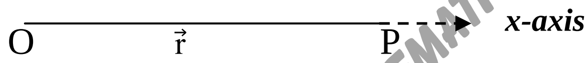
RECTILINEAR MOTION

5

CHAPTER

❖ INTRODUCTION

The motion of a particle along a straight line is called rectilinear motion. Let the particle start from O along a line. We take line along x-axis. Let after time 't' particle be at a point P at a distance 'x' from O.



Let $\vec{r}$ be the position vector of the point P w.r.t origin O. Then

$$\vec{r} = \overline{OP} = x\,\hat{i}$$

Now $\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i}$ and $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2x}{dt^2}\hat{i}$

Let $|\vec{v}| = v$ and $|\vec{a}| = a$

Then $v = \frac{dx}{dt}$ and $a = \frac{d^2x}{dt^2}$

Also $a = \frac{dv}{dt} = \frac{dv}{dx}.\frac{dx}{dt}$

$= \frac{dv}{dx}.v$

$\Rightarrow \quad a = v.\frac{dv}{dx}$

❖ MOTION WITH CONSTANT ACCELERATION

Let the particle start from O with velocity u at time t = 0 with constant acceleration.. Let after time 't' particle be at a point P at a distance 'x' from O. Then

$$a = \frac{dv}{dt} \quad \Rightarrow \quad adt = dv$$

On integrating we get

$$v = at + A \qquad \text{\_\_\_\_\_\_\_\_(i)}$$

Where A is constant of acceleration.

At t = 0, v = u

Using this in (i), we get

$$A = v$$

Using value of A in (i), we get

$$\mathbf{v = u + at} \qquad \text{\_\_\_\_\_\_\_\_\_\_(ii)}$$

Which is 1st equation of motion.

As we know that

$$v = \frac{dx}{dt}$$

$$\Rightarrow \quad \frac{dx}{dt} = u + at \qquad \text{By (ii)}$$

$$\Rightarrow \quad dx = (u + at)dt$$

On integrating we get

$$x = ut + \frac{1}{2}at^2 + B \qquad \text{\_\_\_\_\_\_\_\_\_\_(iii)}$$

At t = 0, x = 0

Using this in (ii), we get B = 0

Using value of B in (ii), we get

$$\mathbf{x = ut + \frac{1}{2}at^2} \qquad \text{\_\_\_\_\_\_\_\_\_\_(iv)}$$

Which is 2nd equation of motion.

As $\quad a = v.\frac{dv}{dx} \quad \Rightarrow \quad a\,.\,dx = v.\,dv$

On integrating, we get

$$ax + C = \frac{v^2}{2} \qquad \text{\_\_\_\_\_\_\_\_\_\_(v)}$$

At t = 0, x = 0, v = u

Using these values in(v), we get

$$C = \frac{u^2}{2}$$

Using value of C in (v), we get

$$ax + \frac{u^2}{2} = \frac{v^2}{2} \quad \Rightarrow \quad 2ax + u^2 = v^2$$

$$\Rightarrow \quad \mathbf{2ax = v^2 - u^2}$$

Which is 3rd equation of motion.

If a particle is moving with constant retardation then $a = -a$

❖ DISTANCE TRAVELLED IN NTH SECOND

Let x_1 and x_2 be the distances traveled in n and n – 1 seconds respectively. Then by 2nd equation of motion we have

$$x_1 = un + \frac{1}{2}an^2$$

and $$x_2 = u(n-1) + \frac{1}{2}a(n-1)^2$$

Distance traveled in nth second $= x_1 - x_2$

$$= un + \frac{1}{2}an^2 - u(n-1) - \frac{1}{2}a(n-1)^2$$

$$= un + \frac{1}{2}an^2 - un + u - \frac{1}{2}a(n^2 - 2n + 1)$$

$$= \frac{1}{2}an^2 + u - \frac{1}{2}an^2 + \frac{1}{2}a(2n-1)$$

$$= u + \frac{1}{2}a(2n-1)$$

❖ QUESTION 1

A particle moving in a straight line starts from rest and is accelerated uniformly to attain a velocity 60 miles per hours in 4 seconds. Finds the acceleration of motion and distance travelled by the particle in the last three seconds.

SOLUTION

Given that

Initial velocity = u = 0

Time = t = 4sec

Final velocity = v = 60 miles/h

$$= \frac{60 \times 1760 \times 3}{3600} = 88 \text{ ft/sec}$$

We know that

$$v = u + at$$

$$\Rightarrow \quad a = \frac{v-u}{t} = \frac{88-0}{4} = 22 \text{ ft/sec}^2$$

Now

x_1 = Distance covered in 1st second

$$= ut + \frac{1}{2}at^2$$

$$= 0 + \frac{1}{2}(22)(1)^2 = 11\text{ft}$$

x_2 = Distance covered in 4 seconds

$= ut + \frac{1}{2}at^2$

$= 0 + \frac{1}{2}(22)(4)^2 = 176ft$

Distance covered in last 3 seconds $= x_2 - x_1$

$= 176 - 11 = 165ft.$

QUESTION 2

Find the distance travelled and velocity attained by a particle moving on a straight line at any timre t. If it starts from rest at t = 0 and subject to an acceleration $\mathbf{t^2 + sint + e^t}$

