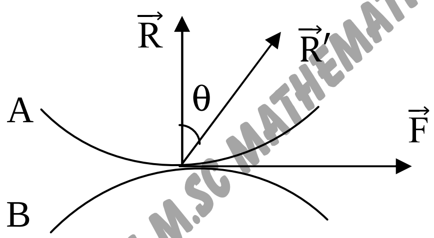
FRICTION

2

CHAPTER

❖ INTRODUCTION

When two smooth surfaces are in contact with each other at the point P. The reaction of one body on the other body is along the common normal. But if the bodies are not smooth then the reaction of the one body on the other is not along the normal but inclined at an angle θ to the common normal at the point of the contact P.



Let $\vec{R}'$ has components $\vec{R}$ along the common normal at P and $\vec{F}$ along the tangent plane at P.
Then $\qquad \vec{R} = R'\cos\theta$ and $\vec{F} = R'\sin\theta$
Where $\vec{R}$ is called normal reaction or normal pressure and $\vec{F}$ is called the force of friction or simply friction. The force of friction prevents the sliding of one body on the surface of other body. $\vec{R}'$ is the resultant of normal reaction $\vec{R}$ and force of friction $\vec{F}$ and also called resultant friction.

❖ TYPES OF FRICTION

Now we discuss some important types of friction:

❖ STATICAL FRICTION

When one body in contact with another is in equilibrium and is not on the point of sliding along the other, the force of friction in this case is called statical friction and the equilibrium in this case is non-limiting.

❖ LIMITING FRICTION

When one body is just at the point of sliding on the other, the friction in this case is maximum and is called limiting friction. The equilibrium in this case is called limiting equilibrium.

❖ DYNAMICAL FRICTION

When one body is sliding on the other body the friction in this case is slightly less than the limiting friction and is called dynamical friction or kinetic friction.

❖ COEFFICIENT OF FRICTION

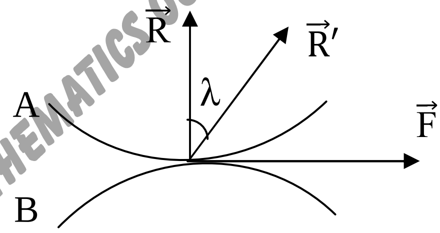
The magnitude of the limiting friction bears a constant ratio μ to the magnitude R of the normal reaction between the surfaces. The constant μ is called coefficient of friction. i.e.

$$\frac{F}{R} = \mu \quad \Rightarrow \quad F = \mu R$$

❖ ANGLE OF FRICTION

The angle λ, which the direction of the resultant reaction $\vec{R}'$ makes with normal reaction $\vec{R}$ when the body is just on the point of the motion, is called the angle of friction. From fig

$$\vec{R} = R'\cos\lambda \quad \text{and} \quad \vec{F} = R'\sin\lambda$$



As $\qquad \dfrac{F}{R} = \mu$

$\Rightarrow \qquad \dfrac{R'\sin\lambda}{R'\cos\lambda} = \mu$

$\Rightarrow \qquad \tan\lambda = \mu$

❖ QUESTION 1

A body weighing 40lb is resting on a rough plane and can just be moved by a force acting horizontally. Find the coefficient of friction.

SOLUTION

For limiting equilibrium,

Horizontally

$F = \mu R$ ________ (i)

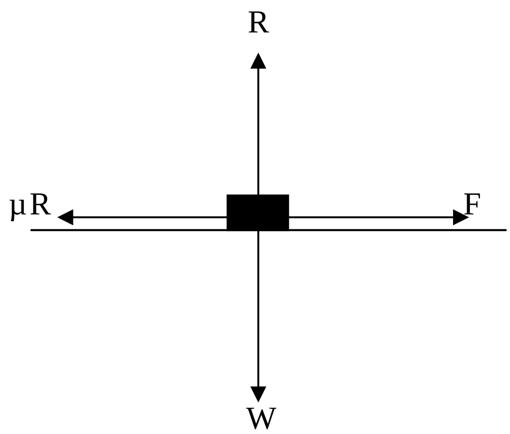
Vertically

$R = W$

Using value of R in (i), we get

$F = \mu W$

$\Rightarrow \qquad \mu = \dfrac{F}{W} = \dfrac{10}{40} = 0.25$



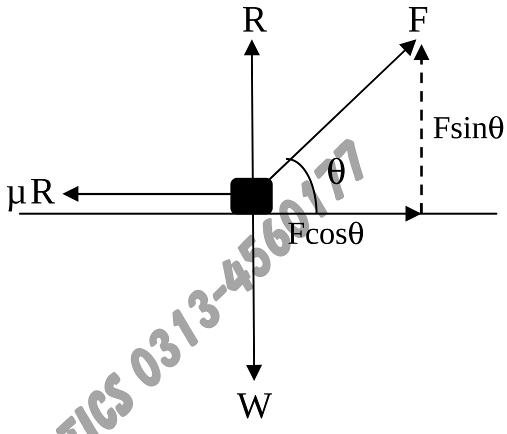
❖ QUESTION 2

Find the least force which will set into motion a particle at rest on a rough horizontal plane.

SOLUTION

Let F be the required least force which makes an angle θ with horizontal plane and W be the weight of the particle.

For limiting equilibrium,



Horizontally

$$F\cos\theta = \mu R \qquad \text{\_\_\_\_\_\_\_\_\_\_ (i)}$$

Vertically

$$W = R + F\sin\theta$$

$$\Rightarrow \quad R = W - F\sin\theta$$

Using value of R in (i), we get

$$F\cos\theta = \mu(W - F\sin\theta)$$

$$\Rightarrow \quad F\cos\theta + \mu\, F\sin\theta = \mu W \quad \Rightarrow \quad F(\cos\theta + \mu\sin\theta) = \mu W$$

$$\Rightarrow \quad F = \frac{\mu W}{\cos\theta + \mu\sin\theta}$$

$$= \frac{\tan\lambda W}{\cos\theta + \tan\lambda\sin\theta} \qquad \because \ \mu = \tan\lambda$$

$$= \frac{\dfrac{\sin\lambda}{\cos\lambda}W}{\cos\theta + \dfrac{\sin\lambda}{\cos\lambda}\sin\theta} = \frac{\dfrac{W\sin\lambda}{\cos\lambda}}{\dfrac{\cos\theta\cos\lambda + \sin\lambda\sin\theta}{\cos\lambda}} = \frac{W\sin\lambda}{\cos\theta\cos\lambda + \sin\lambda\sin\theta}$$

$$= \frac{W\sin\lambda}{\cos(\theta - \lambda)}$$

F will be least when $\cos(\theta - \lambda)$ is maximum and maximum value of $\cos(\theta - \lambda)$ is 1. So F will be least iff $\cos(\theta - \lambda) = 1$

Thus $F_{least} = W\sin\lambda$

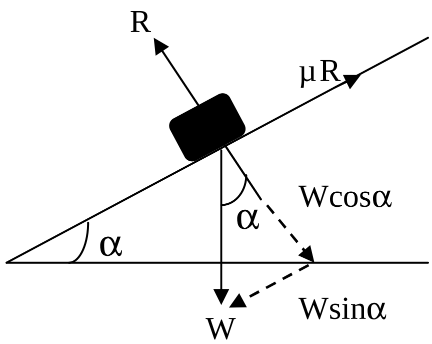
❖ EQUILIBRIUM OF A PARTICLE ON A ROUGH INCLINED PLANE

Show that a particle is in equilibrium on a rough inclined plane under its own weight if the inclination of the plane is equal to the angle of friction.

SOLUTION

Let α be the inclination of the plane and W be the weight acting downward, R is normal reaction and μR is its force of friction.

For limiting equilibrium,



Horizontally

$$W\sin\alpha = \mu R \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_ (i)}$$

Vertically

$$R = W\cos\alpha$$

Using value of R in (i), we get

$$W\sin\alpha = \mu W\cos\alpha$$

$\Rightarrow \quad \tan\alpha = \mu$

$\Rightarrow \quad \tan\alpha = \tan\lambda \qquad \because \tan\lambda = \mu$

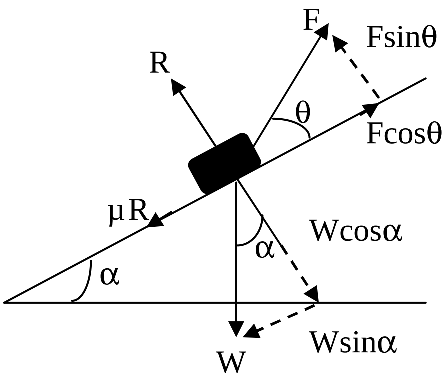
$\Rightarrow \quad \alpha = \lambda$ Where λ is the angle of friction.

❖ QUESTION 2

Find the least force to drag up a particle on a rough inclined plane.

SOLUTION

Let F be the required least force which make angle θ with inclined plane. Let α be the inclination of the plane and W be the weight acting downward , R is normal reaction and μR is its force of friction.



For limiting equilibrium,

Horizontally

$$F\cos\theta = W\sin\alpha + \mu R \qquad \text{\_\_\_\_\_\_\_\_ (i)}$$

Vertically

$$R + F\sin\theta = W\cos\alpha$$

$\Rightarrow \quad R = W\cos\alpha - F\sin\theta$

Using value of R in (i), we get

$$F\cos\theta = W\sin\alpha + \mu(W\cos\alpha - F\sin\theta)$$

$\Rightarrow \quad F\cos\theta + \mu F\sin\theta = W\sin\alpha + \mu W\cos\alpha$

$\Rightarrow \quad F(\cos\theta + \mu\sin\theta) = W(\sin\alpha + \mu\cos\alpha)$

$\Rightarrow \quad F(\cos\theta + \tan\lambda\sin\theta) = W(\sin\alpha + \tan\lambda\cos\alpha) \qquad \because \tan\lambda = \mu$

$\Rightarrow \quad F\left[\cos\theta + \dfrac{\sin\lambda}{\cos\lambda}\sin\theta\right] = W\left[\sin\alpha + \dfrac{\sin\lambda}{\cos\lambda}\cos\alpha\right]$

$\Rightarrow \quad F\left[\dfrac{\cos\theta\cos\lambda + \sin\theta\sin\lambda}{\cos\lambda}\right] = W\left[\dfrac{\sin\alpha\cos\lambda + \sin\lambda\cos\alpha}{\cos\lambda}\right]$

$$\Rightarrow \quad F\left[\frac{\cos(\theta-\lambda)}{\cos\lambda}\right] = W\left[\frac{\sin(\alpha+\lambda)}{\cos\lambda}\right]$$

$$\Rightarrow \quad F = \frac{W\sin(\alpha+\lambda)}{\cos(\theta-\lambda)}$$

F will be least when $\cos(\theta-\lambda)$ is maximum and maximum value of $\cos(\theta-\lambda)$ is 1. So F will be least iff $\cos(\theta-\lambda) = 1$

Thus $F_{least} = W\sin(\alpha+\lambda)$

❖ QUESTION 3

The least force which will move a weight up on an inclined plane is of magnitude P. Show that the least force acting parallel to the plane which will move the weight upward is:

$$P\sqrt{1+\mu^2}$$

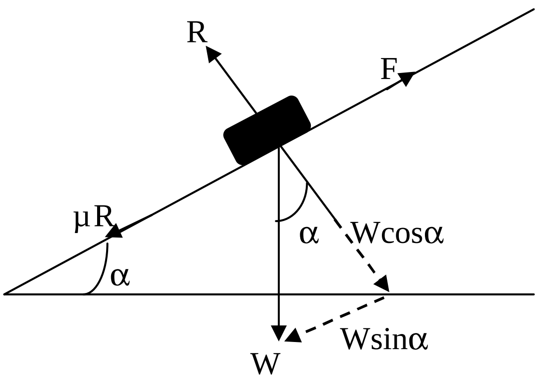
Where μ is the coefficient of friction.

