VIRTUAL WORK

3

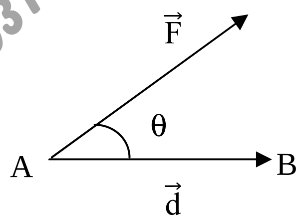
CHAPTER

❖ WORK DONE BY A FORCE

If a constant force $\vec{F}$ acts on a particle and particle is displaced from a point A to B. Let $\overline{AB} = \vec{d}$, then the work done by the force $\vec{F}$ is given by

$$\mathbf{W = \vec{F}.\vec{d} = Fd\cos\theta}$$

Where θ is angle between $\vec{F}$ and $\vec{d}$.



❖ SPECIAL CASES

Now we discuss some special cases of the above article.

(i) **When $\theta = 0^0$**

Work done = $Fd\cos 0^0 = Fd$

So work done will maximum when $\theta = 0^0$

(ii) **When $\theta = 90^0$**

Work done = $Fd\cos 90^0 = 0$

So work done will be zero when applied force is perpendicular to the displacement.

(iii) **When $\theta = 180^0$**

Work done = $Fd\cos 180^0 = -Fd$

Thus,

When $0^0 \leq \theta \leq 90^0$ Then work done will positive because $\cos\theta \geq 0$ when $0^0 \leq \theta \leq 90^0$

When $90^0 \leq \theta \leq 180^0$ Then work done will negative because $\cos\theta \leq 0$ when $90^0 \leq \theta \leq 180^0$

❖ VIRTUAL DISPLACEMENT & VIRTUAL WORK

If a set of particles or a body is in equilibrium under the action of forces then there is no motion and consequently there is no actual displacement. Suppose that the set of particles or the body receives an imaginary displacement, the forces acting thereon being regarded as constant during the displacement. Then such a displacement is called **virtual displacement** and work done by the forces during such a displacement is called **virtual work.**

It may be noted that a virtual displacement is only a hypothetical displacement involving no passage of time and is quite different from actual displacement of a moving body taking place in the course of time.

❖ APPLIED FORCES & FORCES OF CONSTRANIT

Particles or rigid bodies are generally subjected to two types of forces:

(i) ***Internal Forces*** ***(ii)*** ***External Forces***

❖ ***Internal forces***

Internal forces are those forces which the different part of a system exerts on each other and such forces obey Newton's 3^{rd} law of motion.

❖ ***External forces***

External forces are those forces which are not due to any part of a system but which are due to some external agency. External forces are further classified as:

(i) ***Reactive Forces or Forces of Constraint*** ***(ii)*** ***Active or Applied Forces***

❖ ***Reactive Forces or Forces of Constraint***

When a set of particle or a body is made to move along or rest on a curve or surface, the forces are exerted by such curve or surface are called forces of constraint or reactive forces.

❖ ***Active or Applied Forces***

External forces which are not due to any constraint is called active forces or applied forces. If a particle rests on or moves along an inclined plane, the reaction of the plane is a reactive force but the weight of the particle is an active force.

❖ ***Ideal or Workless Constraints***

If the forces of constraint do no virtual work, the constraint are said to be ideal or workless.

Now we state and prove the some important theorems, known as principles of Virtual Work, for a single particle, a set of particles, a rigid body and set of rigid bodies.

❖ PRINCIPLE OF VIRTUAL WORK FOR A SINGLE PARTICLE

STATEMENT:

A particle subject to workless constraints, is in equilibrium if and only if zero virtual work is done by the applied forces in any arbitrary infinitesimal displacement consistent with the constraints.

PROOF:

Let the total applied force on the particle be $\vec{F}_a$ and the total force of constraint is $\vec{F}_c$.
Suppose particle is in equilibrium. Then by definition of equilibrium

$$\vec{F}_a + \vec{F}_c = 0 \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_ (i)}$$

Now we have to show that virtual work done by the applied forces is zero.
Let $\delta\vec{r}$ be an infinitesimal displacement of the particle consistent with constraints. Then by taking dot product of (i) with $\delta\vec{r}$, we get

$$(\vec{F}_a + \vec{F}_c).\delta\vec{r} = \delta\vec{r}.0$$

$$\Rightarrow \quad \vec{F}_a.\delta\vec{r} + \vec{F}_c.\delta\vec{r} = 0 \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_ (ii)}$$

Since the constraint is workless therefore

$$\vec{F}_c.\delta\vec{r} = 0$$

So (ii) becomes

$$\vec{F}_a.\delta\vec{r} = 0$$

Which shows that the virtual work done by the applied forces is zero.
Conversely, suppose that the virtual work done by the applied forces is zero.

i.e. $\quad \vec{F}_a.\delta\vec{r} = 0 \qquad$ ____________ (iii)

Now we have to prove that the particle is in equilibrium. i.e.

i.e. $\quad \vec{F}_a + \vec{F}_c = 0$

Since the constraint is workless therefore

$$\vec{F}_c.\delta\vec{r} = 0 \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_ (iv)}$$

Adding (iii) and (iv), we get

$$\vec{F}_a.\delta\vec{r} + \vec{F}_c.\delta\vec{r} = 0$$

$$\Rightarrow \quad (\vec{F}_a + \vec{F}_c).\delta\vec{r} = 0$$

$$\Rightarrow \quad \vec{F}_a + \vec{F}_c = 0 \qquad \because \ \delta\vec{r} \neq 0$$

Which shows that the particle is in equilibrium. This completes the proof.

PRINCIPLE OF VIRTUAL WORK FOR A SET OF PARTICLES

STATEMENT:

A set of particles subject to workless constraints, is in equilibrium if and only if zero virtual work is done by the applied forces in any arbitrary infinitesimal displacement consistent with the constraints.

PROOF:

Let the total applied force on the set of particles be $\vec{F}_{ia}$ and the total force of constraint is $\vec{F}_{ic}$. Suppose the set of particles is in equilibrium. Then by definition of equilibrium

$$\sum_{i=1}^{n}\left(\vec{F}_{ia}+\vec{F}_{ic}\right)=0 \qquad \text{———— (i)}$$

Now we have to show that virtual work done by the applied forces is zero.
Let $\delta\vec{r}_i$ be an infinitesimal displacement of the particle consistent with constraints. Then by taking dot product of (i) with $\delta\vec{r}_i$, we get

$$\sum_{i=1}^{n}\left(\vec{F}_{ia}+\vec{F}_{ic}\right).\delta\vec{r}_i=\delta\vec{r}_i.0$$

$$\Rightarrow \qquad \sum_{i=1}^{n}\left(\vec{F}_{ia}.\delta\vec{r}_i+\vec{F}_{ic}.\delta\vec{r}_i\right)=0$$

$$\Rightarrow \qquad \sum_{i=1}^{n}\vec{F}_{ia}.\delta\vec{r}_i+\sum_{i=1}^{n}\vec{F}_{ic}.\delta\vec{r}_i=0 \qquad \text{———— (ii)}$$

Since the constraint is workless therefore

$$\sum_{i=1}^{n}\vec{F}_{ic}.\delta\vec{r}_i=0$$

So (ii) becomes

$$\sum_{i=1}^{n}\vec{F}_{ia}.\delta\vec{r}_i=0$$

Which shows that the virtual work done by the applied forces is zero.
Conversely, suppose that the virtual work done by the applied forces is zero.

i.e. $$\sum_{i=1}^{n}\vec{F}_{ia}.\delta\vec{r}_i=0 \qquad \text{———— (iii)}$$

Now we have to prove that the particle is in equilibrium. i.e.

i.e. $$\sum_{i=1}^{n}\left(\vec{F}_{ia}+\vec{F}_{ic}\right)=0$$

Since the constraint is workless therefore

$$\sum_{i=1}^{n}\vec{F}_{ic}.\delta\vec{r}_i=0 \qquad \text{———— (iv)}$$

Adding (iii) and (iv), we get

$$\sum_{i=1}^{n}\vec{F}_{ia}.\delta\vec{r}_i+\sum_{i=1}^{n}\vec{F}_{ic}.\delta\vec{r}_i=0$$

$$\Rightarrow\quad \sum_{i=1}^{n}\left(\vec{F}_{ia}.\delta\vec{r}_i+\vec{F}_{ic}.\delta\vec{r}_i\right)=0$$

$$\Rightarrow\quad \sum_{i=1}^{n}\left(\vec{F}_{ia}+\vec{F}_{ic}\right).\delta\vec{r}_i=0$$

$$\Rightarrow\quad \sum_{i=1}^{n}\left(\vec{F}_{ia}+\vec{F}_{ic}\right)=0 \qquad \because\ \delta\vec{r}_i\neq 0$$

Which shows that the particle is in equilibrium. This completes the proof.

❖ PRINCIPLE OF VIRTUAL WORK FOR A SINGLE RIGID BODY

STATEMENT:

A rigid body subject to workless constraints, is in equilibrium if and only if zero virtual work is done by the applied forces and applied torques in any arbitrary infinitesimal displacement consistent with the constraints.

PROOF:

Note that a system of forces acting on a rigid body can be reduced to a single force $\vec{R}$ at any arbitrary point together with a couple $\vec{G}$.

Let $\quad \vec{R}=\vec{R}_a+\vec{R}_c$

Where $\vec{R}_a$ is sum of applied forces and $\vec{R}_c$ is sum of forces of constraints.

Let $\quad \vec{G}=\vec{G}_a+\vec{G}_c$

Where $\vec{G}_a$ is sum of applied torques and $\vec{G}_c$ is sum of torques of constraints.

Suppose the rigid body is in equilibrium. Then by definition of equilibrium

$$\vec{R}=0 \qquad \text{\_\_\_\_\_\_\_\_\_\_(i)}$$

and $$\vec{G}=0 \qquad \text{\_\_\_\_\_\_\_\_\_\_(ii)}$$

Now we have to show that the virtual work done by applied force and applied torques is zero.

