KINEMATICS

CHAPTER 4

❖ INTRODUCTION

The branch of mechanics which deals with the motion of object is called dynamics. It is divided into two branches:

(i) ***Kinematics*** ***(ii)*** ***Kinetics***

KINEMATICS:

The branch of dynamics which deals with geometry of motion of a body without any reference of the force acting on the body is called kinematics.

KINETICS:

The branch of dynamics which deals with geometry of motion of a body with reference to the force causing motion is called kinetics.

POINTS TO BE REMEMBER

(i) The position of a particle can be specified by a vector $\vec{r}$ whose initial point is at the origin of some fixed coordinate system and the terminal point is at the particle. This vector is called **position vector**. If the particle is moving, the vector $\vec{r}$ changes with time. i.e. it is a function of time.

(ii) The curve traced by a moving particle is called the **trajectory** or the path of the particle.

(iii) The path of the particle can be specified by the vector equation

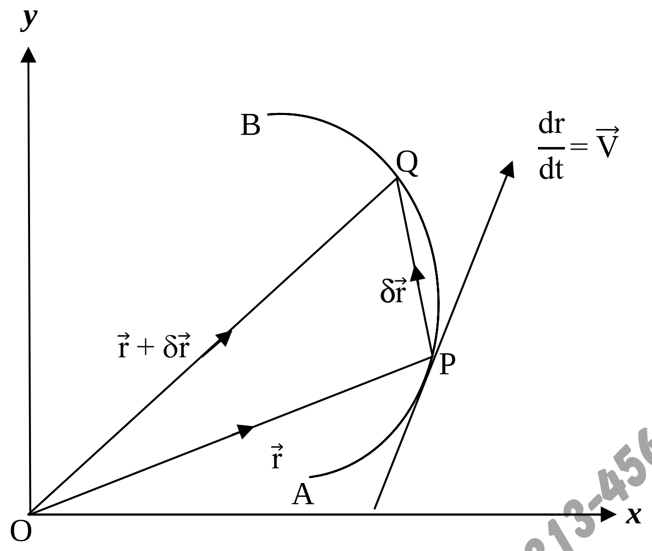
$$\vec{r} = \vec{r}\,(t) \qquad \text{\_\_\_\_\_\_\_\_ (i)}$$

The path of the particle can also be specified by three scalar equations

$$x = x(t), \quad y = y(t), \quad z = z(t) \qquad \text{\_\_\_\_\_\_\_\_\_\_ (ii)}$$

These equations are obtained by equating the components of vectors on two sides of the equation (i). Equation gives the coordinates of the points of the path for different value s of t. We call these as **parametric equations of the path.**

❖ CARTESIAN COMPONENTS OF VELOCITY & ACCELERATION



Let AB be a part of the trajectory of the particle as shown in figure. Let the particle at time t be at the point P whose position vector is $\vec{r}$. After a small time δt, let the particle reach the point Q whose position vector is $\vec{r} + \delta\vec{r}$. The $\overrightarrow{PQ} = \delta\vec{r}$ is the displacement of the particle from the point P in the small time interval δt. The quotient

$$\frac{\delta\vec{r}}{\delta t}$$

gives the average rate of change of displacement of the particle in the interval δt. If we start decreasing the time interval δt, the displacement $\delta\vec{r}$ will go on deceasing and the point Q gets nearer and nearer to P. Thus

$$\lim_{\delta t \to 0} \frac{\delta\vec{r}}{\delta t}$$

can be considered as the instantaneous rate of change of displacement. This is defined as the instantaneous velocity or the simply velocity $\vec{v}$ of the particle at point P.

Thus,

$$\vec{v} = \lim_{\delta t \to 0} \frac{\delta\vec{r}}{\delta t} = \frac{d\vec{r}}{dt}$$

Proceeding in similar way we can see that the acceleration $\vec{a}$ (the instantaneous rate of change of velocity) at time t is given by

$$\vec{a} = \lim_{\delta t \to 0} \frac{\delta\vec{v}}{\delta t} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d^2\vec{r}}{dt^2}$$

In Cartesian coordinates, we can write

$$\vec{r} = x\hat{i} + y\hat{i}$$

Then

$$\vec{v} = \frac{d}{dt}\left(x\hat{i} + y\hat{i}\right) = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}$$

$$\vec{a} = \frac{d^2}{dt^2}\left(x\hat{i} + y\hat{i}\right) = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j}$$