SOLUTION

Given that

$a = t^2 + \sin t + e^t$

$\Rightarrow \quad \frac{d^2x}{dt^2} = t^2 + \sin t + e^t$

On integrating, we get

$\frac{dx}{dt} = \frac{t^3}{3} - \cos t + e^t + A$

Where A is constant of integration

When t = 0 then $\frac{dx}{dt} = 0$

$\Rightarrow \quad A = 0$

Hence velocity is:

$\frac{dx}{dt} = \frac{t^3}{3} - \cos t + e^t$

On integrating again, we get

$x = \frac{t^4}{12} - \sin t + e^t + B$

Where B is constant of integration

When t = 0 then x = 0

$\Rightarrow \quad B = -1$

Hence the distance travelled is given by

$x = \frac{t^4}{12} - \sin t + e^t - 1$

❖ QUESTION 3

Discuss the motion of a particle moving in a straight line if it starts from rest at $t = 0$ and its acceleration is equal to **(i)** $\mathbf{t^n}$ **(ii)** **acost + bsint** **(iii)** $\mathbf{-n^2x}$

SOLUTION

(i)

Given that

$$a = t^n$$

$$\Rightarrow \quad \frac{d^2x}{dt^2} = t^n$$

On integrating, we get

$$\frac{dx}{dt} = \frac{t^{n+1}}{n+1} + A$$

Where A is constant of integration

When $t = 0$ then $\frac{dx}{dt} = 0$

$\Rightarrow \quad A = 0$

Hence velocity is:

$$\frac{dx}{dt} = \frac{t^{n+1}}{n+1}$$

On integrating again, we get

$$x = \frac{t^{n+2}}{(n+1)(n+2)} + B$$

Where B is constant of integration

When $t = 0$ then $x = 0$

$\Rightarrow \quad B = 0$

Hence the distance travelled is given by

$$x = \frac{t^{n+2}}{(n+1)(n+2)}$$

(ii)

Given that

$$a = a\cos t + b\sin t$$

$$\Rightarrow \quad \frac{d^2x}{dt^2} = a\cos t + b\sin t$$

On integrating, we get

$$\frac{dx}{dt} = a\sin t - b\cos t + A$$

Where A is constant of integration

When t = 0 then $\frac{dx}{dt} = 0$

$\Rightarrow$ A = b

Hence velocity is

$$\frac{dx}{dt} = a\sin t - b\cos t + b$$

On integrating again, we get

$$x = -a\cos t - b\sin t + bt + B$$

Where B is constant of integration

When t = 0 then x = 0

$\Rightarrow$ B = a

Hence the distance travelled is given by

$$x = -a\cos t - b\sin t + bt + a$$

$$= a(1 - \cos t) + b(t - \sin t)$$

(iii)

Given that

$$a = -n^2x$$

$\Rightarrow$ $v\frac{dv}{dx} = -n^2x$ $\quad\because a = v\frac{dv}{dx}$

$\Rightarrow$ $v\,dv = -n^2x\,dx$

On integrating, we get

$$\frac{v^2}{2} = -n^2\frac{x^2}{2} + A$$

Where A is constant of integration.

$\Rightarrow$ $v^2 = 2A - n^2x^2$

$\Rightarrow$ $v^2 = B - n^2x^2$

$\Rightarrow$ $v = \sqrt{B - n^2x^2}$

Which is the velocity of the particle.

$\Rightarrow$ $\frac{dx}{dt} = \sqrt{B - n^2x^2}$ $\quad\because v = \frac{dx}{dt}$

$\Rightarrow$ $\frac{dx}{\sqrt{B - n^2x^2}} = dt$

On integrating again, we get

$$\frac{1}{n}\sin^{-1}\left(\frac{nx}{\sqrt{B}}\right) = t + B$$

Where B is constant of integration.

$$\frac{1}{n}\sin^{-1}\left(\frac{nx}{\sqrt{B}}\right) = t + B$$

$$\Rightarrow \quad \sin^{-1}\left(\frac{nx}{\sqrt{B}}\right) = nt + nB$$

$$\Rightarrow \quad \sin^{-1}\left(\frac{nx}{\sqrt{B}}\right) = nt + C$$

$$\Rightarrow \quad x = \frac{\sqrt{B}}{n}\sin(nt + C)$$

❖ QUESTION 4

A particle moves in a straight line with an acceleration kv^3. If its initial velocity is u, then find the velocity and the time spend when the particle has travelled a distance x.

SOLUTION

Given that

$$a = kv^3$$

$$\Rightarrow \quad v\frac{dv}{dx} = kv^3 \qquad \because\ a = v\frac{dv}{dx}$$

$$\Rightarrow \quad v^{-2}dv = kdx$$

On integrating, we get

$$-v^{-1} = kx + A \qquad \text{\_\_\_\_\_\_(i)}$$

Where A is constant of integration.

