Chap # 1

Trigonometry

Complex No. complex number is an element (x,y) of the set $R^2 = \{(x,y) : x, y \in R\}$
OR Number which can be written in form $a+ib$.

Polar Rectangular Components:

Exercise # 1.1

Express each of the following complex no. in polar form. (Problem 1-6)

1. $-\sqrt{3}+i$

$x = -\sqrt{3}$, $y = 1$

$r = |z| = \sqrt{3+1} = \sqrt{4}$

$\boxed{r=2}$

$\cos\theta = -\frac{\sqrt{3}}{2}$, $\sin\theta = \frac{1}{2}$

θ is in 2nd Quardrant

Refrence angle $= \alpha = \frac{\pi}{6}$

$\theta = \pi - \alpha = \pi - \pi/6$

$\theta = 5\pi/6$

$z = r[\cos\theta + i\sin\theta]$

$= 2\left[\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right]$

2. $-i$

$x = 0$ $y = -1$

$r = |z| = \sqrt{0+1} = 1 \Rightarrow \boxed{r=1}$

$\cos\theta = 0$, $\sin\theta = -1$

Here θ is quardrantal angle

$\theta = -\pi/2$

$z = r[\cos\theta + i\sin\theta]$

$z = 1\left[\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right]$

$z = \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)$

(2) $\boxed{r = \sqrt{x^2+y^2}}$

3. $-1-\sqrt{3}i$

$x = -1 \quad , \quad y = -\sqrt{3}$

$|z| = r = \sqrt{1+3} = \sqrt{4}$

$\Rightarrow \boxed{r = 2}$

$\cos\theta = -1/2 \quad , \quad \sin\theta = -\sqrt{3}/2$

θ is in 3rd quardrant

Refrence Angle $= \alpha = \pi/3$

$\theta = \alpha - \pi = \frac{\pi}{3} - \pi$

$\theta = -\frac{2\pi}{3}$

$z = 2\left[\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right]$

4. $-1+i$

$x = -1 \quad , \quad y = 1$

$|z| = r = \sqrt{1+1} \Rightarrow \boxed{r = \sqrt{2}}$

$\cos\theta = \frac{-1}{\sqrt{2}} \quad , \quad \sin\theta = \frac{1}{\sqrt{2}}$

θ is in 2nd quardrant

Refrence angle $= \alpha = \frac{\pi}{4}$

$\theta = \pi - \alpha = \pi - \frac{\pi}{4}$

$\boxed{\theta = 3\pi/4}$

$z = \sqrt{2}\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right]$

5. $(-2+2i)(1-i)$

$= -2+2i+2i-2i^2$

$= -2+4i+2$

$z = 4i$

$x = 0 \quad , \quad y = 4 \quad \boxed{|z| = 4}$

$\cos\theta = 0 \quad , \quad \sin\theta = 1$

Here θ is quardrantal Angle

$\therefore \theta = \pi/2$

$z = r[\cos\theta + i\sin\theta]$

$= 4[\cos(\pi/2) + i\sin(\pi/2)]$

Quardrantal Angle:

Angle which lies on boundries b/w two quardrants like (0, 90, 180, 270, 360)

6. $\frac{-34i}{5-3i}$

$= \frac{-34i}{5-3i} \times \frac{5+3i}{5+3i}$

$= \frac{-34(5i+3i^2)}{(5)^2-(3i)^2}$

$= \frac{-34(5i-3)}{25+9}$

$= \frac{34(3-5i)}{34}$

$z = 3-5i$

$x = 3 \quad , \quad y = -5$

$z = \sqrt{9+25} = \sqrt{34}$

$\cos\theta = 3/\sqrt{34} \quad , \quad \sin\theta = -5/\sqrt{34}$

$\frac{\sin\theta}{\cos\theta} = \frac{-5/\sqrt{34}}{3/\sqrt{34}}$

$\tan\theta = -\frac{5}{3}$

$\theta = \tan^{-1}\left(-\frac{5}{3}\right)$

$z = r[\cos\theta + i\sin\theta]$

$= \sqrt{34}\left[\cos\left(\tan^{-1}\left(-\frac{5}{3}\right)\right) + i\sin\left(\tan^{-1}\left(-\frac{5}{3}\right)\right)\right]$

