2^2 2^6\left[\cos^8\theta + \sin^8\theta\right] = 2\cos 8\theta + 8(\cancel{2\cos 6\theta}) + 28(2\cos 4\theta) + 56(\cancel{2\cos 2\theta})$$
$$+ 70 + 2\cos 8\theta - 8(\cancel{2\cos 6\theta}) + 28(2\cos 4\theta) - 56(\cancel{2\cos 2\theta})$$
$$+ 70$$

$$64\left[\cos^8\theta + \sin^8\theta\right] = \frac{1}{2^2}\left[4\cos 8\theta + 4(28)\cos 4\theta + 2(70)\right]$$

$$= \frac{1}{\cancel{2^2}}(\cancel{2^2})\left[\cos 8\theta + 28\cos 4\theta + 35\right]$$

$$64\left[\cos^8\theta + \sin^8\theta\right] = \cos 8\theta + 28\cos 4\theta + 35$$

PROVED

13. Prove.

$(a+b)^3 = a^3+b^3+3ab(a+b)$

(i) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

(ii) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

Sol. $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$

$\cos 3\theta + i\sin 3\theta = \cos^3\theta + i^3\sin^3\theta + 3(\cos\theta)(i\sin\theta)(\cos\theta + i\sin\theta)$

$= \cos^3\theta - i\sin^3\theta + 3\cos^2\theta\sin\theta\, i - 3\cos\theta\sin^2\theta$

$\cos 3\theta + i\sin 3\theta = (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)$

Equating real and imaginary part.

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

$= \cos^3\theta - 3\cos\theta(1-\cos^2\theta)$

$= \cos^3\theta - 3\cos\theta + 3\cos^3\theta$

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

$= 3(1-\sin^2\theta)\sin\theta - \sin^3\theta$

$= 3\sin\theta - 3\sin^3\theta - \sin^3\theta$

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

(iii) $\sin 4\theta = 4(\cos^3\theta\sin\theta - \cos\theta\sin^3\theta)$

$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$

Sol. $(i\sin 4\theta + \cos 4\theta) = (\cos\theta + i\sin\theta)^4$

$\cos 4\theta + i\sin 4\theta = \binom{4}{0}(\cos\theta)^4(i\sin\theta)^0 + \binom{4}{1}(\cos\theta)^3(i\sin\theta) + \binom{4}{2}(\cos\theta)^2(i\sin\theta)^2 + \binom{4}{3}(\cos\theta)(i\sin\theta)^3 + \binom{4}{4}(\cos\theta)^0(i\sin\theta)^4$

$= \cos^4\theta + 4i\cos^3\theta\sin\theta + 6\cos^2\theta\sin^2\theta\, i^2 + 4\cos\theta\sin^3\theta\, i^3 + i^4\sin^4\theta$

$= \cos^4\theta - 6\cos^2\theta\sin^2\theta + (4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)i + \sin^4\theta$

$\cos 4\theta + i\sin 4\theta = (\cos^4\theta + \sin^4\theta - 6\cos^2\theta\sin^2\theta) + i(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$

Equating real and imaginary part

$\cos 4\theta = \cos^4\theta + \sin^4\theta - 6\cos^2\theta\sin^2\theta$ [Real part]

$= \cos^4\theta + (\sin^2\theta)^2 - 6\cos^2\theta(1-\cos^2\theta)$

$= \cos^4\theta + (1-\cos^2\theta)^2 - 6\cos^2\theta + 6\cos^4\theta$

$= 7\cos^4\theta + 1 + \cos^4\theta - 2\cos^2\theta - 6\cos^2\theta$

$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$ proved

$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

$= 4(\cos^3\theta\sin\theta - \cos\theta\sin^3\theta)$

Proved.

(v) $\dfrac{\sin 5\theta}{\sin\theta} = 16\cos^4\theta - 12\cos^2\theta + 1$

$(\cos 5\theta + i\sin 5\theta) = (\cos\theta + i\sin\theta)^5$

$= \binom{5}{0}(\cos\theta)^5(i\sin\theta)^0 + \binom{5}{1}(\cos\theta)^4(i\sin\theta)^1$

$+ \binom{5}{2}(\cos\theta)^3(i\sin\theta)^2 + \binom{5}{3}(\cos\theta)^2(i\sin\theta)^3 + \binom{5}{4}(\cos\theta)(i\sin\theta)^4 + \binom{5}{5}(\cos\theta)^0(i\sin\theta)^5$

$= \cos^5\theta + 5\cos^4\theta\sin\theta\, i + 10\cos^3\theta\sin^2\theta\, i^2 + 10\cos^2\theta\sin^3\theta\, i^3$
$+ 5\cos\theta\sin^4\theta\, i^4 + \sin^5\theta\, i^5$

