Two bodies are said to be impinge or collide directly when the direction of the motion of each is along their common normal at the point of contact. When the direction of motion of either or both is not along the common normal, the impact is said to be oblique. i.e. the bodies impinge obliquely.

A collision may be elastic or inelastic. In an elastic collision, kinetic energy is same both before and after collision or kinetic energy is conserved in an elastic collision. In an inelastic collision, the initial and final kinetic energies are not equal. All collisions between real objects are more or less inelastic except when the objects are very rigid such as billiard balls. When two bodies stick together after collision , the collision is said to be completely inelastic. The law of conservation of momentum remains valid both in elastics and inelastic collisions.
**IMPACT LAWS**

**NEWTON’S EXPERIMENTAL LAWS**

If two bodies of mass $m_1$ and $m_2$, moving with velocities $u_1$ and $u_2$ respectively, collide directly and let $v_1$ and $v_2$ be their velocities after collision.

Then

$$\frac{v_1 - v_2}{u_1 - u_2} = -e$$

$$\Rightarrow v_1 - v_2 = -e(u_1 - u_2)$$

Where $e$ is called the modulus or coefficient of elasticity or restitution of the bodies.

If two bodies of mass $m_1$ and $m_2$, moving with velocities $u_1$ and $u_2$ in the directions inclined at angles $\alpha$ and $\beta$ respectively to their common normal. Let $v_1$ and $v_2$ be their velocities in the directions inclined at angles $\theta$ and $\phi$ respectively after collision. Then by

$$\frac{v_1 \cos \theta - v_2 \cos \phi}{u_1 \cos \alpha - u_2 \cos \beta} = -e$$

$$\Rightarrow v_1 \cos \theta - v_2 \cos \phi = -e(u_1 \cos \alpha - u_2 \cos \beta)$$
**LAW OF CONSERVATION OF MOMENTUM**

\[ m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2 \]  
(For Direct Collisions)

\[ m_1v_1 \cos \theta + m_2v_2 \cos \varphi = m_1u_1 \cos \alpha + m_2u_2 \cos \beta \]  
(For Oblique Collisions)

**IMPACT OF A SMOOTH SPHERE AGAINST A FIXED PLANE**

Let AB be a fixed plane and C be the point at which the sphere impinges. Let O be the centre of the sphere so that CO is the common normal. Let u and v be the velocities of the sphere before and after impact inclined at angles \( \alpha \) and \( \theta \) respectively to the normal.

By Newton’s experimental law along CO, we have

\[ v \cos \theta = -e(-u \cos \alpha - 0) \]

\[ v \cos \theta = e \cos \alpha \]  
_____(i)

By resolving velocities before and after collision, we have

\[ v \sin \theta = u \sin \alpha \]  
_____(ii)

Squaring and adding (i) and (ii), we get

\[ v^2 = u^2 (e^2 \cos^2 \alpha + \sin^2 \alpha) \]  
_____(iii)

Eq(iii) gives the magnitude of the velocity of the sphere after impact.

Dividing (i) by (ii), we have

\[ \cot \theta = e \cot \alpha \]  
_____(iv)

Eq(iv) gives the direction of motion of the sphere after impact.

We know that

Kinetic Energy = \( \frac{1}{2} \) (mass)(velocity)\(^2\)

Therefore,

Initial K.E. = \( \frac{1}{2} \) \( mu^2 \) and Final K.E. = \( \frac{1}{2} \) \( mv^2 \)

Now
Loss of K.E. = Initial K.E. – Final K.E.

\[
\text{Loss of K.E.} = \frac{1}{2} mu^2 - \frac{1}{2} mv^2
\]

\[
= \frac{1}{2} mu^2 - \frac{1}{2} mu^2 (e^2 \cos^2 \alpha + \sin^2 \alpha) \quad \text{By(iii)}
\]

\[
= \frac{1}{2} mu^2 (1 - (e^2 \cos^2 \alpha + \sin^2 \alpha))
\]

\[
= \frac{1}{2} mu^2 (1 - e^2 \cos^2 \alpha - \sin^2 \alpha)
\]

\[
= \frac{1}{2} mu^2 (\cos^2 \alpha - e^2 \cos^2 \alpha)
\]

\[
= \frac{1}{2} mu^2 (1 - e^2) \cos^2 \alpha
\]

Now we discuss some important cases

**CASE 1**

When \( \alpha = 0 \) i.e. When impact is direct.

Then \( \theta = 0 \) \quad \text{By(iv)}

and \( v = eu \) \quad \text{By(iii)}

Thus, if a sphere impinges directly on a smooth fixed plane with velocity \( u \), it rebounds in the reverse direction with velocity \( eu \).

**CASE 2**

When \( e = 1 \). i.e. sphere is perfectly elastic.

Then \( \theta = \alpha \) \quad \text{By(iv)}

and \( v = u \) \quad \text{By(iii)}

Thus, if a perfectly elastic sphere impinges on a smooth fixed plane, its velocity is unaltered in magnitude by impact and the angle of reflection (\( \theta \)) is equal to the angle of incidence (\( \alpha \)).

**CASE 3**

When \( e = 0 \). i.e. sphere is perfectly inelastic.