Let $\delta\vec{r}$ and $\delta\vec{\theta}$ be an infinitesimal virtual displacement and rotation of the rigid body. Then by taking dot product of (i) with $\delta\vec{r}$ and (ii) with $\delta\vec{\theta}$, we get

$$\vec{R}.\delta\vec{r}=\delta\vec{r}.0 \ \Rightarrow\ \vec{R}.\delta\vec{r}=0 \qquad \text{\_\_\_\_\_\_\_\_\_\_(iii)}$$

and $$\vec{G}.\delta\vec{\theta}=\delta\vec{\theta}.0\Rightarrow\vec{G}.\delta\vec{\theta}=0 \qquad \text{\_\_\_\_\_\_\_\_\_\_(iv)}$$

Adding (iii) and (iv), we get

$$\vec{R}.\delta\vec{r} + \vec{G}.\delta\vec{\theta} = 0$$

$$\Rightarrow \quad (\vec{R}_a + \vec{R}_c).\delta\vec{r} + (\vec{G}_a + \vec{G}_c).\delta\vec{\theta} = 0$$

$$\Rightarrow \quad \vec{R}_a\delta\vec{r} + \vec{R}_c.\delta\vec{r} + \vec{G}_a.\delta\vec{\theta} + \vec{G}_c.\delta\vec{\theta} = 0 \qquad \text{\_\_\_\_\_\_\_\_(v)}$$

Since the constraints are workless therefore

$$\vec{R}_c.\delta\vec{r} = 0 \text{ and } \vec{G}_c.\delta\vec{\theta} = 0$$

So (v) becomes

$$\vec{R}_a\delta\vec{r} + \vec{G}_a.\delta\vec{\theta} = 0$$

Which shows that the virtual work done by the applied forces and applied torques is zero.
Conversely, suppose that the virtual work done by the applied forces and applied torques is zero. i.e. $\vec{R}_a\delta\vec{r} = 0$ and $\vec{G}_a.\delta\vec{\theta} = 0$

$$\Rightarrow \quad \vec{R}_a\delta\vec{r} + \vec{G}_a.\delta\vec{\theta} = 0 \qquad \text{\_\_\_\_\_\_\_(vi)}$$

Now we have to show that the rigid body is in equilibrium.

i.e. $\vec{R} = 0$ and $\vec{G} = 0$

Since the constraints are workless therefore

$$\vec{R}_c.\delta\vec{r} = 0 \text{ and } \vec{G}_c.\delta\vec{\theta} = 0$$

$$\Rightarrow \quad \vec{R}_c.\delta\vec{r} + \vec{G}_c.\delta\vec{\theta} = 0 \qquad \text{\_\_\_\_\_\_\_\_(vii)}$$

Adding (vi) and (vii), we get

$$\vec{R}_a\delta\vec{r} + \vec{G}_a.\delta\vec{\theta} + \vec{R}_c.\delta\vec{r} + \vec{G}_c.\delta\vec{\theta} = 0$$

$$\Rightarrow \quad (\vec{R}_a + \vec{R}_c).\delta\vec{r} + (\vec{G}_a + \vec{G}_c).\delta\vec{\theta} = 0$$

$$\Rightarrow \quad \vec{R}.\delta\vec{r} + \vec{G}.\delta\vec{\theta} = 0$$

Since $\delta\vec{r} \neq 0$ and $\delta\vec{\theta} \neq 0$

Therefore $\vec{R} = 0$ and $\vec{G} = 0$

Which shows that the body is in equilibrium.

This completes the proof.

❖ PRINCIPLE OF VIRTUAL WORK FOR SET OF RIGID BODIES

STATEMENT:

A set of rigid bodies subject to workless constraints, is in equilibrium if and only if zero virtual work is done by the applied forces and applied torques in any arbitrary infinitesimal displacement consistent with the constraints.

PROOF:

Note that a given system of forces acting on i^{th} rigid body can be reduced to a single force $\vec{R}_i$ at any arbitrary point in the plane together with a couple $\vec{G}_i$.

Let $\qquad \vec{R}_i = \vec{R}_{ia} + \vec{R}_{ic}$

Where $\vec{R}_{ia}$ is sum of applied forces and $\vec{R}_{ic}$ is sum of forces of constraints.

Let $\qquad \vec{G}_i = \vec{G}_{ia} + \vec{G}_{ic}$

Where $\vec{G}_{ia}$ is sum of applied torques and $\vec{G}_{ic}$ is sum of torques of constraints.

Suppose the rigid body is in equilibrium. Then by definition of equilibrium

$$\sum_{i=1}^{n} \vec{R}_i = 0 \qquad \text{\_\_\_\_\_\_\_\_\_\_(i)}$$

and
$$\sum_{i=1}^{n} \vec{G}_i = 0 \qquad \text{\_\_\_\_\_\_\_\_\_\_(ii)}$$

Now we have to show that the virtual work done by applied force and applied torques is zero. Let $\delta\vec{r}_i$ and $\delta\vec{\theta}_i$ be an infinitesimal virtual displacement and rotation of the rigid body. Then by taking dot product of (i) with $\delta\vec{r}_i$ and (ii) with $\delta\vec{\theta}_i$, we get

$$\sum_{i=1}^{n} \vec{R}_i.\delta\vec{r}_i = 0 \qquad \text{\_\_\_\_\_\_\_\_\_\_(iii)}$$

and
$$\sum_{i=1}^{n} \vec{G}_i.\delta\vec{\theta}_i = 0 \qquad \text{\_\_\_\_\_\_\_\_\_\_(iv)}$$

Adding (iii) and (iv), we get

$$\sum_{i=1}^{n} \vec{R}_i\delta\vec{r}_i + \sum_{i=1}^{n} \vec{G}_i\delta\vec{\theta}_i = 0$$

$$\Rightarrow \qquad \sum_{i=1}^{n} \left(\vec{R}_{ia} + \vec{R}_{ic}\right).\delta\vec{r}_i + \sum_{i=1}^{n} \left(\vec{G}_{ia} + \vec{G}_{ic}\right).\delta\vec{\theta}_i = 0$$

$$\Rightarrow \qquad \sum_{i=1}^{n} \vec{R}_{ia}.\delta\vec{r}_i + \sum_{i=1}^{n} \vec{R}_{ic}.\delta\vec{r}_i + \sum_{i=1}^{n} \vec{G}_{ia}.\delta\vec{\theta}_i + \sum_{i=1}^{n} \vec{G}_{ic}.\delta\vec{\theta}_i = 0 \qquad \text{\_\_\_\_\_\_\_\_\_\_(v)}$$

Since the constraints are workless therefore

$$\sum_{i=1}^{n} \vec{R}_{ic}.\delta\vec{r}_i = 0 \text{ and } \sum_{i=1}^{n} \vec{G}_{ic}.\delta\vec{\theta}_i = 0$$

So (v) becomes

$$\sum_{i=1}^{n} \vec{R}_{ia}.\delta\vec{r}_i + \sum_{i=1}^{n} \vec{G}_{ia}.\delta\vec{\theta}_i = 0$$

Which shows that the virtual work done by the applied forces and applied torques is zero. **Conversely,** suppose that the virtual work done by the applied forces and applied torques is zero. i.e.

$$\sum_{i=1}^{n} \vec{R}_{ia}.\delta\vec{r}_i = 0 \text{ and } \sum_{i=1}^{n} \vec{G}_{ia}.\delta\vec{\theta}_i = 0$$

$$\Rightarrow \quad \sum_{i=1}^{n} \vec{R}_{ia}.\delta\vec{r}_i + \sum_{i=1}^{n} \vec{G}_{ia}.\delta\vec{\theta}_i = 0 \qquad \text{\_\_\_\_\_\_\_\_(vi)}$$

Now we have to show that the rigid body is in equilibrium. i.e.

$$\sum_{i=1}^{n} \vec{R}_i = 0 \text{ and } \sum_{i=1}^{n} \vec{G}_i = 0$$

Since the constraints are workless therefore

$$\sum_{i=1}^{n} \vec{R}_{ic}.\delta\vec{r}_i = 0 \text{ and } \sum_{i=1}^{n} \vec{G}_{ic}.\delta\vec{\theta}_i = 0$$

$$\Rightarrow \quad \sum_{i=1}^{n} \vec{R}_{ic}.\delta\vec{r}_i + \sum_{i=1}^{n} \vec{G}_{ic}.\delta\vec{\theta}_i = 0 \qquad \text{\_\_\_\_\_\_\_\_(vii)}$$

Adding (vii) and (vii), we get

$$\sum_{i=1}^{n} \vec{R}_{ia}.\delta\vec{r}_i + \sum_{i=1}^{n} \vec{R}_{ic}.\delta\vec{r}_i + \sum_{i=1}^{n} \vec{G}_{ia}.\delta\vec{\theta}_i + \sum_{i=1}^{n} \vec{G}_{ic}.\delta\vec{\theta}_i = 0$$

$$\Rightarrow \quad \sum_{i=1}^{n} \left(\vec{R}_{ia} + \vec{R}_{ic}\right).\delta\vec{r}_i + \sum_{i=1}^{n} \left(\vec{G}_{ia} + \vec{G}_{ic}\right).\delta\vec{\theta}_i = 0$$

$$\Rightarrow \quad \sum_{i=1}^{n} \vec{R}_i\delta\vec{r}_i + \sum_{i=1}^{n} \vec{G}_i\delta\vec{\theta}_i = 0$$

$$\Rightarrow \quad \sum_{i=1}^{n} \vec{R}_i\delta\vec{r}_i + \sum_{i=1}^{n} \delta\vec{\theta}_i = 0$$

$$\Rightarrow \quad \sum_{i=1}^{n} \vec{R}_i\delta\vec{r}_i = 0 \text{ and } \sum_{i=1}^{n} \delta\vec{\theta}_i = 0$$

Since $\delta\vec{r}_i \neq 0$ and $\delta\vec{\theta}_i \neq 0$

$$\Rightarrow \quad \sum_{i=1}^{n} \vec{R}_i = 0 \text{ and } \sum_{i=1}^{n} \vec{G}_i = 0$$

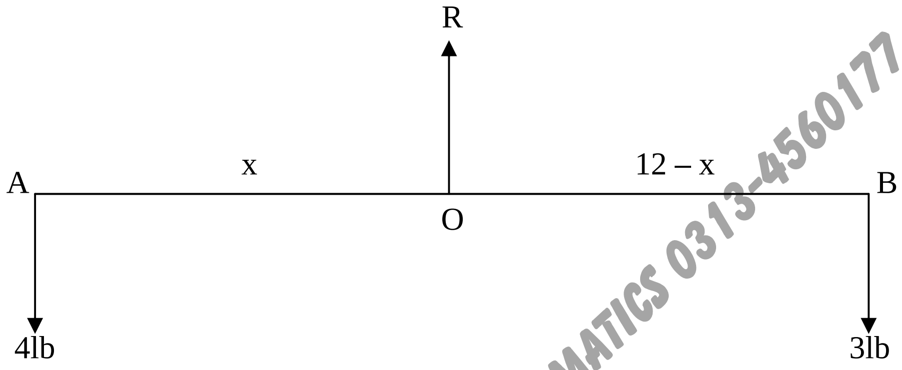
Which shows that the body is in equilibrium. This completes the proof.

