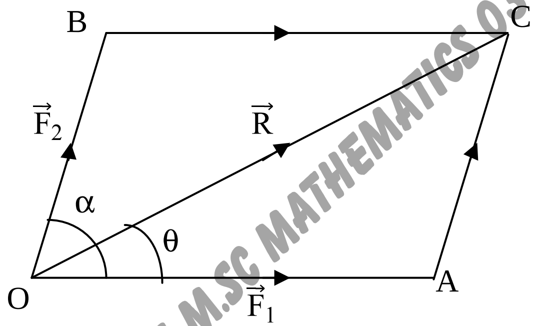
COMPOSITION OF FORCES

1

CHAPTER

❖ Resultant of Two Forces Acting at a Point

Let $\vec{F}_1$ and $\vec{F}_2$ be the two forces acting at a point O. Let α be a angle between $\vec{F}_1$ and $\vec{F}_2$. Let $\vec{R}$ be their resultant which makes an angle θ with $\vec{F}_1$.



Then $\quad \vec{R} = \vec{F}_1 + \vec{F}_2$ __________(i)

Taking dot product with itself, we get

$$\vec{R}.\vec{R} = \left(\vec{F}_1 + \vec{F}_2\right).\left(\vec{F}_1 + \vec{F}_2\right)$$

$$\Rightarrow \quad R^2 = \vec{F}_1.\vec{F}_1 + \vec{F}_1.\vec{F}_2 + \vec{F}_2.\vec{F}_1 + \vec{F}_2.\vec{F}_2$$

$$= F_1{}^2 + 2\vec{F}_1.\vec{F}_2 + F_2{}^2$$

$$= F_1{}^2 + 2F_1F_2\cos\alpha + F_2{}^2$$

$$\Rightarrow \quad R = \sqrt{F_1{}^2 + F_2{}^2 + 2F_1F_2\cos\alpha} \qquad \text{\_\_\_\_\_\_\_\_\_\_(ii)}$$

Which gives the magnitude of the resultant.

Taking dot product of the eq(i) with $\vec{F}_1$, we get

$$\vec{F}_1.\vec{R} = \vec{F}_1.\left(\vec{F}_1 + \vec{F}_2\right)$$

$$\Rightarrow \quad F_1R\cos\theta = F_1{}^2 + \vec{F}_1.\vec{F}_2$$

$$\Rightarrow \quad F_1R\cos\theta = F_1{}^2 + F_1F_2\cos\alpha$$

$$\Rightarrow \quad R\cos\theta = F_1 + F_2\cos\alpha \qquad \text{\_\_\_\_\_\_\_\_\_\_(iii)}$$

Taking cross product of the eq(i) with $\vec{F}_1$, we get

$$\left|\vec{F}_1 \times \vec{R}\right| = \left|\vec{F}_1 \times \left(\vec{F}_1 + \vec{F}_2\right)\right|$$

$$\Rightarrow \quad F_1 R \sin\theta = \left|\vec{F}_1 \times \vec{F}_1 + \vec{F}_1 \times \vec{F}_2\right|$$

$$\Rightarrow \quad F_1 R \sin\theta = \left|\vec{F}_1 \times \vec{F}_2\right| \qquad \because \vec{F}_1 \times \vec{F}_1 = 0$$

$$\Rightarrow \quad R \sin\theta = F_2 \sin\theta \qquad \text{\_\_\_\_\_\_\_\_\_\_(iv)}$$

Dividing eq(iii) by eq(iv), we get

$$\frac{R\sin\theta}{R\cos\theta} = \frac{F_2\sin\theta}{F_1 + F_2\cos\alpha}$$

$$\Rightarrow \quad \tan\theta = \frac{F_2\sin\theta}{F_1 + F_2\cos\alpha}$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{F_2\sin\theta}{F_1 + F_2\cos\alpha}\right) \qquad \text{\_\_\_\_\_\_\_\_\_\_(v)}$$

Which gives the direction of the resultant.

❖ Special Cases

Now we discuss some special cases of the above article.

❖ *Case # 1*

From eq(ii)

$$R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos\alpha}$$

Which shows that R is maximum when $\cos\alpha$ is maximum. But the maximum value of $\cos\alpha$ is 1. i.e. $\cos\alpha = 1 \Rightarrow \alpha = 0^0$.

Thus $\quad R_{max} = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos 0^0} = \sqrt{F_1^2 + F_2^2 + 2F_1F_2}$

$$= \sqrt{(F_1 + F_2)^2} = F_1 + F_2$$

❖ *Case # 2*

From eq(ii)

$$R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos\alpha}$$

When shows that R is minimum when $\cos\alpha$ is minimum. But the minimum value of $\cos\alpha$ is -1. i.e. $\cos\alpha = -1 \Rightarrow \alpha = \pi$.

Thus $\quad R_{min} = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos\pi} = \sqrt{F_1^2 + F_2^2 + 2F_1F_2(-1)}$

$$= \sqrt{F_1{}^2 - 2F_1F_2 + F_2{}^2} = \sqrt{(F_1 - F_2)^2} = F_1 - F_2$$

❖ *Case # 3*

From eq(ii)

$$R = \sqrt{F_1{}^2 + F_2{}^2 + 2F_1F_2\cos\alpha}$$

When $\vec{F}_1$ and $\vec{F}_2$ are perpendicular to each other. i.e. $\alpha = 90^o$

Then $R = \sqrt{F_1{}^2 + F_2{}^2 + 2F_1F_2\cos 90^0}$

$$= \sqrt{F_1{}^2 + F_2{}^2} \qquad \because \cos 90^0 = 0$$

From (v)

$$\theta = \tan^{-1}\left(\frac{F_2\sin 90^0}{F_1 + F_2\cos 90^0}\right)$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{F_2}{F_1}\right) \qquad \because \sin 90^0 = 1$$

❖ QUESTION 1

The greatest resultant that two forces can have is of magnitude P and the least is of magnitude Q. Show that when they act at an angle α their resultant is of magnitude:

$$\sqrt{P^2\cos^2\frac{\alpha}{2} + Q^2\sin^2\frac{\alpha}{2}}$$

SOLUTION

Let $\vec{F}_1$ and $\vec{F}_2$ be two forces and P & Q be magnitude of their greatest and least resultant respectively. Then

$$P = F_1 + F_2 \qquad \text{\_\_\_\_\_\_\_\_(i)}$$

and $$Q = F_1 - F_2 \qquad \text{\_\_\_\_\_\_\_\_(ii)}$$

Adding (i) and (ii),we get

$$2F_1 = P + Q$$

$$\Rightarrow \quad F_1 = \frac{P + Q}{2}$$

Subtracting (ii) from (i), we get

$$2F_2 = P - Q$$

$$\Rightarrow \quad F_2 = \frac{P - Q}{2}$$

Let $\vec{R}$ be the resultant of $\vec{F}_1$ and $\vec{F}_2$ when they act an angle α. Then

$$R = \sqrt{F_1^2 + 2F_1F_2\cos\alpha + F_2^2}$$
$$= \sqrt{\left(\frac{P+Q}{2}\right)^2 + 2\left(\frac{P+Q}{2}\right)\left(\frac{P-Q}{2}\right)\cos\alpha + \left(\frac{P-Q}{2}\right)^2}$$
$$= \sqrt{\frac{(P+Q)^2}{4} + 2\left(\frac{P^2-Q^2}{4}\right)\cos\alpha + \frac{(P-Q)^2}{4}}$$
$$= \sqrt{\frac{1}{4}\left[(P+Q)^2 + (P-Q)^2 + 2\left(P^2-Q^2\right)\cos\alpha\right]}$$
$$= \sqrt{\frac{1}{4}\left[2\left(P^2+Q^2\right) + 2\left(P^2-Q^2\right)\cos\alpha\right]}$$
$$= \sqrt{\frac{1}{2}\left[P^2 + Q^2 + P^2\cos\alpha - Q^2\cos\alpha\right]}$$
$$= \sqrt{\frac{1}{2}\left[P^2(1+\cos\alpha) + Q^2(1-\cos\alpha)\right]}$$
$$= \sqrt{\frac{1}{2}\left[P^2 2\cos^2\frac{\alpha}{2} + Q^2 2\cos^2\frac{\alpha}{2}\right]}$$
$$= \sqrt{P^2\cos^2\frac{\alpha}{2} + Q^2\cos^2\frac{\alpha}{2}}$$

Which is required.

