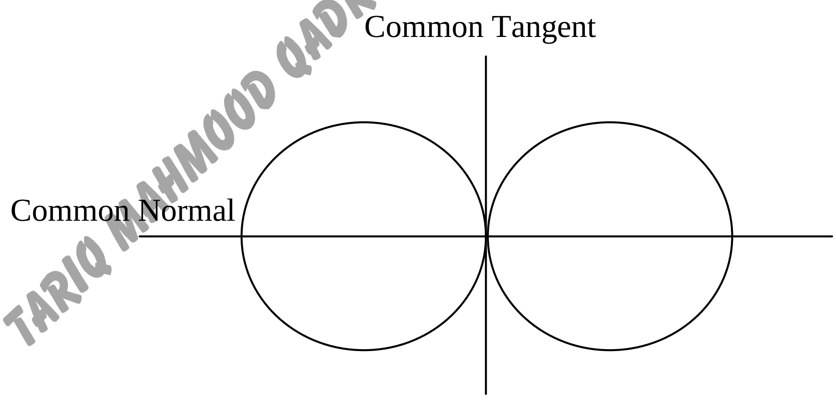
DIRECT COLLISIONS

8

CHAPTER

❖ INTRODUCTION

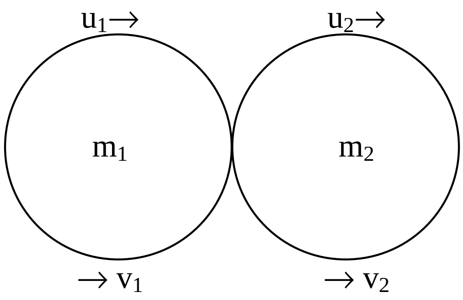
Two bodies are said to be impinge or collide directly when the direction of the motion of each is along their common normal at the point of contact. When the direction of motion of either or both is not along the common normal, the impact is said to be oblique. i.e. the bodies impinge obliquely.



A collision may be elastic or inelastic. In an elastic collision, kinetic energy is same both before and after collision or kinetic energy is conserved in an elastic collision. In an inelastic collision, the initial and final kinetic energies are not equal. All collisions between real objects are more or less inelastic except when the objects are very rigid such as billiard balls. When two bodies stick together after collision , the collision is said to be completely inelastic. The law of conservation of momentum remains valid both in elastics and inelastic collisions.

❖ IMPACT LAWS

NEWTON'S EXPERIMENTAL LAWS



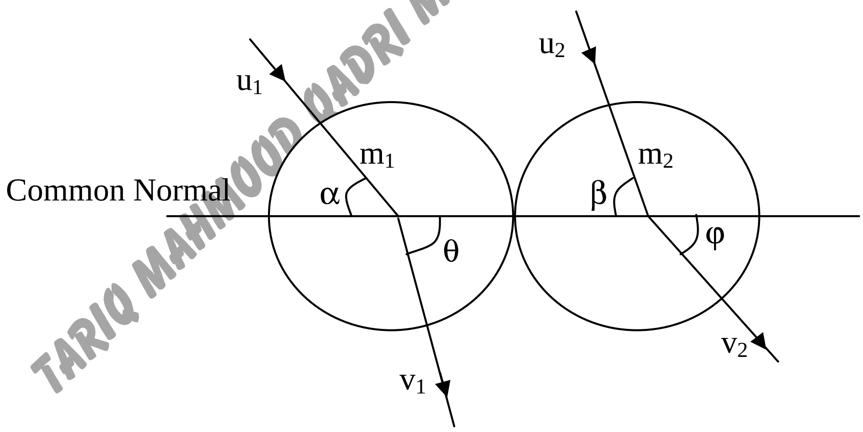
If two bodies of mass m_1 and m_2, moving with velocities u_1 and u_2 respectively, collide directly and let v_1 and v_2 be their velocities after collision.

Then

$$\frac{v_1 - v_2}{u_1 - u_2} = -e$$

$\Rightarrow \quad v_1 - v_2 = -e(u_1 - u_2)$

Where e is called the modulus or coefficient of elasticity or restitution of the bodies.



If two bodies of mass m_1 and m_2, moving with velocities u_1 and u_2 in the directions inclined at angles α and β respectively to their common normal. Let v_1 and v_2 be their velocities in the directions inclined at angles θ and φ respectively after collision. Then by

$$\frac{v_1\cos\theta - v_2\cos\varphi}{u_1\cos\alpha - u_2\cos\beta} = -e$$

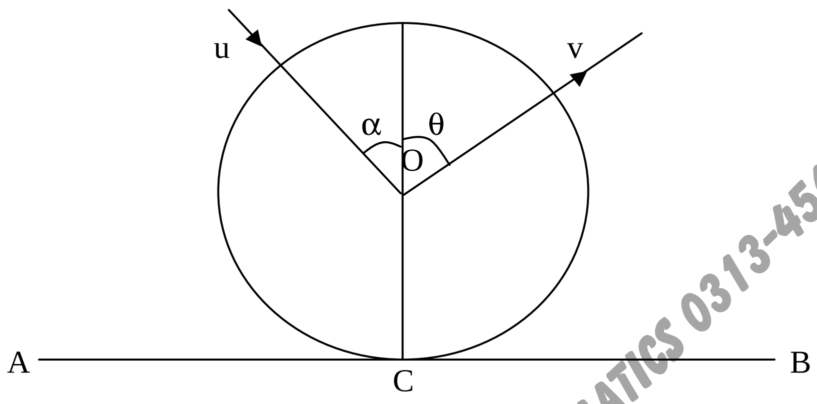
$\Rightarrow \quad v_1\cos\theta - v_2\cos\varphi = -e(u_1\cos\alpha - u_2\cos)$

❖ LAW OF CONSERVATION OF MOMENTUM

$m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2$ (For Direct Collisions)

$m_1v_1 \cos\theta + m_2v_2 \cos\varphi = m_1u_1 \cos\alpha + m_2u_2 \cos\beta$ (For Oblique Collisions)

❖ IMPACT OF A SMOOTH SPHERE AGAINST A FIXED PLANE



Let AB be a fixed plane and C be the point at which the sphere impinges. Let O be the centre of the sphere so that CO is the common normal. Let u and v be the velocities of the sphere before and after impact inclined at angles α and θ respectively to the normal.