Express the given complex numbers in cartesian form and plot on an argand diagram (Problem 7-10)

7. $2\,Cis(\pi/6)$

$= 2[\cos(\pi/6) + i\sin(\pi/6)]$

$= 2\left[\frac{\sqrt{3}}{2} + i\sin(\pi/6)\right]$

$= 2\left(\frac{\sqrt{3}}{2} + i\,\frac{1}{2}\right)$

$= \frac{2}{2}(\sqrt{3} + i)$

$= \sqrt{3} + i$

8. $5\,Cis\left(\frac{3\pi}{4}\right)$

$45 \times 3 = 135$

2nd Quard

Cos −ve
Sin +ve

$= 5\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right]$

$= 5\left[-\frac{1}{\sqrt{2}} + i\,\frac{1}{\sqrt{2}}\right]$

$= -5/\sqrt{2}\,(1 - i)$

9. $\sqrt{3}\,Cis\left(\frac{7\pi}{6}\right)$

$30 \times 7 = 210$

3rd quard.

Sin −ve , Cos −ve

$= \sqrt{3}\left[\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right)\right]$

$= \sqrt{3}\left[-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right]$

$= -\frac{\sqrt{3}}{2}(1 + i)$

10. $\dfrac{5\,Cis(\pi/3)}{2\,Cis(\pi/2)}$

$= \frac{5}{2}\,Cis\left(\frac{\pi}{3} - \frac{\pi}{2}\right)$

$= \frac{5}{2}\,Cis\left(-\frac{\pi}{6}\right)$

$= \frac{5}{2}\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]$

$= \frac{5}{2}\left[\frac{\sqrt{3}}{2} - i\frac{1}{2}\right]$

$= \frac{5\sqrt{3}}{4} - i\frac{5}{4}$

formula if $z = z_1 \cdot z_2 \cdot z_3$
$|z| = |z_1||z_2||z_3|$

$|z| = \sqrt{x^2+y^2}$

(4) **formula:** if $z = \dfrac{z_1 \cdot z_2}{z_3 \cdot z_4}$
then $|z| = \dfrac{|z_1||z_2|}{|z_3||z_4|}$

11. Find $|z|$.

(i) $z = -2i(1+i)(2+4i)(3+i)$

$z_1 = -2i \Rightarrow |z_1| = \sqrt{4} = 2$
$z_2 = 1+i \Rightarrow |z_2| = \sqrt{1+1} = \sqrt{2}$
$z_3 = 2+4i \Rightarrow |z_3| = \sqrt{4+16} = \sqrt{20}$
$z_4 = 3+i \Rightarrow |z_4| = \sqrt{9+1} = \sqrt{10}$

$|z| = |z_1| \cdot |z_2| \cdot |z_3| |z_4|$
$= 2 \cdot \sqrt{2} \cdot \sqrt{20} \cdot \sqrt{10}$
$= 2\sqrt{2 \times 20 \times 10}$
$= 2\sqrt{20 \times 20}$
$= 2(20)$
$|z| = 40$

(ii) $z = \dfrac{(3+4i)(-1+2i)}{(-1-i)(3-i)}$

$z_1 = 3+4i \Rightarrow |z_1| = \sqrt{9+16} = \sqrt{25}$
$|z_1| = 5$
$z_2 = -1+2i \Rightarrow |z_2| = \sqrt{1+4} = \sqrt{5}$
$z_3 = -1-i \Rightarrow |z_3| = \sqrt{1+1} = \sqrt{2}$
$z_4 = 3-i \Rightarrow |z_4| = \sqrt{9+1} = \sqrt{10}$

$|z| = \dfrac{|z_1||z_2|}{|z_3||z_4|}$
$= \dfrac{5 \cdot \sqrt{5}}{\sqrt{2} \cdot \sqrt{10}} = 5\sqrt{\dfrac{\cancel{5}}{2 \times \cancel{10}_2}}$
$= 5\sqrt{\dfrac{1}{4}} = \dfrac{5}{2}$
$|z| = \dfrac{5}{2}$

12. Show that $z = a+ib$ is

(i) real iff $z = \bar{z}$

Suppose
$z = \bar{z} \rightarrow (1$
$z = a+ib \Rightarrow \bar{z} = a-ib$
$\cancel{a}+ib = \cancel{a}-ib$ from (1
$ib+ib = 0$
$2ib = 0$
$\Rightarrow 2 \neq 0, i \neq 0 \Rightarrow b = 0$
put in z
$z = a+0i = a$
$\Rightarrow z$ is real