SOLUTION

We know that

$$P = W\sin(\alpha+\lambda) \qquad \text{\_\_\_\_\_\_\_\_\_\_ (i)}$$

Let F be the required force acting parallel to inclined plane. Let α be the inclination of the plane and W be the weight acting downward , R is normal reaction and μR is its force of friction.



For limiting equilibrium,

Horizontally

$$F = W\sin\alpha + \mu R \qquad \text{\_\_\_\_\_ (ii)}$$

Vertically

$$R = W\cos\alpha$$

Using value of R in (ii), we get

$$F = W\sin\alpha + \mu W\cos\alpha$$

$$= W(\sin\alpha + \mu\cos\alpha)$$

$$= W(\sin\alpha + \tan\lambda\cos\alpha) \qquad \because \tan\lambda = \mu$$

$$= W\left[\sin\alpha + \frac{\sin\lambda}{\cos\lambda}\cos\alpha\right]$$

$$= W\left[\frac{\sin\alpha\cos\lambda + \sin\lambda\cos\alpha}{\cos\lambda}\right]$$

$$= \frac{W\sin(\alpha+\lambda)}{\cos\lambda} = \frac{P}{\cos\lambda} \qquad \text{By(i)}$$

$$= P\sec\lambda$$

$$= P\sqrt{1+\tan^2\lambda}$$

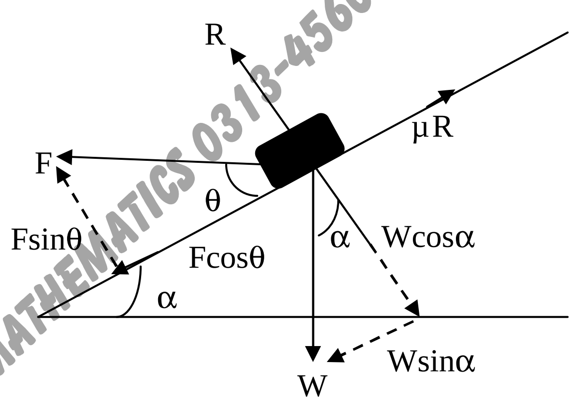
$$= P\sqrt{1+\mu^2} \qquad \because \tan\lambda = \mu$$

❖ QUESTION 4

Find the least force to drag a particle on a rough inclined plane in downward direction.

SOLUTION

Let F be the required least force which make angle θ with inclined plane. Let α be the inclination of the plane and W be the weight acting downward, R is normal reaction and μR is its force of friction.



For limiting equilibrium,

Horizontally

$$F\cos\theta + W\sin\alpha = \mu R \qquad \_\_\_\_\_\_ \text{(i)}$$

Vertically

$$R + F\sin\theta = W\cos\alpha$$

$$\Rightarrow \quad R = W\cos\alpha - F\sin\theta$$

Using value of R in (i), we get

$$F\cos\theta + W\sin\alpha = \mu(W\cos\alpha - F\sin\theta)$$

$$\Rightarrow \quad F\cos\theta + W\sin\alpha = \mu W\cos\alpha - \mu F\sin\theta$$

$$\Rightarrow \quad F(\cos\theta + \mu\sin\theta) = W(\mu\cos\alpha - \sin\alpha)$$

$$\Rightarrow \quad F(\cos\theta + \tan\lambda\sin\theta) = W(\tan\lambda\cos\alpha - \sin\alpha) \qquad \because \tan\lambda = \mu$$

$$\Rightarrow \quad F\left[\cos\theta + \frac{\sin\lambda}{\cos\lambda}\sin\theta\right] = W\left[\frac{\sin\lambda}{\cos\lambda}\cos\alpha - \sin\alpha\right]$$

$$\Rightarrow \quad F\left[\frac{\cos\theta\cos\lambda + \sin\theta\sin\lambda}{\cos\lambda}\right] = W\left[\frac{\sin\lambda\cos\alpha - \sin\alpha\cos\lambda}{\cos\lambda}\right]$$

$$\Rightarrow \quad F\left[\frac{\cos(\lambda-\theta)}{\cos\lambda}\right] = W\left[\frac{\sin(\lambda-\alpha)}{\cos\lambda}\right]$$

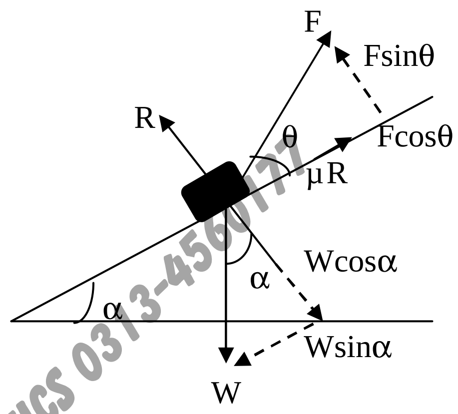
$$\Rightarrow \quad F = \frac{W\sin(\lambda-\alpha)}{\cos(\lambda-\theta)}$$

F will be least when $\cos(\lambda-\theta)$ is maximum and maximum value of $\cos(\lambda-\theta)$ is 1. So F will be least iff $\cos(\lambda-\theta) = 1$ Thus $F_{least} = W\sin(\lambda-\alpha)$

❖ QUESTION 5

Find the force which is necessary just to support a heavy body on an inclined plane with inclination α; $(\alpha > \lambda)$

SOLUTION



Let F be the required force which make angle θ with inclined plane. Let α be the inclination of the plane and W be the weight acting downward, R is normal reaction and μR is its force of friction.

For limiting equilibrium,

Horizontally

$$W\sin\alpha = F\cos\theta + \mu R \qquad \text{\_\_\_\_\_\_\_\_ (i)}$$

Vertically

$$R + F\sin\theta = W\cos\alpha$$

$$\Rightarrow \quad R = W\cos\alpha - F\sin\theta$$

Using value of R in (i), we get

$$W\sin\alpha = F\cos\theta + \mu(W\cos\alpha - F\sin\theta)$$

$$\Rightarrow \quad W\sin\alpha = F\cos\theta + \mu W\cos\alpha - \mu F\sin\theta$$

$$\Rightarrow \quad F\cos\theta - \mu F\sin\theta = W\sin\alpha - \mu W\cos\alpha$$

$$\Rightarrow \quad F(\cos\theta - \mu\sin\theta) = W(\sin\alpha - \mu\cos\alpha)$$

$$\Rightarrow \quad F(\cos\theta - \tan\lambda\sin\theta) = W(\sin\alpha - \tan\lambda\cos\alpha) \qquad \because \tan\lambda = \mu$$

$$\Rightarrow \quad F\left[\cos\theta - \frac{\sin\lambda}{\cos\lambda}\sin\theta\right] = W\left[\sin\alpha - \frac{\sin\lambda}{\cos\lambda}\cos\alpha\right]$$

$$\Rightarrow \quad F\left[\frac{\cos\theta\cos\lambda - \sin\theta\sin\lambda}{\cos\lambda}\right] = W\left[\frac{\sin\alpha\cos\lambda - \sin\lambda\cos\alpha}{\cos\lambda}\right]$$

$$\Rightarrow \quad F\left[\frac{\cos(\theta+\lambda)}{\cos\lambda}\right] = W\left[\frac{\sin(\alpha - \lambda)}{\cos\lambda}\right]$$

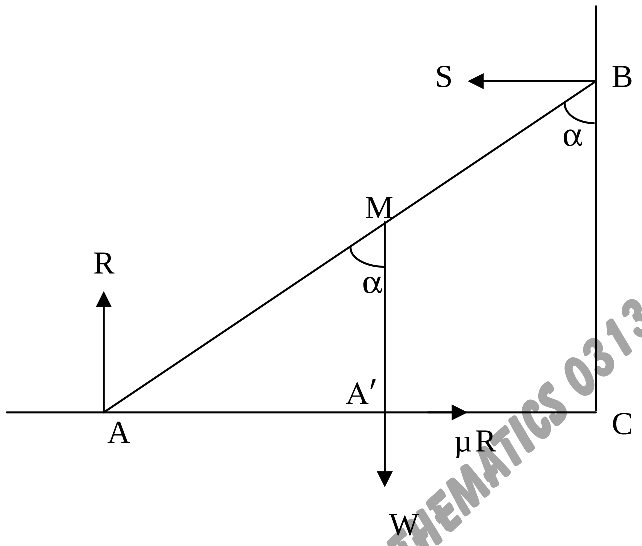
$$\Rightarrow \quad F = \frac{W\sin(\alpha-\lambda)}{\cos(\theta+\lambda)}$$

F will be least when $\cos(\theta+\lambda)$ is maximum and maximum value of $\cos(\theta+\lambda)$ is 1. So F will be least iff $\cos(\theta+\lambda) = 1$ Thus $F_{least} = W\sin(\alpha - \lambda)$

❖ QUESTION 6

A light ladder is supported on a rough floor and leans against a smooth wall. How far a up a man can climb without slipping taking place ?

SOLUTION



Let AB be a light ladder which makes angle α with vertical. Let M be the position of the man of weight W when the ladder is in limiting equilibrium. R and S are normal reactions of ground and wall respectively, μR is force of friction of ground.