❖ QUESTION 1

A light thin rod 12ft long can turn in a vertical plane about one of its point which is attached to a pivot. If weight of 3lb and 4lb are suspended from its ends, it rests in a horizontal position. Find the position of the pivot and its reaction on the rod.

SOLUTION

Let AB be a rod of length 12ft and O be pivot and R be the normal reaction of pivot on rod. Let δy be the infinitesimal displacement vertically upward direction. Then



Equation of virtual work

$$R\delta y - 3\delta y - 4\delta y = 0$$

$\Rightarrow \quad (R - 7)\delta y = 0$

Since δy is arbitrary therefore $\delta y \neq 0$.

So $\quad R - 7 = 0 \;\Rightarrow\; R = 7$

Let AO = x Then BO = 12 – x

The moments of weights about the pivot are 4x and – 3(12 – x). Let $\delta\theta$ be the small angular displacement of the rod about the pivot. Then

Equation of virtual work

$$4x\delta\theta - 3(12 - x)\delta\theta = 0$$

$\Rightarrow \quad (4x - 36 + 3x)\delta\theta = 0$

$\Rightarrow \quad (7x - 36)\delta\theta = 0$

Since $\delta\theta$ is arbitrary therefore $\delta\theta \neq 0$.

So $\quad 7x - 36 = 0$

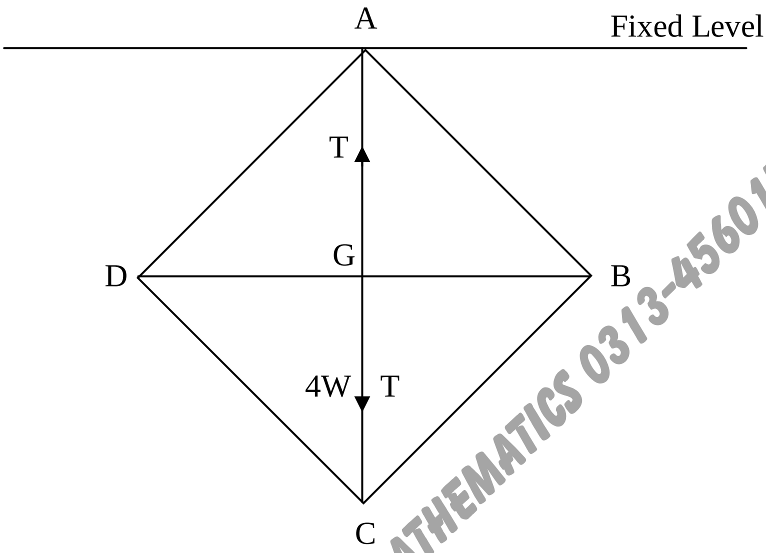
$\Rightarrow \quad x = \dfrac{36}{7} = 5\dfrac{1}{7}\text{ ft} \quad \Rightarrow \quad AO = 5\dfrac{1}{7}\text{ ft}$

and $\quad BO = 12 - \dfrac{36}{7} = \dfrac{48}{7} = 6\dfrac{6}{7}\text{ ft}$

❖ QUESTION 2

Four equal heavy uniform rods are freely jointed to form a rhombus ABCD, which is freely suspended from A and is kept in shape of a square by an inextensible string connecting A and C. Show that the tension in the string is 2W where W is the weight of each rod.

SOLUTION



Let W be the weight of each rod. Then the weight of the four rod is 4W, which acts at a point G, where G is the point of the intersection diagonals AC and BD. Let T be tension in the string. Let $AC = 2y \Rightarrow AG = y$

Then Equation of virtual work

$$4W\,\delta(AG) - T\delta(AC) = 0$$

$\Rightarrow \quad 4W\,\delta(y) - 2T\delta(y) = 0$

$\Rightarrow \quad (4W - 2T)\delta(y) = 0$

$\Rightarrow \quad 4W - 2T = 0 \qquad \because\ \delta(y) \neq 0$

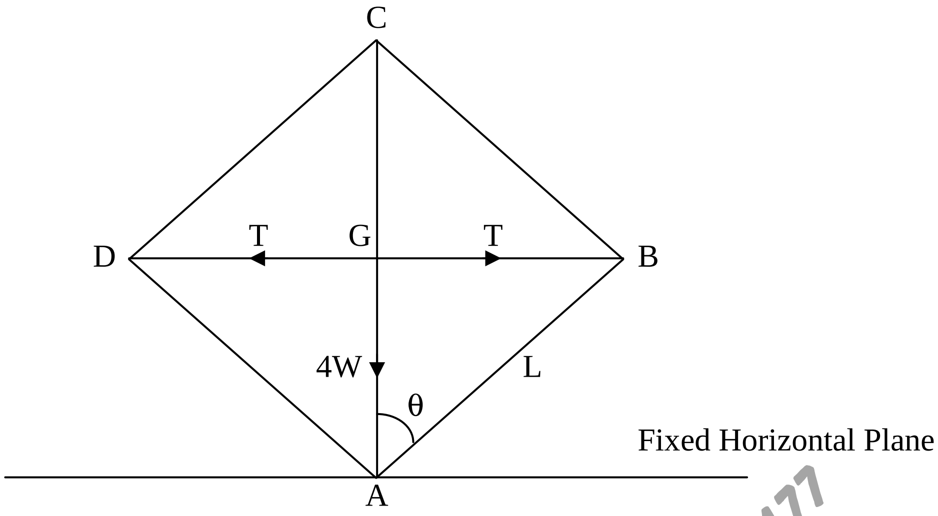
$\Rightarrow \quad 4W = 2T$

$\Rightarrow \quad T = 2W$

❖ QUESTION 3

Four equal uniform rods are smoothly jointed to form a rhombus ABCD, which is placed in a vertical plane with AC vertical and A resting on a horizontal plane. The rhombus is kept in shape, with the measure of angle BAC equal to θ, by a light string joining B and D. Find the tension in the string.

SOLUTION



Let W be the weight of each rod. Then the weight of the four rod is 4W, which acts at a point G. Where G is the point of the intersection diagonals AC and BD.
Let L be length of each rod. Let T be tension in the string. Given $\angle BAC = \theta$

Equation of virtual work

$$-4W\,\delta(AG) - T\delta(BD) = 0$$

$$\Rightarrow \quad 4W\,\delta(AG) + T\delta(BD) = 0 \qquad \text{\_\_\_\_\_\_\_\_(i)}$$

From Fig.

$$AG = L\cos\theta \quad \text{and} \quad BG = L\sin\theta$$

Also $\quad BD = 2(BG) = 2L\sin\theta$

Using values in (i), we get

$$4W\,\delta(L\cos\theta) + T\delta(2L\sin\theta) = 0$$

$$\Rightarrow \quad -4WL\sin\theta\,\delta\theta + 2TL\cos\theta\,\delta\theta = 0$$

$$\Rightarrow \quad (2LT\cos\theta - 4WL\sin\theta)\,\delta\theta = 0$$

Since $\delta\theta$ is an arbitrary therefore $\delta\theta \neq 0$

So $\quad 2LT\cos\theta - 4WL\sin\theta = 0$

$$\Rightarrow \quad 2LT\cos\theta = 4WL\sin\theta$$

$$\Rightarrow \quad T\cos\theta = 2W\sin\theta$$

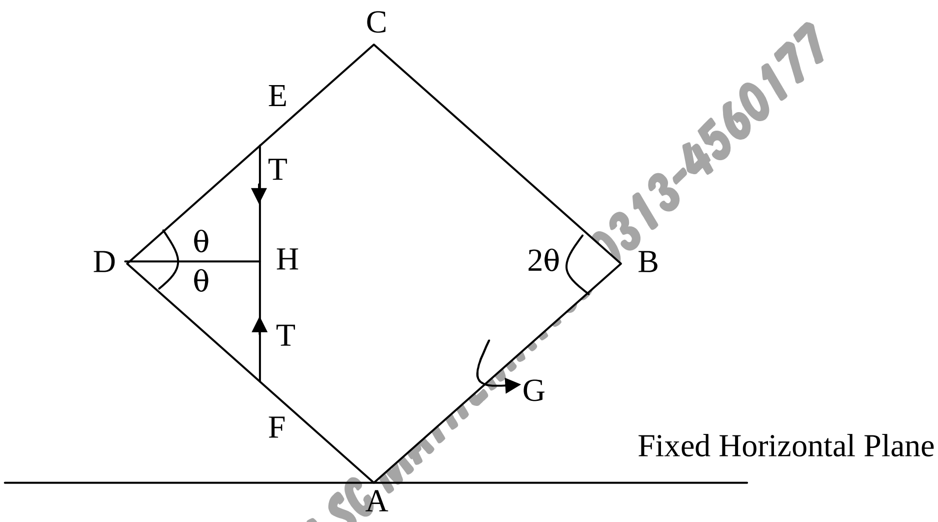
$$\Rightarrow \quad T = 2W\tan\theta$$

❖ QUESTION 4

A rhombus ABCD of smoothly jointed rods, rests on a smooth table with the rod BC fixed in position. The middle points of AD and DC are connected by a string which is kept taut by a couple applied to the rod AB. Prove that the tension of the string is

$$\frac{2G}{AB\cos\left(\frac{1}{2}A\hat{B}C\right)}$$

SOLUTION



Consider a rhombus ABCD of smoothly jointed rods, rests on a smooth table with the rod BC fixed in position as shown in figure. Let T be the tension in the string EF where E and F are the middle points of the rods DC and AD respectively. Let L be the length of each rod and G be a couple applied on the rod AB.

