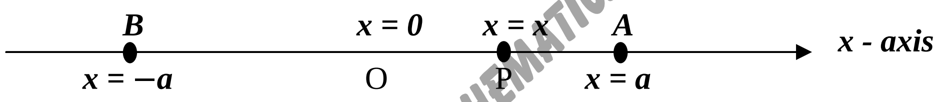
# SIMPLE HARMONIC MOTION

CHAPTER 6

## ❖ SIMPLE HARMONIC MOTION

It is the motion of a particle moving in a straight line with an acceleration which is always directed towards a fixed point in the line and is proportional to the distance of the particle from that point.




Let O be a fixed point on the line along the particle is moving. Let the particle be at a point P a distance x from the point O towards its right as shown in figure. Then the acceleration of the particle is

$$a = \frac{d^2x}{dt^2}$$

For simple harmonic motion this acceleration is proportional to x and is directed towards O i.e.

$$a = -\lambda x$$

Where $\lambda$ is constant of proportionality and negative sign indicates that the acceleration is directed against the direction in which x is increasing. This motion is taking place in such away that when the particle is moving away from the point O, the acceleration is acting against so that as the time progresses, the velocity becomes lesser and lesser.

Since $\quad \text{acceleration} = a = v\dfrac{dv}{dx}$

$\Rightarrow \quad v\dfrac{dv}{dx} = -\lambda x$

$\Rightarrow \quad vdv = -\lambda x dx$

$\Rightarrow \quad \displaystyle\int vdv = -\lambda \int xdx$

$$\Rightarrow \qquad \frac{v^2}{2} = -\lambda \frac{x^2}{2} + A \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_(i)}$$

Where A is constant of acceleration.

When x = a then v = 0

$$\text{(i)} \Rightarrow \qquad A = \lambda \frac{a^2}{2}$$

Using the value of A in (i), we get

$$\frac{v^2}{2} = -\lambda \frac{x^2}{2} + \lambda \frac{a^2}{2}$$

$$\Rightarrow \qquad v^2 = \lambda(a^2 - x^2)$$

$$\Rightarrow \qquad v = \pm\sqrt{\lambda(a^2 - x^2)}$$

Which gives the velocity for a given displacement. If particle is moving to the right then as t increases , x also increases. So the velocity is positive.

$$\text{Thus} \qquad v = \sqrt{\lambda(a^2 - x^2)}$$

$$\Rightarrow \qquad \frac{dx}{dt} = \sqrt{\lambda(a^2 - x^2)} \qquad \because \frac{dx}{dt} = v$$

$$\Rightarrow \qquad \frac{dx}{\sqrt{a^2 - x^2}} = \sqrt{\lambda}\, dt$$

$$\Rightarrow \qquad \int \frac{dx}{\sqrt{a^2 - x^2}} = \sqrt{\lambda} \int dt$$

$$\Rightarrow \qquad \sin^{-1}\left(\frac{x}{a}\right) = \sqrt{\lambda}\, t + B \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_(ii)}$$

Where B is constant of integration.

Initially at t = 0, x = 0

$$\Rightarrow \qquad B = \sin^{-1}(0) = \frac{\pi}{2}$$

Putting value of B in (ii), we get

$$\sin^{-1}\left(\frac{x}{a}\right) = \sqrt{\lambda}\, t + \frac{\pi}{2}$$

$$\Rightarrow \qquad \frac{x}{a} = \sin\left(\sqrt{\lambda}\, t + \frac{\pi}{2}\right)$$

$$\Rightarrow \qquad x = a\cos\sqrt{\lambda}\, t$$

Which gives the displacement at any time t.