❖ QUESTION 2

The resultant of two forces of magnitude P and Q is of magnitude R. If Q is doubled then R is doubled. If Q is reversed then R is again doubled. Show that

$$\mathbf{P^2 : Q^2 : R^2 = 2 : 3 : 2}$$

SOLUTION

Let θ be angle between the forces P & Q. Since R is the magnitude of the resultant of P and Q therefore

$$R = \sqrt{P^2 + 2PQ\cos\alpha + Q^2}$$

$$\Rightarrow \quad R^2 = P^2 + 2PQ\cos\alpha + Q^2 \qquad \text{\_\_\_\_\_\_\_\_\_\_(i)}$$

Since when Q is double then R is double therefore by replacing Q with 2Q and R with 2R eq(i) becomes

$$(2R)^2 = P^2 + 2P(2Q)\cos\alpha + (2Q)^2$$

$\Rightarrow \quad 4R^2 = P^2 + 4PQ\cos\alpha + 4Q^2$ ____________(ii)

Since when Q is reversed then R is double therefore by replacing Q with $-$ Q and R with 2R eq(i) becomes

$$(2R)^2 = P^2 + 2P2(-Q)\cos\alpha + (-Q)^2$$

$\Rightarrow \quad 4R^2 = P^2 - 2PQ\cos\alpha + Q^2$ ____________(iii)

Multiplying (iii) by 2, we get

$8R^2 = 2P^2 - 4PQ\cos\alpha + 2Q^2$ ____________(iv)

Adding (i) and (iii), we get

$$5R^2 = 2P^2 + 2Q^2$$

$\Rightarrow \quad 2P^2 + 2Q^2 - 5R^2 = 0$ ____________(v)

Adding (ii) and (iv), we get

$$12R^2 = 3P^2 + 6Q^2$$

$$\Rightarrow \quad 4R^2 = P^2 + 2Q^2$$

$\Rightarrow \quad P^2 + 2Q^2 - 4R^2 = 0$ ____________(vi)

Solving (v) and (vi) simultaneously, we get

$$\frac{P^2}{-8+10} = \frac{Q^2}{-5+8} = \frac{R^2}{4-2}$$

$$\Rightarrow \quad \frac{P^2}{2} = \frac{Q^2}{3} = \frac{R^2}{2} \quad \Rightarrow \quad P^2 : Q^2 : R^2 = 2 : 3 : 2$$

❖ THEOREM OF RESOLVED PARTS

The algebraic sum of the resolved parts of a system of forces in any direction is equal to the resolved part of the resultant in the same direction.

PROOF

Let $\vec{R}$ be the resultant of forces $\vec{F}_1, \vec{F}_2, \vec{F}_3, \ldots, \vec{F}_n$ and $\hat{a}$ be the unit vector in any direction which makes an angle α with $\vec{R}$ and $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ with $\vec{F}_1, \vec{F}_2, \vec{F}_3, \ldots, \vec{F}_n$ respectively.

Then $\quad \vec{R} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots + \vec{F}_n$ _________(i)

Taking dot product of (i) with $\hat{a}$, we get

$$\hat{a}\cdot\vec{R} = \hat{a}\cdot\left(\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots + \vec{F}_n\right)$$

$$\Rightarrow \quad \hat{a}\cdot\vec{R} = \hat{a}\cdot\vec{F}_1 + \hat{a}\cdot\vec{F}_2 + \hat{a}\cdot\vec{F}_3 + \ldots + \hat{a}\cdot\vec{F}_n$$

$\Rightarrow \quad R\cos\theta = F_1\cos\theta + F_2\cos\theta + F_3\cos\theta + \ldots + F_n\cos\theta$ _________(ii)

Similarly by taking cross product of (i) with $\hat{a}$, we get

$$R\sin\theta = F_1\sin\theta + F_2\sin\theta + F_3\sin\theta + \ldots + F_n\sin\theta \qquad \text{\_\_\_\_\_\_\_\_(iii)}$$

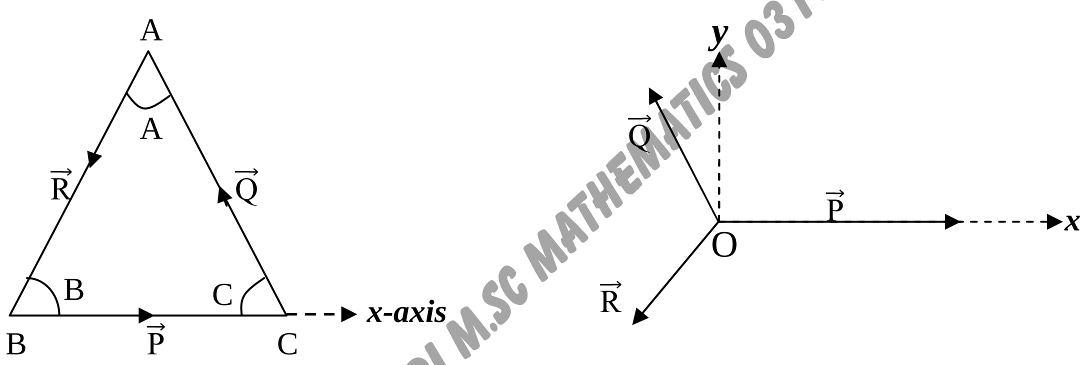
Eq(ii) and eq(iii) shows that the sum of the resolved parts of a system of forces in any direction is equal to the resolved part of the resultant in the same direction.

❖ QUESTION 3

Forces $\vec{P}$, $\vec{Q}$ and $\vec{R}$ act at a point parallel to the sides of a triangle ABC taken in the same order. Show that the magnitude of Resultant is

$$\sqrt{P^2 + Q^2 + R^2 - 2PQ\cos C - 2QR\cos A - 2RP\cos B}$$

SOLUTION



Let the forces $\vec{P}$, $\vec{Q}$ and $\vec{R}$ act along the sides BC, CA and AB of a triangle ABC taking one way round as shown in the figure. We take BC along x-axis.
Let the F be the magnitude of the resultant, then

$$F = \sqrt{F_x^{\,2} + F_y^{\,2}} \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_(i)}$$