By Newton's experimental law along CO, we have

$$v\cos\theta - 0 = -e(-u\cos\alpha - 0)$$

$$v\cos\theta = eu\cos\alpha \qquad \text{\_\_\_\_\_\_\_(i)}$$

By resolving velocities before and after collision, we have

$$v\sin\theta = u\sin\alpha \qquad \text{\_\_\_\_\_\_\_(ii)}$$

Squaring and adding (i) and (ii), we get

$$\mathbf{v^2 = u^2(e^2\cos^2\alpha + \sin^2\alpha)} \qquad \text{\_\_\_\_\_\_(iii)}$$

Eq(iii) gives the magnitude of the velocity of the sphere after impact.

Dividing (i) by (ii), we have

$$\mathbf{\cot\theta = e\cot\alpha} \qquad \text{\_\_\_\_\_\_(iv)}$$

Eq(iv) gives the direction of motion of the sphere after impact.

We know that

$$\text{Kinetic Energy} = \frac{1}{2}(\text{mass})(\text{velocity})^2$$

Therefore,

$$\text{Initial K.E.} = \frac{1}{2}mu^2 \quad \text{and} \quad \text{Final K.E.} = \frac{1}{2}mv^2$$

Now

Loss of K.E. = Initial K.E. – Final K.E.

$$= \frac{1}{2}mu^2 - \frac{1}{2}mv^2$$

$$= \frac{1}{2}mu^2 - \frac{1}{2}mu^2(e^2\cos^2\alpha + \sin^2\alpha) \qquad \text{By(iii)}$$

$$= \frac{1}{2}mu^2(1 - (e^2\cos^2\alpha + \sin^2\alpha))$$

$$= \frac{1}{2}mu^2(1 - e^2\cos^2\alpha - \sin^2\alpha)$$

$$= \frac{1}{2}mu^2(\cos^2\alpha - e^2\cos^2\alpha)$$

$$= \frac{1}{2}mu^2(1 - e^2)\cos^2\alpha$$

Now we discuss some important cases

❖ CASE 1

When $\alpha = 0$ i.e. When impact is direct.

Then $\theta = 0$ By(iv)

and $v = eu$ By(iii)

Thus, if a sphere impinges directly on a smooth fixed plane with velocity u, it rebounds in the reverse direction with velocity eu.

❖ CASE 2

When $e = 1$. i.e. sphere is perfectly elastic.

Then $\theta = \alpha$ By(iv)

and $v = u$ By(iii)

Thus, if a perfectly elastic sphere impinges on a smooth fixed plane, its velocity is unaltered in magnitude by impact and the angle of reflexion (θ) is equal to the angle of incident (α).

❖ CASE 3

When $e = 0$. i.e. sphere is perfectly inelastic.

Then $\theta = 90^0$ By(iv)

and $v = u\sin\alpha$ By(iii)

Thus, if a perfectly inelastic sphere impinges on a smooth fixed plane, it does not rebound but simply slides along the plane, its velocity parallel to the plane remained unaltered.

❖ QUESTION 1

A rubber ball drops from a height h and after rebounding twice from ground it reach a height h/2. Find the coefficient of restitution. What would be the coefficient of restitution had the ball reached a height h/2 after rebounding three times.?

SOLUTION

We know that

$$v^2 - u^2 = 2gh$$

Here $u = 0$

$\Rightarrow \quad v = \sqrt{2gh}$

So, Velocity on reaching the floor $= \sqrt{2gh}$

Let e be the coefficient of restitution. Then

Velocity after 1^{st} rebound = e × velocity before the rebound

$= e\sqrt{2gh}$

$\Rightarrow$ Velocity on reaching the floor 2^{nd} time $= e\sqrt{2gh}$

Now

Velocity after 2^{nd} rebound = e × velocity before the 2^{nd} rebound

$= e^2\sqrt{2gh}$

For the motion after 2^{nd} rebound

Initial velocity $= u = e^2\sqrt{2gh}$

Final Velocity $= v = 0$

Height attained $= h = \dfrac{h}{2}$ and $g = -g$

We know that

$$v^2 - u^2 = 2gh$$

$$\Rightarrow \quad 0 - \left(e^2\sqrt{2gh}\right)^2 = 2(-g)\frac{h}{2}$$

$$\Rightarrow \quad 2ghe^4 = gh$$

$$\Rightarrow \quad e^4 = \frac{1}{2} \quad \Rightarrow \quad e = \left(\frac{1}{2}\right)^{1/4} \quad \Rightarrow \quad e = \frac{1}{\sqrt[4]{2}}$$

Velocity on reaching the floor 3^{rd} time $= e^2\sqrt{2gh}$

Now

Velocity after 3^{rd} rebound = e × velocity before the 3^{rd} rebound $= e^3\sqrt{2gh}$

For the motion after 3rd rebound

Initial velocity = u = $e^3\sqrt{2gh}$

Final Velocity = v = 0

Height attained = h = $\frac{h}{2}$ and g = $-$ g

We know that

$$v^2 - u^2 = 2gh$$

$$\Rightarrow \quad 0 - \left(e^3\sqrt{2gh}\right)^2 = 2(-g)\frac{h}{2}$$

$$\Rightarrow \quad 2ghe^6 = gh$$

$$\Rightarrow \quad e^6 = \frac{1}{2} \quad \Rightarrow \quad e = \left(\frac{1}{2}\right)^{1/6} \quad \Rightarrow \quad e = \frac{1}{\sqrt[6]{2}}$$

❖ QUESTION 2

A heavy elastic ball is dropped upon a horizontal floor from a height of 20ft. and after rebounding twice, it is observed to attain a height 10ft. Find the coefficient of restitution.