Thus

$$\boldsymbol{v_x = x\text{- component of velocity} = \frac{dx}{dt}}$$

$$\boldsymbol{v_y = y\text{- component of velocity} = \frac{dy}{dt}}$$

$$\boldsymbol{a_x = x\text{- component of acceleration} = \frac{d^2x}{dt^2}}$$

$$\boldsymbol{a_y = y\text{- component of acceleration} = \frac{d^2y}{dt^2}}$$

❖ QUESTION 1

A particle is moving in such a way that it position at any time t is specified by

$$\mathbf{\vec{r} = (t^3 + t^2)\hat{i} + (\cos t + \sin^2 t)\hat{j} + (e^t + \log t)\hat{k}}$$

Find the velocity and acceleration.

SOLUTION

If $\vec{v}$ and $\vec{a}$ are velocity and acceleration of particle respectively. Then

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left((t^3 + t^2)\hat{i} + (\cos t + \sin^2 t)\hat{j} + (e^t + \log t)\hat{k}\right)$$

$$= (3t^2 + 2t)\hat{i} + (-\sin t + 2\sin t\cos t)\hat{j} + \left(e^t + \frac{1}{t}\right)\hat{k}$$

$$= (3t^2 + 2t)\hat{i} + (\sin 2t - \sin t)\hat{j} + \left(e^t + \frac{1}{t}\right)\hat{k}$$

and $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left((3t^2 + 2t)\hat{i} + (\sin 2t - \sin t)\hat{j} + \left(e^t + \frac{1}{t}\right)\hat{k}\right)$

$$= (6t + 2)\hat{i} + (2\cos 2t - \cos t)\hat{j} + \left(e^t - \frac{1}{t^2}\right)\hat{k}$$

❖ QUESTION 2

A particle P start from O at t = 0. Find tits velocity and acceleration of particle at any time t if its position at that time is given by

$$\mathbf{\vec{r} = at^2\hat{i} + 4at\hat{j}}$$

SOLUTION

If $\vec{v}$ and $\vec{a}$ are velocity and acceleration of particle respectively. Then

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left(at^2\hat{i} + 4at\hat{j}\right) = 2at\hat{i} + 4a\hat{j}$$

and $$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(2at\hat{i} + 4a\hat{j}\right) = 2a\hat{i}$$

QUESTION 3

At any time t, the position of a particle moving in a plane can be specified by (acoswt, asinwt) where a and w are constants. Find the component of its velocity and acceleration along the coordinates axis.

SOLUTION

Let $\vec{r} = \text{acoswt}\,\hat{i} + \text{asinwt}\,\hat{j}$

Differentiate w.r.t "t", we get

$$\vec{v} = -\text{awsinwt}\,\hat{i} + \text{awcoswt}\,\hat{j}$$

Differentiate again w.r.t "t", we get

$$\vec{a} = -\text{aw}^2\text{coswt}\,\hat{i} - \text{aw}^2\text{sinwt}\,\hat{j}$$

Thus the component of velocity and acceleration are

$$v_x = -\text{awsinwt} \quad , \quad v_y = \text{awcoswt}$$

$$a_x = -\text{aw}^2\text{coswt} \quad , \quad a_y = -\text{aw}^2\text{sinwt}$$

QUESTION 4

The position of particle moving along an ellipse is given by $\vec{\mathbf{r}} = \mathbf{acost}\,\hat{\mathbf{i}} + \mathbf{bsint}\,\hat{\mathbf{j}}$ If a > b, find the position of the particle where velocity has maximum and minimum magnitude.

SOLUTION

As $\vec{r} = \text{acost}\,\hat{i} + \text{bsint}\,\hat{j}$

Differentiate w.r.t "t", we get

$$\vec{v} = -\text{asint}\,\hat{i} + \text{bcost}\,\hat{j}$$

$\Rightarrow$ $$v = \sqrt{(-\text{asint})^2 + (\text{bcost})^2}$$

$$= \sqrt{a^2\sin^2 t + b^2\cos^2 t}$$

$$= \sqrt{a^2\sin^2 t + b^2\left(1 - \sin^2 t\right)}$$

$$= \sqrt{a^2\sin^2 t + b^2 - b^2\sin^2 t}$$

$$= \sqrt{\sin^2 t\left(a^2 - b^2\right) + b^2}$$

v is maximum when $\sin^2 t$ is maximum. i.e. $\sin^2 t = 1 \Rightarrow \sin t = \pm 1 \Rightarrow t = 90, 270$