Now by theorem of resolved parts

F_x = Sum of the resolved parts of the forces along x-axis

$= P\cos 0 + Q\cos(180 - C) + R\cos(180 + B)$

$= P - Q\cos C - R\cos B$

Taking square on both sides, we get

$$F_x^{\,2} = \left(P - Q\cos C - R\cos B\right)^2$$

$$= P^2 + Q^2\cos^2 C + R^2\cos^2 B - 2PQ\cos C + 2QR\cos B\cos C - 2PR\cos B$$

Again by theorem of resolved parts

F_y = Sum of the resolved parts of the forces along y-axis

$= P\sin 0 + Q\sin(180 - C) + R\sin(180 + B)$

$= Q\sin C - R\sin B$

Taking square on both sides, we get

$$F_y^2 = (Q\sin C - R\sin B)^2$$

$$= Q^2\sin^2 C + R^2\sin^2 B - 2QR\sin B\sin C$$

Using values of R_x^2 and R_y^2 in (i), we get

$$F = \sqrt{P^2 + Q^2\cos^2 C + R^2\cos^2 B - 2PQ\cos C + 2QR\cos B\cos C - 2PR\cos B + Q^2\sin^2 C + R^2\sin^2 B - 2QR\sin B\sin C}$$

$$= \sqrt{P^2 + Q^2(\cos^2 C + \sin^2 C) + R^2(\cos^2 B + \sin^2 B) - 2PQ\cos C - 2PR\cos B\ 2QR(\cos B\cos C - \sin B\sin C)}$$

$$= \sqrt{P^2 + Q^2 + R^2 - 2PQ\cos C - 2PR\cos B + 2QR\cos(B + C)}$$

Since $A + B + C = 180^0 \Rightarrow B + C = 180^0 - A$

$$\Rightarrow \quad R = \sqrt{P^2 + Q^2 + R^2 - 2PQ\cos C - 2PR\cos B + 2QR\cos(180^0 - A)}$$

$$= \sqrt{P^2 + Q^2 + R^2 - 2PQ\cos C - 2PR\cos B - 2QR\cos A}$$

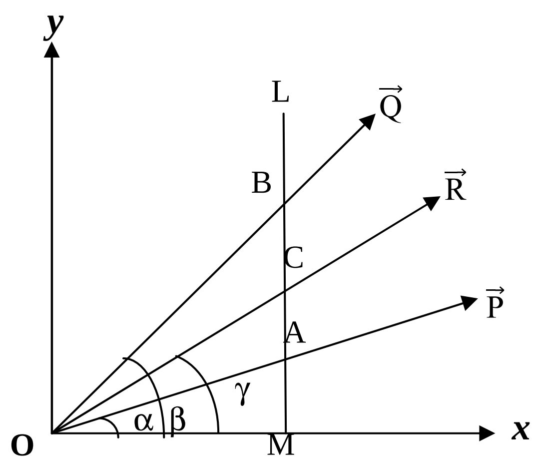
❖ QUESTION 4

Forces $\vec{P}$ and $\vec{Q}$ act at a point O and their resultant is $\vec{R}$. If any transversal cuts the lines of action of forces in the points A, B & C respectively. Prove that

$$\frac{\mathbf{R}}{\mathbf{OC}} = \frac{\mathbf{P}}{\mathbf{OA}} + \frac{\mathbf{Q}}{\mathbf{OB}}$$

SOLUTION

Let $\vec{P}$, $\vec{Q}$ and $\vec{R}$ makes angles α, β and γ with x-axis respectively. The transversal LM cuts the lines of action of the forces at points A, B and C respectively as shown in the figure.



Since $\vec{R}$ is resultant of $\vec{P}$ & $\vec{Q}$ therefore by theorem of resolved parts

$$R\cos\gamma = P\cos\alpha + Q\cos\beta \qquad \text{\_\_\_\_\_\_\_\_(i)}$$

From figure

$$\cos\alpha = \frac{OM}{OA}$$

$$\cos\beta = \frac{OM}{OB} \quad \text{and} \quad \cos\gamma = \frac{OM}{OC}$$

Using These values in (i), we get

$$R\left(\frac{OM}{OC}\right) = P\left(\frac{OM}{OA}\right) + Q\left(\frac{OM}{OB}\right)$$

$$\Rightarrow \quad \frac{R}{OC} = \frac{P}{OA} + \frac{Q}{OB}$$

Which is required.

❖ QUESTION 5

If two forces P and Q act at such an angle that their resultant R = P. Show that if P is doubled, the new resultant is at right angle to Q.

SOLUTION

Let the forces P & Q act at O and makes an angle α with each other. Take Q along x-axis. If R is the magnitude of the resultant then

$$R^2 = P^2 + 2PQ\cos\alpha + Q^2$$

Since R = P therefore

$$P^2 = P^2 + 2PQ\cos\alpha + Q^2$$

$$\Rightarrow \quad 2PQ\cos\alpha + Q^2 = 0$$

$$\Rightarrow \quad 2P\cos\alpha + Q = 0 \qquad \text{\_\_\_\_\_\_\_\_\_\_(i)}$$

By theorem of resolved parts

$$R_x = \text{Sum of the resolved parts of the forces along x-axis}$$

$$= Q\cos 0^0 + P\cos\alpha$$

$$= Q + P\cos\alpha$$

Again by theorem of resolved parts

$$R_y = \text{Sum of the resolved parts of the forces along y-axis}$$

$$= Q\sin 0^0 + P\sin\alpha = P\sin\alpha$$

If P is double. i.e. P = 2P then

$$R_x = Q + 2P\cos\alpha = 0 \qquad \text{By (i)}$$

And $\quad R_y = 2P\sin\alpha$

If the new resultant makes an angle θ with Q then

$$\theta = \tan^{-1}\left(\frac{R_y}{R_x}\right) = \tan^{-1}\left(\frac{2P\sin\alpha}{0}\right)$$

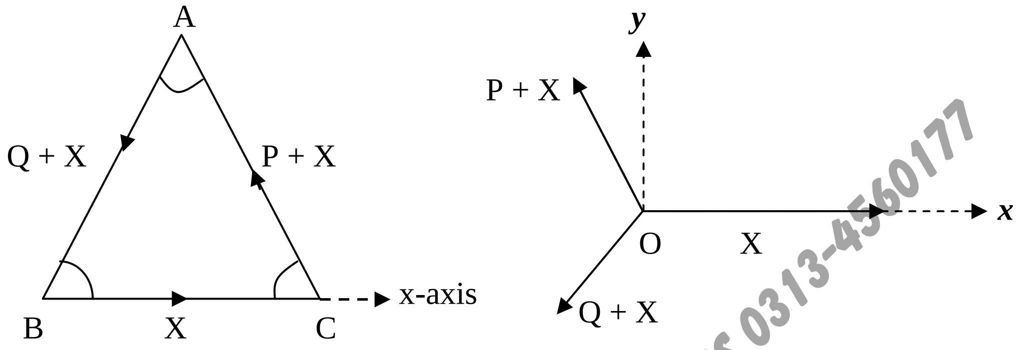
$$= \tan^{-1}(\infty) = \frac{\pi}{2}$$

Which is required.

❖ QUESTION 6

Forces X, P + X and Q + X act at a point in the directions of sides of a equilateral triangle taken one way round. Show that they are equivalent to the forces P & Q acting at an angle of 120^0.

SOLUTION



Let the forces X, X + P and X + Q act along the sides BC, CA and AB of a triangle ABC taking one way round as shown in the figure. We take BC along x-axis.

Let the R be the magnitude if the resultant, then

$$R = \sqrt{R_x^{\,2} + R_y^{\,2}} \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_(i)}$$

Now by theorem of resolved parts

R_x = Sum of the resolved parts of the forces along x-axis

$$= X\cos 0 + (P + X)\cos(180^0 - 60^0) + (Q + X)\cos(180^0 + 60^0)$$

$$= X - (P + X)\cos 60^0 - (Q + X)\cos 60^0$$

$$= X - \frac{1}{2}(P + X) - \frac{1}{2}(Q + X)$$

$$= X - \frac{1}{2}P - \frac{1}{2}X - \frac{1}{2}Q - \frac{1}{2}X$$

$$= -\frac{1}{2}(P + Q)$$

Taking square on both sides, we get

$$R_x^{\,2} = \left(-\frac{1}{2}(P + Q)\right)^2 = \frac{1}{4}\left(P^2 + Q^2 + 2PQ\right)$$