Then \( \theta = 90^0 \) \quad \text{By(iv)}

and \( v = \sin \alpha \) \quad \text{By(iii)}

Thus, if a perfectly inelastic sphere impinges on a smooth fixed plane, it does not rebound but simply slides along the plane, its velocity parallel to the plane remained unaltered.
**QUESTION 1**

A rubber ball drops from a height \( h \) and after rebounding twice from ground it reach a height \( \frac{h}{2} \). Find the coefficient of restitution. What would be the coefficient of restitution had the ball reached a height \( \frac{h}{2} \) after rebounding three times?

**SOLUTION**

We know that

\[
 v^2 - u^2 = 2gh
\]

Here \( u = 0 \)

\[ \Rightarrow v = \sqrt{2gh} \]

So, Velocity on reaching the floor = \( \sqrt{2gh} \)

Let \( e \) be the coefficient of restitution. Then

Velocity after 1\(^{st} \) rebound = \( e \times \) velocity before the rebound

\[ = e \sqrt{2gh} \]

\[ \Rightarrow \text{Velocity on reaching the floor 2\(^{nd} \) time} = e \sqrt{2gh} \]

Now

Velocity after 2\(^{nd} \) rebound = \( e \times \) velocity before the 2\(^{nd} \) rebound

\[ = e^2 \sqrt{2gh} \]

For the motion after 2\(^{nd} \) rebound,

Initial velocity = \( u = e \sqrt{2gh} \)

Final Velocity = \( v = 0 \)

Height attained = \( h = \frac{h}{2} \) and \( g = -g \)

We know that

\[
 v^2 - u^2 = 2gh
\]

\[ \Rightarrow 0 - (e\sqrt{2gh})^2 = 2(-g) \frac{h}{2} \]

\[ \Rightarrow 2geh^4 = gh \]

\[ \Rightarrow e^4 = \frac{1}{2} \quad \Rightarrow \quad e = \left( \frac{1}{2} \right)^{1/4} \quad \Rightarrow \quad e = \frac{1}{\sqrt[4]{2}} \]

Velocity on reaching the floor 3\(^{rd} \) time = \( e^2 \sqrt{2gh} \)

Now

Velocity after 3\(^{rd} \) rebound = \( e \times \) velocity before the 3\(^{rd} \) rebound = \( e^3 \sqrt{2gh} \)
For the motion after 3\textsuperscript{rd} rebound

\begin{align*}
\text{Initial velocity} & = u = e \frac{e^3 \sqrt{2gh}}{g^3} \\
\text{Final Velocity} & = v = 0 \\
\text{Height attained} & = h = \frac{h}{2} \quad \text{and} \quad g = - g
\end{align*}

We know that

\[ v^2 - u^2 = 2gh \]

\[ \Rightarrow \quad 0 - \left(\frac{e^3 \sqrt{2gh}}{g^3}\right)^2 = 2\left(-\frac{h}{2}\right) \]

\[ \Rightarrow \quad 2ge^6 = gh \]

\[ \Rightarrow \quad e^6 = \frac{1}{2} \quad \Rightarrow \quad e = \left(\frac{1}{2}\right)^{1/6} \quad \Rightarrow \quad e = \frac{1}{\sqrt[6]{2}} \]

\section*{QUESTION 2}

A heavy elastic ball is dropped upon a horizontal floor from a height of 20ft. and after rebounding twice, it is observed to attain a height 10ft. Find the coefficient of restitution.

\textbf{SOLUTION}

We know that

\[ v^2 - u^2 = 2gh \]

Here \( u = 0, \ h = 20, \ g = 32\text{ft}. \)

\[ \Rightarrow \quad v = \sqrt{2gh} = \sqrt{2 \times 32 \times 20} = 16\sqrt{5}\text{ft/sec} \]

So, \( \text{Velocity on reaching the floor} = 16\sqrt{5}\text{ft/sec} \)

Let \( e \) be the coefficient of restitution. Then

\( \text{Velocity after 1}^{\text{st}} \text{rebound} = e \times \text{velocity before the rebound} \)

\[ = 16e\sqrt{5}\text{ft/sec} \]

\[ \Rightarrow \quad \text{Velocity on reaching the floor 2}^{\text{nd}} \text{time} = 16e\sqrt{5}\text{ft/sec} \]

Now

\( \text{Velocity after 2}^{\text{nd}} \text{rebound} = e \times \text{velocity before the 2}^{\text{nd}} \text{rebound} \)

\[ = 16e^2\sqrt{5}\text{ft/sec} \]

For the motion after 2\textsuperscript{nd} rebound

\begin{align*}
\text{Initial velocity} & = u = 16e^2\sqrt{5}\text{ft/sec} \\
\text{Final Velocity} & = v = 0 \\
\text{Height attained} & = h = 10\text{ft.} \quad \text{and} \quad g = - 32\text{ft}
\end{align*}
We know that
\[ v^2 - u^2 = 2gh \]
\[ \Rightarrow 0 - (16e^2\sqrt{5})^2 = 2(-32)(10) \]
\[ \Rightarrow 1280 e^4 = 640 \]
\[ \Rightarrow e^4 = \frac{1}{2} \Rightarrow e = \left(\frac{1}{2}\right)^{1/4} \Rightarrow e = \frac{1}{\sqrt{2}} \]

**QUESTION 3**

Two elastic spheres of masses \( m_1 \) and \( m_2 \) moving with velocities \( u_1 \) and \( u_2 \) impinge directly. If \( e \) is the coefficient of restitution, find their velocities after impact.