SOLUTION

We know that

$$v^2 - u^2 = 2gh$$

Here u = 0, h = 20, g = 32ft.

$$\Rightarrow \quad v = \sqrt{2gh} = \sqrt{2 \times 32 \times 20} = 16\sqrt{5}\text{ft/sec}$$

So, Velocity on reaching the floor = $16\sqrt{5}$ft/sec

Let e be the coefficient of restitution. Then

Velocity after 1st rebound = e × velocity before the rebound

= $16e\sqrt{5}$ft/sec

$\Rightarrow$ Velocity on reaching the floor 2nd time = $16e\sqrt{5}$ft/sec

Now

Velocity after 2nd rebound = e × velocity before the 2nd rebound

= $16e^2\sqrt{5}$ft/sec

For the motion after 2nd rebound

Initial velocity = u = $16e^2\sqrt{5}$ft/sec

Final Velocity = v = 0

Height attained = h = 10ft. and g = $-$ 32ft

We know that

$$v^2 - u^2 = 2gh$$

$$\Rightarrow \quad 0 - \left(16e^2\sqrt{5}\right)^2 = 2(-32)(10)$$

$$\Rightarrow \quad 1280\, e^4 = 640$$

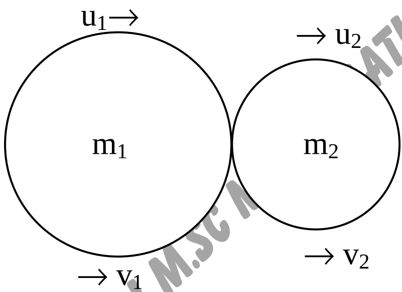
$$\Rightarrow \quad e^4 = \frac{1}{2} \quad \Rightarrow \quad e = \left(\frac{1}{2}\right)^{1/4} \quad \Rightarrow \quad e = \frac{1}{\sqrt[4]{2}}$$

❖ QUESTION 3

Two elastic spheres of masses m_1 and m_2 moving with velocities u_1 and u_2 impinge directly. If e is the coefficient of restitution, find their velocities after impact.

SOLUTION

Let v_1 and v_2 be the velocities of m_1 and m_2 after impact respectively.



By Newton's Experimental Law, we have

$$v_1 - v_2 = -e(u_1 - u_2) \qquad \text{\_\_\_\_\_(i)}$$

By Law of Conservation of Momentum, we have

$$m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2 \qquad \text{\_\_\_\_\_(ii)}$$

Multiplying (i) by m_1 and m_2, we get

$$m_1v_1 - m_1v_2 = -em_1u_1 + em_1u_2 \qquad \text{\_\_\_\_\_(iii)}$$

$$m_2v_1 - m_2v_2 = -em_2u_1 + em_2u_2 \qquad \text{\_\_\_\_\_(iv)}$$

Adding (ii) and (iv), we get

$$m_1v_1 + m_2v_1 = m_1u_1 + m_2u_2 - em_2u_1 + em_2u_2$$

$$\Rightarrow \quad (m_1 + m_2)v_1 = (m_1 - em_2)u_1 + m_2(1 + e)u_2$$

$$\Rightarrow \quad v_1 = \frac{(m_1 - em_2)u_1 + m_2(1 + e)u_2}{m_1 + m_2} \qquad \text{\_\_\_\_\_(v)}$$

Subtracting (iii) from (ii), we get

$$m_2v_2 + m_1v_2 = m_1u_1 + m_2u_2 + em_1u_1 - em_1u_2$$

$$\Rightarrow \quad (m_1 + m_2)v_2 = m_1(1 + em_2)u_1 + (m_2 - e\, m_1)u_2$$

$$\Rightarrow \quad v_2 = \frac{m_1(1 + em_2)u_1 + (m_2 - e\,m_1)u_2}{m_1 + m_2} \qquad \text{\_\_\_\_\_(vi)}$$

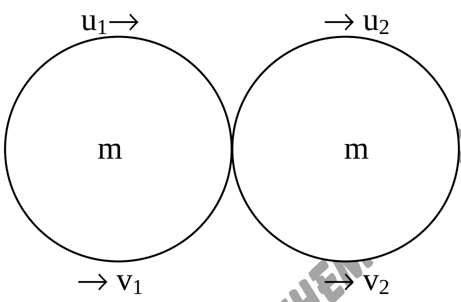
Equations (v) & (vi) gives the velocities after impact.

❖ QUESTION 4

If two equal and perfectly elastic spheres moving with velocities u_1 and u_2 impinge directly. Show that they interchange their velocities after impact.

SOLUTION

Let v_1 and v_2 be the velocities of spheres after impact respectively.



By Newton's Experimental Law, we have

$$v_1 - v_2 = -(u_1 - u_2) \qquad \because e = 1 \text{ for perfectly elastic spheres.}$$

$$\Rightarrow \quad v_1 - v_2 = u_2 - u_1 \qquad \text{\_\_\_\_\_\_(i)}$$

By Law of Conservation of Momentum, we have

$$mv_1 + mv_2 = mu_1 + mu_2$$

$$\Rightarrow \quad v_1 + v_2 = u_1 + u_2 \qquad \text{\_\_\_\_\_\_(ii)}$$

Adding (i) and (ii), we get

$$2v_1 = 2u_2 \quad \Rightarrow \quad v_1 = u_2$$

Subtracting (i) from (ii), we get

$$2v_2 = 2u_1 \quad \Rightarrow \quad v_2 = u_1$$

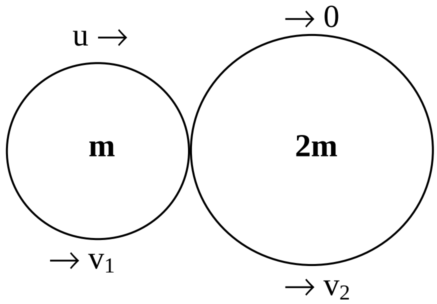
Thus $v_1 = u_2$ and $v_2 = u_1$

❖ QUESTION 5

Three perfectly elastic spheres of masses m, 2m and 3m are placed in a straight line. The first impinges directly on the second with a velocity u and the second impinges on the third. Find the velocity of the third ball after impact.

SOLUTION

Let v_1 and v_2 be the velocities of the ball of masses m and 2m respectively after impact.



Here e = 1

By Newton's Experimental Law, we have

$$v_1 - v_2 = -(u - 0)$$

$\Rightarrow \quad v_1 - v_2 = -u$ ________(i)

By Law of Conservation of Momentum, we have

$$mv_1 + 2mv_2 = mu + 2m(0)$$

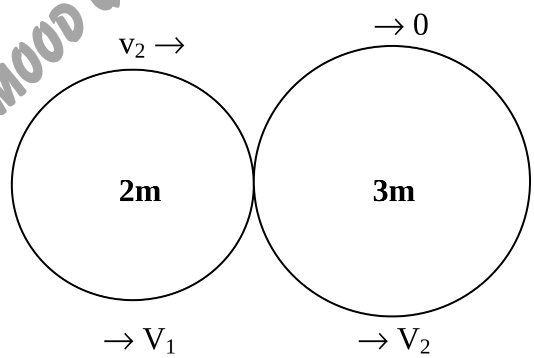
$\Rightarrow \quad v_1 + 2v_2 = u$ _______(ii)

Subtracting (i) from (2), we get

$$3v_2 = 2u$$

$\Rightarrow \quad v_2 = \dfrac{2}{3}u$

Which is the velocity of sphere of ma 2m before it impinges on the third ball. Let V_1 and V_2 be the velocities of masses 2m and 3m respectively after impact.