Initially v = u, x = 0 and t = 0

$$\Rightarrow \quad A = -u^{-1}$$

Using value of A in (i), we get

$$-v^{-1} = kx - u^{-1}$$

$$\Rightarrow \quad \frac{1}{v} = \frac{1}{u} - kx = \frac{1 - kxu}{u}$$

$$\Rightarrow \quad v = \frac{u}{1 - kux}$$

Which is the velocity of the particle.

$$\Rightarrow \quad \frac{dx}{dt} = \frac{u}{1 - kxu} \qquad \because\ v = \frac{dx}{dt}$$

$\Rightarrow \quad (1 - kxu)dx = udt$

On integrating again, we get

$$x - ku\frac{x^2}{2} = ut + B \qquad \text{\_\_\_\_\_\_\_\_\_(ii)}$$

Where B is constant of integration.

Initially, v = u, x = 0 and t = 0

$\Rightarrow \quad B = 0$

Using value of B in (ii), we get

$$x - ku\frac{x^2}{2} = ut$$

$$\Rightarrow \quad ut = \frac{x}{2}(2 - kux)$$

$$\Rightarrow \quad t = \frac{x}{2u}(2 - kux)$$

Which is required time spend when the particle has travelled a distance x.

❖ QUESTION 5

A particle moving in a straight line starts with a velocity u and has acceleration v^3, where v is the velocity of the particle at time t. Find the velocity and the time as functions of the distance travelled by the particle

SOLUTION

Given that

$$a = v^3$$

$$\Rightarrow \quad v\frac{dv}{dx} = v^3 \qquad \because \; a = v\frac{dv}{dx}$$

$$\Rightarrow \quad v^{-2}dv = dx$$

On integrating, we get

$$-v^{-1} = x + A \qquad \text{\_\_\_\_\_\_(i)}$$

Where A is constant of integration.

Initially v = u, x = 0 and t = 0

$\Rightarrow \quad A = -u^{-1}$

Using value of A in (i), we get

$$-v^{-1} = x - u^{-1}$$

$$\Rightarrow \quad \frac{1}{v} = \frac{1}{u} - x = \frac{1 - xu}{u}$$

$\Rightarrow \quad v = \dfrac{u}{1 - ux}$

Which is the velocity of the particle.

$\Rightarrow \quad \dfrac{dx}{dt} = \dfrac{u}{1 - xu} \qquad \because v = \dfrac{dx}{dt}$

$\Rightarrow \quad (1 - xu)dx = udt$

On integrating again, we get

$$x - u\frac{x^2}{2} = ut + B \qquad \text{\_\_\_\_\_\_\_\_\_(ii)}$$

Where B is constant of integration.

Initially, v = u, x = 0 and t = 0

$\Rightarrow \quad B = 0$

Using value of B in (ii), we get

$$x - u\frac{x^2}{2} = ut$$

$\Rightarrow \quad ut = \dfrac{x}{2}(2 - ux)$

$\Rightarrow \quad t = \dfrac{x}{2u}(2 - ux)$

QUESTION 6

A particle starts with a velocity u and moves in a straight line. If it suffers a retardation equal to the square of the velocity. Find the distance travelled by the particle in a time t.

SOLUTION

Given that

Retardation $= v^2$

$\Rightarrow \quad a = -v^2$

$\Rightarrow \quad v\dfrac{dv}{dx} = -v^2 \qquad \because a = v\dfrac{dv}{dx}$

$\Rightarrow \quad \dfrac{dv}{v} = -dx$

On integrating, we get

$$\ln v = -x + A \qquad \text{\_\_\_\_\_\_(i)}$$

Where A is constant of integration.

Initially v = u, x = 0 and t = 0

$\Rightarrow \quad A = \ln u$

Using value of A in (i), we get

$$\ln v = -x + \ln u$$

$\Rightarrow \quad x = \ln u - \ln v$

$\Rightarrow \quad x = \ln\left(\frac{u}{v}\right)$

$\Rightarrow \quad e^x = \frac{u}{v}$

$\Rightarrow \quad v = \frac{u}{e^x}$

Which is the velocity of the particle.

$\Rightarrow \quad \frac{dx}{dt} = \frac{u}{e^x} \qquad \because v = \frac{dx}{dt}$

$\Rightarrow \quad e^x dx = u dt$

On integrating again, we get

$$e^x = ut + B \qquad \text{\_\_\_\_\_\_(ii)}$$

Where B is constant of integration.

Initially, $v = u$, $x = 0$ and $t = 0$

$\Rightarrow \quad B = 1$

Using value of B in (ii), we get

$e^x = ut + 1 \quad \Rightarrow \quad x = \ln(1 + ut)$

❖ QUESTION 7

Discuss the motion of a particle moving in a straight line with an acceleration x^3 where x is the distance of the particle from a fixed point O on the line, if it starts at t = 0 from a point x = c with a velocity $c^2/\sqrt{2}$

SOLUTION

Given that

$$a = x^3$$

$\Rightarrow \quad v\frac{dv}{dx} = x^3 \qquad \because a = v\frac{dv}{dx}$

$\Rightarrow \quad v\,dv = x^3 dx$

On integrating, we get

$$\frac{v^2}{2} = \frac{x^4}{4} + A \qquad \text{\_\_\_\_\_\_\_\_\_(i)}$$

Where A is constant of integration.