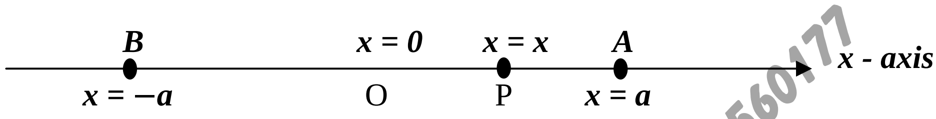
## ❖ NATURE OF SIMPLE HARMONIC MOTION

As the displacement of a particle performing simple harmonic motion at any time t is:

$$x = a\cos\sqrt{\lambda}\,t \qquad \text{\_\_\_\_\_\_\_\_(i)}$$

Also $-1 \le \cos\sqrt{\lambda}\,t \le 1 \;\Rightarrow\; -a \le a\cos\sqrt{\lambda}\,t \le a \;\Rightarrow\; -a \le x \le a$

Thus the maximum displacement from a fixed point O is x = a. The fixed point O is called the **centre of motion.** The maximum displacement from the centre is called the **amplitude of the motion.**




Now

$$x = a\cos\sqrt{\lambda}\,t$$

$$= a\cos\left(\sqrt{\lambda}\,t + 2\pi\right) \qquad \because \cos(2\pi + \theta) = \cos\theta$$

$$= a\cos\sqrt{\lambda}\left(t + \frac{2\pi}{\sqrt{\lambda}}\right)$$

Which shows that the distance at time t and $t + \dfrac{2\pi}{\sqrt{\lambda}}$ is same.

Differentiate (i) w.r.t 't'

$$v = -\,a\sqrt{\lambda}\sin\sqrt{\lambda}\,t$$

$$= -\,a\sqrt{\lambda}\cos\left(\sqrt{\lambda}\,t + 2\pi\right) \qquad \because \cos(2\pi + \theta) = \cos\theta$$

$$= -\,a\sqrt{\lambda}\cos\sqrt{\lambda}\left(t + \frac{2\pi}{\sqrt{\lambda}}\right)$$

Which shows that the velocity at time t and $t + \dfrac{2\pi}{\sqrt{\lambda}}$ is same.

Thus we can say that the motion is repeated after time $t = {}^{2\pi}/_{\sqrt{\lambda}}$ and the particle oscillate between $x = -\,a$ and $x = a$.

**At point A, x = a**

$$\text{velocity} = v = \sqrt{\lambda(\,a^2 -\, a^2)} = 0 \qquad \text{and} \qquad \text{acceleration} = a = -\,\lambda x = -\,\lambda a$$

Thus at point A, (x = a), its velocity is zero but acceleration is maximum and is directed towards O and due to maximum acceleration it moves toward O.

**At point O, x = 0**

$$\text{velocity} = v = \sqrt{\lambda(\,a^2 -\, 0)} = \sqrt{\lambda}\,a$$

And acceleration = a = $-\lambda x = 0$

Thus at point O, (x = 0), its velocity is maximum but acceleration is zero and due to maximum velocity it moves toward B.

**At point B, x = − a**

velocity = v = $\sqrt{\lambda(a^2 - a^2)} = 0$

And acceleration = a = $-\lambda x = \lambda a$

Thus at point B, (x = − a), its velocity is zero but acceleration is maximum and due to maximum acceleration it moves toward O and finally comes to rest at A.

The motion is repeated after time t = $2\pi/\sqrt{\lambda}$. The time taken by a particle to complete one oscillation is called time period and it is usually denoted by T.

i.e. $$T = \frac{2\pi}{\sqrt{\lambda}}$$

The number of vibration or oscillation completed in unit time is called frequency and it is denoted by $\upsilon$.

i.e. $$\upsilon = \frac{1}{T} = \frac{\sqrt{\lambda}}{2\pi}$$

## Summary

Maximum velocity = $\sqrt{\lambda}\,a$

Maximum acceleration = $\lambda a$

Time Period = $\dfrac{2\pi}{\sqrt{\lambda}}$

Frequency = $\dfrac{\sqrt{\lambda}}{2\pi}$



## ❖ QUESTION 1

A particle describes simple harmonic motion with frequency N. If the greatest velocity is V, find the amplitude and maximum value of the acceleration of the particle. Also show that the velocity v at a distance x from the centre of motion is given by

$$v = 2\pi N\sqrt{a^2 - x^2}, \text{ where a is the amplitude.}$$

### SOLUTION

Given that

Frequency = N

But Frequency $= \dfrac{\sqrt{\lambda}}{2\pi}$

$\Rightarrow \quad N = \dfrac{\sqrt{\lambda}}{2\pi}$

$\Rightarrow \quad \sqrt{\lambda} = 2\pi N$ ________(i)