For t = 90

$$\vec{r} = a\cos 90\,\hat{i} + b\sin 90\,\hat{j} = b\hat{j}$$

For t = 270

$$\vec{r} = a\cos 270\,\hat{i} + b\sin 270\,\hat{j} = -\,b\hat{j}$$

So the position of the particle when velocity has maximum magnitude is $\pm\, b\hat{j}$.

v is minimum when $\sin^2 t$ is minimum. i.e. $\sin^2 t = 0 \Rightarrow \sin t = 0 \Rightarrow t = 0, 180$

For t = 0

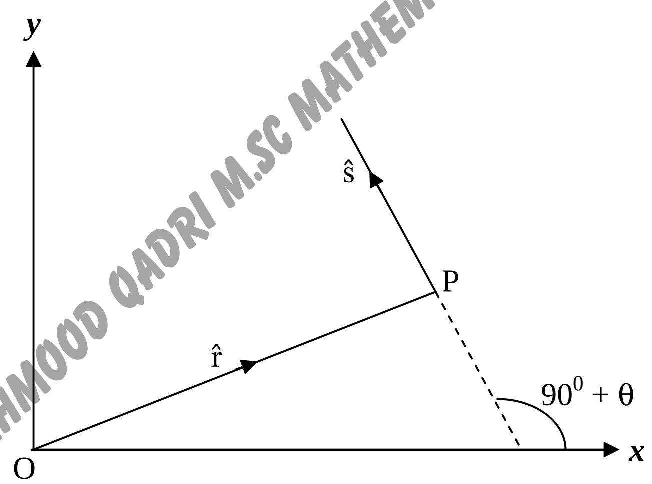
$$\vec{r} = a\cos 0\,\hat{i} + b\sin 0\,\hat{j} = a\hat{i}$$

For t = 180

$$\vec{r} = a\cos 180\,\hat{i} + b\sin 180\,\hat{j} = -\,a\hat{i}$$

So the position of the particle when velocity has minimum magnitude is $\pm\, a\hat{i}$.

❖ RADIAL & TRANSVERSE COMPONENTS OF VELOCITY & ACCELERATION



In polar coordinates, the position of a particle is specified by a radius vector r and the polar angle θ which are related to x and y through the relations

$$x = r\cos\theta$$
$$y = r\sin\theta$$

provided the two coordinates frames have the same origin and the x–axis and the initial line coincide. The direction of radius vector is known as **radial direction** and that perpendicular to it in the direction of increasing θ is called **transverse direction.**

Let $\hat{r}$ and $\hat{s}$ be units vectors in the radial and transverse direction respectively as shown in figure. Then

$$\hat{r} = \cos\theta\hat{i} + \sin\theta\hat{j} \qquad \text{\_\_\_\_\_\_\_\_ (i)}$$

$$\hat{s} = \cos(90^0 + \theta)\hat{i} + \sin(90^0 + \theta)\hat{j} = -\sin\theta\hat{i} + \cos\theta\hat{j} \qquad \text{\_\_\_\_\_\_\_\_ (ii)}$$

Differentiating (i) w.r.t "t"

$$\frac{d\hat{r}}{dt} = \frac{d}{dt}\left(\cos\theta\hat{i} + \sin\theta\hat{j}\right)$$

$$= \left(-\sin\theta\,\hat{i}\left(\frac{d\theta}{dt}\right) + \cos\theta\hat{j}\left(\frac{d\theta}{dt}\right)\right)$$

$$= \frac{d\theta}{dt}\left(-\sin\theta\,\hat{i} + \cos\theta\hat{j}\right)$$

$$= \frac{d\theta}{dt}\hat{s} \qquad \text{By (ii)} \qquad \text{\_\_\_\_\_\_\_\_ (iii)}$$

Differentiating (ii) w.r.t "t"

$$\frac{d\hat{s}}{dt} = \frac{d}{dt}\left(-\sin\theta\hat{i} + \cos\theta\hat{j}\right)$$

$$= \left(-\cos\theta\,\hat{i}\left(\frac{d\theta}{dt}\right) - \sin\theta\hat{j}\left(\frac{d\theta}{dt}\right)\right)$$

$$= -\frac{d\theta}{dt}\left(\sin\theta\,\hat{i} + \cos\theta\hat{j}\right)$$

$$= -\frac{d\theta}{dt}\hat{r} \qquad \text{By (i)} \qquad \text{\_\_\_\_\_\_\_\_ (iv)}$$