Let $AM = x$ and $AB = L$

For limiting equilibrium,

Horizontally

$$S = \mu R \qquad \text{\_\_\_\_\_\_\_ (i)}$$

Vertically

$$R = W \qquad \text{\_\_\_\_\_\_\_ (ii)}$$

From (i) & (ii), we get

$$S = \mu W \qquad \text{\_\_\_\_\_\_\_ (iii)}$$

Taking moment of all forces about A

$$S(BC) = W(AA')$$

$\Rightarrow$ $S(AB\cos\alpha) = W(AM\sin\alpha)$

$\Rightarrow$ $SL\cos\alpha = Wx\sin\alpha$

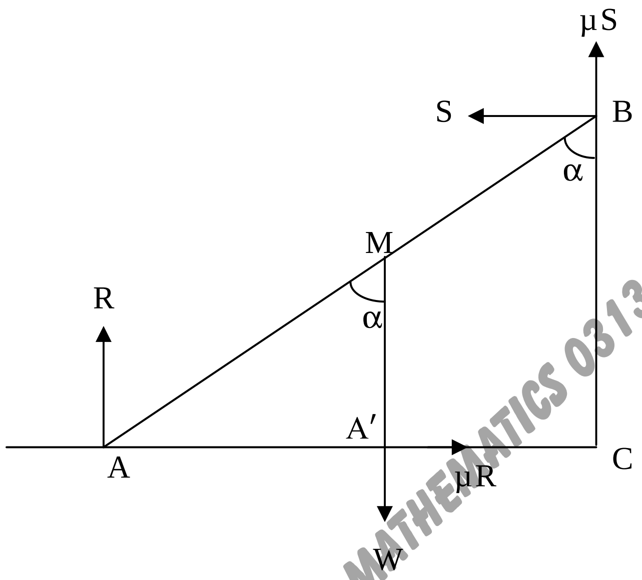
$\Rightarrow$ $\mu WL\cos\alpha = Wx\sin\alpha$ By (iii)

$\Rightarrow$ $x = \mu L\cot\alpha$

❖ QUESTION 7

A light rod is supported on a rough floor and leans against a rough wall. The coefficient of friction of both wall and floor is μ. How far a up a man can climb without slipping taking place ?

SOLUTION



Let AB be a light ladder which makes angle α with vertical. Let M be the position of the man of weight W when the ladder is in limiting equilibrium. R and S are normal reactions of ground and wall respectively, μR and μR are forces of friction of ground and wall respectively.

Let AM = x and AB = L

For limiting equilibrium,

Horizontally

$$S = \mu R \qquad \text{\_\_\_\_\_\_\_ (i)}$$

Vertically

$$R + \mu S = W$$

$$R = W - \mu S \qquad \text{\_\_\_\_\_\_\_ (ii)}$$

From (i) & (ii), we get

$$S = \mu W - \mu^2 S \qquad \Rightarrow \qquad S + \mu^2 S = \mu W$$

$$\Rightarrow \qquad S(1+\mu^2) = \mu W$$

$$\Rightarrow \qquad S = \frac{\mu W}{1+\mu^2} \qquad \text{\_\_\_\_\_\_\_ (iii)}$$

Taking moment of all forces about A

$$S(BC) + \mu S(AC) = W(AA')$$

$\Rightarrow \quad S(AB\cos\alpha) + \mu S(AB\sin\alpha) = W(AM\sin\alpha)$

$\Rightarrow \quad SL\cos\alpha + \mu S(L\sin\alpha) = Wx\sin\alpha$

$\Rightarrow \quad Wx\sin\alpha = SL\cos\alpha + \mu S(L\sin\alpha)$

$$= SL(\cos\alpha + \mu\sin\alpha)$$

$$= \frac{\mu W}{1+\mu^2} L(\cos\alpha + \mu\sin\alpha) \qquad \text{By (iii)}$$

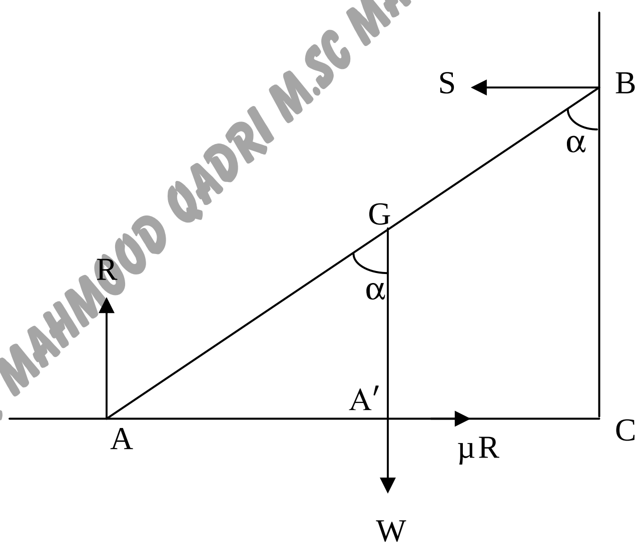
$$= \frac{\mu W\sin\alpha}{1+\mu^2} L(\cot\alpha + \mu)$$

$\Rightarrow \quad x = \dfrac{\mu L}{1+\mu^2}(\mu + \cot\alpha)$

❖ QUESTION 8

A uniform ladder rests in limiting equilibrium with one end on rough floor whose coefficient of friction is μ and with other end on a smooth vertical wall. Show that its inclination to the vertical is $\tan^{-1}(2\mu)$.

SOLUTION



Let AB be a ladder of weight W is its weight is acting downward from midpoint G. R and S are normal reactions of floor and wall respectively, μR is force of friction of floor.

Let AB = 2L then AG = BG = L

For limiting equilibrium,

<u>Horizontally</u>

$$S = \mu R \qquad \text{\_\_\_\_\_\_\_\_ (i)}$$

Vertically

$R = W$ _______ (ii)

From (i) & (ii), we get

$S = \mu W$ _______ (iii)

Taking moment of all forces about A

$S(BC) = W(AA')$

$\Rightarrow \quad S(AB\cos\alpha) = W(AG\sin\alpha)$

$\Rightarrow \quad S(2L\cos\alpha) = W(L\sin\alpha)$

$\Rightarrow \quad 2S\cos\alpha = W\sin\alpha$

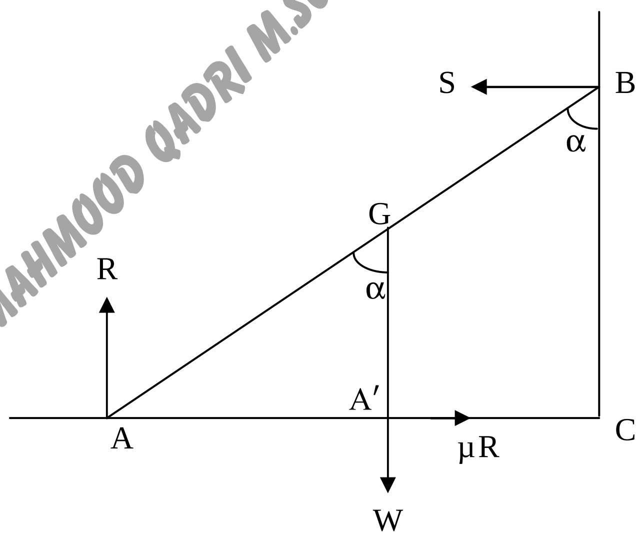
$\Rightarrow \quad 2\mu W\cos\alpha = W\sin\alpha \qquad$ By (iii)

$\Rightarrow \quad 2\mu = \tan\alpha \qquad \Rightarrow \qquad \alpha = \tan^{-1}(2\mu)$

❖ QUESTION 9

A uniform ladder rests in limiting equilibrium with one end on rough horizontal plane and other end on a smooth vertical wall. A man ascends the ladder. Show that he cannot go more than half way up.

SOLUTION



Let AB be a ladder of weight W. Its weight is acting downward from midpoint G. R and S are normal reactions of floor and wall respectively, μR is force of friction of floor. Let AB = 2L then AG = BG = L

For limiting equilibrium,

Horizontally

$S = \mu R$ _______ (i)

<u>Vertically</u>

$$R = W \qquad \_\_\_\_\_\_\_ \text{(ii)}$$

From (i) & (ii), we get

$$S = \mu W \qquad \_\_\_\_\_\_\_ \text{(iii)}$$

<u>Taking moment of all forces about A</u>

$S(BC) = W(AA')$

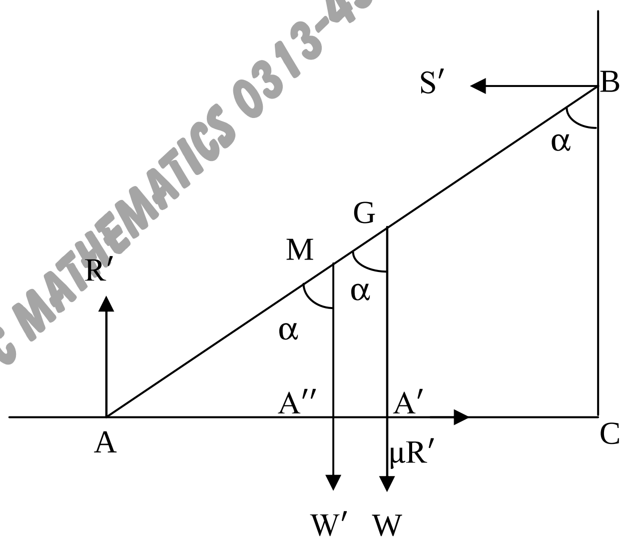
$\Rightarrow \quad S(AB\cos\alpha) = W(AG\sin\alpha)$

$\Rightarrow \quad S(2L\cos\alpha) = W(L\sin\alpha)$

$\Rightarrow \quad 2S\cos\alpha = W\sin\alpha$

$\Rightarrow \quad 2\mu W\cos\alpha = W\sin\alpha$ By (iii)

$\Rightarrow \quad \tan\alpha = 2\mu$



Now a man ascends the ladder and let weight of man be W'. Let M be the position of the man when ladder is in limiting equilibrium. Now normal reactions are R' and S' and force of friction is $\mu R'$.