Initially, t = 0, x = c and v = $c^2/\sqrt{2}$

$\Rightarrow \quad A = 0$

Using value of A in (i), we get

$$\frac{v^2}{2}=\frac{x^4}{4}$$

$$\Rightarrow \quad v^2=\frac{x^4}{2}$$

$$\Rightarrow \quad v=\frac{x^2}{\sqrt{2}}$$

Which is the velocity of the particle.

$$\Rightarrow \quad \frac{dx}{dt}=\frac{x^2}{\sqrt{2}} \qquad \because v=\frac{dx}{dt}$$

$$\Rightarrow \quad \frac{dx}{x^2}=\frac{dt}{\sqrt{2}}$$

$$\Rightarrow \quad x^{-2}dx=\frac{dt}{\sqrt{2}}$$

On integrating again, we get

$$-x^{-1}=\frac{t}{\sqrt{2}}+B \qquad \text{\_\_\_\_\_\_\_\_(ii)}$$

Where B is constant of integration.

Initially, x = c and t = 0

$\Rightarrow \quad B=-c^{-1}$

Using value of B in (ii), we get

$$-x^{-1}=\frac{t}{\sqrt{2}}-c^{-1}$$

$$\Rightarrow \quad c^{-1}-x^{-1}=\frac{t}{\sqrt{2}} \qquad \Rightarrow \quad t=\sqrt{2}(c^{-1}-x^{-1}) \qquad \Rightarrow \quad t=\sqrt{2}\left(\frac{1}{c}-\frac{1}{x}\right)$$

❖ QUESTION 8

Discuss the motion of a particle moving in a straight line if it starts from the rest at a distance a from the point O and moves with an acceleration equal to μ times its distance from O.

SOLUTION

Let x be the distance of particle from O then

$$a=\mu x$$

$\Rightarrow \quad v\dfrac{dv}{dx} = \mu x \qquad \because \; a = v\dfrac{dv}{dx}$

$\Rightarrow \quad vdv = \mu x dx$

On integrating, we get

$$\frac{v^2}{2} = \frac{\mu x^2}{2} + A \qquad \text{\_\_\_\_\_(i)}$$

Where A is constant of integration.

Initially, v = 0, x = a and t = 0

$\Rightarrow \quad A = -\dfrac{\mu a^2}{2}$

Using value of A in (i), we get

$$\frac{v^2}{2} = \frac{\mu x^2}{2} - \frac{\mu a^2}{2}$$

$\Rightarrow \quad v^2 = \mu x^2 - \mu a^2$

$\Rightarrow \quad v = \sqrt{\mu(x^2 - a^2)}$

Which is the velocity of the particle.

$\Rightarrow \quad \dfrac{dx}{dt} = \sqrt{\mu(x^2 - a^2)} \qquad \because \; v = \dfrac{dx}{dt}$

$\Rightarrow \quad \dfrac{dx}{\sqrt{x^2 - a^2}} = \sqrt{\mu}dt$

On integrating again, we get

$$\cosh^{-1}\left(\frac{x}{a}\right) = \sqrt{\mu}t + B \qquad \text{\_\_\_\_\_\_\_\_\_(ii)}$$

Where B is constant of integration.

Initially, x = a and t = 0

$\Rightarrow \quad B = \cosh^{-1}1 = 0$

Using value of B in (ii), we get

$$\cosh^{-1}\left(\frac{x}{a}\right) = \sqrt{\mu}t$$

$\Rightarrow \quad x = a\cosh\left(\sqrt{\mu}t\right)$

❖ QUESTION 9

The acceleration of a particle falling freely under the gravitational pull is equal to ${}^{k}/_{x^2}$, where x is the distance of particle from the centre of the earth. Find the velocity of the particle if it is let fall from an altitude R, on striking the surface of the earth if the radius of earth is r and the air offers no resistance to motion.

SOLUTION

Given that

$$a = -\frac{k}{x^2}$$

Here we measuring distance x from centre O of the earth. The distance and acceleration is in opposite direction. So we take –ive sign. Therefore

$$v\frac{dv}{dx} = -\frac{k}{x^2} \qquad \because\ a = v\frac{dv}{dx}$$

$$\Rightarrow \quad vdv = -\frac{k}{x^2}dx$$

On integrating, we get

$$\frac{v^2}{2} = \frac{k}{x} + A \qquad \text{\_\_\_\_\_\_\_\_(i)}$$

Where A is constant of integration.