Conversly;
Suppose z is real
$\Rightarrow z = a$
$\Rightarrow \bar{z} = a$
here
$z = \bar{z} = a$
proved

(ii) pure imaginary iff $z = -\bar{z}$

Suppose
$z = -\bar{z}$
$z = a+ib \Rightarrow \bar{z} = a-ib$
$\because z = -\bar{z}$
$a+ib = -(a-ib)$
$a+\cancel{ib} = -a+\cancel{ib}$
$a+a = 0$
$2a = 0$
$\because 2 \neq 0 \Rightarrow a = 0$
put in z
$z = 0+ib = ib$
$\Rightarrow z$ is pure imaginary

Conversly, suppose
z is imaginary,
$\Rightarrow z = ib$
$\Rightarrow \bar{z} = -ib$
here
$z = -\bar{z}$
proved

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13. Prove analytically that for complex numbers z_1, z_2.

$$||z_1|-|z_2|| \le |z_1+z_2| \le |z_1|+|z_2|$$

Sol. $|z_1+z_2| \le |z_1|+|z_2|$

$$|z_1+z_2|^2 = (z_1+z_2)(\overline{z_1+z_2})$$
$$= (z_1+z_2)(\bar{z}_1+\bar{z}_2)$$
$$= z_1\bar{z}_1 + z_1\bar{z}_2 + \bar{z}_1 z_2 + z_2\bar{z}_2$$
$$= |z_1|^2 + z_1\bar{z}_2 + \overline{z_1\bar{z}_2} + |z_2|^2$$
$$= |z_1|^2 + 2Re(z_1\bar{z}_2) + |z_2|^2$$
$$\le |z_1|^2 + 2|z_1\bar{z}_2| + |z_2|^2$$
$$= |z_1|^2 + 2|z_1||\bar{z}_2| + |z_2|^2$$
$$= |z_1|^2 + 2|z_1||z_2| + |z_2|^2$$
$$|z_1+z_2|^2 \le (|z_1|+|z_2|)^2$$
$$|z_1+z_2| \le |z_1|+|z_2| \longrightarrow (A)$$ *square root

$\because z\bar{z} = |z|^2$

$\because \bar{\bar{z}}_2 = z_2$

if $z = a+ib$
$\Rightarrow \bar{z} = a-ib$
$z+\bar{z} = 2a$
$= 2Re(z)$

$\because Re(z) < |z|$

$\because a < \sqrt{a^2+b}$

$\because |z_1 z_2| = |z_1||z_2|$

$\because |z_1| = |\bar{z}_1|$

Now for $||z_1|-|z_2|| \le |z_1-z_2|$

$|z_1| = |z_1 - z_2 + z_2|$

$|z_1| \le |z_1-z_2| + |z_2|$ $\quad \because |a+b| \le |a|+|b|$

$|z_1| - |z_2| \le |z_1-z_2| \rightarrow$ (i) $\quad \because |x-y| = |y-x|$

$|z_2| = |z_2 - z_1 + z_1|$

$|z_2| \le |z_2-z_1| + |z_1|$

$-|z_2-z_1| \le |z_1|-|z_2|$

$-|z_1-z_2| \le |z_1|-|z_2| \rightarrow$ (ii)

from (i) & (ii)

$$-|z_1-z_2| \le |z_1|-|z_2| \le |z_1-z_2|$$
$$\Rightarrow ||z_1|-|z_2|| \le |z_1-z_2| \longrightarrow (B)$$

$\because -a < x < a$
$\Rightarrow |x| < a$

Obviously.

$$|z_1-z_2| \le |z_1+z_2| \longrightarrow (C)$$

from (A), (B) & (C).