For limiting equilibrium,

<u>Horizontally</u>

$$S' = \mu R' \qquad \_\_\_\_\_\_\_ \text{(iv)}$$

<u>Vertically</u>

$$R' = W + W' \qquad \_\_\_\_\_\_\_ \text{(v)}$$

From (iv) & (v), we get

$$S' = \mu W + \mu W' \qquad \_\_\_\_\_\_\_ \text{(vi)}$$

<u>Taking moment of all forces about A</u>

$S'(BC) = W(AA') + W'(AA'')$

$\Rightarrow \quad S'(AB\cos\alpha) = W(AG\sin\alpha) + W'(AM\sin\alpha)$

$\Rightarrow \quad S'(2L\cos\alpha) = W(L\sin\alpha) + W'(x\sin\alpha)$

$\Rightarrow \quad 2S'L\cos\alpha = (WL + W'x)\sin\alpha$

$\Rightarrow \quad 2S'L = (WL + W'x)\tan\alpha$

$\Rightarrow \quad 2S'L = (WL + W'x)\,2\mu \qquad [\because \tan\alpha = 2\mu]$

$\Rightarrow \quad 2L(\mu W + \mu W') = (WL + W'x)\,2\mu$ By (vi)

$\Rightarrow \quad 2\mu\,(WL + W'L) = (WL + W'x)\,2\mu \quad \Rightarrow \quad WL + W'L = WL + W'x$

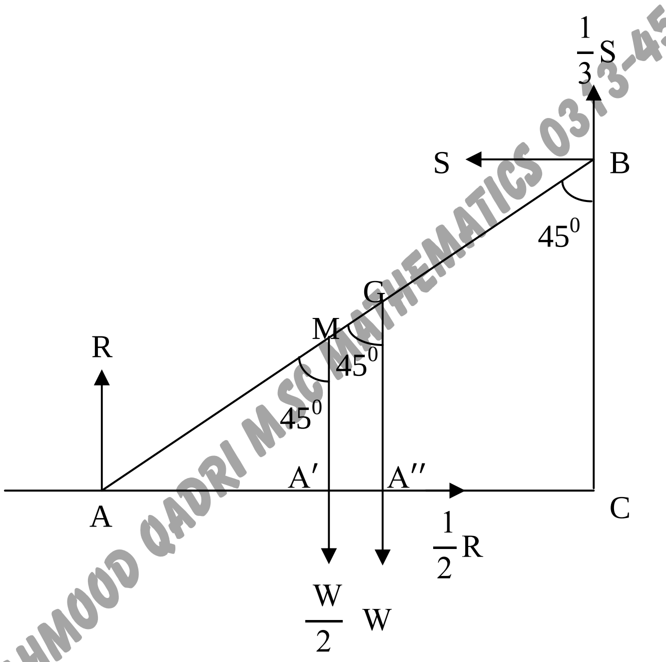
$\Rightarrow \quad W'L = W'x \quad \Rightarrow \quad x = L =$ (Half of the length of Ladder)

❖ QUESTION 10

A uniform ladder of length 70 feet rest against a vertical wall with which it makes an angle of 45^0, the coefficient between ladder & wall & ground respectively being 1/3 and ½. If a man whose weight is one half that of ladder, ascends the ladder, where will he be when the ladder slips.?

If a body now stands on the bottom rung f the ladder what must be his least weight so that the man may go to the top of the ladder. ?

SOLUTION



Let AB be a ladder of length 70 feet makes an angle of 45^0 with vertical. W is its weight acting downward from its midpoint G. R and S are normal reactions of ground and wall respectively. Let M be the position of man when ladder slip. $\frac{1}{2}R$ and $\frac{1}{3}S$ are forces of friction of ground and wall respectively.

Since AB = 70 Therefore AG = BG = 35. Let AM = x

For limiting equilibrium,

Horizontally

$$S = \frac{1}{2}R \qquad \text{\_\_\_\_\_\_\_\_ (i)}$$

Vertically

$$R + \frac{1}{3}S = W + \frac{W}{2}$$

$$\Rightarrow \quad R = \frac{3W}{2} - \frac{1}{3}S \qquad \text{\_\_\_\_\_\_ (ii)}$$

From (i) & (ii), we get

$$S = \frac{1}{2}\left(\frac{3W}{2} - \frac{1}{3}S\right) = \frac{3W}{4} - \frac{S}{6}$$

$\Rightarrow \quad S + \frac{S}{6} = \frac{3W}{4} \qquad \Rightarrow \quad \frac{7S}{6} = \frac{3W}{4} \qquad \Rightarrow \quad S = \frac{18}{28}W$

$\Rightarrow \quad S = \frac{9}{14}W$ ________(iii)

<u>Taking moment of all forces about A</u>

$$S(BC) + \frac{1}{3}S(AC) = \frac{W}{2}(AA') + W(AA'')$$

$\Rightarrow \quad S(AB\cos 45) + \frac{1}{3}S(AB\sin 45) = \frac{W}{2}(AM\sin 45) + W(AG\sin 45)$

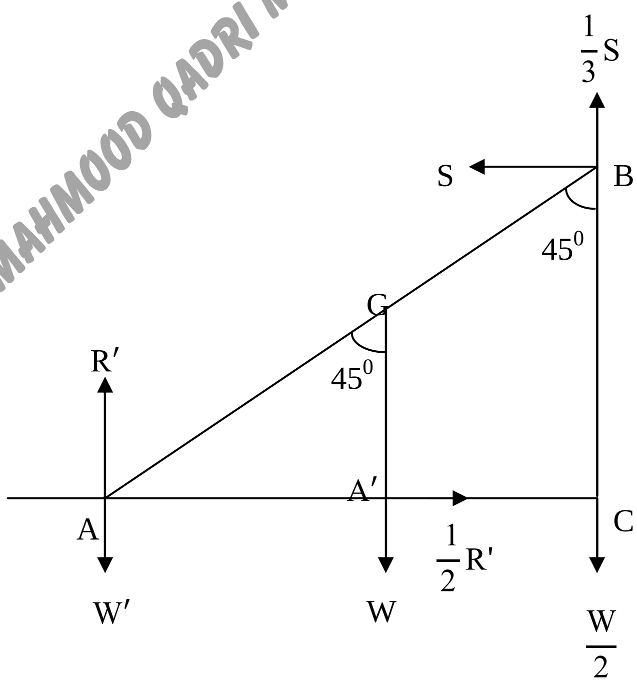
$\Rightarrow \quad S\left(70\frac{1}{\sqrt{2}}\right) + \frac{1}{3}S(70\frac{1}{\sqrt{2}}) = \frac{W}{2}(x\frac{1}{\sqrt{2}}) + W\left(35\frac{1}{\sqrt{2}}\right)$

$\Rightarrow \quad 70S + \frac{70}{3}S = \frac{Wx}{2} + 35W$

$\Rightarrow \quad \left(70 + \frac{70}{3}\right)S = \left(\frac{x}{2} + 35\right)W$

$\Rightarrow \quad \left(70 + \frac{70}{3}\right)\frac{9}{14}W = \left(\frac{x}{2} + 35\right)W$ By (iii)

$\Rightarrow \quad x = 50$ feet



Suppose a man of weight of W' stands on the bottom rung of the ladder and other man reach to the top of the ladder at point B. Now R' is the normal reaction of the ground and $\frac{1}{2}R'$ is the force of friction.

For limiting equilibrium,

<u>Horizontally</u>

$$S = \frac{1}{2}R' \qquad \text{\_\_\_\_\_\_ (iv)}$$

<u>Vertically</u>

$$R' + \frac{1}{3}S = W + W' + \frac{W}{2}$$

$$\Rightarrow \quad R' = \frac{3W}{2} + W' - \frac{1}{3}S \qquad \text{\_\_\_\_\_\_ (v)}$$

From (iv) & (v), we get

$$S = \frac{9W}{14} + \frac{3W'}{7} \qquad \text{\_\_\_\_\_\_ (vi)}$$

<u>Taking moment of all forces about A</u>

$$S(BC) + \frac{1}{3}S(AC) = \frac{W}{2}(AC) + W(AA')$$

$$\Rightarrow \quad S(AB\cos 45) + \frac{1}{3}S(AB\sin 45) = \frac{W}{2}(AB\sin 45) + W(AG\sin 45)$$

$$\Rightarrow \quad S\left(70\frac{1}{\sqrt{2}}\right) + \frac{1}{3}S(70\frac{1}{\sqrt{2}}) = \frac{W}{2}(70\frac{1}{\sqrt{2}}) + W\left(35\frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow \quad 70S + \frac{70}{3}S = 35W + 35W$$

$$\Rightarrow \quad \left(70 + \frac{70}{3}\right)S = 70W$$

$$\Rightarrow \quad \left(\frac{9W}{14} + \frac{3W'}{7}\right)\left(70 + \frac{70}{3}\right) = 70W \qquad \text{By (vi)}$$

$$\Rightarrow \quad \left(\frac{9W + 6W'}{14}\right)\left(1 + \frac{1}{3}\right) = W$$

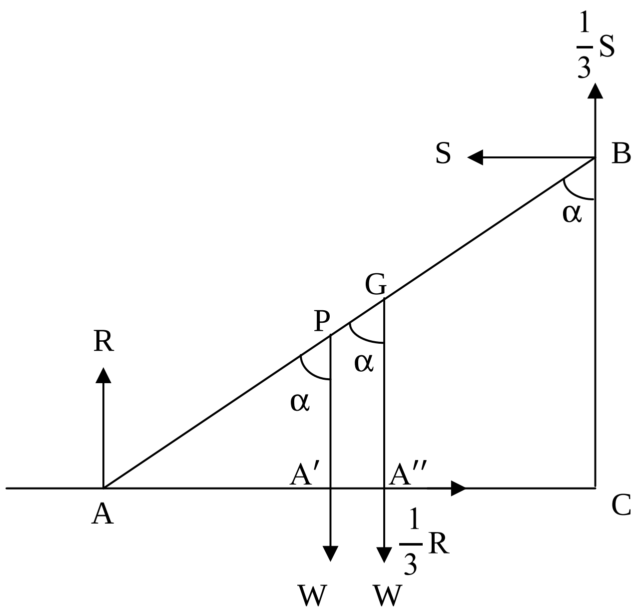
$$\Rightarrow \quad 12W' = 3W$$

$$\Rightarrow \quad W' = \frac{1}{4}W$$

❖ QUESTION 11

The upper end of a uniform ladder rests against a rough vertical wall and lower end on a rough horizontal plane, the coefficient of friction in both cases is 1/3. Prove that if the inclination of the ladder to the vertical is $\tan^{-1}(1/2)$, a weight equal to that of ladder cannot be attached to it at a point more than 9/10 of the distance from the foot of it without destroying the equilibrium.