**SOLUTION**

Let \( v_1 \) and \( v_2 \) be the velocities of \( m_1 \) and \( m_2 \) after impact respectively.

By Newton’s Experimental Law, we have
\[ v_1 - v_2 = -e(u_1 - u_2) \]  
\[ \text{Equation (i)} \]

By Law of Conservation of Momentum, we have
\[ m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2 \]  
\[ \text{Equation (ii)} \]

Multiplying (i) by \( m_1 \) and \( m_2 \), we get
\[ m_1v_1 = -m_1ev_1u_1 + em_1u_2 \]  
\[ \text{Equation (iii)} \]
\[ m_2v_1 = -m_2ev_2u_1 + em_2u_2 \]  
\[ \text{Equation (iv)} \]

Adding (ii) and (iv), we get
\[ m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2 - em_2u_1 + em_2u_2 \]
\[ \Rightarrow (m_1 + m_2)v_1 = (m_1 - em_2)u_1 + m_2(1 + e)u_2 \]
\[ \Rightarrow v_1 = \frac{(m_1 - em_2)u_1 + m_2(1 + e)u_2}{m_1 + m_2} \]  
\[ \text{Equation (v)} \]

Subtracting (iii) from (ii), we get
\[ m_2v_2 + m_1v_2 = m_1u_1 + m_2u_2 + em_1u_1 - em_1u_2 \]
\[ \Rightarrow (m_1 + m_2)v_2 = m_1(1 + em_2)u_1 + (m_2 - e m_1)u_2 \]
\[ v_2 = \frac{m_1(1 + e m_2)u_1 + (m_2 - e m_1)u_2}{m_1 + m_2} \]  

Equations (v) & (vi) gives the velocities after impact.

**QUESTION 4**

If two equal and perfectly elastic spheres moving with velocities \( u_1 \) and \( u_2 \) impinge directly. Show that they interchange their velocities after impact.

**SOLUTION**

Let \( v_1 \) and \( v_2 \) be the velocities of spheres after impact respectively.

By Newton’s Experimental Law, we have

\[ v_1 - v_2 = -(u_1 - u_2) \quad : e = 1 \text{ for perfectly elastic spheres.} \]

\[ v_1 - v_2 = u_2 - u_1 \]  

\[ \text{________(i)} \]

By Law of Conservation of Momentum, we have

\[ m v_1 + m v_2 = m u_1 + m u_2 \]

\[ v_1 + v_2 = u_1 + u_2 \]

\[ \text{________(ii)} \]

Adding (i) and (ii), we get

\[ 2v_1 = 2u_2 \quad \Rightarrow \quad v_1 = u_2 \]

Subtracting (i) from (ii), we get

\[ 2v_2 = 2u_1 \quad \Rightarrow \quad v_2 = u_1 \]

Thus \( v_1 = u_2 \) and \( v_2 = u_1 \)

**QUESTION 5**

Three perfectly elastic spheres of masses \( m \), \( 2m \) and \( 3m \) are placed in a straight line. The first impinges directly on the second with a velocity \( u \) and the second impinges on the third. Find the velocity of the third ball after impact.

**SOLUTION**
Let \( v_1 \) and \( v_2 \) be the velocities of the ball of masses \( m \) and \( 2m \) respectively after impact.

Here \( e = 1 \)

By Newton’s Experimental Law, we have
\[
v_1 - v_2 = -(u - 0)
\]
\[
\Rightarrow v_1 - v_2 = -u \quad \text{(i)}
\]

By Law of Conservation of Momentum, we have
\[
m v_1 + 2m v_2 = mu + 2m(0)
\]
\[
\Rightarrow v_1 + 2v_2 = u \quad \text{(ii)}
\]

Subtracting (i) from (ii), we get
\[
3v_2 = 2u
\]
\[
\Rightarrow v_2 = \frac{2}{3}u
\]

Which is the velocity of sphere of mass \( 2m \) before it impinges on the third ball. Let \( V_1 \) and \( V_2 \) be the velocities of masses \( 2m \) and \( 3m \) respectively after impact.

By Newton’s Experimental Law, we have
\[
V_1 - V_2 = -(v_2 - 0)
\]
\[
\Rightarrow V_1 - V_2 = -v_2 \quad \text{(iii)}
\]

By Law of Conservation of Momentum, we have
\[
2m V_1 + 3m V_2 = 2mv_2 + 3m(0)
\]
\[
\Rightarrow 2V_1 + 3V_2 = 2v_2 \quad \text{(iv)}
\]
Multiplying (iii) by 2, we get
\[ 2V_1 - 2V_2 = -2v_2 \] _______(v)

Subtracting (v) from (iv), we get
\[ 5V_2 = 4v_2 \]
\[ \Rightarrow 5V_2 = 4 \times \frac{2}{3}u \]
\[ \Rightarrow V_2 = \frac{8}{15}u \]

Which is the velocity of the sphere of mass 3m after impact.