$= (\cos^5\theta + 5\cos\theta\sin^4\theta - 10\cos^3\theta\sin^2\theta) + i(5\cos^4\theta\sin\theta$
$- 10\cos^2\theta\sin^3\theta + \sin^5\theta)$

Equating real and imaginary part

<u>Imaginary Part:</u>

$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

$\sin 5\theta = \sin\theta\,(5\cos^4\theta - 10\cos^2\theta\sin^2\theta + \sin^4\theta)$ $\to (\sin^2\theta)^2$

$\sin 5\theta = \sin\theta\,(5\cos^4\theta - 10\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2)$

$= \sin\theta\,[5\cos^4\theta - 10\cos^2\theta + 10\cos^4\theta + 1 + \cos^4\theta - 2\cos^2\theta]$

$\sin 5\theta = \sin\theta\,[16\cos^4\theta - 12\cos^2\theta + 1]$

$\dfrac{\sin 5\theta}{\sin\theta} = 16\cos^4\theta - 12\cos^2\theta + 1$ PROVED

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14. Prove that $\tan 6\theta = 2t\left(\dfrac{3-10t^2+3t^4}{1-15t^2+15t^4-t^6}\right)$, where $t=\tan\theta$

Proof:

$(\cos 6\theta + i\sin 6\theta) = (\cos\theta + i\sin\theta)^6$

$= \binom{6}{0}(\cos\theta)^6(i\sin\theta)^0 + \binom{6}{1}(\cos\theta)^5(i\sin\theta) + \binom{6}{2}(\cos\theta)^4(i\sin\theta)^2$

$\binom{6}{3}(\cos\theta)^3(i\sin\theta)^3 + \binom{6}{4}(\cos\theta)^2(i\sin\theta)^4 + \binom{6}{5}\cos\theta(i\sin\theta)^5$
$+ \binom{6}{6}(i\sin\theta)^6$

$= \cos^6\theta + 6\cos^5\theta\sin\theta\, i + 15\cos^4\theta\sin^2\theta\, i^2 + 20\cos^3\theta\sin^3\theta\, i$
$+ 15\cos^2\theta\sin^4\theta\, i^4 + 6\cos\theta\sin^5\theta\, i^5 + i^6\sin^6\theta$

$= (\cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta)$
$+ i(6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta)$

Equating Real and imaginary part

$\cos 6\theta = \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta \to ①$

$\sin 6\theta = 6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta \to ②$

Dividing (ii) by (i)

$$\frac{\sin 6\theta}{\cos 6\theta} = \frac{6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta}{\cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta}$$

$$= \frac{2\sin\theta}{\cos\theta}\left[\frac{3\cos^5\theta - 10\cos^3\theta\sin^2\theta + 3\cos\theta\sin^4\theta}{\cos^5\theta - 15\cos^3\theta\sin^2\theta + 15\cos\theta\sin^4\theta - \frac{\sin^6\theta}{\cos\theta}}\right]$$

$\because$ dividing Numenator & denumenator by $\cos^5\theta$

$$\tan 6\theta = 2\frac{\sin\theta}{\cos\theta}\left[\frac{\frac{3\cancel{\cos^5\theta}}{\cancel{\cos^5\theta}} - \frac{10\cancel{\cos^3\theta}\sin^2\theta}{\cancel{\cos^3\theta}\cos^2\theta} + \frac{3\cancel{\cos\theta}\sin^4\theta}{\cancel{\cos\theta}\cos^4\theta}}{\frac{\cancel{\cos^5\theta}}{\cancel{\cos^5\theta}} - \frac{15\cancel{\cos^3\theta}\sin^2\theta}{\cancel{\cos^3\theta}\cos^2\theta} + \frac{15\cancel{\cos\theta}\sin^4\theta}{\cancel{\cos\theta}\cos^4\theta} - \frac{\sin^6\theta}{\cos^5\theta\cos\theta}}\right]$$

$$= 2\frac{\sin\theta}{\cos\theta}\left[\frac{3 - 10\tan^2\theta + 3\tan^4\theta}{1 - 15\tan^2\theta + 15\tan^4\theta - \tan^6\theta}\right]$$

$\because \tan\theta = t$

$$\tan 6\theta = 2t\left[\frac{3 - 10t^2 + 3t^4}{1 - 15t^2 + 15t^4 - t^6}\right]$$ **PROVED**

15. **Prove that $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$ and hence solve the equation $1 - 3t^2 = 3t - t^3$ where $t = \tan\theta$.**

Proof

$$\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$$

$$\cos 3\theta + i\sin 3\theta = \cos^3\theta + i^3\sin^3\theta + 3\cos\theta\, i^2\sin^2\theta + 3i\sin\theta\cos^2\theta$$

$$= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\sin\theta\cos^2\theta - \sin^3\theta)$$

Equating Real & Imaginary part.