SOLUTION



Let AB be a ladder of length 2L makes an angle of α with vertical. W is its weight acting downward from its midpoint G. R and S are normal reactions of ground and wall respectively. $\frac{1}{3}R$ and $\frac{1}{3}S$ are forces of friction of ground and wall respectively. Let P be the point where weight W is attached. Since AB = 2L Therefore AG = BG = L. Let AP = x

Now we have to prove that

$$x = \frac{9}{10}(2L)$$

For limiting equilibrium,

<u>Horizontally</u>

$$S = \frac{1}{3}R \qquad \text{\_\_\_\_\_\_\_ (i)}$$

<u>Vertically</u>

$$R + \frac{1}{3}S = W + W$$

$$\Rightarrow \quad R = 2W - \frac{1}{3}S \qquad \text{\_\_\_\_\_\_ (ii)}$$

From (i) & (ii), we get

$$S = \frac{1}{3}\left(2W - \frac{1}{3}S\right)$$

$$\Rightarrow \quad S + \frac{1}{9}S = \frac{2}{3}W$$

$$\Rightarrow \quad S = \frac{3}{5}W \qquad \text{\_\_\_\_\_\_ (iii)}$$

<u>Taking moment of all forces about A</u>

$$S(BC) + \frac{1}{3}S(AC) = W(AA') + W(AA'')$$

$\Rightarrow \quad S(AB\cos\alpha) + \frac{1}{3}S(AB\sin\alpha) = W(AP\sin\alpha) + W(AG\sin\alpha)$

$\Rightarrow \quad S(2L\cos\alpha) + \frac{1}{3}S(2L\sin\alpha) = W(x\sin\alpha) + W(L\sin\alpha)$

$\Rightarrow \quad S\left(2L\cos\alpha + \frac{2}{3}L\sin\alpha\right) = W(x\sin\alpha + L\sin\alpha)$

$\Rightarrow \quad \left(\frac{3}{5}W\right)\left(2L\cos\alpha + \frac{2}{3}L\sin\alpha\right) = W(x\sin\alpha + L\sin\alpha)$ By (iii)

$\Rightarrow \quad \frac{3}{5}\left(2L\cos\alpha + \frac{2}{3}L\sin\alpha\right) = x\sin\alpha + L\sin\alpha$

$\Rightarrow \quad \frac{6}{5}L\cos\alpha + \frac{2}{5}L\sin\alpha = x\sin\alpha + L\sin\alpha$

$\Rightarrow \quad \frac{6}{5}L\cos\alpha + \frac{2}{5}L\sin\alpha - L\sin\alpha = x\sin\alpha$

$\Rightarrow \quad \frac{6}{5}L\cos\alpha + \left(\frac{2}{5} - 1\right)L\sin\alpha = x\sin\alpha$

$\Rightarrow \quad \frac{6}{5}L - \frac{3}{5}L\tan\alpha = x\tan\alpha$ Dividing both sides by $\cos\alpha$

Given that

$\alpha = \tan^{-1}\left(\frac{1}{2}\right) \Rightarrow \tan\alpha = \frac{1}{2}$

$\Rightarrow \quad \frac{6}{5}L - \frac{3}{5}L\left(\frac{1}{2}\right) = x\left(\frac{1}{2}\right)$

$\Rightarrow \quad 12L - 3L = 5x \quad \Rightarrow \quad x = \frac{9}{5}L \quad \Rightarrow \quad x = \frac{9}{10}(2L)$

❖ QUESTION 12

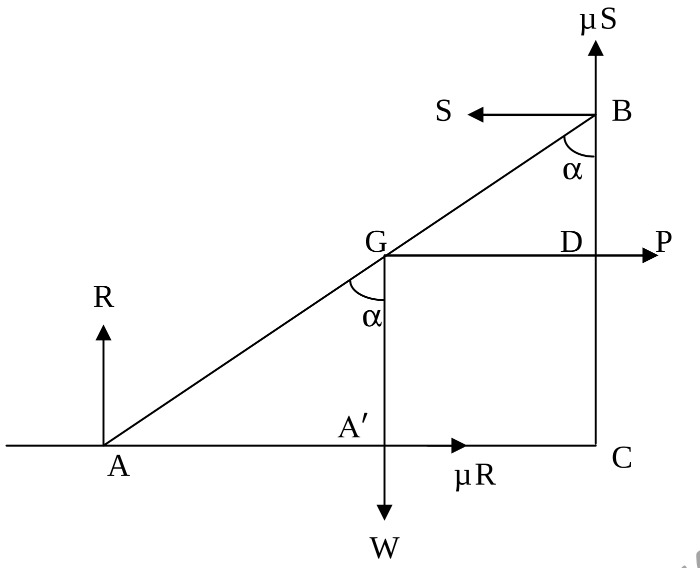
A uniform rod of weight W is place with its lower end on a rough horizontal floor and its upper end against equally rough vertical wall. The rod makes an angle α with the wall and is just prevented from slipping down by a horizontal force P applied at its middle point. Prove that

$$\mathbf{P = W\tan(\alpha - 2\lambda)}$$

Where λ is the angle of friction and $\lambda < \frac{1}{2}\alpha$

SOLUTION

Let AB be a rod of length 2L makes an angle of α with vertical. W is its weight acting downward from its midpoint G. R and S are normal reactions of ground and wall respectively. μR and μs are forces of friction of ground and wall respectively. Let P be the horizontal force applied at middle point G of rod. Since AB = 2L Therefore AG = BG = L.



For limiting equilibrium,

Horizontally

$$S = P + \mu R$$

$$\Rightarrow \quad P = S - \mu R \qquad \text{\_\_\_\_\_\_ (i)}$$

Vertically

$$W = R + \mu S \qquad \text{\_\_\_\_\_\_ (ii)}$$

From (i) & (ii), we get

$$\frac{P}{W} = \frac{S - \mu R}{R + \mu S} \qquad \text{\_\_\_\_\_\_ (iii)}$$

Taking moment of all forces about G

$$R(AA') = \mu R(GA') + S(BD) + \mu S(GD)$$

$$\Rightarrow \quad R(AG\sin\alpha) = \mu R(AG\cos\alpha) + S(GB\cos\alpha) + \mu S(GB\sin\alpha)$$

$$\Rightarrow \quad RL\sin\alpha = \mu RL\cos\alpha + SL\cos\alpha + \mu SL\sin\alpha$$

$$\Rightarrow \quad R(\sin\alpha - \mu\cos\alpha) = S(\cos\alpha + \mu\sin\alpha)$$

$$\Rightarrow \quad R(\sin\alpha - \tan\lambda\cos\alpha) = S(\cos\alpha + \tan\lambda\sin\alpha) \qquad \because \tan\lambda = \mu$$

$$\Rightarrow \quad R\left(\sin\alpha - \frac{\sin\lambda}{\cos\lambda}\cos\alpha\right) = S\left(\cos\alpha + \frac{\sin\lambda}{\cos\lambda}\sin\alpha\right)$$

$$\Rightarrow \quad R\left(\frac{\sin\alpha\cos\lambda - \cos\alpha\sin\lambda}{\cos\lambda}\right) = S\left(\frac{\cos\alpha\cos\lambda + \sin\alpha\sin\lambda}{\cos\lambda}\right)$$

$$\Rightarrow \quad R\sin(\alpha - \lambda) = S\cos(\alpha - \lambda)$$

$$\Rightarrow \quad S = R\tan(\alpha - \lambda)$$

Using value of S in (iii), we get

$$\frac{P}{W} = \frac{R\tan(\alpha - \lambda) - \mu R}{R + \mu R\tan(\alpha - \lambda)}$$

$$= \frac{\tan(\alpha - \lambda) - \mu}{1 + \mu\tan(\alpha - \lambda)} = \frac{\tan(\alpha - \lambda) - \tan\lambda}{1 + \tan(\alpha - \lambda)\tan\lambda} = \tan(\alpha - \lambda - \lambda) = \tan(\alpha - 2\lambda)$$

$\Rightarrow \quad P = W\tan(\alpha - 2\lambda)$

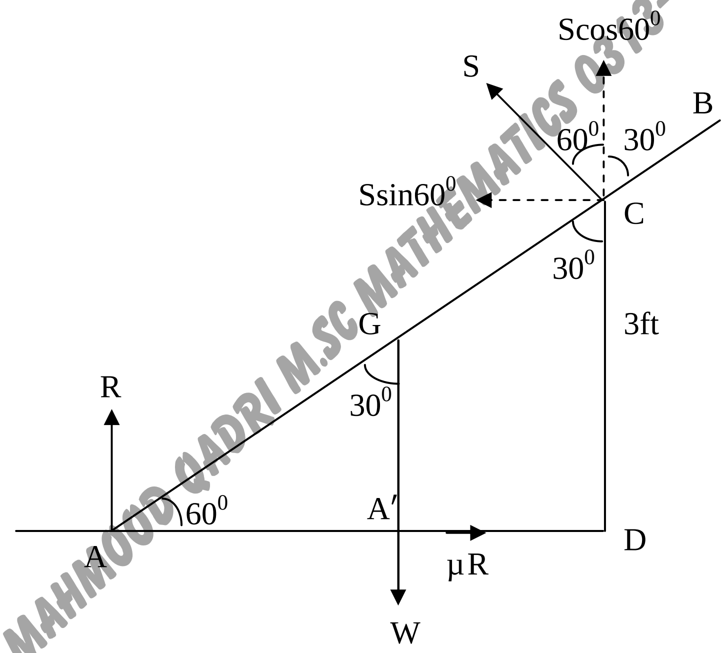
Since P is a force and force is always positive, therefore $\alpha - 2\lambda$ must be positive.

i.e. $\quad \alpha - 2\lambda > 0 \quad \Rightarrow \quad \alpha > 2\lambda \quad \Rightarrow \quad \lambda < \frac{\alpha}{2}$

❖ QUESTION 13

A rod 4ft long rests on a rough floor against the smooth edge of a table of height 3ft. If the rod is on the point of slipping when inclined at an angle of 60^0 to the horizontal, find the coefficient of friction.

SOLUTION



Let AB be a rod of length 4ft makes an angle of 60^0 with horizon. W is its weight acting downward from its midpoint G. R and S are normal reactions of ground and edge of the table respectively. μR is forces of friction of ground. Since AB = 4ft Therefore AG = BG = 2ft.

For limiting equilibrium,

Horizontally

$S\sin60^0 = \mu R$ ________ (i)

Vertically

$W = R + S\cos60^0$

$\Rightarrow \quad R = W - S\cos60^0$ ________ (ii)

From (i) & (ii), we get

$S\sin60^0 = \mu W - \mu S\cos60^0$

$\Rightarrow \quad S\sin60^0 + \mu S\cos60^0 = \mu W$

$\Rightarrow \quad S(\sin60^0 + \mu\cos60^0) = \mu W$

$\Rightarrow \quad S\left(\frac{\sqrt{3}}{2} + \frac{\mu}{2}\right) = \mu W$

$\Rightarrow \quad S = \frac{2\mu W}{\sqrt{3} + \mu}$ _______ (iii)

<u>Taking moment of all forces about A</u>

$S(AC) = W(AA')$

$\Rightarrow \quad S(AC) = W(AG\cos60)$

$\Rightarrow \quad S(AC) = W\left(2\frac{1}{2}\right)$

$\Rightarrow \quad S(AC) = W$

From fig.