**QUESTION 6**

If the masses of two balls be as 2:1 and their respective velocities be as 1:2 in opposite direction. Show that if the coefficient of restitution is 5/6, each ball moves back after impact, with 5/6th of its original velocity.

**SOLUTION**

Let masses of two balls be 2m, m and their velocities before impact be u and 2u. Let \( v_1 \) and \( v_2 \) be their velocities after impact.

\[ \text{u} \rightarrow v_1 \rightarrow 2m \rightarrow -2u \rightarrow v_2 \]

By Newton’s Experimental Law, we have
\[ v_1 - v_2 = \frac{5}{6}[u - ( -2u)] \]
\[ \Rightarrow v_1 - v_2 = -\frac{5}{2}u \] _______(i)

By Law of Conservation of Momentum, we have
\[ 2mv_1 + mv_2 = 2mu + m(-2u) \]
\[ \Rightarrow 2v_1 + v_2 = 0 \] _______(ii)

Adding (i) and (ii), we get
\[ 3v_1 = -\frac{5}{2}u \]
\[ \Rightarrow v_1 = -\frac{5}{6}u \] _______(iii)
From (i), we get

\[ v_2 = v_1 + \frac{5}{2}u \]

\[ = -\frac{5}{6}u + \frac{5}{2}u \quad \text{By(iii)} \]

\[ = -\frac{5}{6}(u - 3u) \]

\[ = \frac{5}{6}(2u) \quad \text{_____ (iv)} \]

Equations (iii) and (iv) show that each ball moves back after impact with 5/6th of its original velocity.

**QUESTION 7**

Two elastic spheres impinge directly with equal and opposite velocities. Find ratio of their masses so that one of them may be reduced to rest by the impact, the coefficient of elasticity being \( e \).

**SOLUTION**

Let \( m_1 \) and \( m_2 \) be masses of two spheres and \( u \) be the velocity of each sphere. Let the sphere of mass \( m_1 \) come to rest after impact and \( v \) be the velocity of sphere of mass \( m_2 \) after impact.

We have to find \( m_1 : m_2 \)

By Newton’s Experimental Law, we have

\[ 0 - v = -e[u - (-u)] \]

\[ \Rightarrow \quad v = 2eu \quad \text{_____ (i)} \]

By Law of Conservation of Momentum, we have

\[ 0 + m_2v = m_1u + m_2(-u) \]

\[ \Rightarrow \quad m_22eu = m_1u + m_2(-u) \quad \text{By(i)} \]

\[ \Rightarrow \quad m_22e = m_1 - m_2 \]

\[ \Rightarrow \quad m_22e + m_2 = m_1 \quad \Rightarrow \quad (1 + 2e)m_2 = m_1 \]
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⇒ \[ \frac{m_1}{m_2} = \frac{1 + 2e}{1} \]
⇒ \[ m_1 : m_2 = 1 + 2e : 1 \]

**QUESTION 8**

A ball A moving with velocity \( u \) impinges directly on an equal ball B moving with velocity \( v \) in the opposite direction. If A be brought to rest by impact, show that

\[ u : v = 1 + e : 1 - e \]

Where \( e \) is the coefficient of restitution.

**SOLUTION**

Let \( m \) be the mass of each sphere and \( V \) be the velocity of the sphere B after impact.

[Diagram of two spheres colliding]

By Newton’s Experimental Law, we have

\[ 0 - V = e[u - (-v)] \]
⇒ \[ V = e(u + v) \] \[ \text{(i)} \]

By Law of Conservation of Momentum, we have

\[ 0 + mV = mu + m(-v) \]
⇒ \[ V = u - v \]
⇒ \[ u - v = e(u + v) \] By(i)
⇒ \[ u - eu = v + ev \]
⇒ \[ u(1 - e) = v(1 + e) \]
⇒ \[ \frac{u}{v} = \frac{1 + e}{1 - e} \]
⇒ \[ u : v = 1 + e : 1 - e \]
**QUESTION 9**

A sphere impinges directly on an equal sphere at rest. If the coefficient of restitution is $e$, show that their velocities after impact are as $1 + e : 1 - e$. If the mass of first sphere is $m$ and second sphere is $M$, show that the first cannot have its velocity reversed if $m > eM$.

**SOLUTION**

Let $m$ be the mass of each sphere and $u$ be the velocity of the first sphere and let $v_1, v_2$ are their velocities after impact. We have to show that $v_1 : v_2 = 1 + e : 1 - e$

![Diagram](image)

By Newton’s Experimental Law, we have

$$v_1 - v_2 = -e(u - 0) = -eu$$

_____ (i)

By Law of Conservation of Momentum, we have

$$mu + 0 = mv_1 + mv_2$$

$$\Rightarrow \quad v_1 + v_2 = u$$

_____ (ii)

From (i) and (ii), we get

$$v_1 - v_2 = -e(v_1 + v_2)$$

$$\Rightarrow \quad v_1 + ev_1 = v_2 - ev_2$$

$$\Rightarrow \quad v_1(1 + e) = v_2(1 - e)$$

$$\Rightarrow \quad \frac{v_1}{v_2} = \frac{1 - e}{1 + e}$$

$$\Rightarrow \quad v_1 : v_2 = 1 - e : 1 + e$$

Let the mass of second sphere is $M$. 