Again by theorem of resolved parts

R_y = Sum of the resolved parts of the forces along y-axis

$$= X\sin 0^0 + (P + X)\sin(180^0 - 60^0) + (Q + X)\sin(180^0 + 60^0)$$

$$= (P + X)\sin 60^0 - (Q + X)\sin 60^0$$

$$= \frac{\sqrt{3}}{2}(P + X) - \frac{\sqrt{3}}{2}(Q + X) = \frac{\sqrt{3}}{2}P + \frac{\sqrt{3}}{2}X - \frac{\sqrt{3}}{2}Q - \frac{\sqrt{3}}{2}X$$

$$= \frac{\sqrt{3}}{2}(P - Q)$$

Taking square on both sides, we get

$$R_y^{\ 2} = \left(\frac{\sqrt{3}}{2}(P - Q)\right)^2 = \frac{3}{4}\left(P^2 + Q^2 - 2PQ\right)$$

Using values of $R_x^{\ 2}$ and $R_y^{\ 2}$ in (i), we get

$$R = \sqrt{\frac{1}{4}\left(P^2 + Q^2 + 2PQ\right) + \frac{3}{4}\left(P^2 + Q^2 - 2PQ\right)}$$

$$= \sqrt{\frac{P^2 + Q^2 + 2PQ + 3P^2 + 3Q^2 - 6PQ}{4}} = \sqrt{\frac{4P^2 + 4Q^2 - 4PQ}{4}}$$

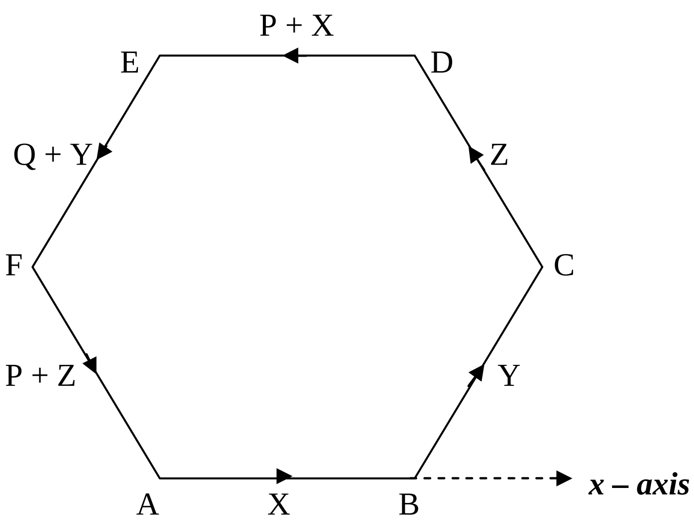
$$= \sqrt{P^2 + Q^2 - PQ} = \sqrt{P^2 + Q^2 + 2PQ\left(-\frac{1}{2}\right)} = \sqrt{P^2 + Q^2 + 2PQ\cos 120^0}$$

This result shows that the given forces are equivalent to the forces P and Q acting an angle of 120^0.

❖ QUESTION 7

Forces X, Y, Z, P + X, Q + Y and P + Z act at a point in the directions of sides of a regular hexagon taken one way round. Show that their resultant is equivalent to the force P + Q in the direction of the force Q + Y.

SOLUTION



Let the force X, Y, Z, P + X, Q + Y and P + Z act along the sides AB, BC, CD, DE, EF, FA of a regular hexagon taken one way round as shown in figure. Take AB along x-axis.

Let the R be the magnitude if the resultant and R_x and R_y be the resolved parts of the resultant, then

$$R = \sqrt{R_x^2 + R_y^2} \qquad \text{\_\_\_\_\_\_\_\_\_\_(i)}$$

Now by theorem of resolved parts

R_x = Sum of the resolved parts of the forces along x-axis

$$= X\cos 0^0 + Y\cos 60^0 + Z\cos 120^0 + (P + X)\cos 180^0 + (Q + Y)\cos 240^0 + (P + Z)\cos 300^0$$

$$= X + \frac{1}{2}Y - \frac{1}{2}Z - (P + X) - \frac{1}{2}(Q + Y) + \frac{1}{2}(P + Z)$$

$$= X + \frac{1}{2}Y - \frac{1}{2}Z - P - X - \frac{1}{2}Q - \frac{1}{2}Y + \frac{1}{2}P + \frac{1}{2}Z$$

$$= -\frac{1}{2}(P + Q)$$

Taking square on both sides, we get

$$R_x^2 = \left(-\frac{1}{2}(P + Q)\right)^2 = \frac{1}{4}(P + Q)^2$$

Again by theorem of resolved parts

R_y = Sum of the resolved parts of the forces along y-axis

$$= X\sin 0^0 + Y\sin 60^0 + Z\sin 120^0 + (P + X)\sin 180^0 + (Q + Y)\sin 240^0 + (P + Z)\sin 300^0$$

$$= 0 + \frac{\sqrt{3}}{2}Y + \frac{\sqrt{3}}{2}Z - 0\,(P + X) - \frac{\sqrt{3}}{2}(Q + Y) - \frac{\sqrt{3}}{2}(P + Z)$$

$$= \frac{\sqrt{3}}{2}Y + \frac{\sqrt{3}}{2}Z - \frac{\sqrt{3}}{2}Q - \frac{\sqrt{3}}{2}Y - \frac{\sqrt{3}}{2}P - \frac{\sqrt{3}}{2}Z$$

$$= -\frac{\sqrt{3}}{2}(P + Q)$$

Taking square on both sides, we get

$$R_y^2 = \left(-\frac{\sqrt{3}}{2}(P + Q)\right)^2 = \frac{3}{4}(P + Q)^2$$

Using values of R_x^2 and R_y^2 in (i), we get

$$R = \sqrt{\frac{1}{4}(P + Q)^2 + \frac{3}{4}(P + Q)^2} = \sqrt{(P + Q)^2}$$

$\Rightarrow \quad R = P + Q$

If resultant makes angle θ with x-axis then

$$\theta = \tan^{-1}\left(\frac{R_y}{R_x}\right)$$

$$= \tan^{-1}\left(\frac{-\frac{\sqrt{3}}{2}(P+Q)}{-\frac{1}{2}(P+Q)}\right)$$

$$= \tan^{-1}\left(\frac{-\sqrt{3}}{-1}\right) = 240^0$$

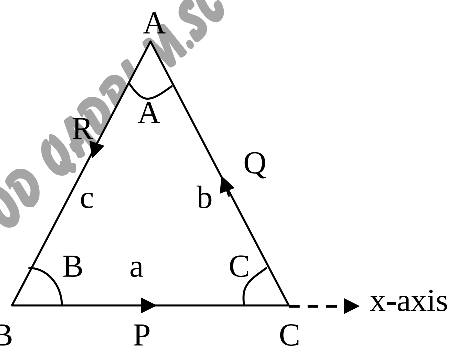
Which shows that the resultant is a force P + Q in the direction of Q + Y because Q + Y makes an angle 240 with x-axis.

❖ QUESTION 8

Forces P, Q and R act along the sides BC, CA, AB of a triangle ABC. Find the condition that their resultant is parallel to BC and determine its magnitude.

SOLUTION

Let the forces P, Q and R act along the sides BC, CA and AB of a triangle ABC taking one way round as shown in the figure. We take BC along x-axis.



Let the F be the magnitude of the resultant, then

$$F = \sqrt{F_x^{\,2} + F_y^{\,2}} \qquad \text{\_\_\_\_\_\_\_\_(i)}$$

Now by theorem of resolved parts

F_x = Sum of the resolved parts of the forces along x-axis

$= P\cos 0^0 + Q\cos(180 - C) + R\cos(180 + B)$

$= P - Q\cos C - R\cos B$

And $\quad F_y$ = Sum of the resolved parts of the forces along y-axis

$= P\sin 0^0 + Q\sin(180 - C) + R\sin(180 + B)$

$= Q\sin C - R\sin B$

If the resultant makes an angle θ with x – axis , then

$$\tan\theta = \frac{F_y}{F_x}$$

Since the resultant is parallel to BC therefore θ must be zero.