Given that

Maximum velocity = V

But Maximum velocity = $\sqrt{\lambda}\, a$

$\Rightarrow \quad V = \sqrt{\lambda}\, a$ ________(ii)

Where a is the amplitude

From (i) and (ii), we get

$V = 2\pi a N$

$\Rightarrow \quad a = \dfrac{V}{2\pi N}$

So Amplitude = $\dfrac{V}{2\pi N}$

Now

Maximum acceleration = $\lambda a$

$$= (2\pi N)^2 \frac{V}{2\pi N}$$

$$= 2\pi N V$$

Velocity at distance x is given by

$$v = \sqrt{\lambda(a^2 - x^2)}$$

$$= \sqrt{(2\pi N)^2(a^2 - x^2)}$$

$$= 2\pi N\sqrt{(a^2 - x^2)}$$

## ❖ QUESTION 2

A particle describing simple harmonic motion has velocities 5ft/sec. and 4ft/sec. when its distances from the centre are 12ft. and 13ft. respectively. Find the time-period of motion.

### SOLUTION

The time-period of a particle describing simple harmonic motion is given by

Time Period = $\dfrac{2\pi}{\sqrt{\lambda}}$ ________(i)

We know that

$v^2 = \lambda(a^2 - x^2)$

When x = 12ft then v = 5ft/sec and when x = 13ft then v = 4ft/sec

So $25 = \lambda(a^2 - 144)$ ________(ii) and $16 = \lambda(a^2 - 169)$ _______(iii)

Subtracting (iii) from (ii), we get

$$9 = \lambda 25$$

$$\Rightarrow \quad \sqrt{\lambda} = \frac{3}{5}$$

Using value of $\sqrt{\lambda}$ in (i), we get

$$\text{Time Period} = \frac{10\pi}{3}$$

## ❖ QUESTION 3

The maximum velocity that a particle executing simple harmonic motion of amplitude a attains, is v. If it is disturbed in such a way that its maximum velocity becomes nv, find the change in the amplitude and the time period of motion.

### SOLUTION

Given that

Maximum velocity = v

But Maximum velocity = $\sqrt{\lambda}\,a$

$$\Rightarrow \quad v = \sqrt{\lambda}\,a$$

$$\Rightarrow \quad a = \frac{v}{\sqrt{\lambda}} \qquad \text{\_\_\_\_\_\_\_\_(i)}$$

Where a is the amplitude. Suppose that A is the new amplitude when the velocity is nv. Then from (i), we get

$$A = \frac{nv}{\sqrt{\lambda}}$$

Change in amplitude = A – a

$$= \frac{nv}{\sqrt{\lambda}} - \frac{v}{\sqrt{\lambda}} = \frac{v}{\sqrt{\lambda}}(n-1) = a(n-1)$$

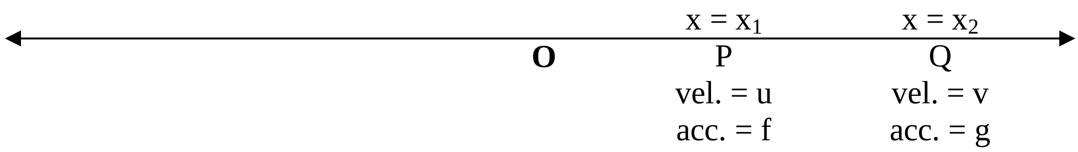
The time period in both cases is $\frac{2\pi}{\sqrt{\lambda}}$

## ❖ QUESTION 4

A point describes simple harmonic motion in such a way that its velocity and acceleration at a point P are u and f respectively and the corresponding quantities at another point Q are v and g. Find the distance PQ.

### SOLUTION

Let O be the centre of motion and OP = $x_1$ and PQ = $x_2$ as shown in figure.