![Diagram](image)
By Law of Conservation of Momentum, we have
\[ mu + 0 = mv_1 + Mv_2 \]
\[ \Rightarrow \quad mv_1 + Mv_2 = mu \quad \text{____}(iii) \]
Multiplying (i) by \( M \), we get
\[ Mv_1 - Mv_2 = -Mvu \quad \text{____}(iv) \]
Adding (iii) and (iv), we get
\[ mv_1 + Mv_1 = mu - Mvu \]
\[ \Rightarrow \quad (m + M)v_1 = (m - Me)u \]
\[ \Rightarrow \quad v_1 = \left( \frac{m - Me}{m + M} \right) u \]
First sphere cannot have its velocity reversed if \( v_1 > 0 \).
i.e.
\[ \frac{m - Me}{m + M} u > 0 \quad \Rightarrow \quad m - Me > 0 \quad \Rightarrow \quad m > Me \]

**QUESTION 10**

Two spheres of masses \( M, m \) impinge directly when moving in opposite directions with velocities \( u, v \) respectively and the sphere of mass \( m \) is brought to rest by the collision. Prove that
\[ v(m - eM) = M(1 + e)u \]
After collision, the sphere of mass \( M \) is acted on by a constant retarding force which brings it to rest after travelling a distance \( a \). Prove that the magnitude of this force is
\[ \frac{Me^2(u + v)^2}{2a} \]

**SOLUTION**

Let \( V \) be the velocity of the sphere of mass \( M \) after impact.

\[ \rightarrow u \quad \rightarrow v \]
\( M \)
\( m \)
\( \rightarrow V \quad \rightarrow 0 \)

By Newton’s Experimental Law, we have
\[ V - 0 = -e[u - (-v)] \]
\[ V = - e(u + v) \]  \quad \text{(i)}

By Law of Conservation of Momentum, we have

\[ MV = Mu - mv \]
\[ - Me(u + v) = Mu - mv \quad \text{By (i)} \]
\[ - Meu - Mev = Mu - mv \]
\[ mv - Mev = Mu + Meu \]
\[ v(m - Me) = M(1 + e)u \]

Which is required.

Initial velocity of sphere of mass M after impact = \[ V = - e(u + v) \]

We know that

\[ v^2 - u^2 = 2as \]  \quad \text{(ii)}

Here

Final velocity = \( v = 0 \)
Initial velocity = \( u = - e(u + v) \)
Distance covered = \( s = a \) (given)

Suppose that

\[ \text{Retardation} = a = - f \]

Using these values in (ii), we get

\[ 0 - (-e(u + v))^2 = 2(-f)a \]
\[ f = \frac{e^2(u + v)^2}{2a} \]

Thus,

\[ \text{Retarding force} = \text{mass} \times \text{retardation} \]
\[ = \frac{Me^2(u + v)^2}{2a} \]

**QUESTION 11**

An imperfectly elastics sphere of mass \( m \) moving velocity \( u \) impinges on another sphere of mass \( M \) at rest. The second sphere afterwards strikes a vertical plane at right angle to its path. Show that there will be no further impact of the spheres if

\[ m(1 + e' + ee') < eM \]

Where \( e \) and \( E \) are the coefficients of restitution between the spheres and between the sphere and the plane respectively.
**SOLUTION**

Let $v_1$ and $v_2$ be the velocities of the sphere of mass $m$ and $M$ respectively after impact.

By Newton’s Experimental Law, we have

$$v_1 - v_2 = -e(u - 0) = -eu$$

_____ (i)

By Law of Conservation of Momentum, we have

$$mu + 0 = mv_1 + Mv_2$$

$$\Rightarrow mv_1 + Mv_2 = mu$$

_____ (ii)

Multiplying (i) by $M$, we get

$$Mv_1 - Mv_2 = -Meu$$

_____ (iii)

Adding (ii) and (iii), we get

$$mv_1 + Mv_1 = mu - Meu$$

$$\Rightarrow (m + M)v_1 = (m - Me)u$$

$$\Rightarrow v_1 = \left(\frac{m - Me}{m + M}\right) u$$

From (i), we get

$$v_2 = v_1 + eu$$

$$= \left(\frac{m - Me}{m + M}\right) u + eu$$

$$= \frac{mu - Meu + meu + Meu}{m + M}$$

$$= \frac{(1 + e)mu}{m + M}$$

The sphere of mass $M$ strikes with plane with velocity $v_2$ and it then rebound with velocity $e'v_2$ ($\because e'$ is the coefficient of restitution between sphere and plane). Also the velocity of the sphere of mass $m$ away from the plane is $-v_1$. Thus there will be no further impact if

$$e'v_2 < -v_1$$
Taking square of (ii), we get

\[ \Rightarrow \quad m \left( m - M \right) \left( 1 - e^2 \right) \left( u_1 - u_2 \right)^2 < (1 - e^2)(u_1 - u_2)^2 \]

or
\[ (1 + e^2 - e^2) m + m < M \]

or
\[ (1 + e^2) m < eM \]

Which is required.