When $x = R$ then $v = 0$

$$\Rightarrow \quad A = -\frac{k}{R}$$

Using value of A in (i), we get

$$\frac{v^2}{2} = \frac{k}{x} - \frac{k}{R}$$

$$\Rightarrow \quad v^2 = 2k\left(\frac{1}{x} - \frac{1}{R}\right)$$

$$\Rightarrow \quad v = \sqrt{2k\left(\frac{1}{x} - \frac{1}{R}\right)}$$

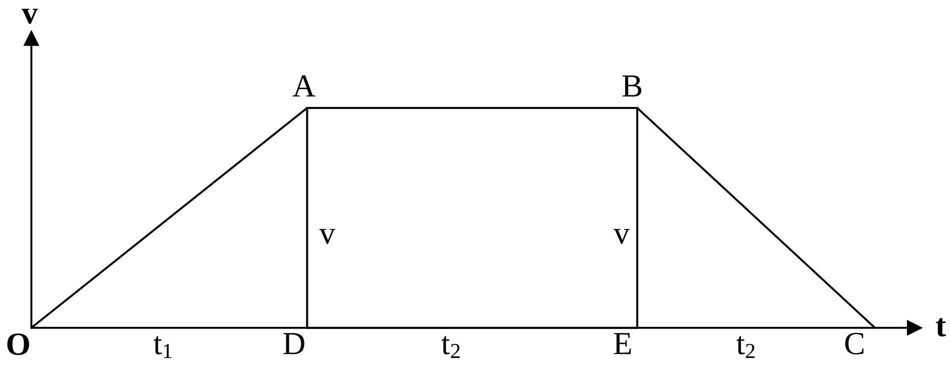
❖ QUESTION 10

A particle starts from rest with a constant acceleration a. When its velocity acquires a certain value v, it moves uniformly and then its velocity starts decreasing with a constant retardation 2a till it comes to rest. Find the distance travelled by the particle, if the time taken from rest to rest is t.

SOLUTION

Let t_1, t_2 and t_3 be the times for acceleration, uniform motion and retardation motion respectively. Then

$$t = t_1 + t_2 + t_3 \qquad \text{\_\_\_\_\_\_\_(i)}$$



Now

acceleration = slope of OA

$\Rightarrow \quad a = \dfrac{v}{t_1}$

$\Rightarrow \quad t_1 = \dfrac{v}{a}$

Similarly

retardation = slope of BC

$\Rightarrow \quad 2a = \dfrac{v}{t_3}$

$\Rightarrow \quad t_3 = \dfrac{v}{2a}$

From (i), we have

$t_2 = t - t_1 - t_3$

$= t - \dfrac{v}{a} - \dfrac{v}{2a}$

$= t - \dfrac{3v}{2a}$

Distance = Area under the velocity-time curve

= Area of trapezium OABC

$= \dfrac{1}{2}(OC + AB)(AD)$

$= \dfrac{1}{2}(t_1 + t_2 + t_3 + t_2)v$

$= \dfrac{1}{2}(t + t_2)v$

$= \dfrac{1}{2}\left(t + t - \dfrac{3v}{2a}\right)v$

$= \dfrac{1}{2}v\left(2t - \dfrac{3v}{2a}\right)$

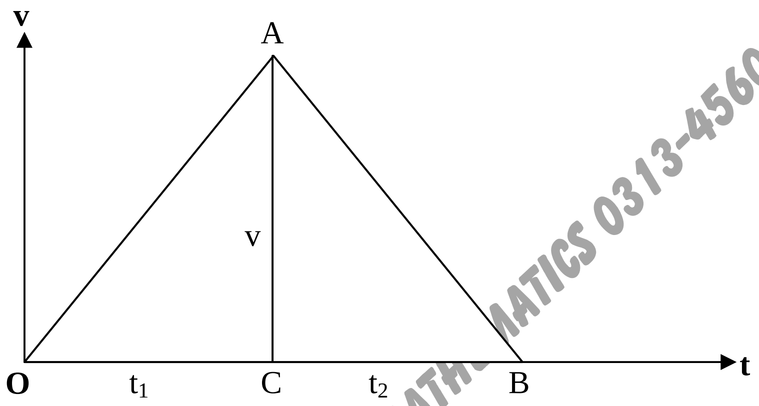
❖ QUESTION 11

A particle moving along a straight line starts from rest and is accelerated uniformly until it attains a velocity v. The motion is then retarded and the particle comes to rest after traversing a total distance x. If acceleration is f, find the retardation and the total time taken by the particle from rest to rest.

SOLUTION

Let t_1 and t_2 be the times for acceleration and retardation respectively. Then

$$t = t_1 + t_2 \qquad \text{(i)}$$



Now

acceleration = slope of OA

$\Rightarrow \quad f = \dfrac{v}{t_1}$

$\Rightarrow \quad t_1 = \dfrac{v}{f}$

Let g be the retardation. Then

retardation = slope of BC

$\Rightarrow \quad g = \dfrac{v}{t_2}$

$\Rightarrow \quad t_2 = \dfrac{v}{g}$

Distance = Area under the velocity-time curve

$\Rightarrow \quad x = \text{Area of } \Delta ABC$

$$= \frac{1}{2}(OB)(AC)$$

$$= \frac{1}{2}(t_1 + t_2)v \qquad \text{(ii)}$$

$$= \frac{1}{2}tv$$

$\Rightarrow \quad t = \dfrac{2x}{v}$

Thus

$$\text{Total time} = \frac{2x}{v}$$

From (ii), we have

$$x = \frac{1}{2}(t_1 + t_2)v$$
$$= \frac{1}{2}\left(\frac{v}{f} + \frac{v}{g}\right)v$$
$$= \frac{v^2}{2}\left(\frac{1}{f} + \frac{1}{g}\right)$$