Equation of virtual work

$$G\delta(2\theta) - T\delta(EF) = 0 \qquad \text{\_\_\_\_\_\_\_\_(i)}$$

From figure.

$$EF = 2(HF)$$

$$= 2\left(\frac{L}{2}\sin\theta\right)$$

$$= L\sin\theta$$

Using value of EF in (i), we get

$$G\delta(2\theta) - T\delta(L\sin\theta) = 0$$

$$\Rightarrow \quad 2G\delta\theta - TL\cos\theta\,\delta\theta = 0$$

$$\Rightarrow \quad (2G - TL\cos\theta)\,\delta\theta = 0$$

$\Rightarrow \quad 2G - TL\cos\theta = 0 \qquad \because \delta\theta \neq 0$

$\Rightarrow \quad T = \dfrac{2G}{L\cos\theta}$

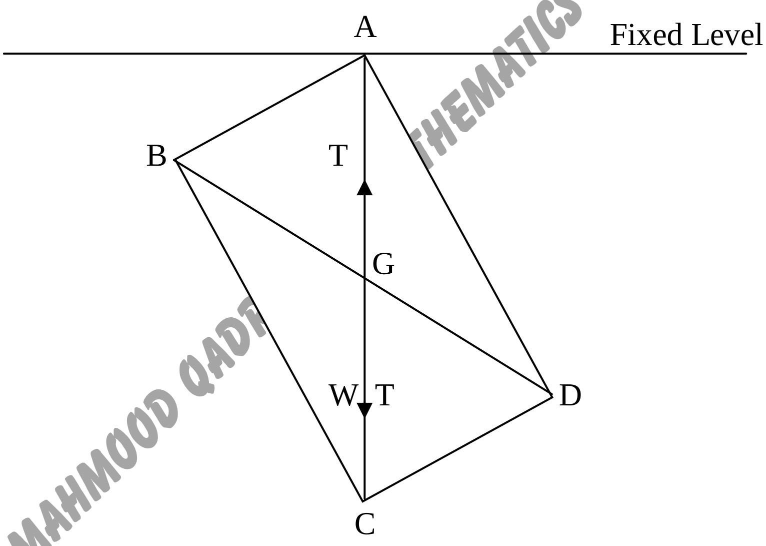
Since $L = AB$ and $\theta = \dfrac{1}{2}A\hat{B}C$

$\Rightarrow \quad T = \dfrac{2G}{AB\cos\left(\frac{1}{2}A\hat{B}C\right)}$

❖ QUESTION 5

Four uniform rods are freely jointed at their extremities and form a parallel-gram ABCD, which is suspended from the joint A and is kept in shape by an inextensible string AC. Prove that the tension in the string is equal to half the whole weight.

SOLUTION



Let W be the weight of the four rods acting at point G. Where G is the point of intersection of the diagonals AC and BD. Suppose the T be the tension in the string AC.

Let $AC = 2y$ Then $AG = y$

Equation of virtual Work

$W\delta(AG) - T\delta(AC) = 0$

$\Rightarrow \quad W\delta(y) - T\delta(2y) = 0$

$\Rightarrow \quad (W - 2T)\delta y = 0$

$\Rightarrow \quad W - 2T = 0 \qquad \because \delta y \neq 0$

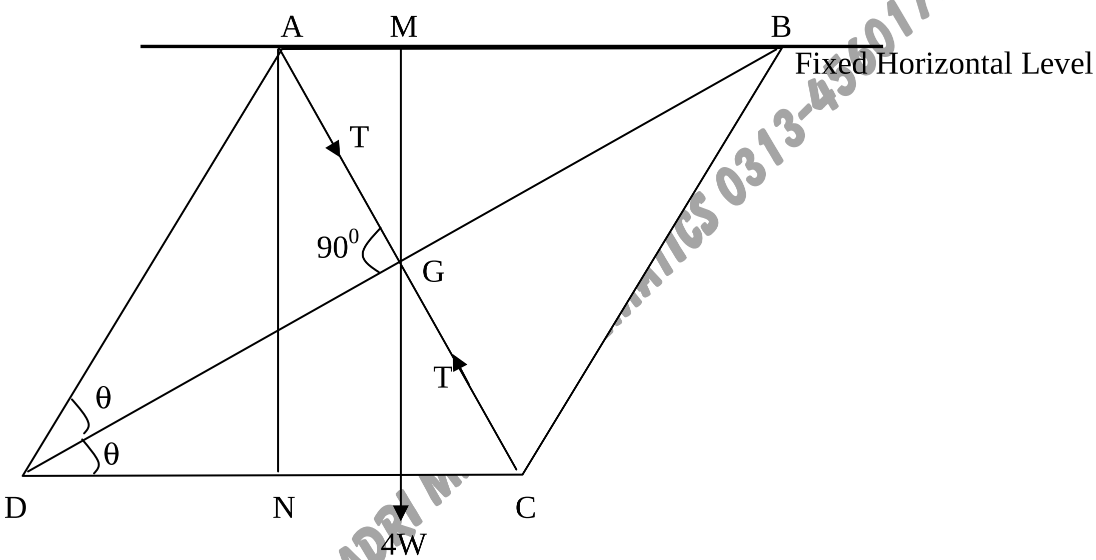
$\Rightarrow \quad T = \dfrac{W}{2}$

❖ QUESTION 6

A string of length a forms the shorter diagonal of a rhombus formed by four uniform rods, each of length b and weight W, which are hanged together. If one of the rod is supported in a horizontal position. Prove that the tension in the string is

$$\frac{2W(2b^2 - a^2)}{b\sqrt{4b^2 - a^2}}$$

SOLUTION



Since W is the weight of each rod therefore the weight of four rods is 4W acting at point G. Where G is the point of intersection of the diagonals AC and BD. Let T be the tension in the string.

Given $AB = BC = CD = DA = b$ and $AC = a$

Let $\angle ADC = 2\theta$ Then $\angle ADG = \theta$ and $\angle BDC = \theta$

Equation of virtual work

$$4W\delta(MG) - T\delta(AC) = 0 \qquad \text{\_\_\_\_\_\_\_\_\_\_(i)}$$

From fig.

$$MG = \frac{1}{2}AN = \frac{1}{2}AD\sin 2\theta = \frac{1}{2}b\sin 2\theta$$

Also

$$AC = 2AG = 2AD\sin\theta = 2b\sin\theta$$

But $AC = a \Rightarrow a = 2b\sin\theta \Rightarrow \sin\theta = {}^{a}/_{2b}$

Using values of MG and AC in (i), we get

$$4W\delta\left(\frac{1}{2}b\sin 2\theta\right) - T\delta(2b\sin\theta) = 0$$

$\Rightarrow \quad 4Wb\cos2\theta\delta\theta - 2Tb\cos\theta\delta\theta = 0$

$\Rightarrow \quad (4Wb\cos2\theta - 2Tb\cos\theta)\delta\theta = 0$

$\Rightarrow \quad 4Wb\cos2\theta - 2Tb\cos\theta = 0 \qquad \because \delta\theta \neq 0$

$\Rightarrow \quad 4Wb\cos2\theta = 2Tb\cos\theta$

$$\Rightarrow \quad T = \frac{2W\cos2\theta}{\cos\theta}$$

$$= \frac{2W(\cos^2\theta - \sin^2\theta)}{\cos\theta} = \frac{2W(1 - \sin^2\theta - \sin^2\theta)}{\sqrt{1 - \sin^2\theta}}$$

$$= \frac{2W(1 - 2\sin^2\theta)}{\sqrt{1 - \sin^2\theta}} = \frac{2W\left(1 - 2\left(\frac{a}{2b}\right)^2\right)}{\sqrt{1 - \left(\frac{a}{2b}\right)^2}}$$

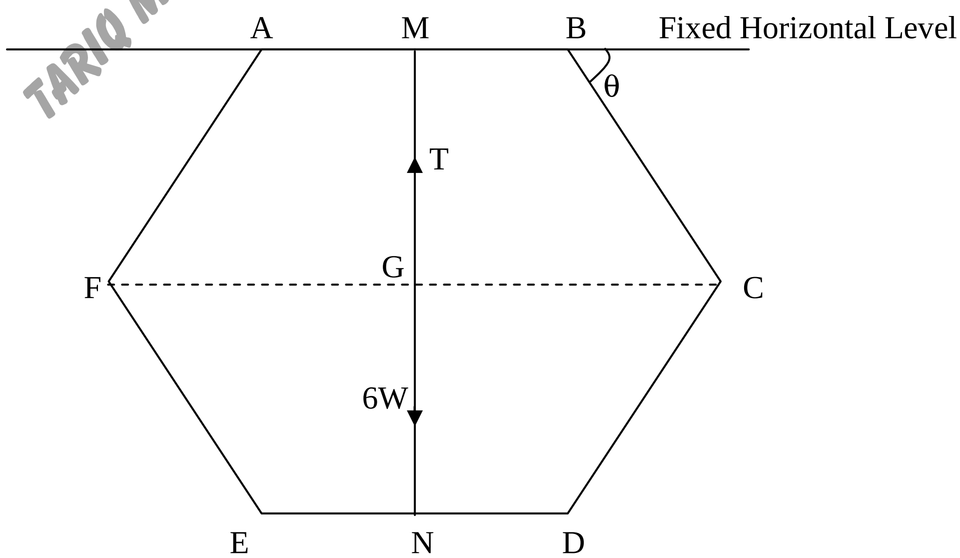
$$= \frac{2W\left(1 - 2\frac{a^2}{4b^2}\right)}{\sqrt{1 - \frac{a^2}{4b^2}}} = \frac{2W\left(\frac{4b^2 - a^2}{2b^2}\right)}{\sqrt{\frac{4b^2 - a^2}{4b^2}}}$$

$$= \frac{2W\left(2b^2 - a^2\right)}{b\sqrt{4b^2 - a^2}}$$

❖ QUESTION 7

Six equal rods AB, BC, CD, DE, EF and FA are each of weight W and are freely jointed at their extremities so as to form a hexagon. The rod AB is fixed in a horizontal position and the middle points of AB and DE are jointed by a string. Prove that its tension is 3W.