**LOSS OF KINETIC ENERGY DUE TO DIRECT IMPACT**

Two spheres of given masses with given velocities impinge directly. Show that there is always loss of kinetic energy unless the elasticity is perfect.

**SOLUTION**

Let \( m_1 \) and \( m_2 \) be the masses of spheres and \( u_1 \) and \( u_2 \) be their velocities before impact. Let \( v_1 \) and \( v_2 \) be their velocities after impact and \( e \) be the coefficient of restitution. Then

\[
\text{Total K.E. before impact} = \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 = \frac{1}{2} (m_1 u_1^2 + m_2 u_2^2)
\]

\[
\text{Total K.E. after impact} = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} (m_1 v_1^2 + m_2 v_2^2)
\]

By Newton’s Experimental Law, we have

\[ v_1 - v_2 = - e(u_1 - u_2) \quad \text{____(i)} \]

By Law of Conservation of Momentum, we have

\[ m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2 \quad \text{____(ii)} \]

Taking square and then multiplying (i) by \( m_1 m_2 \), we get

\[ m_1 m_2 (v_1 - v_2)^2 = m_1 m_2 e^2 (u_1 - u_2)^2 \quad \text{____(iii)} \]

Taking square of (ii), we get

\[ (m_1 v_1 + m_2 v_2)^2 = (m_1 u_1 + m_2 u_2)^2 \quad \text{____(iv)} \]

Adding (iii) and (iv), we get

\[
\begin{align*}
(m_1 v_1 + m_2 v_2)^2 + m_1 m_2 (v_1 - v_2)^2 &= (m_1 u_1 + m_2 u_2)^2 + m_1 m_2 e^2 (u_1 - u_2)^2 \\
\Rightarrow m_1^2 v_1^2 + m_2^2 v_2^2 + m_1 m_2 v_1^2 + m_1 m_2 v_2^2 &= (m_1 u_1 + m_2 u_2)^2 + m_1 m_2 e^2 (u_1 - u_2)^2 \\
&\quad \quad \quad + m_1 m_2 (u_1 - u_2)^2 - m_1 m_2 (u_1 - u_2)^2 \\
&\Rightarrow m_1 (m_1 + m_2) v_1^2 + m_2 (m_1 + m_2) v_2^2 = (m_1 u_1 + m_2 u_2)^2 + m_1 m_2 (u_1 - u_2)^2 \\
&\quad \quad \quad - m_1 m_2 (1 - e^2)(u_1 - u_2)^2 \\
&\Rightarrow (m_1 + m_2) (m_1 v_1^2 + m_2 v_2^2) = m_1^2 u_1^2 + m_2^2 u_2^2 + m_1 m_2 u_1^2 + m_1 m_2 u_2^2 \\
&\quad \quad \quad - m_1 m_2 (1 - e^2)(u_1 - u_2)^2 \\
&\quad \quad \quad = m_1 (m_1 + m_2) u_1^2 + m_2 (m_1 + m_2) u_2^2 - m_1 m_2 (1 - e^2)(u_1 - u_2)^2
\end{align*}
\]
\[
( m_1 + m_2)( m_1 v_1^2 + m_2 v_2^2) = ( m_1 + m_2)( m_1 u_1^2 + m_2 u_2^2) - m_1 m_2(1 - e^2)(u_1 - u_2)^2
\]

\[
m_1 v_1^2 + m_2 v_2^2 = m_1 u_1^2 + m_2 u_2^2 - \frac{m_1 m_2}{m_1 + m_2}(1 - e^2)(u_1 - u_2)^2
\]

\[
\frac{1}{2}( m_1 v_1^2 + m_2 v_2^2) = \frac{1}{2}( m_1 u_1^2 + m_2 u_2^2) - \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2}(1 - e^2)(u_1 - u_2)^2
\]

\[
\frac{1}{2}( m_1 v_1^2 + m_2 v_2^2) - \frac{1}{2}( m_1 u_1^2 + m_2 u_2^2) = - \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2}(1 - e^2)(u_1 - u_2)^2
\]

\[
\text{Total K.E. after impact} - \text{Total K.E. before impact} = - \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2}(1 - e^2)(u_1 - u_2)^2
\]

\[
\text{Change in K.E.} = - \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2}(1 - e^2)(u_1 - u_2)^2
\]

Since there is a negative sign on the R.H.S and \( e < 1 \Rightarrow 1 - e^2 > 0 \)

Therefore, there is loss of K.E. due to impact.

Thus,

\[
\text{K.E. Lost due to impact} = \frac{1}{4} \frac{m_1 m_2}{m_1 + m_2}(1 - e^2)(u_1 - u_2)^2
\]

When \( e = 1 \), i.e. elasticity is perfect.

Then, \( \text{K.E. Lost due to impact} = 0 \)

Thus there is no loss of K.E. when elasticity is perfect.