By Newton's Experimental Law, we have

$$V_1 - V_2 = -(v_2 - 0)$$

$\Rightarrow \quad V_1 - V_2 = -v_2$ ________(iii)

By Law of Conservation of Momentum, we have

$$2mV_1 + 3mV_2 = 2mv_2 + 3m(0)$$

$\Rightarrow \quad 2V_1 + 3V_2 = 2v_2$ _______(iv)

Multiplying (iii) by 2, we get

$$2V_1 - 2V_2 = -2v_2 \qquad \text{\_\_\_\_\_\_\_(v)}$$

Subtracting (v) from (iv), we get

$$5V_2 = 4v_2$$

$$\Rightarrow \quad 5V_2 = 4 \times \frac{2}{3}u$$

$$\Rightarrow \quad V_2 = \frac{8}{15}u$$

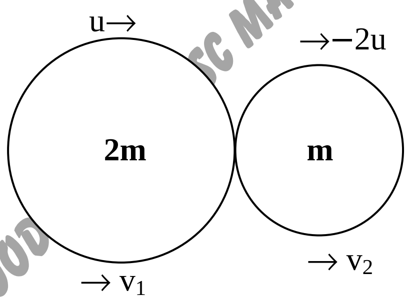
Which is the velocity of the sphere of mass 3m after impact.

❖ QUESTION 6

If the masses of two balls be as 2:1 and their respective velocities be as 1:2 in opposite direction. Show that if the coefficient of restitution is 5/6, each ball moves back after impact, with 5/6th of its original velocity.

SOLUTION

Let masses of two balls be 2m, m and their velocities before impact be u and 2u. Let v_1 and v_2 be their velocities after impact.



By Newton's Experimental Law, we have

$$v_1 - v_2 = -\frac{5}{6}[u - (-2u)]$$

$$\Rightarrow \quad v_1 - v_2 = -\frac{5}{2}u \qquad \text{\_\_\_\_\_\_\_(i)}$$

By Law of Conservation of Momentum, we have

$$2mv_1 + mv_2 = 2mu + m(-2u)$$

$$\Rightarrow \quad 2v_1 + v_2 = 0 \qquad \text{\_\_\_\_\_\_(ii)}$$

Adding (i) and (ii), we get

$$3v_1 = -\frac{5}{2}u$$

$$\Rightarrow \quad v_1 = -\frac{5}{6}u \qquad \text{\_\_\_\_\_\_(iii)}$$

From (i), we get

$$v_2 = v_1 + \frac{5}{2}u$$

$$= -\frac{5}{6}u + \frac{5}{2}u \qquad \text{By(iii)}$$

$$= -\frac{5}{6}(u - 3u)$$

$$= \frac{5}{6}(2u) \qquad \text{\_\_\_\_\_\_(iv)}$$

Equations (iii) and (iv) shows that each ball moves back after impact with 5/6th of its original velocity.

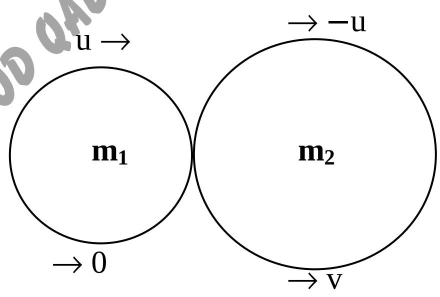
❖ QUESTION 7

Two elastic spheres impinge directly with equal and opposite velocities. Find ratio of their masses so that one of them may be reduced to rest by the impact, the coefficient of elasticity being e.

SOLUTION

Let m_1 and m_2 be masses of two spheres and u be the velocity of each sphere. Let the sphere of mass m_1 come tom rest after impact and v be the velocity of sphere of mass m_2 after impact.

We have to find $m_1 : m_2$



By Newton's Experimental Law, we have

$$0 - v = -e[u - (-u)]$$

$$\Rightarrow \quad v = 2eu \qquad \text{\_\_\_\_\_\_\_(i)}$$

By Law of Conservation of Momentum, we have

$$0 + m_2v = m_1u + m_2(-u)$$

$$\Rightarrow \quad m_2 2eu = m_1u + m_2(-u) \qquad \text{By(i)}$$

$$\Rightarrow \quad m_2 2e = m_1 - m_2$$

$$\Rightarrow \quad m_2 2e + m_2 = m_1 \quad \Rightarrow \quad (1 + 2e)m_2 = m_1$$

$$\Rightarrow \quad \frac{m_1}{m_2} = \frac{1+2e}{1}$$

$$\Rightarrow \quad m_1 : m_2 = 1 + 2e : 1$$

❖ QUESTION 8

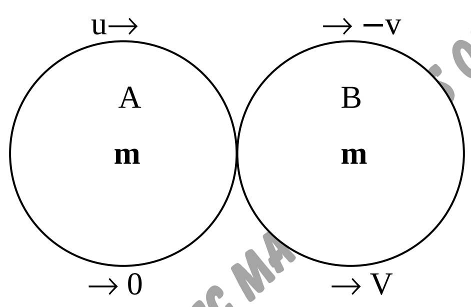
A ball A moving with velocity u impinges directly on an equal ball B moving with velocity v in the opposite direction. If A be brought to rest by impact, show that

$$\mathbf{u : v = 1 + e : 1 - e}$$

Where e is the coefficient of restitution.

SOLUTION

Let m be the mass of each sphere and V be the velocity of the sphere B after impact.