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta \rightarrow (1$$

$$\sin 3\theta = 3\sin\theta\cos^2\theta - \sin^3\theta \rightarrow (2$$

dividing (1 & (2

$$\frac{\sin 3\theta}{\cos 3\theta} = \frac{3\sin\theta\cos^2\theta - \sin^3\theta}{\cos^3\theta - 3\cos\theta\sin^2\theta}$$

Dividing Numerator and denuminator by $\cos^3\theta$

$$= \frac{\frac{3\sin\theta\cancel{\cos^2\theta}}{\cancel{\cos^2\theta}\cos\theta} - \frac{\sin^3\theta}{\cos^3\theta}}{\frac{\cos^3\theta}{\cos^3\theta} - \frac{3\cos\theta\sin^2\theta}{\cos\theta\cos^2\theta}}$$

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$ $\because \tan\theta = t$

$$\tan 3\theta = \frac{3t - t^3}{1 - 3t^2}$$

Put $\tan 3\theta = 1$

$$1 = \frac{3t - t^3}{1 - 3t^2}$$

$$\therefore \quad 3t - t^3 = 1 - 3t^2$$

when

$$t = \tan 15$$
$$t = \tan(45 - 30)$$
$$t = \frac{\tan 45 - \tan 30}{1 + \tan 45 \tan 30}$$
$$= \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}}$$
$$t = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} \times \frac{\sqrt{3} - 1}{\sqrt{3} - 1}$$
$$= \frac{(\sqrt{3})^2 + (1)^2 - 2\sqrt{3}}{(\sqrt{3})^2 - (1)^2}$$
$$= \frac{\overset{4}{3+1} - 2\sqrt{3}}{3 - 1}$$
$$= \frac{\cancel{2}(2 - \sqrt{3})}{\cancel{2}}$$
$$\boxed{t = 2 - \sqrt{3}}$$

$\Rightarrow \tan 3\theta = 1$

$$3\theta = 45, 225$$
$$\theta = \frac{45}{3}, \frac{225}{3}$$
$$\theta = 15, 75$$

when

$$t = \tan 35$$
$$t = \tan(45 + 30)$$
$$t = \frac{\tan 45 + \tan 30}{1 - \tan 45 \tan 30}$$
$$= \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}}$$
$$t = \frac{\sqrt{3} + 1}{\sqrt{3} - 1} \times \frac{\sqrt{3} + 1}{\sqrt{3} + 1}$$
$$= \frac{(\sqrt{3})^2 + (1)^2 + 2\sqrt{3}}{(\sqrt{3})^2 - (1)}$$
$$= \frac{3 + 1 + 2\sqrt{3}}{3 - 1}$$
$$= \frac{\cancel{2}(2 + \sqrt{3})}{\cancel{2}}$$
$$\boxed{t = 2 + \sqrt{3}}$$

16. Prove that $\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} = \frac{1}{2}$

Proof: Let $x^7 - 1 = 0$

$$x^7 = 1$$
$$x = (1)^{\frac{1}{7}}$$

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~~Here~~ $z = 1 + 0i \qquad |z| = 1$

$\left.\begin{array}{l}\cos\theta = 1 \\ \sin\theta = 0\end{array}\right\}$ quardrantal angle $\theta = 0°$

$$x_k = (1)^{1/7}\left[\cos\left(\frac{0 + 2k\pi}{7}\right) + i\sin\left(\frac{0 + 2k\pi}{7}\right)\right] \quad k = -3, -2, -1, 0, 1, 2, 3$$

$$x_k = 1\left[\cos\left(\frac{2k\pi}{7}\right) + i\sin\left(\frac{2k\pi}{7}\right)\right] \quad k = -3, -2, -1, 0, 1, 2, 3$$

$$x_k = \sum_{k=-3}^{3}\left[\cos\left(\frac{2k\pi}{7}\right) + i\sin\left(\frac{2k\pi}{7}\right)\right] = 0 + 0i$$

Equating real part

$$\therefore \sum_{k=-3}^{3} \cos\left(\frac{2k\pi}{7}\right) = 0$$

$$\cos\left(-\frac{6\pi}{7}\right) + \cos\left(-\frac{4\pi}{7}\right) + \cos\left(\frac{-2\pi}{2}\right) + \cos 0 + \cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{6\pi}{7}\right) = 0$$

$$\cos\left(\frac{6\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{2\pi}{7}\right) + 1 + \cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{6\pi}{7}\right) = 0$$

$$2\cos\left(\frac{6\pi}{7}\right) + 2\cos\left(\frac{4\pi}{7}\right) + 2\cos\left(\frac{2\pi}{7}\right) = -1$$

$$\cos\left(\frac{6\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{2\pi}{7}\right) = -\frac{1}{2}$$

$$\cos\left(\pi - \frac{\pi}{7}\right) + \cos\left(\pi - \frac{3\pi}{7}\right) + \cos\left(\frac{2\pi}{7}\right) = -\frac{1}{2}$$

$$-\cos\left(\frac{\pi}{7}\right) - \cos\left(\frac{3\pi}{7}\right) + \cos\left(\frac{2\pi}{7}\right) = -\frac{1}{2}$$

$$\cos\left(\frac{\pi}{7}\right) + \cos\left(\frac{3\pi}{7}\right) - \cos\left(\frac{2\pi}{7}\right) = \frac{1}{2} \quad \text{Proved}$$