SOLUTION



Let M and N be the midpoints of the sides AB and DE of a hexagon ABCDEF. Let W be the weight of the each rod. Then weight of the six rods is 6W which acts at point G. Where G is the centre of the gravity of hexagon. Let T be the tension in the string MN.

Let MN = 2y Then MG = y

Equation of virtual work

$6W\delta(MG) - T\delta(MN) = 0$

$\Rightarrow \quad 6W\delta(y) - T\delta(2y) = 0$

$\Rightarrow \quad (6W - 2T)\delta y = 0$

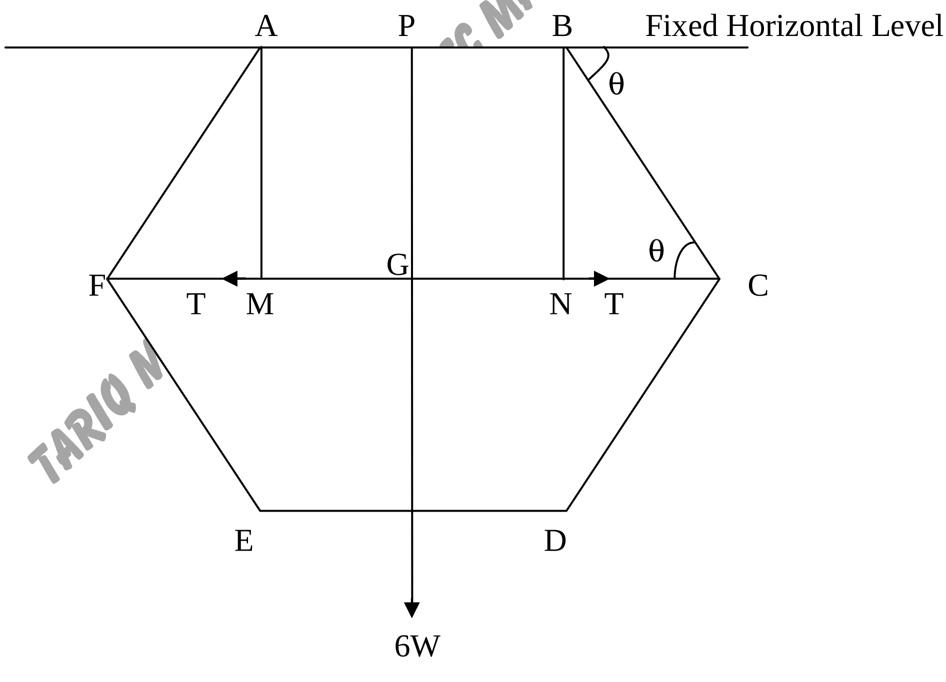
$\Rightarrow \quad 6W - 2T = 0 \qquad \because\ \delta y \neq 0$

$\Rightarrow \quad T = 3W$

❖ QUESTION 8

Six equal uniform rods AB, BC, CD, DE, EF and FA are each of weight W are freely jointed to form a regular hexagon. The rod AB is fixed in a horizontal position and the shape of the hexagon is maintained by a light rod joining C and F. Show that the thrust in this rod is $\sqrt{3}$W.

SOLUTION



Since W be weight of each rod therefore weight of six rod is 6W which acts at the centre of gravity G of hexagon. Let L be the length of each rod and θ be the angle which rod BC makes with horizontal. Let AM and BN are perpendicular on CF. Let P be the midpoint of the AB. Let T be the thrust in the rod FC.

Equation of virtual work

$$6W\,\delta(PG) + T\,\delta(FC) = 0 \qquad \text{—————(i)}$$

From figure.

$PG = AM = BN = BC\sin\theta = L\sin\theta$ and $CN = L\cos\theta$

Also $FC = FM + MN + CN$

$= CN + AB + CN \qquad \because\ FM = CN \text{ and } MN = AB$

$= 2CN + AB$

$= 2L\cos\theta + L$

Using values of PG and FC in (i), we get

$$6W\,\delta(L\sin\theta) + T\,\delta(2L\cos\theta + L) = 0$$

$\Rightarrow\quad 6WL\cos\theta\delta\theta - 2TL\sin\theta\delta\theta = 0 \quad\Rightarrow\quad (6WL\cos\theta - 2TL\sin\theta)\delta\theta = 0$

$\Rightarrow\quad 6WL\cos\theta - 2TL\sin\theta = 0 \qquad \because\ \delta\theta \neq 0$

$\Rightarrow\quad 6WL\cos\theta = 2TL\sin\theta \quad\Rightarrow\quad T = 3W\cot\theta$

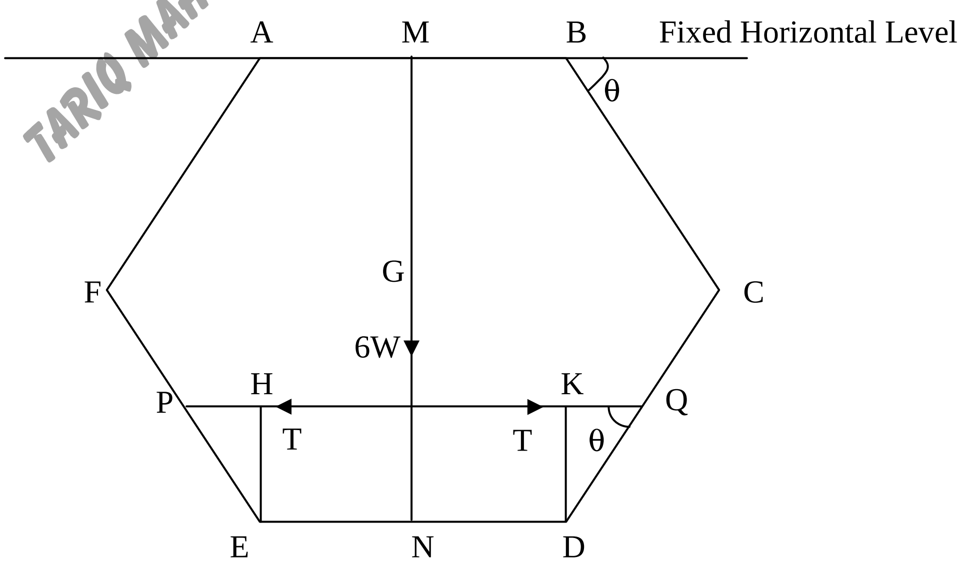
For a regular hexagon $\theta = 60^0$

So $\quad T = 3W\cot 60^0 = 3W\dfrac{1}{\sqrt{3}} = \sqrt{3}W$

❖ QUESTION 9

A hexagon ABCDEF, consisting of six equal heavy rods, of weight W, freely jointed together, hangs in a vertical plane with AB horizontal and the frame is kept in the form of regular hexagon by a light rod connecting the midpoints of CD and EF. Show that the thrust in the light rod is $2\sqrt{3}W$.

SOLUTION



Since W be weight of each rod therefore weight of six rod is 6W which acts at the centre of gravity G of hexagon. Let L be the length of each rod and θ be the angle which rod AB makes with horizontal. Let EH and DK are perpendicular on PQ where P and Q are the midpoints of the CD and EF. Let T be the thrust in the rod PQ.

Equation of virtual work

$$6W\,\delta(MG) + T\,\delta(PQ) = 0 \qquad \text{\_\_\_\_\_\_\_\_(i)}$$

From figure.

$$MG = GN = 2DK = DQ\sin\theta = 2\left(L/2\right)\sin\theta = L\sin\theta \quad \text{and} \quad KQ = L/2\cos\theta$$

Also $\quad PQ = PH + HK + KQ$

$$= KQ + ED + KQ \qquad \because\ HK = ED \text{ and } KQ = PH$$

$$= 2KQ + ED$$

$$= L\cos\theta + L$$

Using values of PQ and MG in (i), we get

$$6W\,\delta(L\sin\theta) + T\,\delta(L\cos\theta + L) = 0$$

$\Rightarrow \quad 6WL\cos\theta\delta\theta - TL\sin\theta\delta\theta = 0$

$\Rightarrow \quad (6WL\cos\theta - TL\sin\theta)\delta\theta = 0$

$\Rightarrow \quad 6WL\cos\theta - TL\sin\theta = 0 \qquad \because\ \delta\theta \neq 0$

$\Rightarrow \quad 6WL\cos\theta = TL\sin\theta$

$\Rightarrow \quad T = 6W\cot\theta$

For a regular hexagon $\theta = 60^0$

So $\quad T = 6W\cot 60^0 = 6W\dfrac{1}{\sqrt{3}} = 2\sqrt{3}W$

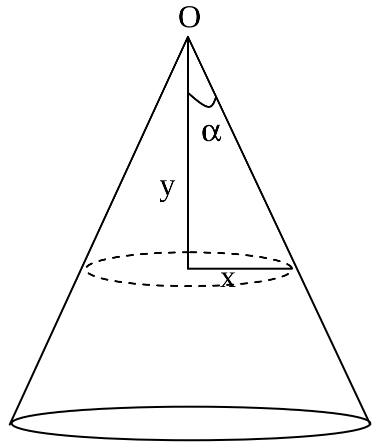
❖ QUESTION 10

A heavy elastic string whose natural length is $2\pi a$, is placed around a smooth cone whose axes is vertical and whose semi-vertical angle has measure α. If W be the weight and λ the modulus of string. Prove that it will be in equilibrium when in the form of a circle of radius

$$a\left(1+\frac{W}{2\pi\lambda}\cot\alpha\right)$$

SOLUTION

Let the string be in equilibrium at a depth y below the vertex of the cone. Let the radius of the circle formed by the string in the equilibrium position be x. The circumference of the circle is $2\pi x$.



From fig.

$$\tan\alpha = \frac{x}{y}$$

$\Rightarrow \quad x = y\tan\alpha$

Let T be the tension in the string then by Hook's law

$$T = \lambda\left(\frac{\text{Change in length}}{\text{Original Length}}\right)$$

$$= \lambda\left(\frac{2\pi x - 2\pi a}{2\pi a}\right)$$

$$= \lambda\left(\frac{x - a}{a}\right)$$

Let the string be displaced downward by displacement δy.