By Newton's Experimental Law, we have

$$0 - V = -\ e[u - (-v)]$$

$$\Rightarrow \quad V = e(u + v) \qquad \_\_\_\_\_\_\_\_(i)$$

By Law of Conservation of Momentum, we have

$$0 + mV = mu + m(-v)$$

$$\Rightarrow \quad V = u - v$$

$$\Rightarrow \quad u - v = e(u + v) \quad \text{By(i)}$$

$$\Rightarrow \quad u - eu = v + ev$$

$$\Rightarrow \quad u(1 - e) = v(1 + e)$$

$$\Rightarrow \quad \frac{u}{v} = \frac{1+e}{1-e}$$

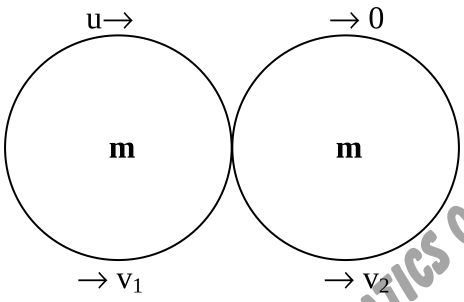
$$\Rightarrow \quad u : v = 1 + e : 1 - e$$

❖ QUESTION 9

A sphere impinges directly on an equal sphere at rest. If the coefficient of restitution is e, show that their velocities after impact are as **1 + e : 1 – e**. If the mass of first sphere is m and second sphere is M, show that the first cannot have its velocity reversed if $m > eM$.

SOLUTION

Let m be the mass of each sphere and u be the velocity of the first sphere and let v_1, v_2 are their velocities after impact. **We have to show that $v_1 : v_2 = 1 + e : 1 - e$**



By Newton's Experimental Law, we have

$$v_1 - v_2 = -\ e(u - 0) = -\ eu \qquad \text{\_\_\_\_\_\_(i)}$$

By Law of Conservation of Momentum, we have

$$mu + 0 = mv_1 + mv_2$$

$$\Rightarrow \quad v_1 + v_2 = u \qquad \text{\_\_\_\_\_\_(ii)}$$

From (i) and (ii), we get

$$v_1 - v_2 = -\ e(v_1 + v_2)$$

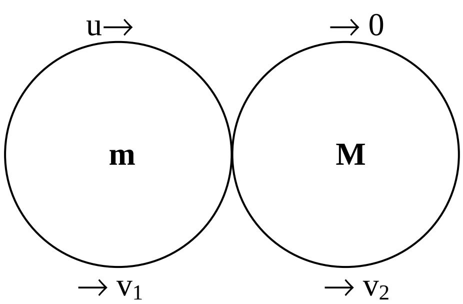
$$\Rightarrow \quad v_1 + ev_1 = v_2 - ev_2$$

$$\Rightarrow \quad v_1(1 + e) = v_2(1 - e)$$

$$\Rightarrow \quad \frac{v_1}{v_2} = \frac{1 - e}{1 + e}$$

$$\Rightarrow \quad v_1 : v_2 = 1 - e : 1 + e$$

Let the mass of second sphere is M.



By Law of Conservation of Momentum, we have

$$mu + 0 = mv_1 + Mv_2$$

$$\Rightarrow \quad mv_1 + Mv_2 = mu \qquad \_\_\_\_\_\_(iii)$$

Multiplying (i) by M, we get

$$Mv_1 - Mv_2 = -Meu \qquad \_\_\_\_\_\_(iv)$$

Adding (iii) and (iv), we get

$$mv_1 + Mv_1 = mu - Meu$$

$$\Rightarrow \quad (m + M)v_1 = (m - Me)u$$

$$\Rightarrow \quad v_1 = \left(\frac{m - Me}{m + M}\right)u$$

First sphere cannot have its velocity reversed if $v_1 > 0$.

i.e.

$$\left(\frac{m - Me}{m + M}\right)u > 0 \;\; \Rightarrow m - Me > 0 \;\; \Rightarrow \;\; m > Me$$

❖ QUESTION 10

Two spheres of masses M, m impinge directly when moving in opposite directions with velocities u, v respectively and the sphere of mass m is brought to rest by the collision. Prove that

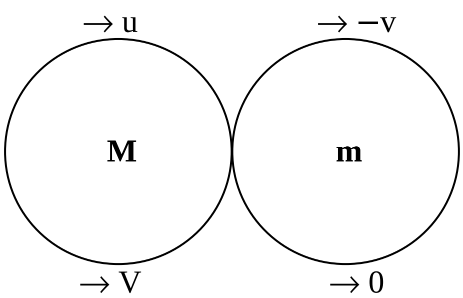
$$v(m - eM) = M(1 + e)u$$

After collision, the sphere of mass M is acted on by a constant retarding force which brings it to rest after travelling a distance a. Prove that the magnitude of this force is

$$\frac{Me^2(u + v)^2}{2a}$$

SOLUTION

Let V be the velocity of the sphere of mass M after impact.



By Newton's Experimental Law, we have

$$V - 0 = -e[u - (-v)]$$

$\Rightarrow \quad V = -e(u + v)$ ________(i)

By Law of Conservation of Momentum, we have

$$MV + 0 = Mu + m(-v)$$

$\Rightarrow \quad MV = Mu - mv$

$\Rightarrow \quad -Me(u + v) = Mu - mv \quad$ By(i)

$\Rightarrow \quad -Meu - Mev = Mu - mv$

$\Rightarrow \quad mv - Mev = Mu + Meu$

$\Rightarrow \quad v(m - Me) = M(1 + e)u$

Which is required.

Initial velocity of sphere of mass M after impact = $V = -e(u + v)$

We know that

$$v^2 - u^2 = 2as$$ ________(ii)

Here

Final velocity = $v = 0$

Initial velocity = $u = -e(u + v)$

Distance covered = $s = a$ (given)

Suppose that

Retardation = $a = -f$

Using these values in (ii), we get

$$0 - (-e(u+v))^2 = 2(-f)a$$

$\Rightarrow \quad f = \dfrac{e^2(u+v)^2}{2a}$

Thus,

Retarding force = mass × retardation

$$= \frac{Me^2(u+v)^2}{2a}$$

❖ QUESTION 11

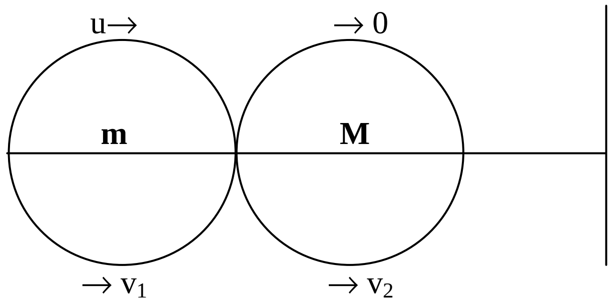
An imperfectly elastics sphere of mass m moving velocity u impinges on another sphere of mass M at rest. The second sphere afterwards strikes a vertical plane at right angle to its path. Show that there will be no further impact of the spheres if

$$m(1 + e' + ee') < eM$$

Where e and E are the coefficients of restitution between the spheres ad between the sphere and the plane respectively.