We know that

$$\hat{r} = \frac{\vec{r}}{r} \qquad \Rightarrow \qquad \vec{r} = r\,\hat{r}$$

Now $\vec{v} = \dfrac{d\vec{r}}{dt}$

$$= \frac{d}{dt}(r\hat{r}) = \frac{dr}{dt}.\hat{r} + r\frac{d\hat{r}}{dt} = \frac{dr}{dt}.\hat{r} + r\frac{d\theta}{dt}\hat{s}$$

Thus,

$$v_r = \textit{Radial component of velocity} = \frac{dr}{dt} = \dot{r}$$

$$v_\theta = \textit{Transverse component of velocity} = r\frac{d\theta}{dt} = r\dot{\theta}$$

Where dot denotes the differentiation with respect to time "t".

Let $\vec{a}$ be the acceleration Then

$$\vec{a} = \frac{d\vec{v}}{dt}$$
$$= \frac{d}{dt}\left(\frac{dr}{dt}.\hat{r} + r\frac{d\theta}{dt}\hat{s}\right)$$
$$= \frac{d}{dt}\left(\frac{dr}{dt}.\hat{r}\right) + \frac{d}{dt}\left(r\frac{d\theta}{dt}\hat{s}\right)$$
$$= \frac{d}{dt}\left(\frac{dr}{dt}\right)\hat{r} + \frac{dr}{dt}\frac{d\hat{r}}{dt} + \frac{dr}{dt}\left(\frac{d\theta}{dt}\hat{s}\right) + \frac{d}{dt}\left(\frac{d\theta}{dt}\right) r\hat{s} + \frac{d\hat{s}}{dt}\left(r\frac{d\theta}{dt}\right)$$
$$= \frac{d^2r}{dt^2}\hat{r} + \frac{dr}{dt}\frac{d\hat{r}}{dt} + \frac{dr}{dt}\left(\frac{d\theta}{dt}\right)\hat{s} + \frac{d^2\theta}{dt^2} r\hat{s} + \frac{d\hat{s}}{dt}\left(\frac{d\theta}{dt}\right) r$$
$$= \frac{d^2r}{dt^2}\hat{r} + \frac{dr}{dt}\left(\frac{d\theta}{dt}\hat{s}\right) + \frac{dr}{dt}\left(\frac{d\theta}{dt}\right)\hat{s} + \frac{d^2\theta}{dt^2} r\,\hat{s} + \left(-\frac{d\theta}{dt}\hat{r}\right)\left(\frac{d\theta}{dt}\right) r \qquad \text{By (iii)\& (iv)}$$
$$= \frac{d^2r}{dt^2}\hat{r} - r\left(\frac{d\theta}{dt}\right)^2\hat{r} + 2\frac{dr}{dt}\left(\frac{d\theta}{dt}\right)\hat{s} + \frac{d^2\theta}{dt^2} r\,\hat{s}$$
$$= \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{r} + \left[2\frac{dr}{dt}\left(\frac{d\theta}{dt}\right) + \frac{d^2\theta}{dt^2} r\right]\hat{s}$$

Thus,

a_r = Radial component of acceleration $= \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \ddot{r} - r\left(\dot{\theta}\right)^2$

a_θ = Transverse component of acceleration $= 2\frac{dr}{dt}\left(\frac{d\theta}{dt}\right) + r\frac{d^2\theta}{dt^2} = 2\dot{r}\dot{\theta} + r\ddot{\theta}$

❖ QUESTION 5

A particle P moves in a plane in such away that at any time t, its distance from point O is $r = at + bt^2$ and the line connecting O and P makes an angle $\theta = ct^{3/2}$ with a fixed line OA. Find the radial and transverse components of velocity and acceleration of particle at t = 1

SOLUTION

Given that

$r = at + bt^2$ and $\theta = ct^{3/2}$

Differentiate w.r.t "t", we get

$$\frac{dr}{dt} = a + 2bt \quad \text{and} \quad \frac{d\theta}{dt} = \frac{3}{2}ct^{1/2}$$

Differentiate again w.r.t "t", we get

$$\frac{d^2r}{dt^2} = 2b \quad \text{and} \quad \frac{d^2\theta}{dt^2} = \frac{3}{4}ct^{-1/2}$$

At t = 1

$r = a + b$ and $\theta = c$

$\frac{dr}{dt} = a + 2b$, $\frac{d\theta}{dt} = \frac{3}{2}c$, $\frac{d^2r}{dt^2} = 2b$ and $\frac{d^2\theta}{dt^2} = \frac{3}{4}c$