Then equation of virtual work

$$W\delta y - T\delta(2\pi x) = 0$$

$\Rightarrow \quad W\delta y - T\delta(2\pi y\tan\alpha) = 0$

$\Rightarrow \quad W\delta y - 2\pi T\tan\alpha\delta y = 0$

$\Rightarrow \quad (W - 2\pi T\tan\alpha)\delta y = 0$

$\Rightarrow \quad W - 2\pi T\tan\alpha = 0 \qquad \because \ \delta y \neq 0$

$\Rightarrow \quad W = 2\pi T\tan\alpha$

$\Rightarrow \quad W = 2\pi\lambda\left(\frac{x - a}{a}\right)\tan\alpha$

$\Rightarrow \quad Wa\cot\alpha = 2\pi\lambda(x - a)$

$\Rightarrow \quad Wa\cot\alpha = 2\pi x\lambda - 2\pi a\lambda$

$\Rightarrow \quad Wa\cot\alpha + 2\pi a\lambda = 2\pi x\lambda$

$\Rightarrow \quad x = \dfrac{Wa\cot\alpha + 2\pi a\lambda}{2\pi\lambda}$

$$= a + \frac{Wa\cot\alpha}{2\pi\lambda}$$

$$= a\left(1 + \frac{W\cot\alpha}{2\pi\lambda}\right)$$

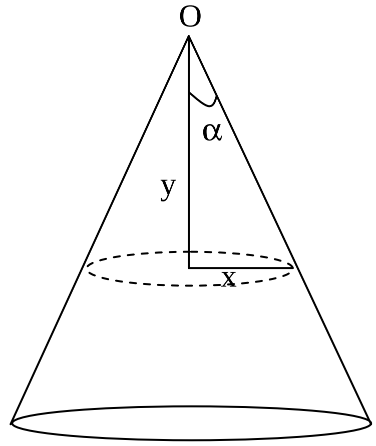
❖ QUESTION 11

Two equal particles are connected by two given weightless strings, which are placed like a necklace on a smooth cone whose axis is vertical and whose vertex is uppermost. Show that the tension in the string is

$$\mathbf{\frac{W}{\pi}\cot\alpha}$$

Where W is the weight of each particle and 2α the measure of the vertical angle of the cone.

SOLUTION



Let both particles be in equilibrium at a depth y below the vertex O. Let the radius of the circle formed by the string in the equilibrium position be x. The circumference of the circle is $2\pi x$.

From fig.

$$\tan\alpha = \frac{x}{y}$$

$\Rightarrow \quad x = y\tan\alpha$

Let T be the tension in the string. Then equation of virtual work

$$2W\delta y - T\delta(2\pi x) = 0 \quad \Rightarrow \quad 2W\delta y - T\delta(2\pi y\tan\alpha) = 0$$

$\Rightarrow \quad 2W\delta y - 2\pi T\tan\alpha\delta y = 0 \quad \Rightarrow \quad (2W - 2\pi T\tan\alpha)\delta y = 0$

$\Rightarrow \quad 2W - 2\pi T\tan\alpha = 0 \quad \because \delta y \neq 0$

$\Rightarrow \quad 2W = 2\pi T\tan\alpha$

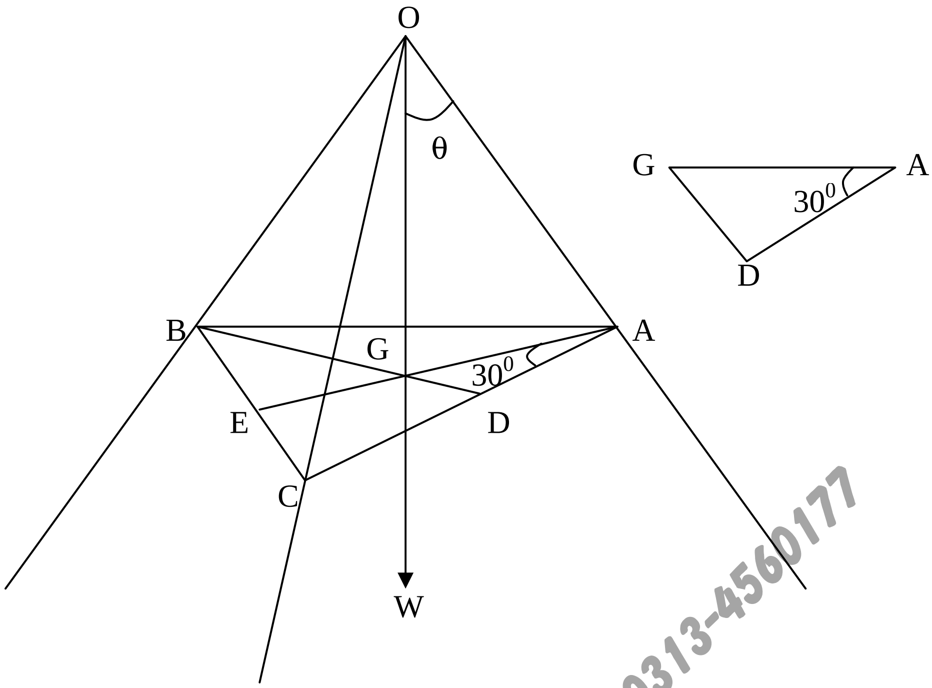
$\Rightarrow \quad T = \dfrac{W}{\pi}\cot\alpha$

❖ QUESTION 12

A weightless tripod, consisting of three legs of equal length l, smoothly jointed at the vertex, stands on a smooth horizontal plane. A weight W hangs from the apex. The tripod is prevented from collapsing by three inextensible strings each of length is $^1/_2$, joining the mid-points of the legs. Show that the tension in each string is

$$\frac{\sqrt{2}}{3\sqrt{3}}\mathbf{W}$$

SOLUTION



Since the length of each rod is l. Therefore OA = OB = OC = AB = BC = CA = $l/2$ The vertical line through O meets the plane of ΔABC at point G. Where G is the point of intersection of the medians AE and BD. Let each rod makes an angle θ with OG. Let T be the tension in each string. Then

Equation of virtual work

$$-W\,\delta(2OG) - 3T\delta(AC) = 0 \quad \{\because \text{The height of O above the horizontal plane} = 2OG\}$$

$$\Rightarrow \quad W\,\delta(2OG) + 3T\delta(AC) = 0 \qquad \text{\_\_\_\_\_\_\_\_(i)}$$

From ΔAOG

$$OG = OA\cos\theta = \frac{l}{2}\cos\theta$$

and $$AG = OA\sin\theta = \frac{l}{2}\sin\theta$$

$$AC = 2AD = 2AG\cos 30^0 = 2\,\frac{l}{2}\sin\theta\frac{\sqrt{3}}{2} = \sqrt{3}\frac{l}{2}\sin\theta$$

$$\Rightarrow \quad \sqrt{3}\frac{l}{2}\sin\theta = \frac{l}{2} \qquad \because AC = \frac{l}{2}$$

$$\Rightarrow \quad \sin\theta = \frac{1}{\sqrt{3}}$$

As $$\cos\theta = \sqrt{1-\sin^2\theta}$$

$$= \sqrt{1-\left(\frac{1}{\sqrt{3}}\right)^2} = \sqrt{\frac{2}{3}}$$

$\Rightarrow \quad \tan\theta = \dfrac{1}{\sqrt{2}}$

Using value of OG and AC, we get

$$W\delta(l\cos\theta)+3T\delta\left(\sqrt{3}\frac{l}{2}\ \sin\theta\right)=0 \quad \Rightarrow \quad -Wl\sin\theta\ \delta\theta + 3\sqrt{3}\frac{l}{2}\cos\theta T\delta\theta = 0$$

$$\Rightarrow \quad \left(-Wl\sin\theta + 3\sqrt{3}\frac{l}{2}\cos\theta T\right)\delta\theta = 0 \quad \Rightarrow \quad -Wl\sin\theta + 3\sqrt{3}\frac{l}{2}\cos\theta T = 0 \qquad \because\ \delta\theta \neq 0$$

$$\Rightarrow \quad Wl\sin\theta = 3\sqrt{3}\frac{l}{2}\cos\theta T$$

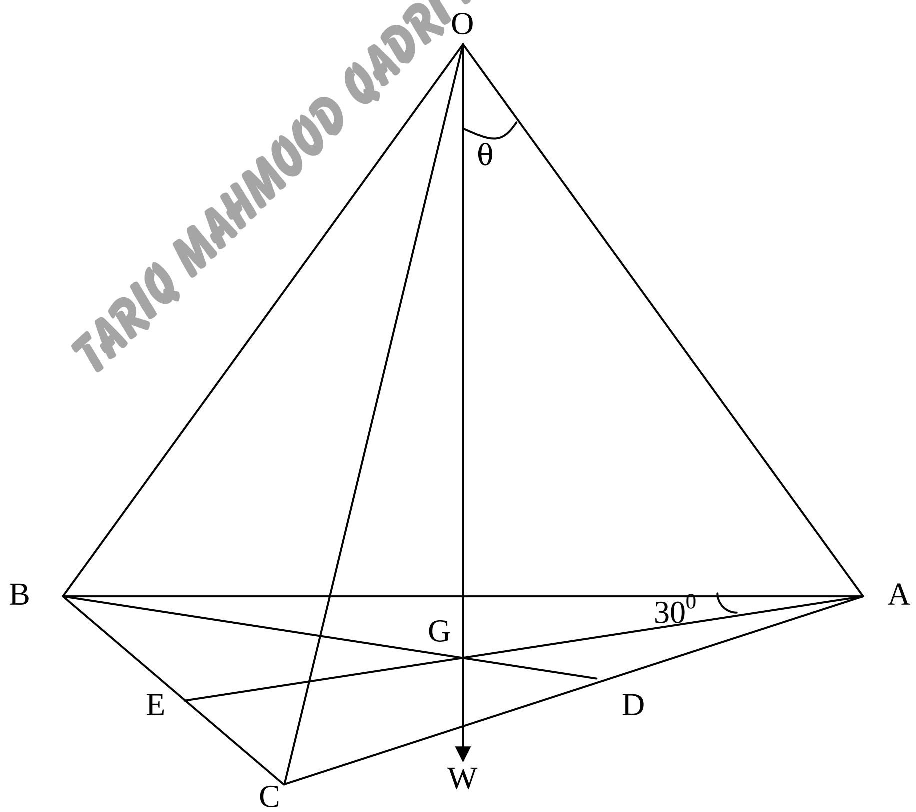
$$\Rightarrow \quad T = \frac{2}{3\sqrt{3}}W\tan\theta = \frac{2}{3\sqrt{3}}W\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{3\sqrt{3}}W$$