SOLUTION

Let v_1 ad v_2 be the velocities of the sphere of mass m and M respectively after impact.



By Newton's Experimental Law, we have

$$v_1 - v_2 = -e(u - 0) = -eu \quad \text{\_\_\_\_\_\_(i)}$$

By Law of Conservation of Momentum, we have

$$mu + 0 = mv_1 + Mv_2$$

$$\Rightarrow \quad mv_1 + Mv_2 = mu \quad \text{\_\_\_\_\_\_(ii)}$$

Multiplying (i) by M, we get

$$Mv_1 - Mv_2 = -Meu \quad \text{\_\_\_\_\_(iii)}$$

Adding (ii) and (iii), we get

$$mv_1 + Mv_1 = mu - Meu$$

$$\Rightarrow \quad (m + M)v_1 = (m - Me)u$$

$$\Rightarrow \quad v_1 = \left(\frac{m - Me}{m + M}\right)u$$

From (i), we get

$$v_2 = v_1 + eu$$

$$= \left(\frac{m - Me}{m + M}\right)u + eu$$

$$= \frac{mu - Meu + meu + Meu}{m + M}$$

$$= \frac{(1 + e)mu}{m + M}$$

The sphere of mass M strikes with plane with velocity v_2 and it then rebound with velocity $e'v_2$ ($\because e'$ is the coefficient of restitution between sphere and plane). Also the velocity of the sphere of mass m away from the plane is $-v_1$.

Thus there will be no further impact if

$$e'v_2 < -v_1$$

i.e. if $e'\dfrac{(1+e)mu}{m+M} < -\left(\dfrac{m-Me}{m+M}\right)u$

or if $(1+e)me' < -(m-Me)$

or if $(e' + ee')m + m < Me$

or if $(1+e'+ee')m < eM$

Which is required.

❖ LOSS OF KINETIC ENERGY DUE TO DIRECT IMPACT

Two spheres of given masses with given velocities impinges directly. Show that there is always loss of kinetic energy unless the elasticity is perfect.

SOLUTION

Let m_1 and m_2 be the masses of spheres and u_1 and u_2 be their velocities before impact. Let v_1 and v_2 be their velocities after impact and e be the coefficient of restitution. Then

$$\text{Total K.E. before impact} = \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}(m_1u_1^2 + m_2u_2^2)$$

$$\text{Total K.E. after impact} = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}(m_1v_1^2 + m_2v_2^2)$$

By Newton's Experimental Law, we have

$$v_1 - v_2 = -e(u_1 - u_2) \qquad \text{\_\_\_\_\_\_(i)}$$

By Law of Conservation of Momentum, we have

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \qquad \text{\_\_\_\_\_\_(ii)}$$

Taking square and then multiplying (i) by m_1m_2, we get

$$m_1m_2(v_1 - v_2)^2 = m_1m_2e^2(u_1 - u_2)^2 \qquad \text{\_\_\_\_\_\_(iii)}$$

Taking square of (ii), we get

$$(m_1v_1 + m_2v_2)^2 = (m_1u_1 + m_2u_2)^2 \qquad \text{\_\_\_\_\_\_(iv)}$$

Adding (iii) and (iv), we get

$$(m_1v_1 + m_2v_2)^2 + m_1m_2(v_1 - v_2)^2 = (m_1u_1 + m_2u_2)^2 + m_1m_2e^2(u_1 - u_2)^2$$

$$\Rightarrow \quad m_1^2v_1^2 + m_2^2v_2^2 + m_1m_2v_1^2 + m_1m_2v_2^2 = (m_1u_1 + m_2u_2)^2 + m_1m_2e^2(u_1 - u_2)^2$$

$$m_1^2v_1^2 + m_2^2v_2^2 + m_1m_2v_1^2 + m_1m_2v_2^2 = (m_1u_1 + m_2u_2)^2 + m_1m_2e^2(u_1 - u_2)^2$$
$$+ \mathbf{m_1m_2(u_1 - u_2)^2 - m_1m_2(u_1 - u_2)^2}$$

$$\Rightarrow \quad m_1(m_1 + m_2)v_1^2 + m_2(m_1 + m_2)v_2^2 = (m_1u_1 + m_2u_2)^2 + m_1m_2(u_1 - u_2)^2$$
$$- m_1m_2(1 - e^2)(u_1 - u_2)^2$$

$$\Rightarrow \quad (m_1 + m_2)(m_1v_1^2 + m_2v_2^2) = m_1^2u_1^2 + m_2^2u_2^2 + m_1m_2u_1^2 + m_1m_2u_2^2$$
$$- m_1m_2(1 - e^2)(u_1 - u_2)^2$$
$$= m_1(m_1 + m_2)u_1^2 + m_2(m_1 + m_2)u_2^2 - m_1m_2(1 - e^2)(u_1 - u_2)^2$$

$\Rightarrow \quad (m_1+m_2)(m_1v_1^2+m_2v_2^2) = (m_1+m_2)(m_1u_1^2+m_2u_2^2) - m_1m_2(1-e^2)(u_1-u_2)^2$

$\Rightarrow \quad m_1v_1^2+m_2v_2^2 = m_1u_1^2+m_2u_2^2 - \dfrac{m_1m_2}{m_1+m_2}(1-e^2)(u_1-u_2)^2$

$\Rightarrow \quad \dfrac{1}{2}(m_1v_1^2+m_2v_2^2) = \dfrac{1}{2}(m_1u_1^2+m_2u_2^2) - \dfrac{1}{2}.\dfrac{m_1m_2}{m_1+m_2}(1-e^2)(u_1-u_2)^2$

$\Rightarrow \quad \dfrac{1}{2}(m_1v_1^2+m_2v_2^2) - \dfrac{1}{2}(m_1u_1^2+m_2u_2^2) = -\dfrac{1}{2}.\dfrac{m_1m_2}{m_1+m_2}(1-e^2)(u_1-u_2)^2$

$\Rightarrow$ Total K.E. after impact − Total K.E. before impact $= -\dfrac{1}{2}.\dfrac{m_1m_2}{m_1+m_2}(1-e^2)(u_1-u_2)^2$

$\Rightarrow$ Change in K.E. $= -\dfrac{1}{2}.\dfrac{m_1m_2}{m_1+m_2}(1-e^2)(u_1-u_2)^2$

Since there is a negative sign on R.H.S and $e < 1 \Rightarrow 1 - e^2 > 0$

Therefore, there is loss of K.E. due to impact.