Radial component of velocity = $v_r = \frac{dr}{dt} = a + 2b$

Transverse component of velocity = $v_\theta = r\frac{d\theta}{dt} = \frac{3}{2}c(a+b)$

Radial component of acceleration = $a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$

$= 2b - (a+b)\left(\frac{3}{2}c\right)^2$

$= 2b - \frac{9}{4}c^2(a+b)$

$= \frac{1}{4}\left(8b - 9c^2(a+b)\right)$

Transverse component of acceleration = $a_\theta = 2\frac{dr}{dt}\left(\frac{d\theta}{dt}\right) + r\frac{d^2\theta}{dt^2}$

$= 2(a+2b)\left(\frac{3}{2}c\right) + (a+b)\left(\frac{3}{4}c\right)$

$= \frac{3}{4}c(5a+9b)$

❖ QUESTION 6

Find the radial and transverse components of velocity of a particle moving along the curve

$$\mathbf{ax^2 + by^2 = 1}$$

at any time t if the polar angle is $\theta = ct^2$

SOLUTION

Given that

$\theta = ct^2$

Differentiate w.r.t "t", we get

$\frac{d\theta}{dt} = 2ct$

Also given that

$ax^2 + by^2 = 1$

First we change this into polar form by putting $x = r\cos\theta$ and $y = r\sin\theta$

$$ar^2\cos^2\theta + br^2\sin^2\theta = 1$$

$\Rightarrow$ $$r^2(a\cos^2\theta + b\sin^2\theta) = 1$$

$\Rightarrow$ $$r\sqrt{a\cos^2\theta + b\sin^2\theta} = 1$$

$\Rightarrow$ $$r = (a\cos^2\theta + b\sin^2\theta)^{-\frac{1}{2}}$$

Differentiate w.r.t "t", we get

$$\frac{dr}{dt} = -\frac{1}{2}(a\cos^2\theta + b\sin^2\theta)^{-\frac{3}{2}}\left(-a2\cos\theta\sin\theta\frac{d\theta}{dt} + b2\sin\theta\cos\theta\frac{d\theta}{dt}\right)$$

$$= \frac{1}{2}(a\cos^2\theta + b\sin^2\theta)^{-\frac{3}{2}}(a - b)\sin2\theta\frac{d\theta}{dt}$$

$$= \frac{1}{2}(a\cos^2\theta + b\sin^2\theta)^{-\frac{3}{2}}(a - b)\sin2\theta.2ct$$

$$= \frac{ct(a - b)\sin2\theta}{(a\cos^2\theta + b\sin^2\theta)^{\frac{3}{2}}}$$

Radial component of velocity $= \frac{dr}{dt} = \frac{ct(a - b)\sin2\theta}{(a\cos^2\theta + b\sin^2\theta)^{\frac{3}{2}}}$

Transverse component of velocity $= r\frac{d\theta}{dt} = \frac{2ct}{(a\cos^2\theta + b\sin^2\theta)^{\frac{1}{2}}}$

❖ QUESTION 7

Find the radial and transverse components of acceleration of a particle moving along the circle $\mathbf{x^2 + y^2 = a^2}$ with constant velocity c.

SOLUTION

Given that

$$\frac{d\theta}{dt} = c$$

Differentiate w.r.t "t", we get

$$\frac{d^2\theta}{dt^2} = 0$$

Also given that

$$x^2 + y^2 = a^2$$

First we change this into polar form by putting x = rcosθ and y = rsinθ

$$r^2\cos^2\theta + r^2\sin^2\theta = a^2$$

$\Rightarrow$ $$r^2(\cos^2\theta + \sin^2\theta) = a^2$$

$\Rightarrow$ $$r^2 = a^2$$

$\Rightarrow \quad r = a$

$\Rightarrow \quad \dfrac{dr}{dt} = 0 \quad \Rightarrow \quad \dfrac{d^2r}{dt^2} = 0$

Radial component of acceleration $= a_r = \dfrac{d^2r}{dt^2} - r\left(\dfrac{d\theta}{dt}\right)^2$