$$\cancel{||z_1|-|z_2|| \le |z_1-z_2| \le |z_1+z_2|} < |z_1|+|z_2|$$

$$\Rightarrow ||z_1|-|z_2|| \le |z_1+z_2| \le |z_1|+|z_2|$$ **Proved.**

14. Let $z_1 = 24+7i$ and $|z_2| = 6$. Find the greatest and least value of $|z_1+z_2|$.

Sol. $\because$ greatest value of $|z_1+z_2| = |z_1|+|z_2| =$

$|z_1| = \sqrt{(24)^2+(7^2)} = \sqrt{576+49} = \sqrt{625} = 25$

$|z_2| = \sqrt{0+6} = 6$

greatest value $= 25+6 = 31$

least value of $|z_1+z_2| = \big||z_1|-|z_2|\big|$

$$= |25-6| = 19$$

15. if z_1, z_2 are complex numbers. show that

$$|z_1+z_2|^2 + |z_1-z_2|^2 = 2\left(|z_1|^2+|z_2|^2\right)$$

Sol:

$$|z_1+z_2|^2 = (z_1+z_2)\overline{(z_1+z_2)}$$
$$= (z_1+z_2)(\bar{z}_1+\bar{z}_2)$$
$$= (z_1\bar{z}_1 + z_2\bar{z}_2 + \bar{z}_1 z_2 + z_2\bar{z}_2)$$
$$|z_1+z_2|^2 = \left(|z_1|^2 + z_1\bar{z}_2 + \bar{z}_1 z_2 + |z_2|^2\right) \quad (1)$$

$$|z_1-z_2|^2 = (z_1-z_2)\overline{(z_1-z_2)}$$
$$= (z_1-z_2)(\bar{z}_1-\bar{z}_2)$$
$$= (z_1\bar{z}_1 + z_2\bar{z}_2 - z_1\bar{z}_2 - z_2\bar{z}_1)$$
$$|z_1-z_2|^2 = \left(|z_1|^2+|z_2|^2 - z_1\bar{z}_2 - \bar{z}_1 z_2\right) \quad (2)$$

adding (1) and (2)

$$|z_1+z_2|^2 + |z_1-z_2|^2 = |z_1|^2 + \cancel{z_1\bar{z}_2} + \cancel{\bar{z}_1 z_2} + |z_2|^2$$
$$+ |z_1|^2 + |z_2|^2 - \cancel{z_1\bar{z}_2} - \cancel{\bar{z}_1 z_2}$$
$$= 2|z_1|^2 + 2|z_2|^2$$
$$|z_1+z_2|^2 + |z_1-z_2|^2 = 2\left(|z_1|^2+|z_2|^2\right)$$

16. Prove that $\left|\dfrac{az+b}{\bar{b}z+\bar{a}}\right| = 1$ for $|z|=1$

$$\left|\frac{az+b}{\bar{b}z+a}\right| = \frac{|az+b|}{|\bar{b}z+\bar{a}|} = \frac{|az+b|}{|\overline{\bar{b}z+a}|} \qquad \because |z| = |\bar{z}|$$

$$= \frac{|az+b|}{1\,|\bar{\bar{b}}\bar{z}+\bar{\bar{a}}|} \qquad \overline{a\pm b} = \bar{a}\pm\bar{b}, \quad \overline{ab} = \bar{a}\bar{b}, \quad \bar{\bar{a}} = a$$

$$= \frac{|az+b|}{|z|\,|b\bar{z}+\bar{\bar{a}}|}$$

$$= \frac{|az+b|}{|z(b\bar{z}+a)|}$$

$$= \frac{|az+b|}{|bz\bar{z}+az|}$$

$$= \frac{|az+b|}{|b|z|^2+az|} \qquad z\bar{z} = |z|^2, \quad |z| = 1, \quad |z|^2 = 1$$

$$= \frac{|az+b|}{|az+b|}$$

$$\left|\frac{az+b}{\bar{b}z+a}\right| = 1$$

17. Find locus of points in the plane satisfying each of the given conditions.

(i) $|z-5|=6$

$= |x+iy-5| = 6$

$= |(x-5)+iy| = 6$

$\sqrt{(x-5)^2+y^2} = 6$

$(x-5)^2+(y-0)^2 = 36 = (6)^2$

locus of circle with centre $(5,0)$ and radius 6

(iii) $\text{Re}(z+2) = -1$

$\text{Re}(x+iy+2) = -1$

$\text{Re}(x+2+iy) = -1$

$x+2 = -1$

$x = -3$

locus is line parallel to y-axis

(v) $|z+i| = |z-i|$

$|x+iy+i| = |x+iy-i|$

$|x+i(y+1)| = |x+i(y-1)|$

$\sqrt{x^2+(y+1)^2} = \sqrt{x^2+(y-1)^2}$

Squaring both sides

$\cancel{x^2}+(y+1)^2 = \cancel{x^2}+(y-1)^2$

$(y+1)^2 = (y-1)^2$

$\cancel{y^2}+\cancel{1}+2y = \cancel{y^2}+\cancel{1}-2y$

$4y = 0$

$y = 0$

x-axis.