$\sin60^0 = \frac{3}{AC} \Rightarrow \frac{\sqrt{3}}{2} = \frac{3}{AC} \Rightarrow AC = \frac{2.3}{\sqrt{3}} \Rightarrow AC = 2\sqrt{3}$

So $\quad S(2\sqrt{3}) = W$

$\Rightarrow \quad S = \frac{W}{2\sqrt{3}}$ _______ (iv)

From (iii) & (iv), we have

$\frac{2\mu W}{\sqrt{3} + \mu} = \frac{W}{2\sqrt{3}}$

$\Rightarrow \quad 4\mu\sqrt{3} = \sqrt{3} + \mu$

$\Rightarrow \quad 4\mu\sqrt{3} - \mu = \sqrt{3}$

$\Rightarrow \quad \mu(4\sqrt{3} - 1) = \sqrt{3}$

$\Rightarrow \quad \mu = \frac{\sqrt{3}}{4\sqrt{3} - 1}$

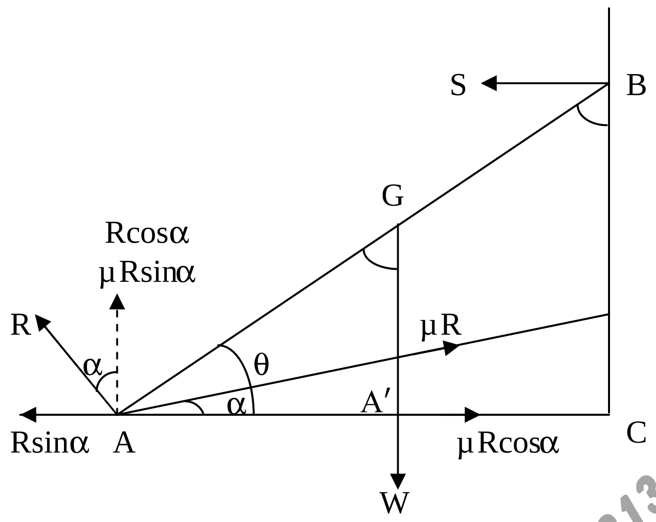
$\Rightarrow \quad \mu = 0.292$

❖ QUESTION 14

One end of a uniform ladder of weight W rests against a smooth wall, and other end on a rough ground, which slopes down from the wall at an angle α to the horizon. Find the inclination of the ladder to the horizontal when it is at the point of slipping, and show that the reaction of the wall is then **W tan ($\lambda - \alpha$)** where λ is the angle of friction.

SOLUTION

Let AB be a ladder of length 2L makes an angle of θ with horizon. W is its weight acting downward from its midpoint G. R and S are normal reactions of ground and wall respectively. μR is forces of friction of ground. Let α be the inclination of ground to the horizon. Since AB = 2L Therefore AG = BG = L.



For limiting equilibrium,

Horizontally

$$S + R\sin\alpha = \mu R\cos\alpha$$

$\Rightarrow \quad S = R(\mu\cos\alpha - \sin\alpha)$ ________ (i)

Vertically

$$W = \mu R\sin\alpha + R\cos\alpha$$

$\Rightarrow \quad W = R(\mu\sin\alpha + \cos\alpha)$ ________ (ii)

From (i) & (ii), we get

$$\frac{S}{W} = \frac{\mu\cos\alpha - \sin\alpha}{\mu\sin\alpha + \cos\alpha}$$

$$= \frac{\tan\lambda\cos\alpha - \sin\alpha}{\tan\lambda\sin\alpha + \cos\alpha} \qquad \because \tan\lambda = \mu$$

$$= \frac{\left(\dfrac{\sin\lambda\cos\alpha - \cos\lambda\sin\alpha}{\cos\lambda}\right)}{\left(\dfrac{\sin\alpha\sin\lambda + \cos\alpha\cos\lambda}{\cos\lambda}\right)} = \frac{\sin(\lambda - \alpha)}{\cos(\lambda - \alpha)}$$

$\Rightarrow \quad S = W\tan(\lambda - \alpha)$ ________ (iii)

Thus the normal reaction of wall is $W\tan(\lambda - \alpha)$.

Taking moment of all forces about A

$$S(BC) = W(AA')$$

$\Rightarrow \quad S(AB\sin\theta) = W(AG\cos\theta)$

$\Rightarrow \quad S(2L\sin\theta) = W(L\cos\theta) \quad \Rightarrow \quad 2S = W\cot\theta$

$\Rightarrow \quad 2W\tan(\lambda - \alpha) = W\cot\theta \quad$ By(iii)

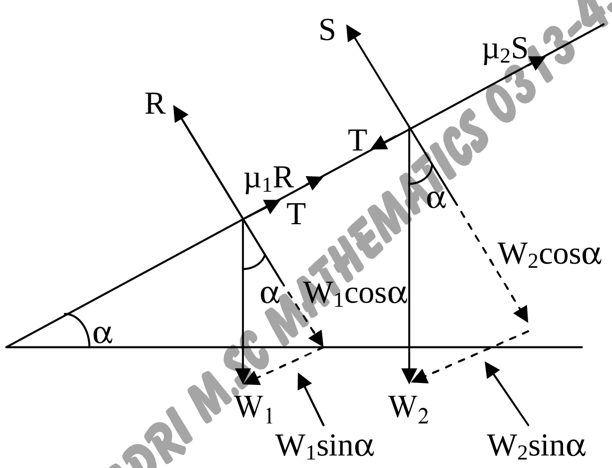
$\Rightarrow \quad \cot\theta = 2\tan(\lambda - \alpha)$

$\Rightarrow \quad \theta = \cot^{-1}\left(2\tan(\lambda - \alpha)\right)$

❖ QUESTION 15

Two bodies, weight W_1, W_2 are placed on an inclined plane are connected by a light string which coincides with a line of greatest slope of the plane. If the coefficient of friction between the bodies and the plane be respectively μ_1 and μ_2. Find the inclination of the plane to the horizon when the bodies are on the point of motion, it is being assumed that the smoother body is below the other.

SOLUTION



R and S are normal reactions and μ_1R and μ_2S are forces of friction. Let T be the tension in the string. Let α be the inclination of plane to the horizontal.

For W_1 : For limiting equilibrium,

Horizontally

$\mu_1R + T = W_1\sin\alpha$

$\Rightarrow \quad T = W_1\sin\alpha - \mu_1R \qquad$ ________ (i)

Vertically

$R = W_1\cos\alpha \qquad$ ________ (ii)

From (i) & (ii), we get

$T = W_1\sin\alpha - \mu_1W_1\cos\alpha \qquad$ ________ (iii)

For W_2 : For limiting equilibrium,

Horizontally

$T + W_2\sin\alpha = \mu_2S$

$\Rightarrow \quad T = \mu_2S - W_2\sin\alpha \qquad$ ________ (iv)

Vertically

$$S = W_2\cos\alpha \qquad \text{\_\_\_\_\_\_\_ (v)}$$

From (iv) & (v), we get

$$T = \mu_2 W_2\cos\alpha - W_2\sin\alpha \qquad \text{\_\_\_\_\_\_\_ (vi)}$$

From (iii) & (vi), we get

$$W_1\sin\alpha - \mu_1 W_1\cos\alpha = \mu_2 W_2\cos\alpha - W_2\sin\alpha$$

$$\Rightarrow \quad W_1\sin\alpha + W_2\sin\alpha = \mu_1 W_1\cos\alpha + \mu_2 W_2\cos\alpha$$

$$\Rightarrow \quad (W_1 + W_2)\sin\alpha = (\mu_1 W_1 + \mu_2 W_2)\cos\alpha$$

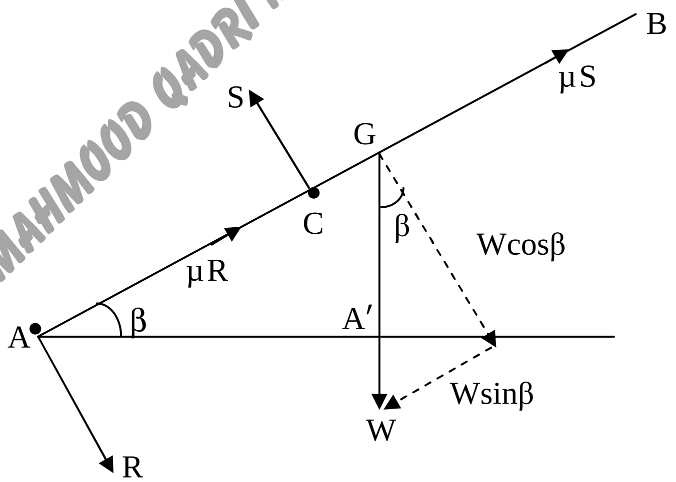
$$\Rightarrow \quad \tan\alpha = \frac{\mu_1 W_1 + \mu_2 W_2}{W_1 + W_2} \qquad \Rightarrow \qquad \alpha = \tan^{-1}\left(\frac{\mu_1 W_1 + \mu_2 W_2}{W_1 + W_2}\right)$$

❖ QUESTION 16

A thin uniform rod passes over a peg and under another, the coefficient of friction between each peg and the rod being μ. The distance between the pegs is a, and the straight line joining them makes an angle β with the horizon. show that the equilibrium is not possible unless the length of the rod is greater than

$$\frac{\mathbf{a}}{\mu}(\mu + \tan\beta)$$

SOLUTION



Let AB be a rod of length 2L makes an angle of β with horizon. W is its weight acting downward from its midpoint G. R and S are normal reactions of pegs at A and C respectively. μR and μS are forces of friction.

Since AB = 2L Therefore AG = BG = L.