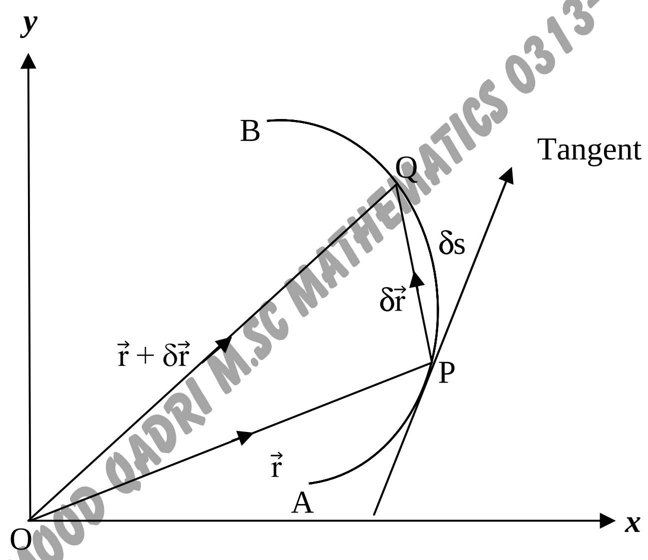
$= 0 - ac^2$

$= -ac^2$

Transverse component of acceleration $= a_\theta = 2\dfrac{dr}{dt}\left(\dfrac{d\theta}{dt}\right) + r\dfrac{d^2\theta}{dt^2}$

$= 0$

❖ TANGENTIAL & NORMAL COMPONENTS OF VELOCITY & ACCELERATION



Let AB be a part of the trajectory of the particle as shown in figure. Let the particle at time t be at the point P whose position vector is $\vec{r}$. After a small time δt, let the particle reach the point Q whose position vector is $\vec{r} + \delta\vec{r}$. Then $\overrightarrow{PQ} = \delta\vec{r}$ and arcPQ $= \delta s$

Now $\quad \vec{v} = \dfrac{d\vec{r}}{dt} = \dfrac{d\vec{r}}{ds} \cdot \dfrac{ds}{dt} = v.\dfrac{d\vec{r}}{ds}$ $\qquad$ ______________(i)

Here $\dfrac{d\vec{r}}{ds}$ is a unit tangent at point P.

Let $\hat{t}$ be a unit vector along the tangent at P and $\hat{n}$ unit vector along normal at the point P. Then

$$\frac{d\vec{r}}{ds} = \hat{t}$$

Using this in (i), we get

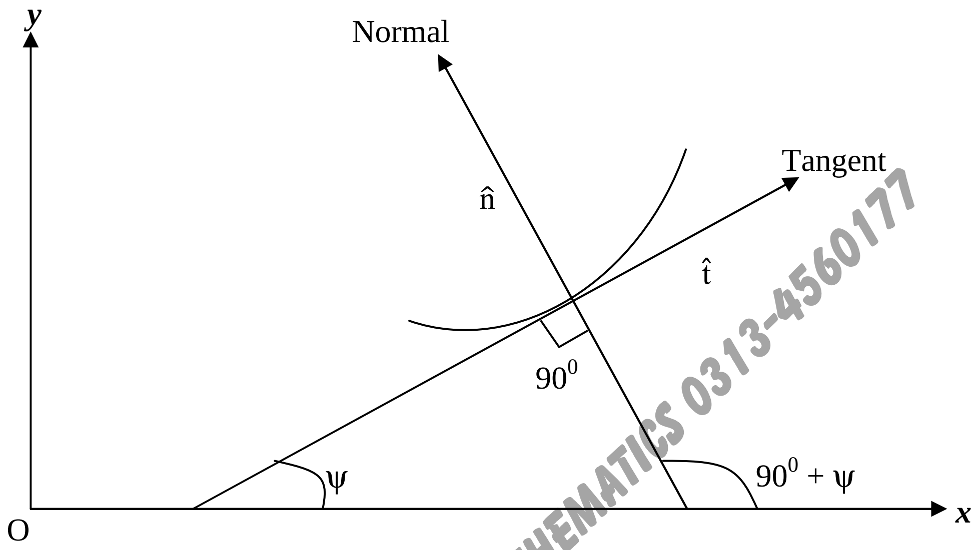
$$\vec{v} = v\,\hat{t} + 0.\hat{n}$$

Thus,

v_t = Tangential component of velocity = v

v_n = Normal component of velocity = 0

Hence the velocity is along the tangent to the path.