(ii) $|z-2i| \geq 1$

$|x+iy-2i| \geq 1$

$|x+(y-2)i| \geq 1$

$\sqrt{x^2+(y-2)^2} \geq 1$

$(x-0)^2+(y-2)^2 \geq 1$

locus is set of points on and outside the circle with center $(0,2)$ and radius 1.

(iv) $\text{Re}(i\bar{z}) = 3$

$\text{Re}(i(x+iy)) = 3$

$\text{Re}(i(x-yi)) = 3$

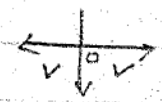
$\text{Re}(ix+y^2) = 3$

$y = 3$

locus is line parallel to x-axis.

(vii) $\text{Im}(z) < 0$

$\text{Im}(x+iy) < 0$



$y < 0$

3rd & 4rth quardrant.

(vii) $-1 < \text{Re}(z) < 1$

$-1 < \text{Re}(x+iy) < 1$

$-1 < x < 1$

$x \in]-1, 1[$

locus is interval on real line

(vi) $|z+3|+|z+1|=4$

$|x+iy+3|+|x+iy+1| = 4 \quad = |x+3+iy| + |x+1+iy| = 4$

$\sqrt{(x+3)^2+y^2}+\sqrt{(x+1)^2+y^2}=4$

$\sqrt{(x+3)^2+y^2} = 4-\sqrt{(x+1)^2+y^2}$

$\sqrt{x^2+9+6x+y^2} = 4-\sqrt{x^2+2x+1+y^2}$

Squaring both sides.

using Formula $(x+y)^2 = x^2+y^2+2xy$

$\cancel{x^2}+9+6x+\cancel{y^2} = 16+\cancel{x^2}+2x+1+\cancel{y^2}-2(4)\sqrt{x^2+2x+1+y^2}$

$6x-2x+9-16-1 = -8\sqrt{x^2+2x+1+y^2}$

$4x-8 = -8\sqrt{x^2+2x+1+y^2}$

$4(x-2) = -8\sqrt{x^2+2x+1+y^2}$

$x-2 = -2\sqrt{x^2+2x+1+y^2}$

$x^2-4x+4 = +4(x^2+2x+1+y^2)$ Squaring both sides

$x^2-4x+\cancel{4} = 4x^2+8x+\cancel{4}+4y^2$

$4x^2-x^2+8x+4x+4y^2=0$

$3x^2+12x+4y^2=0$

$3(x^2+4x)+4(y^2)=0$

$\because$ Completing Square

$x^2+4x = x^2+4x+4-4$

$= (x+2)^2-4$

$3\left[(x+2)^2-4\right]+4(y^2)=0$

$3(x+2)^2-12+4y^2=0$

$3(x+2)^2+4y^2 = 12$

divide both sides by 12.

$\dfrac{(x+2)^2}{4}+\dfrac{y^2}{3} = 1$

locus is ellipse.

(IX): $Arg(z) = \pi/3$

$Arg(x+iy) = \pi/3$

$\tan^{-1}(y/x) = \pi/3$

$\tan(\pi/3) = y/x$

$y/x = \sqrt{3}$

$y = \sqrt{3}x$

locus is Straigh line

(X) $Arg(z-1) = \dfrac{-3\pi}{4}$

$Arg(x+iy-1) = -3\pi/4$

$Arg((x-1)+iy) = -3\pi/4$

$\Rightarrow \tan^{-1}\left(\dfrac{y}{x-1}\right) = -\dfrac{3\pi}{4}$

$\dfrac{y}{x-1} = \tan\left(-\dfrac{3\pi}{4}\right)$

$\dfrac{y}{x-1} = -\tan(3\pi/4)$

$\dfrac{y}{x-1} = -(-1) = 1$

$\dfrac{y}{x-1} = 1$

$y = x-1$

$x-y-1$

locus is straight line