❖ QUESTION 13

Three equal rods each of weight W are freely jointed together at one extremity of each to form a tripod and rest with their other extremities on a smooth horizontal plane. Each rod is inclined at angle of θ to the vertical, equilibrium is being maintained by three light strings each joining two these extremities. Prove that the tension in each string is

$$\mathbf{W\frac{\tan\theta}{2\sqrt{3}}}$$

SOLUTION



Let the length of each rod is L and length of each string is x. Then

$$OA = OB = OC = L \text{ and } AB = BC = CA = x$$

The vertical line through O meets the plane of ΔABC at point G. Where G is the point of intersection of the medians AE and BD. Let each rod makes an angle θ with OG. Let T be the tension in each string. Then

Equation of virtual work

$$-3W\delta\left(\frac{OG}{2}\right) - 3T\delta(AC) = 0$$

$$\Rightarrow \quad 3W\delta\left(\frac{OG}{2}\right) + 3T\delta(AC) = 0 \qquad \text{\_\_\_\_\_\_\_\_\_(i)}$$

Where $\frac{OG}{2}$ is equal the height of midpoint of each rod above the horizontal plane.

From ΔAOG

$$OG = OA\cos\theta = L\cos\theta$$

and $\quad AG = OA\sin\theta = L\sin\theta$

$$AC = 2AD = 2AG\cos 30^0 = 2\,L\sin\theta\frac{\sqrt{3}}{2} = \sqrt{3}L\sin\theta$$

Using value of OG and AC, we get

$$3W\delta\left(\frac{L}{2}\cos\theta\right) + 3T\delta\left(\sqrt{3}L\sin\theta\right) = 0$$

$$\Rightarrow \quad -\frac{3}{2}WL\sin\theta\,\delta\theta + 3\sqrt{3}L\cos\theta T\delta\theta = 0$$

$$\Rightarrow \quad \left(-\frac{W}{2}\sin\theta + \sqrt{3}\cos\theta T\right)\delta\theta = 0$$

$$\Rightarrow \quad -\frac{W}{2}\sin\theta + \sqrt{3}\cos\theta T = 0 \qquad \because\ \delta\theta \neq 0$$

$$\Rightarrow \quad \frac{W}{2}\sin\theta = \sqrt{3}\cos\theta T$$

$$\Rightarrow \quad T = \frac{W}{2\sqrt{3}}\tan\theta$$

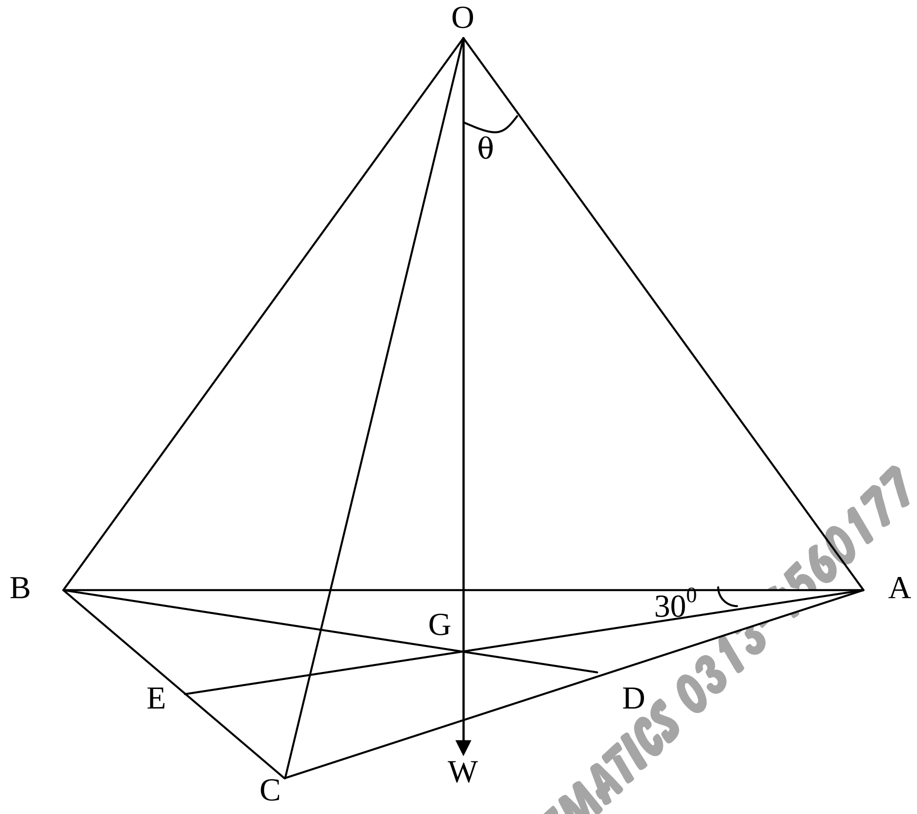
❖ QUESTION 14

Six equal uniform rods are freely jointed at their extremities to form a tetrahedron. If this tetrahedron is place with one face on a smooth horizontal table. Prove that the thrust along a horizontal rod is

$$\mathbf{\frac{W}{2\sqrt{6}}}$$

Where W is the weight of each rod.

SOLUTION



Let the length of each rod is L. Therefore OA = OB = OC = AB = BC = CA = L

The vertical line through O meets the plane of ΔABC at point G. Where G is the point of intersection of the medians AE and BD. Let each rod makes an angle θ with OG. Let T be the thrust in each lower rod. Then

Equation of virtual work

$$-3W\delta\left(\frac{OG}{2}\right) - 3T\delta(AC) = 0$$

$$\Rightarrow \quad 3W\delta\left(\frac{OG}{2}\right) + 3T\delta(AC) = 0 \qquad \text{\_\_\_\_\_\_\_\_(i)}$$

Where $\frac{OG}{2}$ is equal the height of midpoint of each rod above the horizontal plane.

From ΔAOG

$$OG = OA\cos\theta = L\cos\theta$$

and $$AG = OA\sin\theta = L\sin\theta$$

$$AC = 2AD = 2AG\cos30^0 = 2L\sin\theta\frac{\sqrt{3}}{2} = \sqrt{3}L\sin\theta$$

$$\Rightarrow \quad \sqrt{3}L\sin\theta = L \qquad \because AC = L$$

$$\Rightarrow \quad \sin\theta = \frac{1}{\sqrt{3}}$$

Now $\quad \cos\theta = \sqrt{1-\sin^2\theta}$

$$= \sqrt{1-\left(\frac{1}{\sqrt{3}}\right)^2}$$

$$= \sqrt{\frac{2}{3}}$$

$\Rightarrow \quad \tan\theta = \dfrac{1}{\sqrt{2}}$

Using value of OG and AC, we get

$$3W\delta\left(\frac{L}{2}\cos\theta\right) + 3T\delta\left(\sqrt{3}L\sin\theta\right) = 0$$

$\Rightarrow \quad -\dfrac{3}{2}WL\sin\theta\,\delta\theta + 3\sqrt{3}L\cos\theta T\delta\theta = 0$

$\Rightarrow \quad \left(-\dfrac{W}{2}\sin\theta + \sqrt{3}\cos\theta T\right)\delta\theta = 0$

$\Rightarrow \quad -\dfrac{W}{2}\sin\theta + \sqrt{3}\cos\theta T = 0 \qquad \because\ \delta\theta \neq 0$

$\Rightarrow \quad \dfrac{W}{2}\sin\theta = \sqrt{3}\cos\theta T$

$\Rightarrow \quad T = \dfrac{W}{2\sqrt{3}}\tan\theta$

$$= \frac{W}{2\sqrt{3}}\frac{1}{\sqrt{2}}$$

$$= \frac{W}{2\sqrt{6}}$$

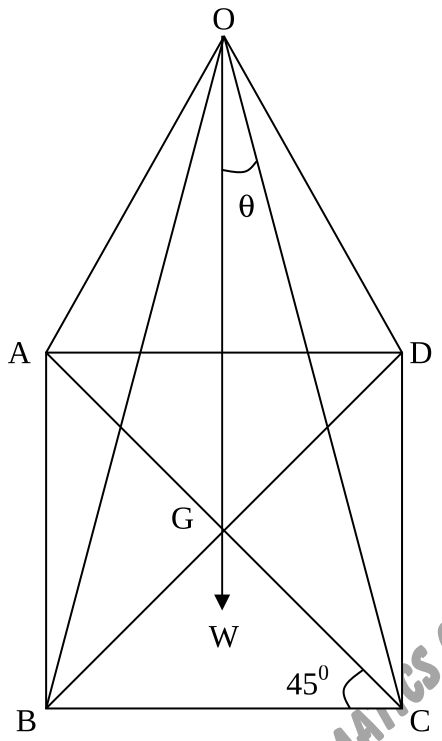
❖ QUESTION 15

Four equal uniform rods, each of weight w, are connected at one end of each by means of a smooth joint, and other ends rest on a smooth table and are connected by equal strings. A weight W is suspended from the joint. Show that the tension in each string is

$$\left(\frac{\mathbf{W+2w}}{\mathbf{4}}\right)\left(\frac{\mathbf{a}}{\sqrt{\mathbf{4l^2-2a^2}}}\right)$$

Where l is the length of each rod and a is the length of each string.