Let $\vec{a}$ be the acceleration. Then

$$\vec{a} = \frac{d\vec{v}}{dt}$$

$$= \frac{d}{dt}(v\,\hat{t})$$

$$= \frac{dv}{dt}\,\hat{t} + v\frac{d\hat{t}}{dt}$$

$$= \frac{dv}{dt}\,\hat{t} + v\frac{d\hat{t}}{d\psi}\frac{d\psi}{ds}\frac{ds}{dt}$$

$$= \frac{dv}{dt}\,\hat{t} + v\frac{d\hat{t}}{d\psi}(Kv) \qquad \because \quad \frac{ds}{dt} = v$$

Where $\frac{d\psi}{ds} = K$ is called curvature and $K = \frac{1}{\rho}$

So $\quad \vec{a} = \frac{dv}{dt}\,\hat{t} + v\frac{d\hat{t}}{d\psi}.\frac{v}{\rho}$

$$= \frac{dv}{dt}\,\hat{t} + \frac{v^2}{\rho}\frac{d\hat{t}}{d\psi}$$

Since $\hat{t}$ and $\hat{n}$ are unit vectors along tangent and normal at P Therefore

$$\hat{t} = \cos\psi\,\hat{i} + \sin\psi\,\hat{j}$$

$$\hat{n} = \cos(90^0 + \psi)\hat{i} + \sin(90^0 + \psi)\hat{j} = -\sin\psi\,\hat{i} + \cos\psi\,\hat{j}$$

Now $\quad \frac{d\hat{t}}{d\psi} = \frac{d}{d\psi}\left(\cos\psi\,\hat{i} + \sin\psi\,\hat{j}\right)$

$$= \left(-\sin\psi\,\hat{i} + \cos\psi\,\hat{j}\right)$$

$= \hat{n}$

So $\quad \vec{a} = \frac{dv}{dt}\hat{t} + \frac{v^2}{\rho}\hat{n}$

Thus,

Tangential component of acceleration $= a_t = \frac{dv}{dt}$

Normal component of acceleration $= a_n = \frac{v^2}{\rho}$

Where

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|}$$

❖ QUESTION 8

A particle is moving along the parabola $x^2 = 4ay$ with constant speed. Determine tangential and normal components of its acceleration when it reaches the point whose abscissa is $\sqrt{5}a$.

SOLUTION

Given that

$$x^2 = 4ay$$

Differentiate w.r.t "x", we get

$$2x = 4a\frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{x}{2a}$$

Differentiate again w.r.t "x", we get

$$\frac{d^2y}{dx^2} = \frac{1}{2a}$$

Given that $x = \sqrt{5}a$ therefore

$$\frac{dy}{dx} = \frac{\sqrt{5}a}{2a} = \frac{\sqrt{5}}{2}$$

We know that

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|} = \frac{\left[1 + \left(\frac{\sqrt{5}}{2}\right)^2\right]^{3/2}}{\frac{1}{2a}} = 2a\left[1 + \frac{5}{4}\right]^{3/2} = 2a\left[\frac{9}{4}\right]^{3/2} = 2a\left[\frac{3}{2}\right]^3 = \frac{27a}{4}$$

Since the particle is moving with constant speed therefore

$$\frac{dv}{dt} = 0$$

Tangential component of acceleration = $a_t = \frac{dv}{dt} = 0$

Normal component of acceleration = $a_n = \frac{v^2}{\rho} = \frac{v^2}{\frac{27a}{4}} = \frac{4v^2}{27a}$

❖ QUESTION 9

Find the tangential and normal component of acceleration of a point describing ellipse

$$\mathbf{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

With uniform speed v when the particle is at (0, b).

SOLUTION

Given that

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$\Rightarrow \quad x^2b^2 + y^2a^2 = a^2b^2$

Differentiate w.r.t "x", we get

$$2b^2x + 2a^2y\frac{dy}{dx} = 0$$

$\Rightarrow \quad \frac{dy}{dx} = -\frac{b^2x}{a^2y}$

Differentiate again w.r.t "x", we get

$$\frac{d^2y}{dx^2} = -\frac{b^2}{a^2}\left(\frac{y - x\frac{dy}{dx}}{y^2}\right)$$

$$= -\frac{b^2}{a^2}\left(\frac{y - x\left(-\frac{b^2x}{a^2y}\right)}{y^2}\right)$$

$$= -\frac{b^2}{a^2}\left(\frac{1}{y} + \frac{x^2b^2}{a^2y^3}\right)$$

At (0, b)

$$\frac{dy}{dx} = -\frac{b^2 0}{a^2b} = 0$$

and $\quad \frac{d^2y}{dx^2} = -\frac{b^2}{a^2}\left(\frac{1}{b} + \frac{0.b^2}{a^2b^3}\right) = -\frac{b}{a^2}$