**QUESTION 12**

Two elastic spheres each of mass \( m \) collide directly. Show that the energy lost during impact is

\[
\frac{1}{4} m(U^2 - V^2)
\]

Where \( U \) and \( V \) are the relative velocities before and after impact.

**SOLUTION**

We know that if two spheres of mass \( m_1 \) and \( m_2 \) moving with velocities \( u_1 \) and \( u_2 \) collide directly then

\[
\text{Loss of K.E.} = \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2}(1 - e^2)(u_1 - u_2)^2
\]

Here \( m_1 = m_2 = m \) and \( U = \text{relative velocity before impact} = u_1 - u_2 \)

\[
\Rightarrow \text{Loss of K.E.} = \frac{1}{2} \frac{m m}{m + m}(1 - e^2)U^2
\]

\[
= \frac{1}{4} m(1 - e^2)U^2
\]
By Newton’s Experimental Law, we have
\[ v_1 - v_2 = - e(u_1 - u_2) \]
\[ \Rightarrow V = - eU \]
Where \( V \) = relative velocity after impact. = \( v_1 - v_2 \)
\[ \Rightarrow V^2 = e^2U^2 \]
Using this value in (i), we get
\[ \text{Loss of K.E.} = \frac{1}{4} m(U^2 - V^2) \]
Which is required.

**Question 13**
If two elastic spheres have direct impact. Show that the energy lost during impact is equal to the energy of the body whose mass is half of the harmonic mean between those of the spheres and whose velocity is equal to relative velocity before impact.

**Solution**
We know that if two spheres of mass \( m_1 \) and \( m_2 \) moving with velocities \( u_1 \) and \( u_2 \) collide directly then
\[ \text{Loss of K.E.} = \frac{1}{2} \cdot \frac{m_1m_2}{m_1+m_2} (1 + e^2)(u_1 - u_2)^2 \]
Here \( e = 0 \) since spheres are inelastic.
\[ \Rightarrow \text{Loss of K.E.} = \frac{1}{2} \cdot \frac{m_1m_2}{m_1+m_2} (u_1 - u_2)^2 \]
We know that
\[ \text{Half of the harmonic mean between } m_1 \text{ and } m_2 = \frac{1}{2} \cdot \frac{2m_1m_2}{m_1+m_2} = \frac{m_1m_2}{m_1+m_2} \]
Relative velocity before impact = \( u_1 - u_2 \)
Therefore
\[ \text{K.E. of the body whose mass is } \frac{m_1m_2}{m_1+m_2} \text{ and velocity is } u_1 - u_2 \]
\[ = \frac{1}{2} \cdot \frac{m_1m_2}{m_1+m_2} (u_1 - u_2)^2 \]
Which is same as the K.E. loss of K.E. during impact. Hence the result.
**QUESTION 14**

A ball impinges directly on another ball at rest and is itself reduced to rest by impact. If half of initial K.E. is destroyed in collision, find the coefficient of restitution.

**SOLUTION**

Let \( m_1 \) and \( m_2 \) be masses of two spheres and \( u \) be the velocity of sphere of mass \( m_1 \) before impact \( v \) be the velocity of sphere of mass \( m_2 \) after impact.

![Diagram of two spheres colliding](image)

By Newton’s Experimental Law, we have

\[
0 - v = -e(u - 0)
\]

\[
\Rightarrow v = eu
\]

By Law of Conservation of Momentum, we have

\[
0 + m_2 v = m_1 u + m_2 0
\]

\[
\Rightarrow m_2 v = m_1 u
\]

Now

Initial K.E. = \[
\frac{1}{2} m_1 u^2 + \frac{1}{2} m_2 0 = \frac{1}{2} m_1 u^2
\]

Final K.E. = \[
\frac{1}{2} m_1 0 + \frac{1}{2} m_2 v = \frac{1}{2} m_2 v^2
\]

So

Loss of K.E. = Initial K.E. – Final K.E.

\[
= \frac{1}{2} m_1 u^2 - \frac{1}{2} m_2 v^2
\]

According to given condition,

Loss of K.E. = \[
\frac{1}{2} \text{ (Initial K.E.)}
\]

\[
\Rightarrow \frac{1}{2} m_1 u^2 - \frac{1}{2} m_2 v^2 = \frac{1}{4} m_1 u^2
\]

\[
\Rightarrow 2m_1 u^2 - 2m_2 v^2 = m_1 u^2
\]

\[
\Rightarrow m_1 u^2 = 2m_2 v^2 = 2m_2 v \cdot v
\]

\[
\Rightarrow m_1 u^2 = 2m_1 u v \quad \text{By(ii)}
\]

\[
\Rightarrow u = 2v \quad \Rightarrow \quad u = 2eu \quad \Rightarrow \quad e = \frac{1}{2}
\]
**QUESTION 15**

Two spheres of masses 4lb and 8lb moving with velocities 9ft/sec and 3ft/sec in opposite directions collide. If A rebounds with velocity of 1ft/sec, find the velocity of B after impact, the coefficient of elasticity and loss of K.E.

**SOLUTION**

Let $v$ be the velocity of sphere B after impact.