Thus,

$$\text{K.E. Lost due to impact} = \frac{1}{2}.\frac{m_1m_2}{m_1+m_2}(1-e^2)(u_1-u_2)^2$$

When e = 1. i.e. elasticity is perfect.

Then, K.E. Lost due to impact = 0

Thus there is no loss of K.E. when elasticity is perfect.

❖ QUESTION 12

Two elastic spheres each of masses m collide directly. Show that the energy lost during impact is

$$\mathbf{\frac{1}{4}m(U^2 - V^2)}$$

Where U and V are the relative velocities before and after impact.

SOLUTION

We know that if two spheres of mass m_1 and m_2 moving with velocities u_1 and u_2 collide directly then

$$\text{Loss of K.E.} = \frac{1}{2}.\frac{m_1m_2}{m_1+m_2}(1-e^2)(u_1-u_2)^2$$

Here $m_1 = m_2 = m$ and U = relative velocity before impact = $u_1 - u_2$

$\Rightarrow \quad$ Loss of K.E. $= \dfrac{1}{2}.\dfrac{mm}{m+m}(1-e^2)U^2$

$$= \frac{1}{4}.m(1-e^2)U^2$$

$$= \frac{1}{4}.\, m\left(U^2 - e^2U^2\right) \qquad \_\_\_\_\_\_\_\_\_\_ \text{(i)}$$

By Newton's Experimental Law, we have

$$v_1 - v_2 = -\ e(u_1 - u_2)$$

$\Rightarrow \quad V = -\, eU$

Where V = relative velocity after impact. = $v_1 - v_2$

$\Rightarrow \quad V^2 = e^2U^2$

Using this value in (i), we get

$$\text{Loss of K.E.} = \frac{1}{4}.\, m\left(U^2 - V^2\right)$$

Which is required.

❖ QUESTION 13

If two elastic spheres have direct impact. Show that the energy lost during impact is equal to the energy of the body whose mass is half of the harmonic mean between those of the spheres and whose velocity is equal to relative velocity before impact.

SOLUTION

We know that if two spheres of mass m_1 and m_2 moving with velocities u_1 and u_2 collide directly then

$$\text{Loss of K.E.} = \frac{1}{2}.\frac{m_1m_2}{m_1+m_2}(1-e^2)(u_1-u_2)^2$$

Here e = 0 since spheres are inelastic.

$$\Rightarrow \quad \text{Loss of K.E.} = \frac{1}{2}.\frac{m_1m_2}{m_1+m_2}(u_1-u_2)^2$$

We know that

Half of the harmonic mean between m_1 and m_2 = $\frac{1}{2}.\frac{2m_1m_2}{m_1+m_2} = \frac{m_1m_2}{m_1+m_2}$

Relative velocity before impact = $u_1 - u_2$

Therefore

K.E. of the body whose mass is $\frac{m_1m_2}{m_1+m_2}$ and velocity is $u_1 - u_2$

$$= \frac{1}{2}.\frac{m_1m_2}{m_1+m_2}(u_1-u_2)^2$$

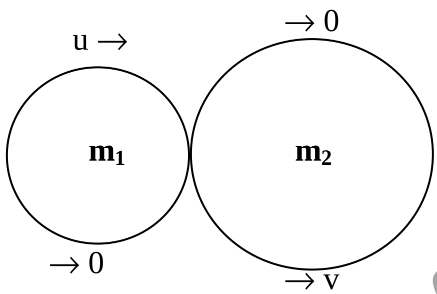
Which is same as the K.E. loss of K.E. during impact. Hence the result.

❖ QUESTION 14

A ball impinges directly on another ball at rest and is itself reduced to rest by impact. If half of initial K.E. is destroyed in collision, find the coefficient of restitution.

SOLUTION

Let m_1 and m_2 be masses of two spheres and u be the velocity of sphere of mass m_1 before impact v be the velocity of sphere of mass m_2 after impact.



By Newton's Experimental Law, we have

$$0 - v = -\ e(u - 0)$$

$\Rightarrow \quad v = eu \qquad$ ________(i)

By Law of Conservation of Momentum, we have

$$0 + m_2v = m_1u + m_2 0$$

$\Rightarrow \quad m_2v = m_1u \qquad$ ________(ii)

Now $\quad$ Initial K.E. $= \frac{1}{2}m_1u^2 + \frac{1}{2}m_2 0 = \frac{1}{2}m_1u^2$

$\quad$ Final K.E. $= \frac{1}{2}m_1 0 + \frac{1}{2}m_2 v = \frac{1}{2}m_2v^2$

So $\quad$ Loss of K.E. = Initial K.E. – Final K.E.

$$= \frac{1}{2}m_1u^2 - \frac{1}{2}m_2v^2$$

According to given condition,

$$\text{Loss of K.E.} = \frac{1}{2}(\text{Initial K.E.})$$

$\Rightarrow \quad \frac{1}{2}m_1u^2 - \frac{1}{2}m_2v^2 = \frac{1}{4}m_1u^2$

$\Rightarrow \quad 2m_1u^2 - 2m_2v^2 = m_1u^2$

$\Rightarrow \quad m_1u^2 = 2m_2v^2 = 2m_2v.v$

$\Rightarrow \quad m_1u^2 = 2m_1uv \qquad$ By(ii)

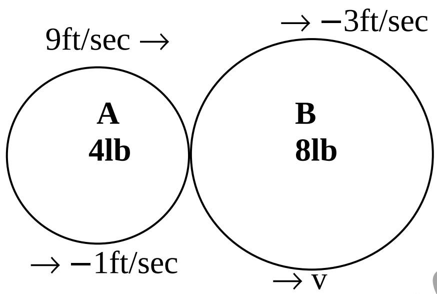
$\Rightarrow \quad u = 2v \quad \Rightarrow \quad u = 2eu \quad \Rightarrow \quad e = \frac{1}{2}$

❖ QUESTION 15

Two spheres of masses 4lb and 8lb moving with velocities 9ft/sec and 3ft/sec in opposite directions collide. If A rebounds with velocity of 1ft/sec, find the velocity of B after impact, the coefficient of elasticity and loss of K.E.