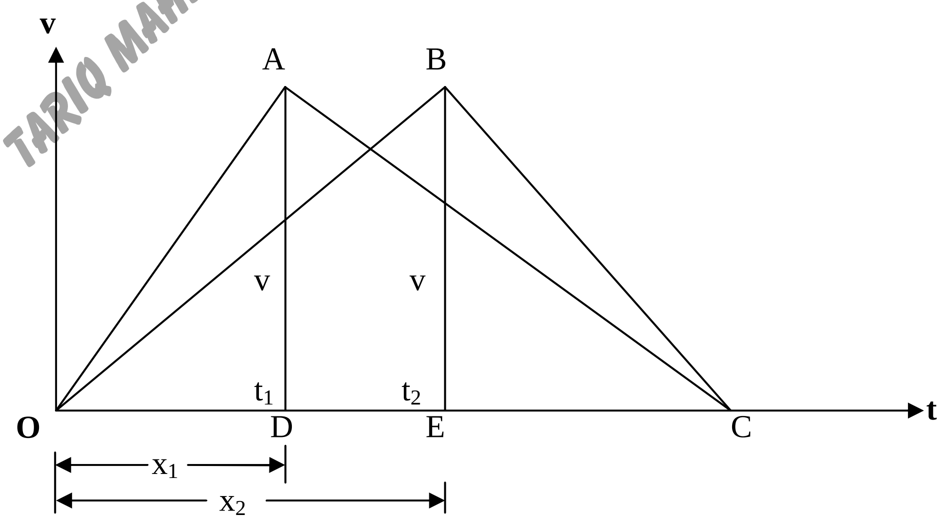
$\Rightarrow \quad \dfrac{2x}{v^2} = \dfrac{1}{f} + \dfrac{1}{g} \quad \Rightarrow \quad \dfrac{1}{g} = \dfrac{2x}{v^2} - \dfrac{1}{f} \quad \Rightarrow \quad \dfrac{1}{g} = \dfrac{2xf - v^2}{fv^2}$

$\Rightarrow \quad g = \dfrac{fv^2}{2xf - v^2}$

❖ QUESTION 12

Two particles travel along a straight line. Both start at the same time and are accelerated uniformly at different rates. The motion is such that when a particle attains the maximum velocity v, its motion is retarded uniformly. Two particles come to rest simultaneously at a distance x from the starting point. If the acceleration of the first is a and that of second is $\frac{1}{2}$a. Find the distance between the point where the two particles attain their maximum velocities.

SOLUTION



Let both particle attain maximum velocity at t_1 and t_2 respectively. Then

For 1st Particle

Acceleration = slope of OA

$\Rightarrow \quad a = \frac{v}{t_1} \Rightarrow t_1 = \frac{v}{a}$

For 2nd Particle

Acceleration = slope of OB

$\Rightarrow \quad \frac{1}{2}a = \frac{v}{t_2} \Rightarrow t_2 = \frac{2v}{a}$

Let x_1 and x_2 be distances covered by the 1st and 2nd particles to attain velocity v. Then

x_1 = Area of ΔOAD

$= \frac{1}{2}(OD)(AD)$

$= \frac{1}{2}vt_1 = \frac{1}{2}v\left(\frac{v}{a}\right) = \frac{v^2}{2a}$

Similarly

x_2 = Area of ΔOBE

$= \frac{1}{2}(OE)(BE)$

$= \frac{1}{2}vt_2 = \frac{1}{2}v\left(\frac{2v}{a}\right) = \frac{v^2}{a}$

Required Distance = $x_2 - x_1$

$= \frac{v^2}{a} - \frac{v^2}{2a} = \frac{v^2}{2a}$

❖ QUESTION 13

Two particles start simultaneously from point O and move in a straight line one with velocity of 45 mile/h and an acceleration 2ft/sec^2 and other with a velocity of 90mile/h and a retardation of 8ft/sec^2. Find the time after which the velocities of particles are same and the distance of O from the point where they meet again.

SOLUTION

For 1st Particle

Given that

u = 45 mile/h

$= \frac{45 \times 1760 \times 30}{60 \times 60} = 66$ft/sec

a = 2ft/sec^2

We know that

$v = u + at$

$= 66 + 2t$ ______(i)

For 2nd Particle

Given that

$u = 90$ mile/h

$= \frac{90 \times 1760 \times 30}{60 \times 60} = 132\text{ft/sec}$

$a = -8\text{ft/sec}^2$

We know that

$v = u + at$

$= 132 - 8t$ ______(ii)

From (i) and (ii), we get

$66 + 2t = 132 - 8t$

$\Rightarrow$ $10t = 66$

$\Rightarrow$ $t = 6.6\text{sec}$

So after 6.6sec velocities of particles will same. Let both particle meet after a distance x.

Then

For 1st Particle

$x = ut + \frac{1}{2}at^2$

$= 66t + \frac{1}{2}(2)t^2$

$= 66t + t^2$

______(iii)

For 2nd Particle

$x = ut + \frac{1}{2}at^2$

$= 132t + \frac{1}{2}(-8)t^2$

$= 132t - 4t^2$ ______(iv)

From (iii) and (iv), we get

$66t + t^2 = 132t - 4t^2$

$\Rightarrow$ $5t^2 = 66t$

$\Rightarrow$ $t = 13.2$

Putting value of t in (iii), we get

$x = 10.4544\text{ft}$

❖ VERTICAL MOTION UNDER GRAVITY

For a falling body, the acceleration is constant. It is called acceleration due to gravity and is denoted by "g".