SOLUTION



Given that

$$OA = OB = OC = OD = l$$

and $$AB = BC = CD = DA = a$$

Let the suspended weight W makes an angle θ with each rod. Let T be the tension in each string. Then **equation of virtual work**

$$-4w\delta\left(\frac{OG}{2}\right) - W\delta(OG) - 4T\delta(BC) = 0 \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_(i)}$$

In ΔOCG

$$CG = OC\sin\theta = l\sin\theta \quad \text{and} \quad OG = l\cos\theta$$

Also $\quad AC = 2CG = 2l\sin\theta$

In ΔABC

$$BC = AC\cos 45^0$$

$$= 2l\sin\theta\, {}^1\!/\!_{\sqrt{2}}$$

$$= \sqrt{2}\, l\sin\theta$$

$\Rightarrow \quad \sqrt{2}\, l\sin\theta = a \qquad \because BC = a$

$\Rightarrow \quad \sin\theta = \dfrac{a}{l\sqrt{2}}$

Now $\cos\theta = \sqrt{1-\sin^2\theta}$

$$= \sqrt{1-\left(\frac{a}{l\sqrt{2}}\right)^2}$$

$$= \frac{\sqrt{2l^2-a^2}}{l\sqrt{2}}$$

$\Rightarrow \quad \tan\theta = \dfrac{a}{\sqrt{2l^2-a^2}}$

Using values of OG and BC in (i), we get

$$-4w\delta\left(\frac{l\cos\theta}{2}\right) - W\delta(l\cos\theta) - 4T\delta(\sqrt{2}l\sin\theta) = 0$$

$\Rightarrow \quad 2w\sin\theta\delta\theta + W\sin\theta\delta\theta - 4\sqrt{2}\,T\cos\theta\delta\theta = 0$

$\Rightarrow \quad (2w\sin\theta + W\sin\theta - 4\sqrt{2}\,T\cos\theta)\delta\theta = 0$

$\Rightarrow \quad 2w\sin\theta + W\sin\theta - 4\sqrt{2}\,T\cos\theta = 0 \qquad \because\ \delta\theta \neq 0$

$\Rightarrow \quad 2w\sin\theta + W\sin\theta = 4\sqrt{2}\,T\cos\theta$

$\Rightarrow \quad T = \dfrac{(2w+W)\tan\theta}{4\sqrt{2}}$

$$= \frac{(2w+W)\tan\theta}{4\sqrt{2}}$$

$$= \frac{(2w+W)}{4\sqrt{2}}\frac{a}{\sqrt{2l^2-a^2}}$$

$$= \frac{(2w+W)}{4}\frac{a}{\sqrt{4l^2-2a^2}}$$

❖ QUESTION 16

A regular octahedron formed of twelve equal rods, each of weight w, freely jointed together is suspended from one corner. Show that the thrust in each horizontal rod is

$$\frac{3}{2}\sqrt{2}w$$

SOLUTION

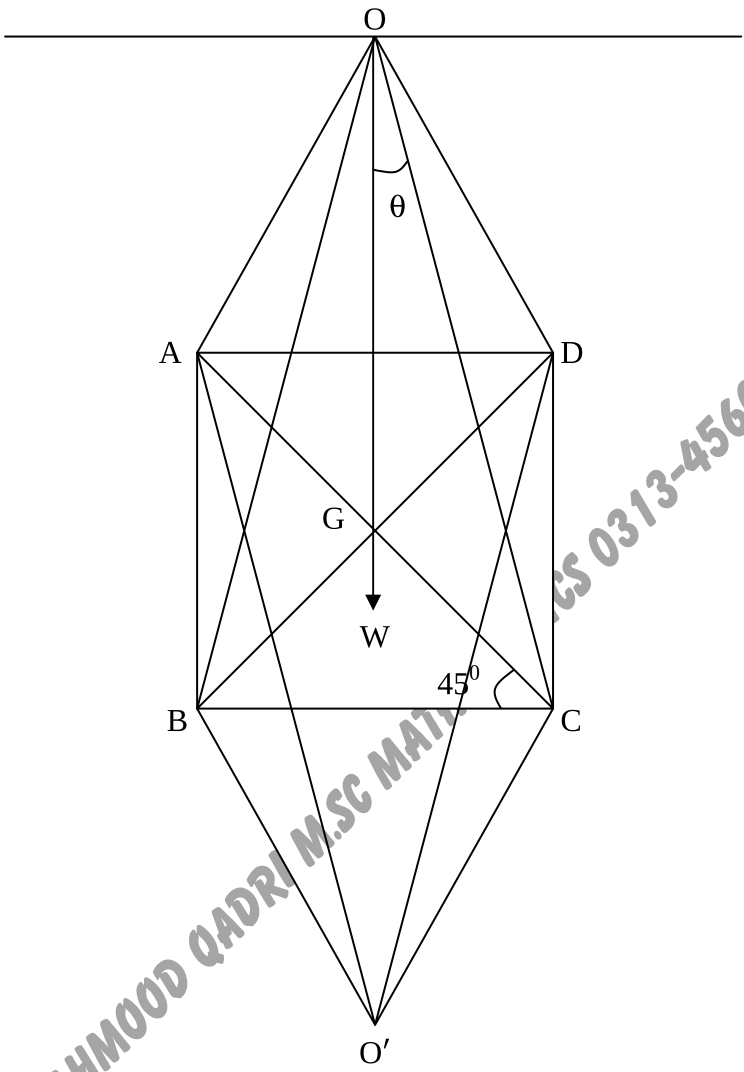
Let length of each rod be L. Then

$$OA = OB = OC = OD = AB = BC = CD = DA = O'A = O'B = O'C = O'D = L$$

Let each upper rod makes an angle θ with vertical OG. Where G is the point of intersection of diagonals AC and BD. Let T be the thrust in each horizontal rod.

Then Equation of virtual work

$$12W\delta(OG) + 4T\delta(BC) = 0 \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_(i)}$$



In ΔCOG

$$OG = OC\cos\theta = L\cos\theta$$

And $\quad CG = OC\sin\theta = L\sin\theta$

Also $\quad AC = 2CG = 2L\sin\theta$

In ΔABC

$$BC = AC\cos 45^0 = 2L\sin\theta \; {}^{1}\!/_{\sqrt{2}} = \sqrt{2}L\sin\theta$$

$\Rightarrow \quad \sqrt{2}L\sin\theta = L \qquad \because \; BC = L$

$\Rightarrow \quad \sin\theta = \dfrac{1}{\sqrt{2}} \quad \Rightarrow \quad \theta = 45^0$

Using values of OG and BC in (i), we get

$$12W\delta(L\cos\theta) + 4T\delta(\sqrt{2}L\sin\theta) = 0$$

$\Rightarrow \quad -3W\sin\theta\,\delta\theta + T\sqrt{2}\cos\theta\,\delta\theta = 0$

$\Rightarrow \quad \left(-3W\sin\theta + T\sqrt{2}\cos\theta\right)\delta\theta = 0$

$\Rightarrow \quad -3W\sin\theta + T\sqrt{2}\cos\theta = 0 \qquad \because \delta\theta \neq 0$

$\Rightarrow \quad 3W\sin\theta = T\sqrt{2}\cos\theta$

$\Rightarrow \quad T = \dfrac{3}{\sqrt{2}}W\tan\theta$

$= \dfrac{3}{\sqrt{2}}W\tan 45$

$= \dfrac{3}{\sqrt{2}}W \times \dfrac{2}{2}$

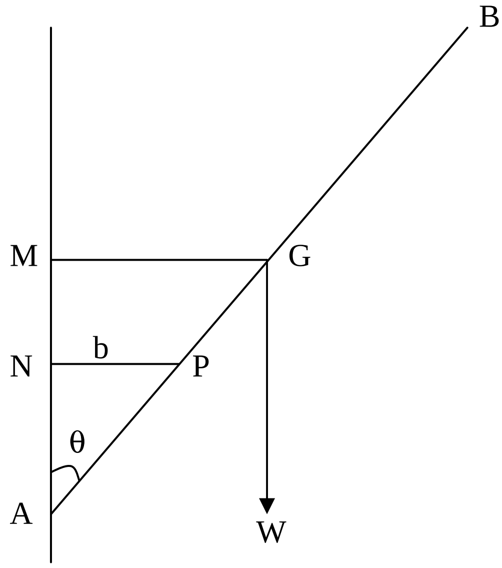
$= \dfrac{3}{2}\sqrt{2}W$

❖ QUESTION 17

A uniform rod of length 2a rest in equilibrium against a smooth vertical wall and upon a smooth peg at a distance b from the wall. Show that, in the position of equilibrium, the beam is inclined to the wall at an angle

$$\sin^{-1}\left(\frac{b}{a}\right)^{1/3}$$

SOLUTION



Let AB be the rod of length 2a. W is its weight acting downward from its centre G. Then

$AB = 2a$ and $AG = a$

Let P be a peg at distance b from the wall.

Then $NP = b$

Equation of virtual work

$-W\delta(MN) = 0$ ______(i)

From figure.

$MN = AM - AN$

$= AG\cos\theta - NP\cot\theta$

$= a\cos\theta - b\cot\theta$

Using value of MN in (i), we get

$-W\delta(a\cos\theta - b\cot\theta) = 0$

$\Rightarrow \quad -W(-a\sin\theta\delta\theta + b\,\text{cosec}^2\theta\delta\theta) = 0$

$\Rightarrow \quad Wa\sin\theta\delta\theta - Wb\,\text{cosec}^2\theta\delta\theta = 0$

$\Rightarrow \quad (Wa\sin\theta - Wb\,\text{cosec}^2\theta)\delta\theta = 0$

$\Rightarrow \quad Wa\sin\theta - Wb\,\text{cosec}^2\theta = 0 \qquad \because \delta\theta \neq 0$

$\Rightarrow \quad Wa\sin\theta = Wb\,\text{cosec}^2\theta$

$\Rightarrow \quad \dfrac{\sin\theta}{\text{cosec}^2\theta} = \dfrac{b}{a}$

$\Rightarrow \quad \sin^3\theta = \dfrac{b}{a}$

$\Rightarrow \quad \sin\theta = \left(\dfrac{b}{a}\right)^{1/3}$

$\Rightarrow \quad \theta = \sin^{-1}\left(\dfrac{b}{a}\right)^{1/3}$

%%%%% End of The Chapter # 3 %%%%%