So $$\tan 0 = \frac{F_y}{F_x}$$

$$\Rightarrow \quad \frac{F_y}{F_x} = 0$$

$$\Rightarrow \quad F_y = 0$$

$$\Rightarrow \quad Q\sin C - R\sin B = 0$$

$$\Rightarrow \quad \frac{\sin C}{\sin B} = \frac{R}{Q} \qquad \text{\_\_\_\_\_\_\_\_(ii)}$$

Let a, b and c are the lengths of the sides BC, CA and AB respectively. Then by Law of sine

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\Rightarrow \quad \frac{\sin C}{\sin B} = \frac{c}{b} \qquad \text{\_\_\_\_\_\_\_\_(iii)}$$

From (ii) and (iii), we get

$$\frac{R}{Q} = \frac{c}{b}$$

$$\Rightarrow \quad R = Q\left(\frac{c}{b}\right) \qquad \text{\_\_\_\_\_\_\_\_(iv)}$$

$$\Rightarrow \quad Qc = Rb$$

Which is required condition.

Using values of F_x and F_y in (i), we get

$$F^2 = (P - Q\cos C - R\cos B)^2 + (Q\sin C - R\sin B)^2$$

$$= P^2 + Q^2\cos^2 C + R^2\cos^2 B - 2PQ\cos C + 2QR\cos B\cos C - 2PR\cos B$$
$$+ Q^2\sin^2 C + R^2\sin^2 B - 2QR\sin B\sin C$$

$$= P^2 + Q^2(\cos^2 C + \sin^2 C) + R^2(\cos^2 B + \sin^2 B) - 2PQ\cos C - 2PR\cos B$$
$$+ 2QR(\cos B\cos C - \sin B\sin C)$$

$$= P^2 + Q^2 + R^2 - 2PQ\cos C - 2PR\cos B + 2QR\cos(B + C)$$

$$= P^2 + Q^2 + R^2 - 2PQ\cos C - 2PR\cos B + 2QR\cos(180 - A) \quad \because A + B + C = 180$$

$$= P^2 + Q^2 + R^2 - 2PQ\cos C - 2PR\cos B - 2QR\cos A$$

$$= P^2 + Q^2 + \left(Q\left(\frac{c}{b}\right)\right)^2 - 2PQ\cos C - 2P\left(Q\left(\frac{c}{b}\right)\right)\cos B - 2Q\left(Q\left(\frac{c}{b}\right)\right)\cos A$$

$$= P^2 + Q^2 + Q^2\left(\frac{c}{b}\right)^2 - 2PQ\cos C - 2PQ\left(\frac{c}{b}\right)\cos B - 2Q^2\left(\frac{c}{b}\right)\cos A$$

$$= P^2 + Q^2\left(\frac{b^2}{b^2}\right) + Q^2\left(\frac{c^2}{b^2}\right) - 2PQ\cos C - 2PQ\left(\frac{c}{b}\right)\cos B - 2Q^2\left(\frac{bc}{b^2}\right)\cos A$$

$$= P^2 + Q^2\left(\frac{b^2}{b^2}\right) + Q^2\left(\frac{c^2}{b^2}\right) - 2Q^2\left(\frac{bc}{b^2}\right)\cos A - 2PQ\left(\frac{b}{b}\right)\cos C - 2PQ\left(\frac{c}{b}\right)\cos B$$

$$= P^2 + \frac{Q^2}{b^2}\left(b^2 + c^2 - 2bc\cos A\right) - 2P\frac{Q}{b}(b\cos C - c\cos B)$$

$$= P^2 + \frac{Q^2}{b^2}(a^2) - 2P\frac{Q}{b}(a) \quad \because a^2 = b^2 + c^2 - 2bc\cos A \;\&\; a = b\cos C - c\cos B$$

$$= \left(P - \frac{Q}{b}a\right)^2$$

$$\Rightarrow \quad F = \left(P - \frac{Q}{b}a\right)$$

Thus the magnitude of the resultant $= \left(P - \dfrac{Q}{b}a\right)$

❖ (λ, μ) THEOREM

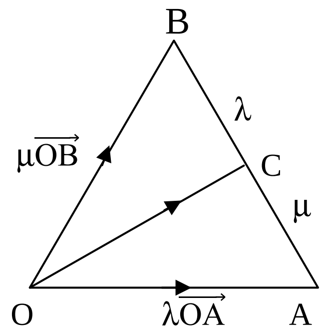
If two concurrent forces are represented by $\lambda\overrightarrow{OA}$ and $\mu\overrightarrow{OB}$. Then their resultant is given by $(\lambda + \mu)\overrightarrow{OC}$ where C divides AB such that

$$AC : CB = \mu : \lambda$$

PROOF

Let R be the resultant of the forces λOA and μOB. Then

$$\vec{R} = \lambda\overrightarrow{OA} + \mu\overrightarrow{OB} \qquad \text{\_\_\_\_\_\_\_\_\_\_(i)}$$



Given that

$$AC : CB = \mu : \lambda$$

$$\Rightarrow \quad \frac{AC}{CB} = \frac{\mu}{\lambda}$$

$$\Rightarrow \quad \lambda AC = \mu CB \quad \Rightarrow \quad \lambda AC - \mu CB = 0$$

$$\Rightarrow \quad \lambda AC + \mu BC = 0 \qquad \text{\_\_\_\_\_\_\_\_\_\_(ii)}$$

Again from fig.

$$\overrightarrow{OA} = \overrightarrow{OC} + \overrightarrow{CA}$$

$$\Rightarrow \quad \lambda\overrightarrow{OA} = \lambda\overrightarrow{OC} + \lambda\overrightarrow{CA} \qquad \text{\_\_\_\_\_\_\_\_\_\_(iii)}$$

From fig.

$$\overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CB}$$

$$\Rightarrow \quad \mu\overrightarrow{OB} = \mu\overrightarrow{OC} + \mu\overrightarrow{CB} \qquad \text{\_\_\_\_\_\_\_\_\_\_(iv)}$$

Using eq(iii) and eq(iv) in (i), we get

$$\vec{R} = \lambda\overrightarrow{OC} + \lambda\overrightarrow{CA} + \mu\overrightarrow{OC} + \mu\overrightarrow{CB} = (\lambda + \mu)\overrightarrow{OC} + \lambda\overrightarrow{CA} + \mu\overrightarrow{CB}$$

$$= (\lambda + \mu)\overrightarrow{OC} - \lambda\overrightarrow{CA} - \mu\overrightarrow{BC}$$

$$= (\lambda + \mu)\overrightarrow{OC} - (\lambda\overrightarrow{CA} + \mu\overrightarrow{CB})$$

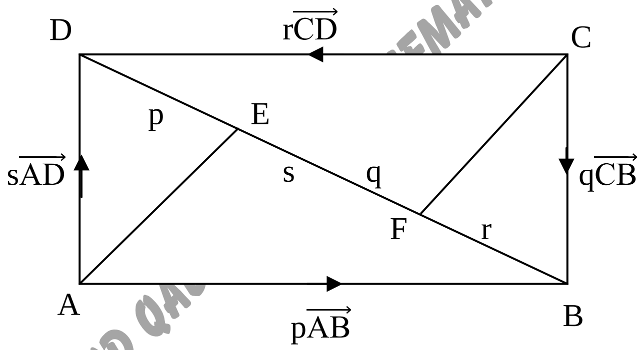
$$= (\lambda + \mu)\overrightarrow{OC} - 0 \quad \text{By(ii)}$$

Thus $\vec{R} = (\lambda + \mu)\overrightarrow{OC}$

❖ QUESTION 9

If forces $p\overrightarrow{AB}$, $q\overrightarrow{CB}$, $r\overrightarrow{CD}$ and $s\overrightarrow{AD}$ acting along the sides of a plane quadrilateral are in equilibrium. Show that **pr = qs**

SOLUTION



Let ABCD be a plane quadrilateral and force $p\overrightarrow{AB}$, $q\overrightarrow{CB}$, $r\overrightarrow{CD}$ and $s\overrightarrow{AD}$ acting along its sides as shown in figure.