In FPS system value of g is 32ft/sec^2

In CGS system value of g is 981cm/sec^2

In MKS system value of g is 9.81m/sec^2

If the body is projected vertically upward then $g = -g$. For a falling body equations of motion are

$$v = u + gt$$
$$x = ut + \frac{1}{2}gt^2$$
$$2gx = v^2 - u^2$$

Note:

If $ax^2 + bx + c = 0$ be a quadratic equation and α, β be the roots of this equation. Then

$$\alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}$$

❖ QUESTION 14

A particle is projected vertically upward at $t = 0$ with a velocity u, passes a point at a height h at $t = t_1$ and $t = t_2$. Show that

$$\mathbf{t_1 + t_2 = \frac{2u}{g} \quad \text{and} \quad t_1 t_2 = \frac{2h}{g}}$$

SOLUTION

The distance travelled by the particle in time t is given by

$$x = ut - \frac{1}{2}gt^2$$

Put $x = h$

$$h = ut - \frac{1}{2}gt^2$$

$\Rightarrow \quad 2h = 2ut - gt^2$

$\Rightarrow \quad gt^2 - 2ut + 2h = 0$

The time t_1 and t_2 when the particle is at a height h from the point of projection, are roots of the quadratic equation

$$gt^2 - 2ut + 2h = 0$$

We know that

$$\text{Sum of the roots} = -\frac{\text{coefficient of } t}{\text{coefficient of } t^2}\ ,\quad \text{Product of the roots} = \frac{\text{coefficient of } t^0}{\text{coefficient of } t^2}$$

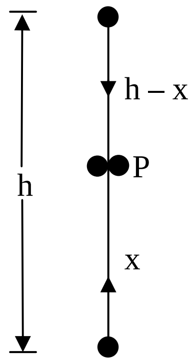
$$\Rightarrow \quad t_1 + t_2 = \frac{2u}{g} \quad \text{and} \quad t_1 t_2 = \frac{2h}{g}$$

❖ QUESTION 15

A particle is projected vertically upward with a velocity $\sqrt{2gh}$ and another is let fall from a height h at the same time. Find the height of the point where they meet each other.

SOLUTION

Let both particles meet at point P at height x. Then



For 1st Particle

$$x = ut - \frac{1}{2}gt^2 \qquad \text{\_\_\_\_\_\_(i)}$$

Put $u = \sqrt{2gh}$

$$x = \sqrt{2gh}\,t - \frac{1}{2}gt^2$$

For 2nd Particle

$$x = ut + \frac{1}{2}gt^2$$

Put $u = 0$ and $x = h - x$

$$h - x = \frac{1}{2}gt^2$$

$$x = h - \frac{1}{2}gt^2 \qquad \text{\_\_\_\_\_\_(ii)}$$

From (i) and (ii), we get

$$h - \frac{1}{2}gt^2 = \sqrt{2gh}\,t - \frac{1}{2}gt^2$$

$$\Rightarrow \quad h = \sqrt{2gh}\,t \quad \Rightarrow \quad t = \frac{h}{\sqrt{2gh}}$$

Using value of t in (i), we get

$$x = \sqrt{2gh}\,\frac{h}{\sqrt{2gh}} - \frac{1}{2}g\left(\frac{h}{\sqrt{2gh}}\right)^2 = h - \frac{1}{2}g\left(\frac{h^2}{2gh}\right) = h - \frac{h}{4} = \frac{3h}{4}$$

❖ QUESTION 16

A particle is projected vertically upwards. After a time t, another particle is sent up from the same point with the same velocity and meets the first at height h during the downward flight of the first. Find the velocity of the projection.

SOLUTION

Let u be the velocity of projection and v be the velocity at height h. Then

$$v^2 - u^2 = -2gh$$

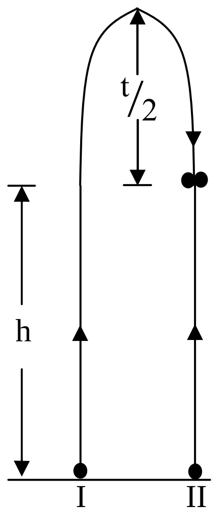
$$\Rightarrow \quad v^2 = u^2 - 2gh$$

$$\Rightarrow \quad v = \sqrt{u^2 - 2gh} \qquad \text{\_\_\_\_\_\_\_(i)}$$

Since time taken by 1st particle from height h to the maximum point and back to height h is t therefore time taken from the height h to the heights point is t/2. Velocity at the highest point is zero and at the height h the velocity is v.

We know that

$$v = u - gt$$



Since the velocity at the highest point is zero and at the height h the velocity is v. therefore

Put v = 0, u = v and t = t/2

$$0 = v - \frac{gt}{2}$$

$$\Rightarrow \quad v = \frac{gt}{2} \qquad \text{\_\_\_\_\_\_\_(ii)}$$

From (i) and (ii), we get

$$\frac{gt}{2} = \sqrt{u^2 - 2gh}$$

$$\Rightarrow \quad \frac{g^2t^2}{4} = u^2 - 2gh \quad \Rightarrow \quad g^2t^2 = 4u^2 - 8gh$$

$$\Rightarrow \quad 4u^2 = g^2t^2 + 8gh \quad \Rightarrow \quad 2u = \sqrt{g^2t^2 + 8gh} \quad \Rightarrow \quad u = \frac{\sqrt{g^2t^2 + 8gh}}{2}$$