Let $x_2 > x_1$ Then

$$PQ = x_2 - x_1 \qquad \text{\_\_\_\_\_\_\_\_(i)}$$

We know that

$$v^2 = \lambda(a^2 - x^2) \text{ and } a = -\lambda x$$

At point P

$$u^2 = \lambda(a^2 - x_1^2)$$

and $\quad f = -\lambda x_1$

At point Q

$$v^2 = \lambda(a^2 - x_2^2)$$

and $\quad g = -\lambda x_2$

Now

$$u^2 - v^2 = \lambda(a^2 - x_1^2) - \lambda(a^2 - x_2^2)$$
$$= \lambda(x_2^2 - x_1^2)$$
$$= \lambda(x_2 - x_1)(x_2 + x_1) \qquad \text{\_\_\_\_\_\_\_\_(ii)}$$

Now $\quad f + g = -\lambda x_1 - \lambda x_2$

$$= -\lambda(x_1 + x_2)$$

$$\Rightarrow \quad \lambda = -\frac{f+g}{x_1+x_2}$$

Using value of $\lambda$ in (ii), we get

$$u^2 - v^2 = \left(-\frac{f+g}{x_1+x_2}\right)(x_2 - x_1)(x_2 + x_1)$$

$$\Rightarrow \quad x_2 - x_1 = -\frac{u^2 - v^2}{f+g}$$

$$\Rightarrow \quad x_2 - x_1 = \frac{v^2 - u^2}{f+g}$$

$$\Rightarrow \quad PQ = \frac{v^2 - u^2}{f+g} \qquad \text{By(i)}$$

## ❖ QUESTION 5

If a point P moves with a velocity v given by

$$v^2 = n^2(ax^2 + 2bx + c)$$

Show that P executes a simple harmonic motion. Find the centre, the amplitude and the time-period of the motion.

### SOLUTION

Given that

$$v^2 = n^2(ax^2 + 2bx + c)$$

Diff. w.r.t 'x', we get

$$2v\frac{dv}{dx} = n^2(2ax + 2b)$$

$$\Rightarrow \quad v\frac{dv}{dx} = n^2a\left(x + \frac{b}{a}\right)$$

$$\Rightarrow \quad \text{acceleration} = n^2a\left(x + \frac{b}{a}\right) \qquad \because v\frac{dv}{dx} = \text{acceleration}$$

$$\Rightarrow \quad \text{acceleration} = -\,n^2a\left[-\left(x + \frac{b}{a}\right)\right]$$

Put $X = -\left(x + \frac{b}{a}\right)$

$$\Rightarrow \quad \text{acceleration} = -\,n^2aX$$

$$\Rightarrow \quad \text{acceleration} \propto -\,X$$

Which shows that P executes the simple harmonic motion.



To find centre put X = 0

$$\Rightarrow \quad x + \frac{b}{a} = 0$$

$$\Rightarrow \quad x = -\frac{b}{a}$$

To find amplitude put v = 0

$$\Rightarrow \quad n^2(ax^2 + 2bx + c) = 0$$

$$\Rightarrow \quad ax^2 + 2bx + c = 0$$

$$\Rightarrow \quad x = \frac{-2b \pm \sqrt{4b^2 - 4ac}}{2a}$$

$$= \frac{-b \pm \sqrt{b^2 - ac}}{a}$$

Let O be the origin, then

$$OA = \frac{-b + \sqrt{b^2 - ac}}{a} \quad \text{and} \quad OB = \frac{-b - \sqrt{b^2 - ac}}{a}$$

Let C be the centre then

$$OC = -\frac{b}{a}$$

$$\begin{aligned}\text{Amplitude} &= CA \\ &= OA - OC \\ &= \frac{-b + \sqrt{b^2 - ac}}{a} + \frac{b}{a} \\ &= \frac{\sqrt{b^2 - ac}}{a}\end{aligned}$$

$$\text{Time Period} = \frac{2\pi}{\sqrt{\lambda}}$$

Here $\lambda = n^2 a$

$$\Rightarrow \quad \text{Time Period} = \frac{2\pi}{n\sqrt{a}}$$

**B** **C** **O** **A**

$x = \dfrac{-b - \sqrt{b^2 - ac}}{a}$ $\qquad x = -\dfrac{b}{a}$ $\qquad x = \dfrac{-b + \sqrt{b^2 - ac}}{a}$

***<u>End of The Chapter # 6.</u>***