By (λ, μ) theorem

$$p\overrightarrow{AB} + s\overrightarrow{AD} = (p + s)\,\overrightarrow{AE}$$

Where E is the point on BD such that

$$\frac{BE}{ED} = \frac{s}{p} \qquad \text{\_\_\_\_\_\_\_\_\_\_(i)}$$

Again by (λ, μ) theorem

$$q\overrightarrow{CB} + r\overrightarrow{CD} = (q + r)\,\overrightarrow{CF}$$

Where F is the point on BD such that

$$\frac{BF}{FD} = \frac{r}{q} \qquad \text{\_\_\_\_\_\_\_\_\_\_(ii)}$$

Since forces are in equilibrium therefore point E & F must coincides. So eq(i) & eq(ii) must equal. Thus

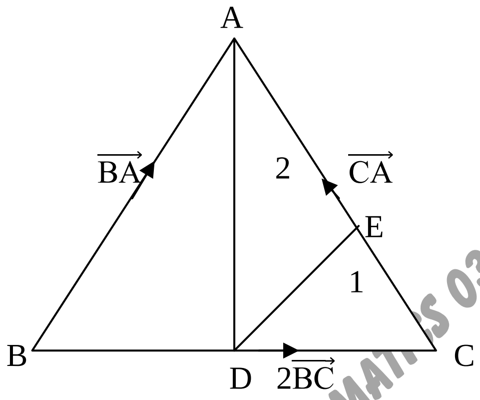
$$\frac{BE}{ED} = \frac{BF}{FD} \quad \Rightarrow \quad \frac{s}{p} = \frac{r}{q} \quad \Rightarrow \quad pr = sq$$

❖ QUESTION 10

If forces $2\overrightarrow{BC}$, $\overrightarrow{CA}$, and $\overrightarrow{BA}$ acting along the sides of triangle ABC. Show that their resultant is $6\overrightarrow{DE}$ where D bisect BC and E is a point on CA such that

$$\mathbf{CE = \frac{1}{3}CA}$$

SOLUTION



Let $\vec{R}$ be resultant of forces then

$$\vec{R} = 2\overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{BA}$$

$$= 2\overrightarrow{BC} - \overrightarrow{AC} - \overrightarrow{AB}$$

$$= 2\overrightarrow{BC} - (\overrightarrow{AC} + \overrightarrow{AB}) \qquad \_\_\_\_\_\_\_\_(i)$$

By (λ, μ) theorem,

$$\overrightarrow{AC} + \overrightarrow{AB} = (1 + 1)\overrightarrow{AD} = 2\overrightarrow{AD}$$

Using this value in (i), we get

$$\vec{R} = 2\overrightarrow{BC} - 2\overrightarrow{AD} = 2(\overrightarrow{BC} - \overrightarrow{AD})$$

$$= 2(2\overrightarrow{DC} - \overrightarrow{AD}) \qquad \because \ \overrightarrow{BC} = 2\overrightarrow{DC}$$

$$= 2(2\overrightarrow{DC} + \overrightarrow{DA}) \qquad \_\_\_\_\_\_\_\_(ii)$$

By (λ, μ) theorem,

$$2\overrightarrow{DC} + \overrightarrow{DA} = (2 + 1)\overrightarrow{DE} = 3\overrightarrow{DE}$$

Using this value in (ii), we get

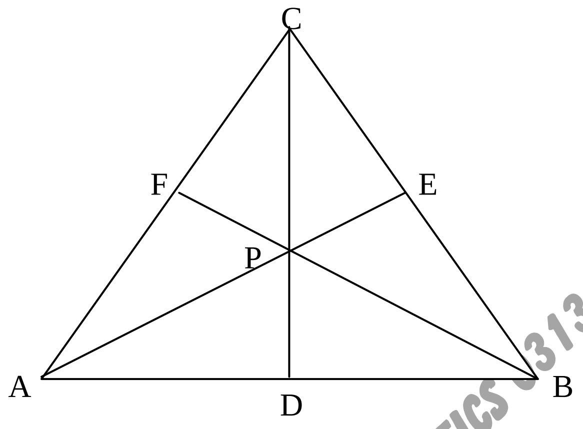
$$\vec{R} = 2(3\overrightarrow{DE})$$

$$= 6\overrightarrow{DE}$$

❖ QUESTION 11

P is any point in the plane of a triangle ABC and D, E, F are middle points of its sides. Prove that forces $\overrightarrow{AP}$, $\overrightarrow{BP}$, $\overrightarrow{CP}$, $\overrightarrow{PD}$, $\overrightarrow{PE}$, $\overrightarrow{PF}$ are in equilibrium.

SOLUTION



Applying (λ, μ) theorem, we get

$$\overrightarrow{AP} + \overrightarrow{BP} = 2\overrightarrow{DP}$$

$$\overrightarrow{BP} + \overrightarrow{CP} = 2\overrightarrow{EP}$$

$$\overrightarrow{CP} + \overrightarrow{AP} = 2\overrightarrow{FP}$$

Adding above equations, we get

$$\overrightarrow{AP} + \overrightarrow{BP} + \overrightarrow{BP} + \overrightarrow{CP} + \overrightarrow{CP} + \overrightarrow{AP} = 2\overrightarrow{DP} + 2\overrightarrow{EP} + 2\overrightarrow{FP}$$

$\Rightarrow \quad 2\overrightarrow{AP} + 2\overrightarrow{BP} + 2\overrightarrow{CP} = 2\overrightarrow{DP} + 2\overrightarrow{EP} + 2\overrightarrow{FP}$

$\Rightarrow \quad \overrightarrow{AP} + \overrightarrow{BP} + \overrightarrow{CP} = \overrightarrow{DP} + \overrightarrow{EP} + \overrightarrow{FP}$

$\Rightarrow \quad \overrightarrow{AP} + \overrightarrow{BP} + \overrightarrow{CP} - \overrightarrow{DP} - \overrightarrow{EP} - \overrightarrow{FP} = 0$

$\Rightarrow \quad \overrightarrow{AP} + \overrightarrow{BP} + \overrightarrow{CP} + \overrightarrow{PD} + \overrightarrow{PE} + \overrightarrow{PF} = 0$

Since the vector sum of all forces is zero therefore the given forces are in equilibrium.

❖ LAMI'S THEOREM

If a particle is in equilibrium under the action of three forces then the forces are coplanar and each force has magnitude proportional to the sine of the angle between the other two.

PROOF

Let $\vec{F}_1, \vec{F}_2$ and $\vec{F}_3$act at a point O and are in equilibrium.

Then $\vec{F}_1 + \vec{F}_2 + \vec{F}_3 = 0$ ________________(i)

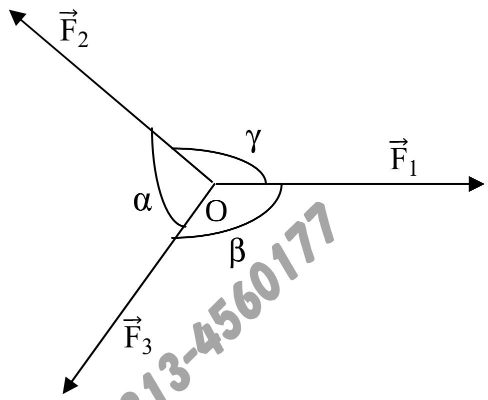
Taking dot product of eq(i) with $\vec{F}_2 \times \vec{F}_3$, we get

$$(\vec{F}_1 + \vec{F}_2 + \vec{F}_3).(\vec{F}_2 \times \vec{F}_3) = 0$$

$$\Rightarrow \quad \vec{F}_1.(\vec{F}_2 \times \vec{F}_3) + \vec{F}_2.(\vec{F}_2 \times \vec{F}_3) + \vec{F}_3.(\vec{F}_2 \times \vec{F}_3) = 0$$

$$\Rightarrow \quad \vec{F}_1.(\vec{F}_2 \times \vec{F}_3) + 0 + 0 = 0$$

$$\Rightarrow \quad [\vec{F}_1 \ \vec{F}_2 \ \vec{F}_3] = 0$$



Which shows that the forces $\vec{F}_1$, $\vec{F}_2$ and $\vec{F}_3$ are coplanar.