17. Prove the following Relations

(i) $z^m z^n = z^{m+n}$

$z = r\,\text{Cis}\,\theta$

$z^m = r^m\,\text{Cis}(m\theta)$ $\qquad$ $z^n = r^n\,\text{Cis}\,n\theta$

$$\begin{aligned} z^m . z^n &= r^m\,\text{Cis}(m\theta) . r^n\,\text{Cis}(n\theta) \\ &= r^{m+n}\,\text{Cis}(m\theta + n\theta) \\ &= r^{m+n}\,\text{Cis}\,(m+n)\theta \\ &= z^{m+n} \end{aligned}$$

(ii) $(z^m)^n = z^{mn}$

$z = r\,\text{Cis}\,\theta$

$z^m = (r\,\text{Cis}\,\theta)^m$

$z^m = r^m\,\text{Cis}(m\theta)$ $\qquad$ By De-Moivre's Theorem

$$\begin{aligned} (z^m)^n &= (r^m\,\text{Cis}(m\theta))^n \\ &= (r^m)^n\,\text{Cis}\,(n(m\theta)) \quad // \\ &= r^{mn}\,\text{Cis}\,(mn)\theta \\ (z^m)^n &= z^{mn} \quad \text{Proved} \end{aligned}$$

(iii) $(z_1 z_2)^n = z_1^n z_2^n$

$z_1 = r_1\,\text{Cis}\,\theta_1$ $\qquad$ $z_2 = r_2\,\text{Cis}\,\theta$

$z_1^n = r_2^n\,\text{Cis}\,n\theta_1$ $\qquad$ $z_2^n = r_2^n\,\text{Cis}\,n\theta_2$

$$z_1 z_2 = r_1 Cis\theta_1 \cdot r_2 Cis\theta_2$$
$$z_1 z_2 = r_1 r_2 \; Cis(\theta_1+\theta_2)$$
$$(z_1 z_2)^n = [r_1 r_2 \; Cis(\theta_1+\theta_2)]^n$$
$$(z_1 z_2)^n = r_1^n r_2^n \; Cis\, n(\theta_1+\theta_2) \rightarrow (1)$$
$$z_1^n z_2^n = r_1^n Cis\, n\theta_1 \cdot r_2^n Cis\, n\theta_2$$
$$= r_1^n r_2^n \; Cis(n\theta_1+n\theta_2)$$
$$z_1^n z_2^n = r_1^n r_2^n \; Cis(\theta_1+\theta_2)n \rightarrow (2)$$

from (1) & (2)

$$(z_1 z_2)^n = (z_1)^n (z_2)^n$$

17.(IV) $\dfrac{z^m}{z^n} = z^{m-n}$

$$z = rCis\theta$$
$$z^m = r^m Cis\, m\theta \qquad z^n = r^n Cis\, n\theta$$
$$\frac{z^m}{z^n} = \frac{r^m Cis\, m\theta}{r^n \; Cis\, n\theta}$$
$$= r^{m-n} \; Cis(m\theta - n\theta)$$
$$= r^{m-n} \; Cis(m-n)\theta$$
$$\frac{z^m}{z^n} = z^{m-n}$$

(V) $\left(\dfrac{z_1}{z_2}\right)^n = \dfrac{z_1^n}{z_2^n}$ Provided $z_2 \neq 0$

$$z_1 = r_1 Cis\theta_1 \; , \; z_2 = r_2 Cis\theta_2$$
$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \frac{Cis\theta_1}{Cis\theta_2}$$
$$= \frac{r_1}{r_2} Cis(\theta_1-\theta_2) \rightarrow (1)$$
$$z_1^n = r_1^n Cis\, n\theta_1 \; , \; z_2^n = r_2^n Cis\, n\theta_2$$
$$\frac{z_1^n}{z_2^n} = \frac{r_1^n}{r_2^n} \cdot \frac{Cis\, n\theta_1}{Cis\, n\theta_2}$$
$$= \frac{r_1^n}{r_2^n} Cis(n\theta_1 - n\theta_2)$$
$$= \frac{r_1^n}{r_2^n} Cis(\theta_1-\theta)n \rightarrow (2)$$

from (1) & (2)

$$\left(\frac{z_1}{z_n}\right)^n = \frac{z_1^n}{z_2^n}$$