For limiting equilibrium,

Horizontally

$$\mu R + \mu S = W\sin\beta \qquad \text{\_\_\_\_\_\_\_ (i)}$$

Vertically

$$S = R + W\cos\beta$$

$\Rightarrow \quad R = S - W\cos\beta \qquad$ _______ (ii)

From (i) & (ii), we get

$$\mu(S - W\cos\beta) + \mu S = W\sin\beta \quad \Rightarrow \quad \mu S - \mu W\cos\beta + \mu S = W\sin\beta$$

$\Rightarrow \quad 2\mu S - \mu W\cos\beta = W\sin\beta$

$\Rightarrow \quad 2\mu S = \mu W\cos\beta + W\sin\beta$

$\Rightarrow \quad S = \dfrac{W}{2\mu}(\mu\cos\beta + \sin\beta) \qquad$ _____ (iii)

Taking moment of all forces about A

$$S(AC) = W(AA') \quad \Rightarrow \quad S(a) = W(AG\cos\beta) \quad \Rightarrow \quad S(a) = W\,L\cos\beta$$

$\Rightarrow \quad S = \dfrac{WL\cos\beta}{a} \qquad$ _____ (iv)

From (iii) & (iv), we get

$$\frac{W}{2\mu}(\mu\cos\beta + \sin\beta) = \frac{WL\cos\beta}{a}$$

$\Rightarrow \quad a(\mu\cos\beta + \sin\beta) = 2\mu L\cos\beta$

$\Rightarrow \quad a(\mu + \tan\beta) = 2\mu L \qquad$ By dividing both sides by $\cos\beta$

$\Rightarrow \quad 2L = \dfrac{a}{\mu}(\tan\beta + \mu)$

$\Rightarrow \quad \text{Length of rod} = \dfrac{a}{\mu}(\tan\beta + \mu)$

Which gives the least length of the rod.
For equilibrium,

$$2L > \frac{a}{\mu}(\tan\beta + \mu)$$

or $\quad \text{Length of rod} > \dfrac{a}{\mu}(\tan\beta + \mu)$

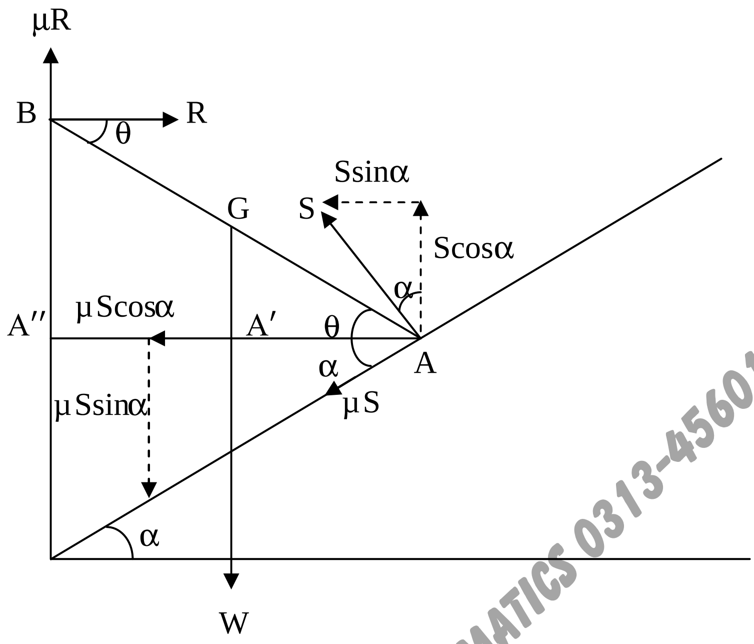
❖ QUESTION 17

A uniform rod slides with its ends on two fixed equally rough rods, one being vertical and the other inclined at an angle α to the horizontal. Show that the angle θ to the horizontal of the movable rod, when it is on the point of sliding, is given by

$$\mathbf{\tan\theta = \frac{1 - 2\mu\tan\alpha - \mu^2}{2(\mu + \tan\alpha)}}$$

SOLUTION

Let AB be a rod of length 2L makes an angle α with horizon. W is its weight acting downward from its midpoint G. R and S are normal reactions. μR and μS are forces of friction. Since AB = 2L Therefore AG = BG = L.



For limiting equilibrium,

Horizontally

$$R = \mu S\cos\alpha + S\sin\alpha \qquad \_\_\_\_\_\_\_\_ \text{(i)}$$

Vertically

$$\mu R + S\cos\alpha = W + \mu S\sin\alpha \qquad \_\_\_\_\_\_\_\_ \text{(ii)}$$

From (i) & (ii), we get

$$\mu(\mu S\cos\alpha + S\sin\alpha) + S\cos\alpha = W + \mu S\sin\alpha$$

$$\Rightarrow \quad \mu^2 S\cos\alpha + \mu S\sin\alpha + S\cos\alpha = W + \mu S\sin\alpha$$

$$\Rightarrow \quad W = S(\mu^2\cos\alpha + \cos\alpha) \qquad \_\_\_\_\_\_\_\_ \text{(iii)}$$

Taking moment of all forces about A

$$W(AA') = R(BA'') + \mu R(AA'')$$

$$\Rightarrow \quad W(AG\cos\theta) = R(AB\sin\theta) + \mu R(AB\cos\theta)$$

$$\Rightarrow \quad WL\cos\theta = 2LR\sin\theta + 2\mu RL\cos\theta$$

$$\Rightarrow \quad W\cos\theta = 2R\sin\theta + 2\mu R\cos\theta$$

$$\Rightarrow \quad W = 2R\tan\theta + 2\mu R \qquad \text{By dividing both sides by } \cos\theta$$

$$\Rightarrow \quad W = R(2\tan\theta + 2\mu) \qquad \_\_\_\_\_\_\_\_ \text{(iv)}$$

From (i) and (iv), we get

$$W = (\mu S\cos\alpha + S\sin\alpha)(2\tan\theta + 2\mu)$$

$\Rightarrow \quad W = S(\mu\cos\alpha + \sin\alpha)(2\tan\theta + 2\mu)$ ________ (v)

From (iii) and (v), we get

$$S(\mu^2\cos\alpha + \cos\alpha) = S(\mu\cos\alpha + \sin\alpha)(2\tan\theta + 2\mu)$$

$\Rightarrow \quad \mu^2\cos\alpha + \cos\alpha = (\mu\cos\alpha + \sin\alpha)(2\tan\theta + 2\mu)$

$\Rightarrow \quad \mu^2\cos\alpha + \cos\alpha = \tan\theta(2\mu\cos\alpha + 2\sin\alpha) + 2\mu^2\cos\alpha + 2\mu\sin\alpha$

$\Rightarrow \quad \tan\theta(2\mu\cos\alpha + 2\sin\alpha) = \mu^2\cos\alpha + \cos\alpha - 2\mu^2\cos\alpha - 2\mu\sin\alpha$

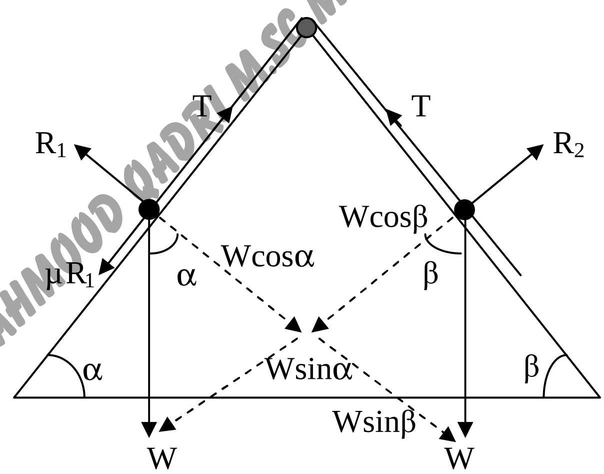
$\Rightarrow \quad \tan\theta = \dfrac{\cos\alpha - 2\mu\sin\alpha - \mu^2\cos\alpha}{2\sin\alpha + 2\mu\cos\alpha}$

$\Rightarrow \quad \tan\theta = \dfrac{1 - 2\mu\tan\alpha - \mu^2}{2(\mu + \tan\alpha)}$

QUESTION 18

Two inclined planes have a common vertex, and a string, passing over a small smooth pulley at the vertex, supports two equal weights. If one of the plane be rough and the other is smooth, find the relation between the inclinations of the planes when the weight on the smooth plane is on the point of moving down.

SOLUTION



Let W be the weight of either particle, α and β be the inclinations of the planes and R_1 and R_2 are normal reactions of planes. μR_1 is force of friction and T is tension in string.

First Particle: For limiting equilibrium,

<u>Horizontally</u>

$T = \mu R_1 + W\sin\alpha$ ________ (i)

<u>Vertically</u>

$R_1 = W\cos\alpha$ ________ (ii)

From (i) & (ii), we get

$$T = \mu W\cos\alpha + W\sin\alpha$$

$\Rightarrow$ $T = W(\mu\cos\alpha + \sin\alpha)$ _______ (iii)

Second Particle: For limiting equilibrium,

Horizontally

$$T = W\sin\beta \qquad \text{\_\_\_\_\_\_\_ (iv)}$$

From (iii) & (iv), we get

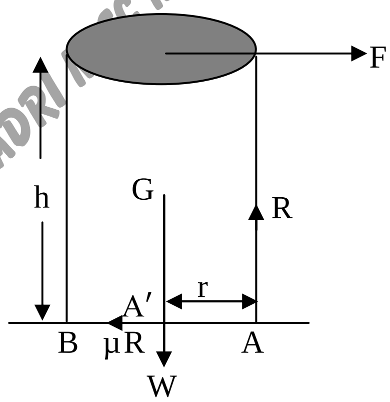
$$W\sin\beta = W(\mu\cos\alpha + \sin\alpha)$$

$\Rightarrow$ $\sin\beta = \sin\alpha + \mu\cos\alpha$ Which is required.

❖ QUESTION 19

A solid cylinder rests on a rough horizontal plane with one of its flat ends on the plane, and is acted on by a horizontal force through the centre of its upper end. If this force be just sufficient to move the solid, show that it will slide, and is not topple over, if the coefficient of friction be less than the ratio of the radius of the base of the cylinder to its height.

SOLUTION



Let r be radius of the base of the cylinder and h be the height of the cylinder. W be the weight acting downward, R is normal reaction and μR is its force of friction. Let F be a force at the top of cylinder.

For limiting equilibrium,

Horizontally

$F = \mu R$ _______ (i)

Vertically

$$R = W$$

Using value of R in (i), we get

$F = \mu W$ _______ (ii)

Taking moment of all forces about A

$$W(AA') = P(h) \quad \Rightarrow \quad W(r) = F(h)$$

$\Rightarrow \quad W(r) = \mu Wh \qquad$ By (ii)

$\Rightarrow \quad r = \mu h$

$\Rightarrow \quad \mu = \dfrac{r}{h}$

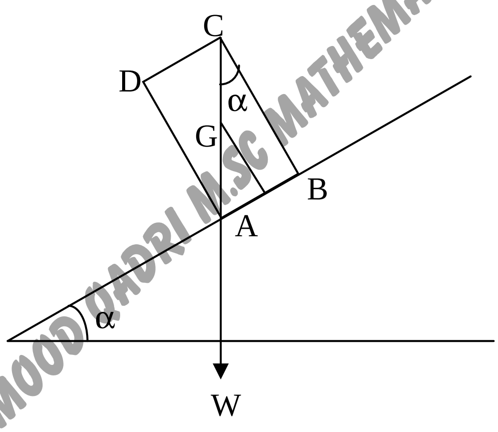
Which shows that the cylinder will slide and not topple over if

$$\mu < \frac{r}{h}$$

❖ QUESTION 20

A uniform rectangular block of height h whose base is a square of side a, rests on a rough horizontal plane. The plane is gradually tilted about a line parallel to two edges of the base. Show that the block will slide or topple over according as $a \gtrless \mu h$, where μ is the coefficient of friction.