Taking cross product of eq(i) with $\vec{F}_3$, we get

$$(\vec{F}_1 + \vec{F}_2 + \vec{F}_3) \times \vec{F}_3 = 0$$

$$\Rightarrow \quad \vec{F}_1 \times \vec{F}_3 + \vec{F}_2 \times \vec{F}_3 + \vec{F}_3 \times \vec{F}_3 = 0$$

$$\Rightarrow \quad \vec{F}_1 \times \vec{F}_3 + \vec{F}_2 \times \vec{F}_3 = 0$$

$$\Rightarrow \quad \vec{F}_1 \times \vec{F}_3 = -\vec{F}_2 \times \vec{F}_3$$

$$\Rightarrow \quad |\vec{F}_1 \times \vec{F}_3| = |\vec{F}_2 \times \vec{F}_3|$$

$$\Rightarrow \quad F_1 F_3 \sin\beta = F_2 F_3 \sin\alpha$$

$$\Rightarrow \quad \frac{F_1}{\sin\alpha} = \frac{F_2}{\sin\beta} \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_(ii)}$$

Similarly by taking cross product of eq(i) with $\vec{F}_1$, we get

$$\frac{F_2}{\sin\beta} = \frac{F_3}{\sin\gamma} \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_(iii)}$$

From eq(ii) and (iii), we get

$$\frac{F_1}{\sin\alpha} = \frac{F_2}{\sin\beta} = \frac{F_3}{\sin\gamma}$$

Which shows that each force has magnitude proportional to the sine of the angle between the other two.

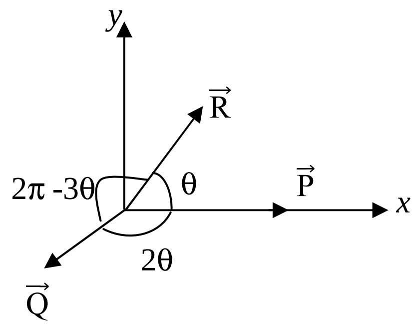
❖ QUESTION 12

Three forces $\vec{P}$, $\vec{Q}$ & $\vec{R}$ acting at a point, are in equilibrium and the angle between $\vec{P}$ & $\vec{Q}$ is double of the angle between $\vec{P}$ & $\vec{R}$. Prove that $\mathbf{R^2 = Q(Q - P)}$

SOLUTION

Let angle between $\vec{P}$& $\vec{R}$ is θ and angle between $\vec{P}$ & $\vec{Q}$ is 2θ. Since forces are in equilibrium therefore by Lami's theorem we have

$$\frac{P}{\sin(2\pi - 3\theta)} = \frac{Q}{\sin\theta} = \frac{R}{\sin 2\theta}$$



$\Rightarrow \quad -\frac{P}{\sin 3\theta} = \frac{Q}{\sin\theta} = \frac{R}{\sin 2\theta}$ ______________(i)

$\Rightarrow \quad -\frac{P}{\sin 3\theta} = \frac{Q}{\sin\theta}$

$\Rightarrow \quad -P\sin\theta = Q\sin 3\theta$

$\Rightarrow \quad P\sin\theta + Q(3\sin\theta - 4\sin^3\theta) = 0$

$\Rightarrow \quad P\sin\theta + Q\sin\theta(3 - 4\sin^2\theta) = 0$

$\Rightarrow \quad P + Q(3 - 4\sin^2\theta) = 0$

$\Rightarrow \quad P + Q[3 - 4(1 - \cos^2\theta)] = 0$

$\Rightarrow \quad P + Q(3 - 4 + 4\cos^2\theta) = 0$

$\Rightarrow \quad P + Q(4\cos^2\theta - 1) = 0$ ------- (ii)

From (i), we have

$$\frac{Q}{\sin\theta} = \frac{R}{\sin 2\theta}$$

$\Rightarrow \quad Q\sin 2\theta = R\sin\theta$

$\Rightarrow \quad Q2\sin\theta\cos\theta = R\sin\theta$

$\Rightarrow \quad Q2\cos\theta = R$

$\Rightarrow \quad \cos\theta = \frac{R}{2Q}$

Using value of cosθ in (ii), we get

$\Rightarrow \quad P + Q\left[4\left(\frac{R}{2Q}\right)^2 - 1\right] = 0$

$\Rightarrow \quad P + Q\left(\frac{R^2}{Q^2} - 1\right) = 0$

$\Rightarrow \quad P + Q\left(\frac{R^2 - Q^2}{Q^2}\right) = 0$

$\Rightarrow \quad PQ + R^2 - Q^2 = 0$

$\Rightarrow \quad R^2 = Q^2 - PQ$

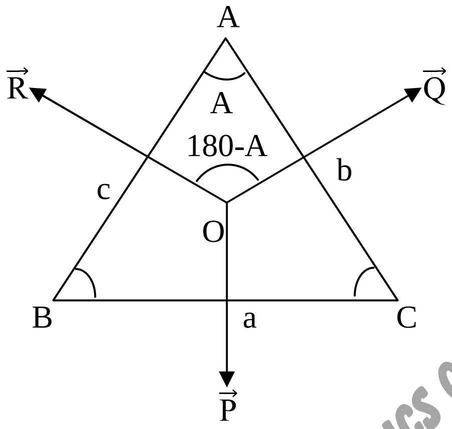
$\Rightarrow \quad R^2 = Q(Q - P)$

Hence Proved.

❖ QUESTION 13

Three forces act perpendicularly to the sides of a triangle at their middle points and are proportional to the sides. Prove that they are in equilibrium.

SOLUTION



Let a, b and c are the lengths of the sides of the triangle ABC. Then given that

$$\frac{P}{a}=\frac{Q}{b}=\frac{R}{c} \qquad \text{\_\_\_\_\_\_\_\_\_\_(i)}$$

By law of sine

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C} \qquad \text{\_\_\_\_\_\_\_\_\_\_(ii)}$$

Comparing (i) and (ii), we get

$$\frac{P}{\sin A}=\frac{Q}{\sin B}=\frac{R}{\sin C} \qquad \text{\_\_\_\_\_\_\_\_\_\_(iii)}$$

From fig.

$$\angle QOR = 180 - A, \quad \angle POQ = 180 - B, \quad \angle POR = 180 - C$$

$\Rightarrow$ $\sin\angle QOR = \sin(180 - A) = \sin A$

$\sin\angle POQ = \sin(180 - B) = \sin B$

$\sin\angle POR = \sin(180 - C) = \sin C$

From (iii)

$$\frac{P}{\sin\angle QOR}=\frac{Q}{\sin\angle POQ}=\frac{R}{\sin\angle POR}$$

Thus, by Lami's theorem forces are in equilibrium.

❖ MOMENT OR TORQUE OF A FORCE

The tendency of a force to rotate a body about a point is called moment or torque of that force.