By Law of Conservation of Momentum, we have

$$4 \times 9 + 8 \times (-3) = 4 \times (-1) + 8 \times v$$

$$\Rightarrow 36 - 24 = -4 + 8v$$

$$\Rightarrow 12 + 4 = 8v \Rightarrow v = 2\text{ft/sec}$$

Newton’s Experimental Law, we have

$$-1 - v = -e(9 - (-3))$$

$$\Rightarrow -1 - 2 = -12e$$

$$\Rightarrow e = \frac{1}{4}$$

Now Initial K.E. = \(\frac{1}{2}(4)(9)^2 + \frac{1}{2}(8)(-3)^2\)

= 198

Final K.E. = \(\frac{1}{2}(4)(-1)^2 + \frac{1}{2}(8)(2)^2\)

= 18

So Loss of K.E. = Initial K.E. – Final K.E.

= 198 – 18

= 180\text{ft.poundals.}
A series of \( n \) elastic spheres whose masses are 1, \( e \), \( e^2 \) etc. are at rest separated by intervals, with their centers on a straight line. The first is made to impinge directly on the second with velocity \( u \). Show that finally the first \((n-1)\) spheres will be moving with the same velocity \((1 - e)u\) and the last with velocity \( u \). Prove that the final K.E. of the system is

\[
\frac{1}{2} (1 - e + 2^n)u^2
\]

**SOLUTION**

Consider the impact of the first and second ball.

Let \( v_1 \) and \( v_2 \) be the velocities of the spheres after impact.

By Newton’s Experimental Law, we have

\[ v_1 - v_2 = -e(u - 0) \]

\[ \Rightarrow v_1 - v_2 = -eu \]  \hspace{1cm} (i)

By Law of Conservation of Momentum, we have

\[ 1.u + 0 = 1.v_1 + ev_2 \]

\[ \Rightarrow v_1 + ev_2 = u \]  \hspace{1cm} (ii)

Subtracting (i) from (ii), we get

\[ v_2 + ev_2 = u + eu \]

\[ \Rightarrow v_2(1 + e) = (1 + e)u \]

\[ \Rightarrow v_2 = u \]

Using value of \( v_2 \) in (ii), we get

\[ v_1 = (1 - e)u \]

Thus, we conclude

Velocity of impinging ball after impact = \((1 - e)u\)

\[ = (1 - e) \text{velocity of impinging ball before impact} \]

Velocity of the ball at rest after impact = \( u \) = velocity of impinging ball before impact
Now we consider the impact of the 2\textsuperscript{nd} and 3\textsuperscript{rd} ball.

Let $V_1$ and $V_2$ be the velocities of the spheres after impact.

By Newton's Experimental Law, we have

$$V_1 - V_2 = -e(u - 0)$$

$\Rightarrow$ $V_1 - V_2 = -eu$ \hspace{1cm} (iii)

By Law of Conservation of Momentum, we have

$$1u + 0 = 1V_1 + eV_2$$

$\Rightarrow$ $V_1 + eV_2 = u$ \hspace{1cm} (iv)

Subtracting (iii) from (iv), we get

$$V_2 + eV_2 = u + eu$$

$\Rightarrow$ $V_2(1 + e) = (1 + e)u$

$\Rightarrow$ $V_2 = u$

Using value of $V_2$ in (iv), we get

$$V_1 = (1 - e)u$$

Thus, we conclude

Velocity of impinging ball after impact = $(1 - e)u$

= $(1 - e)$velocity of impinging ball before impact

Velocity of the ball at rest after impact = $u$ = velocity of impinging ball before impact

Proceeding in the same way,

**Velocity of (n – 1)\textsuperscript{th} ball after it impinges on n\textsuperscript{th} ball = (1 – e)u**

and **Velocity of n\textsuperscript{th} ball after impact = u**

Final K.E. of the system

$$= \frac{1}{2} 1 [(1 - e)u]^2 + \frac{1}{2} e [(1 - e)u]^2 + \ldots + \frac{1}{2} e^{n-2} [(1 - e)u]^2 + \frac{1}{2} e^{n-1}u^2$$

$$= \frac{1}{2} (1 - e)^2 u^2 [1 + e + e^2 + \ldots + e^{n-2}] + \frac{1}{2} e^{n-1}u^2$$

$$= \frac{1}{2} (1 - e)^2 u^2 \left[\frac{e^{n-1} - 1}{e - 1}\right] + \frac{1}{2} e^{n-1}u^2$$
\[ \frac{1}{2} (1 - e)^2 u^2 \left[ \frac{1 - e^{n-1}}{1 - e} \right] + \frac{1}{2} e^{n-1} u^2 \]
\[ = \frac{1}{2} (1 - e)(1 - e^{n-1}) u^2 + \frac{1}{2} e^{n-1} u^2 \]
\[ = \frac{1}{2} \left[ (1 - e)(1 - e^{n-1}) + e^{n-1} \right] u^2 \]
\[ = \frac{1}{2} \left[ 1 - e^{n-1} - e + e^n + e^{n-1} \right] u^2 \]
\[ = \frac{1}{2} (1 - e + e^n) u^2 \]

Which is required.

%% End of The Chapter # 8 %%%