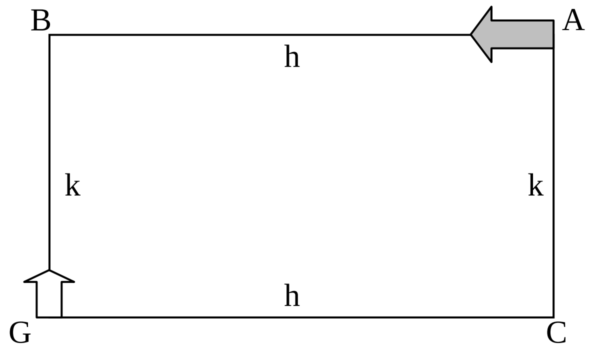
❖ QUESTION 17

A gunner detects a plane at t = 0 approaching him with a velocity v, the horizontal and the vertical distances of the plane being h and k respectively. His gun can fire a shell vertically upwards with an initial velocity u. Find the time when he should fire the gun and the condition on u so that he may be able to hit the plane if it continuous its flight in the same horizontal line.

SOLUTION

Let G be a gun and A be the position of plane at t = 0. Let gun hits the plane at point B and AB = h. Let time taken by plane from A to B is t_1. Then

$$t_1 = \frac{\text{Distance}}{\text{Velocity}} = \frac{h}{v}$$



Let t_2. be time taken by shell to reach at point B.
We know that

$$x = ut - \frac{1}{2}gt^2$$

Putting x = k and t = t_2, we get

$$k = ut_2 - \frac{1}{2}gt_2^2$$

$\Rightarrow$ $2k = 2ut_2 - gt_2^2$

$\Rightarrow$ $gt_2^2 - 2ut_2 + 2k = 0$

$\Rightarrow$ $t_2 = \dfrac{2u \pm \sqrt{4u^2 - 8gk}}{2g} = \dfrac{u \pm \sqrt{u^2 - 2gk}}{g}$

Let T be the time after which gun should be fired. Then

$$T = t_1 - t_2$$

$$= \frac{h}{v} - \frac{u \pm \sqrt{u^2 - 2gk}}{g}$$

For a shell to reach at B, the maximum velocity at B is zero.
Since

$$v^2 - u^2 = 2ax$$

Putting v = 0, a = $-$g and x = k, we get

$$-u^2 = -2gk \quad \Rightarrow \quad u^2 = 2gk$$

Which gives the least value of u. Hence $u^2 > 2gk$

❖ QUESTION 18

Two particles are projected simultaneously in the vertically upward direction with velocities $\sqrt{2gh}$ and $\sqrt{2gk}$ (k > h). After time t, when the two particles are still in flight, another particle is projected upwards with velocity u. Fin the condition so that the third particle may meet the first two during their upward flight.

SOLUTION

For 1st particle

$$v^2 - u^2 = 2ax$$

For maximum height put v = 0, a = −g and u = $\sqrt{2gh}$

$$2gh = 2gx$$

$\Rightarrow \quad a = h$

Thus maximum height attained by 1st particle is h. Similarly maximum height attained by 2nd particle is k.

Let t_1 be time take by the 1st particle to attain the maximum height h then

$$v = u + at$$

Put v = 0, u = $\sqrt{2gh}$, a = −g and t = t_1

$$0 = \sqrt{2gh} - gt_1$$

$\Rightarrow \quad t_1 = \dfrac{\sqrt{2gh}}{g}$

$\Rightarrow \quad t_1 = \sqrt{\dfrac{2h}{g}}$

Similarly time t_2 taken by the 2nd particle to attain the maximum height k is

$$t_2 = \sqrt{\frac{2k}{g}}$$

Since k > h therefore $t_2 > t_1$

Thus the 1st particle reach the maximum height earlier then 2nd.

If the 3rd particle is projected after time t then t must be less than t_1 in order to meet the 1st two particles during their upward flight. i.e. $t < t_1$

or $\quad t < \sqrt{\dfrac{2h}{g}}$

Now time left with 3rd particle is

$$\sqrt{\frac{2h}{g}} - t$$

and during this time it has to meet both the particles. i.e. It may have to cover a distance k.

Since

$$x = ut - \frac{1}{2}gt^2$$

When x = k, time = $\sqrt{\dfrac{2h}{g}} - t$ Then

$$k = u\left(\sqrt{\frac{2h}{g}} - t\right) - \frac{1}{2}g\left(\sqrt{\frac{2h}{g}} - t\right)^2$$

$$\Rightarrow \quad k + \frac{1}{2}g\left(\sqrt{\frac{2h}{g}} - t\right)^2 = u\left(\sqrt{\frac{2h}{g}} - t\right)$$

$$\Rightarrow \quad u = \frac{k}{\sqrt{\frac{2h}{g}} - t} + \frac{1}{2}g\left(\sqrt{\frac{2h}{g}} - t\right)$$

$$\Rightarrow \quad u = \frac{k}{\sqrt{\frac{2h}{g}} - t} + \frac{1}{2}\left(\sqrt{2hg} - t\right)$$

Thus the third particle meet the tow 1st particles if

$$u > \frac{k}{\sqrt{\frac{2h}{g}} - t} + \frac{1}{2}\left(\sqrt{2hg} - t\right)$$

%%%%% End of The Chapter # 5 %%%%%