We know that

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|}$$

$$= \frac{[1+(0)^2]^{3/2}}{\left|-\frac{b}{a^2}\right|} = \frac{a^2}{b}$$

Since the particle is moving with uniform speed therefore

$$\frac{dv}{dt} = 0$$

Thus, Tangential component of acceleration $= a_t = \frac{dv}{dt} = 0$

Normal component of acceleration $= a_n = \frac{v^2}{\rho} = \frac{v^2}{\frac{a^2}{b}} = \frac{bv^2}{a^2}$\

❖ QUESTION 10

A particle is moving with uniform speed along the curve

$$\mathbf{x^2y = a\left(x^2 + \frac{a^2}{\sqrt{5}}\right)}$$

Show that acceleration has maximum value $\mathbf{\frac{10v^2}{9a}}$

SOLUTION

Given that

$$x^2y = a\left(x^2 + \frac{a^2}{\sqrt{5}}\right)$$

$$\Rightarrow \quad y = a + \frac{a^3}{\sqrt{5}}x^{-2}$$

Differentiate w.r.t "x", we get

$$\frac{dy}{dx} = -\frac{2a^3}{\sqrt{5}}x^{-3}$$

Differentiate again w.r.t "x", we get

$$\frac{d^2y}{dx^2} = \frac{6a^3}{\sqrt{5}}x^{-4}$$

We know that

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|}$$

$$= \frac{\left[1+\left(-\frac{2a^3}{\sqrt{5}}x^{-3}\right)^2\right]^{3/2}}{\frac{6a^3}{\sqrt{5}}x^{-4}} = \frac{\left[1+\frac{4a^6}{5x^6}\right]^{3/2}}{\frac{6a^3}{\sqrt{5}x^4}} = \frac{\sqrt{5}x^4}{6a^3}\left[\frac{5x^6+4a^6}{5x^6}\right]^{3/2} = \frac{[5x^6+4a^6]^{3/2}}{30a^3x^5} \quad \_\_\_\_(i)$$

We know that

$$\vec{a} = \frac{dv}{dt}\hat{t} + \frac{v^2}{\rho}\hat{n}$$

Since the particle is moving with constant speed therefore

$$\frac{dv}{dt} = 0$$

$$\Rightarrow \quad \vec{a} = \frac{v^2}{\rho}\hat{n}$$

$$\Rightarrow \quad |\vec{a}| = \frac{v^2}{\rho}|\hat{n}| = \frac{v^2}{\rho} \qquad \because |\hat{n}| = 1$$

$|\vec{a}|$ will maximum when ρ is minimum.

Differentiate (i) w.r.t "x", we get

$$\frac{d\rho}{dx} = \frac{30a^3x^5\left[\frac{3}{2}(5x^6+4a^6)^{1/2}30x^5\right] - [5x^6+4a^6]^{3/2}(150a^3x^4)}{(30a^3x^5)^2}$$

$$= \frac{(5x^6+4a^6)^{1/2}}{(30)^2a^6x^{10}}[30a^3x^5[45x^5] - [5x^6+4a^6](150a^3x^4)]$$

$$= \frac{(5x^6+4a^6)^{1/2}}{30a^3x^6}[45x^6 - 5(5x^6+4a^6)]$$

$$= \frac{(5x^6+4a^6)^{1/2}}{30a^3x^6}[45x^6 - 25x^6 - 20a^6]$$

$$= \frac{(5x^6+4a^6)^{1/2}}{30a^3x^6}[20x^6 - 20a^6]$$

$$= \frac{20(5x^6+4a^6)^{1/2}}{30a^3x^6}[x^6 - a^6]$$

$$= \frac{20(5x^6+4a^6)^{1/2}}{30a^3x^6}(x^2-a^2)(x^4+x^2a^2+a^4)$$

Putting $\frac{d\rho}{dx} = 0$, we get

$$x = \pm a$$

Since

$\frac{d\rho}{dx} < 0$ before x = a and $\frac{d\rho}{dx} > 0$ after x = a

Therefore ρ is minimum when x = a

Thus

$$\rho_{min} = \frac{[5a^6 + 4a^6]^{3/2}}{30a^3a^5} = \frac{[9a^6]^{3/2}}{30a^8} = \frac{27a}{30} = \frac{9}{10}a$$

Maximum value of acceleration $= \frac{v^2}{\rho_{min}} = \frac{v^2}{\frac{9}{10}a} = \frac{10v^2}{9a}$

%%%%% End of The Chapter # 4 %%%%%