SOLUTION



Let α be inclination of the plane to the horizon when the block is on the point of toppling over. The vertical through G must fall just within the base of length a as shown in figure.
From fig.

$$\tan\alpha = \frac{AB}{BC} = \frac{a}{h}$$

Also the inclination θ of the plane to the horizontal when the block is about to slide is given by

$$\tan\theta = \mu$$

The block will slide or topple over accordingly as

$$\theta \lesseqgtr \alpha$$

$\Rightarrow \quad \tan\theta \lesseqgtr \tan\alpha$

$\Rightarrow \quad \mu \lesseqgtr \dfrac{a}{h} \quad \Rightarrow \quad a \gtreqless \mu h$

Which is required.

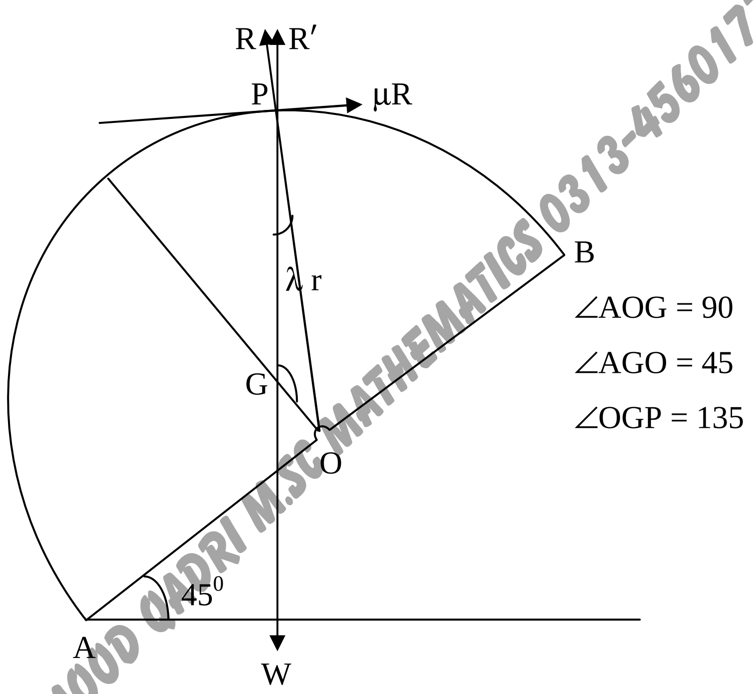
❖ QUESTION 21

A uniform semi-circular wire hangs on a rough peg, the line joining its extremities making an angle of 45^0 with the horizontal. If it is just on the point of slipping, find the coefficient of friction between the wire and the pegs.

SOLUTION

Let P be the peg and APB be the semi-circular wire with centre at O and radius r. Then OP = r. Let G be the centre of gravity. Then

$$OG = \frac{2r}{\pi}$$



From fig.

$$\angle OGP = 135^0 \text{ and } \angle OPG = \lambda$$

In triangle OGP,

$$\frac{OG}{\sin\lambda} = \frac{OP}{\sin 135} \qquad \text{By Law of Sine}$$

$$\Rightarrow \quad \frac{2r/\pi}{\sin\lambda} = \frac{r}{1/\sqrt{2}}$$

$$\Rightarrow \quad \frac{2}{\pi\sin\lambda} = \sqrt{2} \quad \Rightarrow \quad \sin\lambda = \frac{2}{\pi\sqrt{2}}$$

$$\Rightarrow \quad \sin\lambda = \frac{\sqrt{2}}{\pi}$$

$$\Rightarrow \quad \frac{\sin\lambda}{\cos\lambda} = \frac{\sqrt{2}}{\pi\cos\lambda} \qquad \text{Dividing both sides by } \cos\lambda$$

$\Rightarrow \quad \tan\lambda = \dfrac{\sqrt{2}}{\pi}\sec\lambda$

$\Rightarrow \quad \tan\lambda = \dfrac{\sqrt{2}}{\pi}\sqrt{1+\tan^2\lambda} \qquad \because \; 1+\tan^2\lambda = \sec^2\lambda$

$\Rightarrow \quad \mu = \dfrac{\sqrt{2}}{\pi}\sqrt{1+\mu^2} \qquad \because \; \tan\lambda = \mu$

$\Rightarrow \quad \mu^2 = \dfrac{2}{\pi^2}(1+\mu^2)$

$\Rightarrow \quad \mu^2\pi^2 = 2(1+\mu^2)$

$\Rightarrow \quad \mu^2\pi^2 - 2\mu^2 = 2 \Rightarrow \quad \mu^2(\pi^2 - 2) = 2$

$\Rightarrow \quad \mu = \sqrt{\dfrac{2}{\pi^2 - 2}} = 0.504$

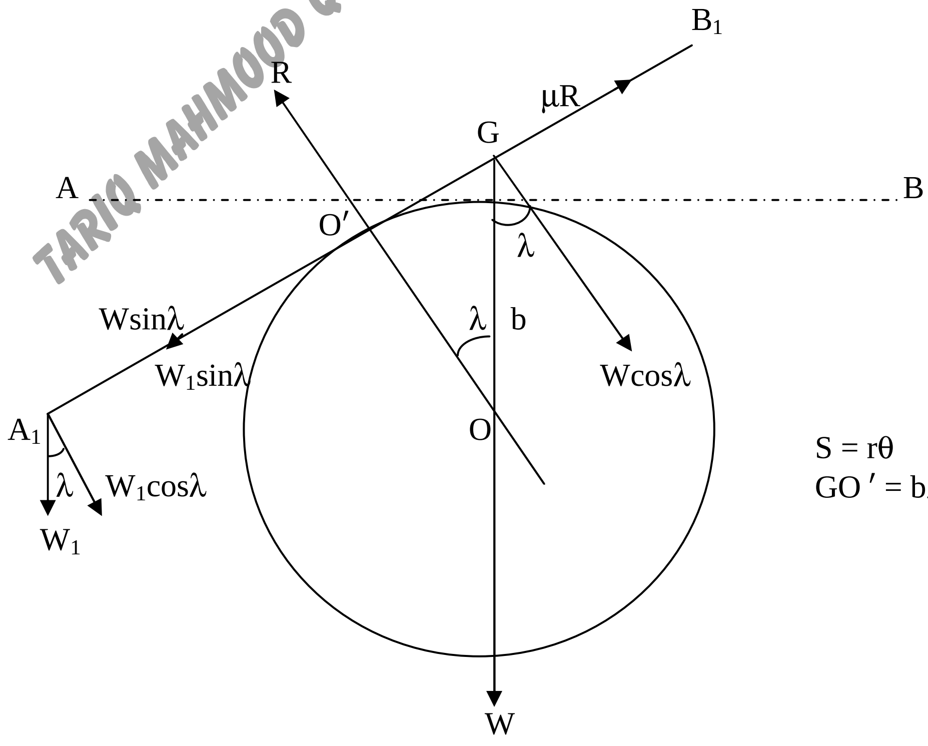
❖ QUESTION 22

A uniform rod of length 2a and weight W rests with its middle point upon a rough horizontal cylinder whose axis is perpendicular to the rod. show that the greatest weight that can be attached to one end of the rod without slipping it off the cylinder is

$$\frac{\mathbf{b\lambda}}{\mathbf{a - b\lambda}}\mathbf{W}$$

Where b is the radius of the cylinder and λ is the angle of friction.

SOLUTION



S = rθ
GO ′ = bλ

Let AB be the rod of the length 2a, weight W and b be the radius of the cylinder. Let R is normal reaction and μR is force of friction. Let W_1 is the weight attached to one end of the rod. A_1 and B_1 are new positions of the rod in the limiting equilibrium.

Since AB = 2a Therefore AG = a

Taking moment of all forces about O′

$$W_1\cos\lambda(A_1O') = W\cos\lambda(GO')$$

$$\Rightarrow \quad W_1(A_1O') = W(GO')$$

From fig.

$$GO' = b\lambda \quad \text{and} \quad A_1O' = AG - GO' = a - b\lambda$$

$$\Rightarrow \quad W_1(a - b\lambda) = W(b\lambda)$$

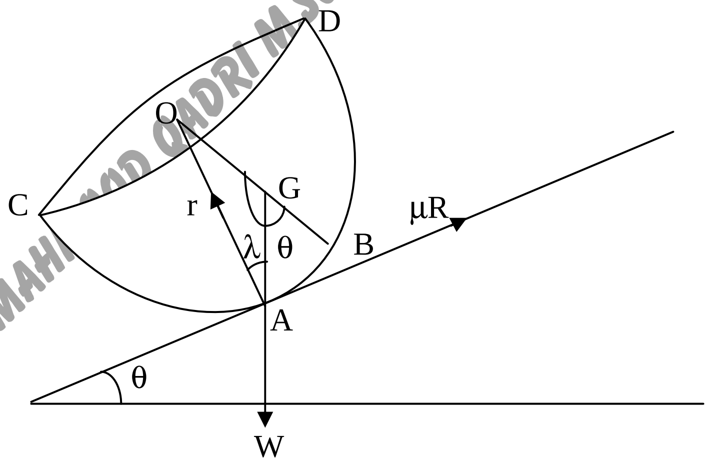
$$\Rightarrow \quad W_1 = \frac{b\lambda}{a - b\lambda}W$$

❖ QUESTION 23

A hemispherical shell rests on a rough inclined plane whose angle of friction is λ. Show that the inclination of the plane base to the horizontal cannot be greater than

$$\sin^{-1}(2\sin\lambda)$$

SOLUTION



Let G be the centre of the gravity of hemispherical shell, W is its weight acting downward from G. Let θ be angle that the plane base makes with the horizontal then we have to show that

$$\theta = \sin^{-1}(2\sin\lambda)$$

From fig.

$$OA = r \text{ (radius)}$$

$$OG = \frac{r}{2}$$

$$\angle OAG = \lambda$$

$\angle AGB = \theta$

$\angle AGO = \pi - \theta$

In triangle OAG,

$$\frac{OG}{\sin\lambda} = \frac{OA}{\sin(\pi - \theta)}$$

$\Rightarrow \quad \dfrac{r/2}{\sin\lambda} = \dfrac{r}{\sin\theta} \qquad \because \ \sin(\pi - \theta) = \sin\theta$

$\Rightarrow \quad \sin\theta = 2\sin\lambda$

$\Rightarrow \quad \theta = \sin^{-1}(2\sin\lambda)$

%%%%%% End of The Chapter # 2 %%%%%%