SOLUTION

Let v be the velocity of sphere B after impact.



By Law of Conservation of Momentum, we have

$$4 \times 9 + 8 \times (-3) = 4 \times (-1) + 8 \times v$$

$\Rightarrow \quad 36 - 24 = -4 + 8v$

$\Rightarrow \quad 12 + 4 = 8v \quad \Rightarrow \quad v = 2\text{ft/sec}$

Newton's Experimental Law, we have

$$-1 - v = -e(9 - (-3))$$

$\Rightarrow \quad -1 - 2 = -12e$

$\Rightarrow \quad e = \dfrac{1}{4}$

Now Initial K.E. $= \frac{1}{2}(4)(9)^2 + \frac{1}{2}(8)(-3)^2$

$= 198$

Final K.E. $= \frac{1}{2}(4)(-1)^2 + \frac{1}{2}(8)(2)^2$

$= 18$

So Loss of K.E. = Initial K.E. – Final K.E.

$= 198 - 18$

$= 180$ft.poundals.

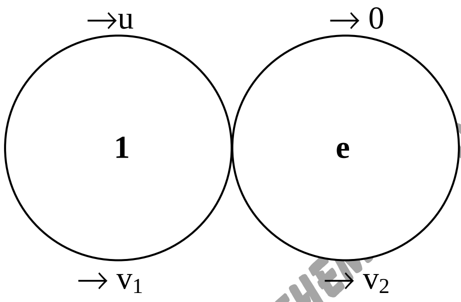
❖ QUESTION 16

A series of n elastic spheres whose masses are 1, e, e^2 etc. are at rest separated by intervals, with their centers on a straight line. The first is made to impinge directly on the second with velocity u. Show that finally the first $(n - 1)$spheres will be moving with the same velocity $(1 - e)u$ and the last with velocity u. Prove that the final K.E. of the system is

$$\frac{1}{2}(1 - e + 2^n)u^2$$

SOLUTION

Consider the impact of the first and second ball.



Let v_1 and v_2 be the velocities of the spheres after impact.

By Newton’s Experimental Law, we have

$$v_1 - v_2 = -e(u - 0)$$

$\Rightarrow \quad v_1 - v_2 = -eu \qquad$ _______(i)

By Law of Conservation of Momentum, we have

$$1.u + 0 = 1.v_1 + ev_2$$

$\Rightarrow \quad v_1 + ev_2 = u \qquad$ _______(ii)

Subtracting (i) from (ii), we get

$$v_2 + ev_2 = u + eu$$

$\Rightarrow \quad v_2(1 + e) = (1 + e)u$

$\Rightarrow \quad v_2 = u$

Using value of v_2 in (ii), we get

$$v_1 = (1 - e)u$$

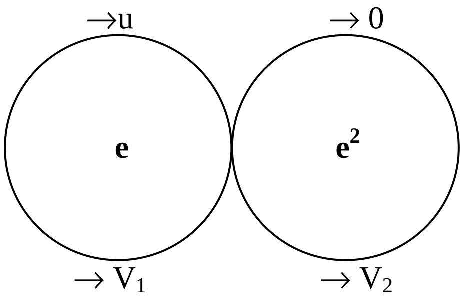
Thus, we conclude

Velocity of impinging ball after impact = $(1 - e)u$

= $(1 - e)$velocity of impinging ball before impact

Velocity of the ball at rest after impact = u = velocity of impinging ball before impact

Now we consider the impact of the 2nd and 3rd ball.



Let V_1 and V_2 be the velocities of the spheres after impact.

By Newton's Experimental Law, we have

$$V_1 - V_2 = -e(u - 0)$$

$$\Rightarrow \quad V_1 - V_2 = -eu \qquad \text{\_\_\_\_\_\_\_(iii)}$$

By Law of Conservation of Momentum, we have

$$1.u + 0 = 1.V_1 + eV_2$$

$$\Rightarrow \quad V_1 + eV_2 = u \qquad \text{\_\_\_\_\_\_\_(iv)}$$

Subtracting (iii) from (iv), we get

$$V_2 + eV_2 = u + eu$$

$$\Rightarrow \quad V_2(1 + e) = (1 + e)u$$

$$\Rightarrow \quad V_2 = u$$

Using value of V_2 in (iv), we get

$$V_1 = (1 - e)u$$

Thus, we conclude

Velocity of impinging ball after impact = $(1 - e)u$

= $(1 - e)$velocity of impinging ball before impact

Velocity of the ball at rest after impact = u = velocity of impinging ball before impact

Proceeding in the same way,

Velocity of $(n - 1)^{th}$ ball after it impinges on n^{th} ball = $(1 - e)u$

and **Velocity of n^{th} ball after impact = u**

Final K.E. of the system

$$= \frac{1}{2}1.[(1-e)u]^2 + \frac{1}{2}e.[(1-e)u]^2 + \ldots + \frac{1}{2}e^{n-2}[(1-e)u]^2 + \frac{1}{2}e^{n-1}u^2$$

$$= \frac{1}{2}(1-e)^2u^2[1 + e + e^2 + \ldots + e^{n-2}] + \frac{1}{2}e^{n-1}u^2$$

$$= \frac{1}{2}(1-e)^2u^2\left[\frac{e^{n-1} - 1}{e - 1}\right] + \frac{1}{2}e^{n-1}u^2$$

$$=\frac{1}{2}(1-e)^2u^2\left[\frac{1-e^{n-1}}{1-e}\right]+\frac{1}{2}e^{n-1}u^2$$

$$=\frac{1}{2}(1-e)\left(1-e^{n-1}\right)u^2+\frac{1}{2}e^{n-1}u^2$$

$$=\frac{1}{2}\left[(1-e)\left(1-e^{n-1}\right)+e^{n-1}\right]u^2$$

$$=\frac{1}{2}[1-e^{n-1}-e+e^n+e^{n-1}]u^2$$

$$=\frac{1}{2}(1-e+e^n)u^2$$

Which is required.

%%%%% End of The Chapter # 8 %%%%%