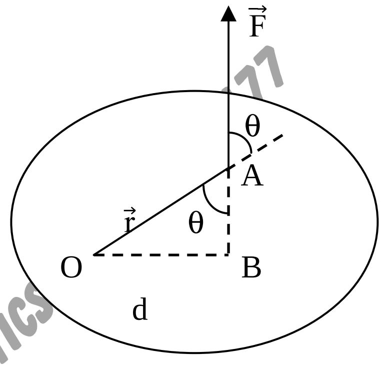
EXPLANATION

Consider a force $\vec{F}$ acting on a rigid body which tends to rotate the body about O. Take any point A on the line of action of force $\vec{F}$. Let $\vec{r}$ be the position vector of A with respect to O. Then moment of force about O is defined as

$$\vec{M} = \vec{r} \times \vec{F}$$

$$\Rightarrow \quad |\vec{M}| = |\vec{r} \times \vec{F}|$$

$$\Rightarrow \quad M = rF\sin\theta = F(r\sin\theta) \qquad \text{\_\_\_\_\_\_\_\_\_\_(i)}$$



Where θ is the angle between $\vec{r}$ and $\vec{F}$.

From fig.

$$d = r\sin\theta$$

Using this in eq(i), we get

$$M = Fd$$

Where d is the perpendicular distance from O to the line of the action of force $\vec{F}$.

Note :

1. Moment will be negative if body rotates in clockwise direction.
2. Moment will be positive if body rotates in anticlockwise direction.

❖ VARIGNON'S THEOREM

The moment about a point of the resultant of a system of concurrent force is equal to the sum of the moments of these forces about the same point.

PROOF

Let forces $\vec{F}_1, \vec{F}_2, \vec{F}_3, \ldots, \vec{F}_n$ be concurrent at a point A. Let $\vec{F}$ be their resultant and $\vec{r}$ be the position vector of A with respect to O. Then

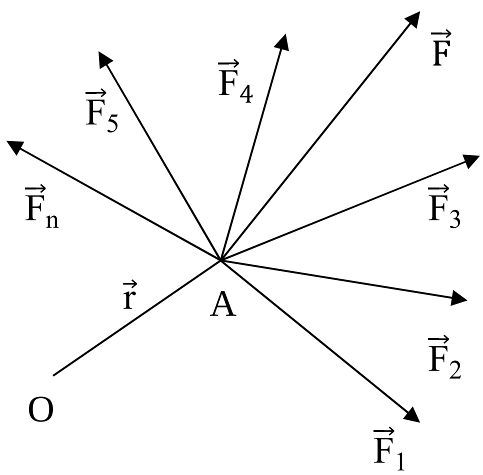
$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots + \vec{F}_n \qquad \text{\_\_\_\_\_\_ (i)}$$

$$\Rightarrow \quad \vec{r} \times \vec{F} = \vec{r} \times \left(\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots + \vec{F}_n\right)$$

$$\Rightarrow \quad \vec{M} = \vec{r} \times \vec{F}_1 + \vec{r} \times \vec{F}_2 + \ldots + \vec{r} \times \vec{F}_n$$

$$\Rightarrow \quad \vec{M} = \vec{M}_1 + \vec{M}_2 + \vec{M}_3 + \ldots + \vec{M}_n$$

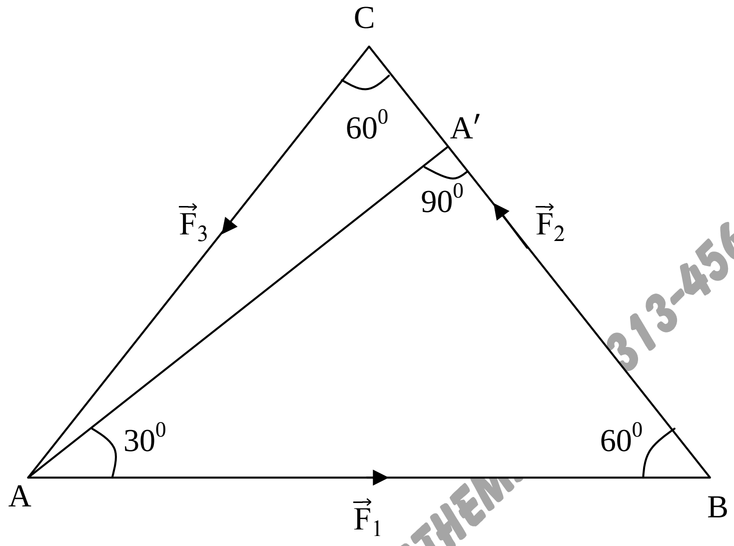
Where $\vec{M}$ is the moment of resultant of forces about point O and $\vec{M}_1 + \vec{M}_2 + \vec{M}_3 + \ldots + \vec{M}_n$ is the sum of the moments of forces about the same point. This completes the proof.



❖ QUESTION 14

A system of forces acts on a plate in the from of an equilateral triangle of side 2a. The moments of the forces about the three vertices are G_1, G_2, and G_3 respectively. Find the magnitude of the resultant.

SOLUTION



Let $\vec{F}_1$, $\vec{F}_2$ and $\vec{F}_3$ be the forces acting along the sides AB, BC and CA taking one way round.

Take AB along x–axis.

Let the R be the magnitude of the resultant of the forces, then

$$R = \sqrt{R_x^{\ 2} + R_y^{\ 2}} \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_(i)}$$

Now by theorem of resolved parts

R_x = Sum of the resolved parts of the forces along x-axis

$= F_1\cos 0^0 + F_2\cos(180 - 60) + F_3\cos(180 + 60)$

$= F_1 - F_2\cos 60 - F_3\cos 60$

$$= F_1 - F_2\left(\frac{1}{2}\right) - F_3\left(\frac{1}{2}\right) = \frac{2F_1 - F_2 - F_3}{2}$$

And R_y = Sum of the resolved parts of the forces along y-axis

$= F_1\sin 0^0 + F_2\sin(180 - 60) + F_3\sin(180 + 60)$

$= 0 + F_2\sin 60 - F_3\sin 60$

$$= F_2\left(\frac{\sqrt{3}}{2}\right) - F_3\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{\sqrt{3}}{2}(F_2 - F_3)$$

Using values of F_x and F_y in (i), we get

$$R = \sqrt{\left(\frac{2F_1 - F_2 - F_3}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}(F_2 - F_3)\right)^2}$$

$$= \sqrt{\frac{4F_1^2 + F_2^2 + F_3^2 - 4F_1F_2 + 2F_2F_3 - 4F_3F_1}{4} + \frac{3F_2^2 + 3F_3^2 - 6F_2F_3}{4}}$$

$$= \sqrt{\frac{4F_1^2 + 4F_2^2 + 4F_3^2 - 4F_1F_2 - 4F_2F_3 - 4F_3F_1}{4}}$$

$$= \sqrt{F_1^2 + F_2^2 + F_3^2 - F_1F_2 - F_2F_3 - F_3F_1} \qquad \text{(ii)}$$

Take moments of all forces about A.

$$G_1 = F_2(AA')$$

$$= F_2(AB\cos 30) = 2aF_2\frac{\sqrt{3}}{2} = a\sqrt{3}F_2$$

$$\Rightarrow \qquad F_2 = \frac{G_1}{a\sqrt{3}}$$

Similarly by taking moments of all forces about B and C, we get

$$F_3 = \frac{G_2}{a\sqrt{3}} \quad \text{and} \quad F_1 = \frac{G_3}{a\sqrt{3}}$$

Using values of F_1, F_2 and F_3 in (ii), we get

$$R = \sqrt{\frac{G_3^2 + G_1^2 + G_2^2 - G_1G_3 - G_1G_2 - G_2G_3}{3a^2}}$$

$$= \sqrt{\frac{G_1^2 + G_2^2 + G_3^2 - G_1G_2 - G_2G_3 - G_3G_1}{3a^2}}$$

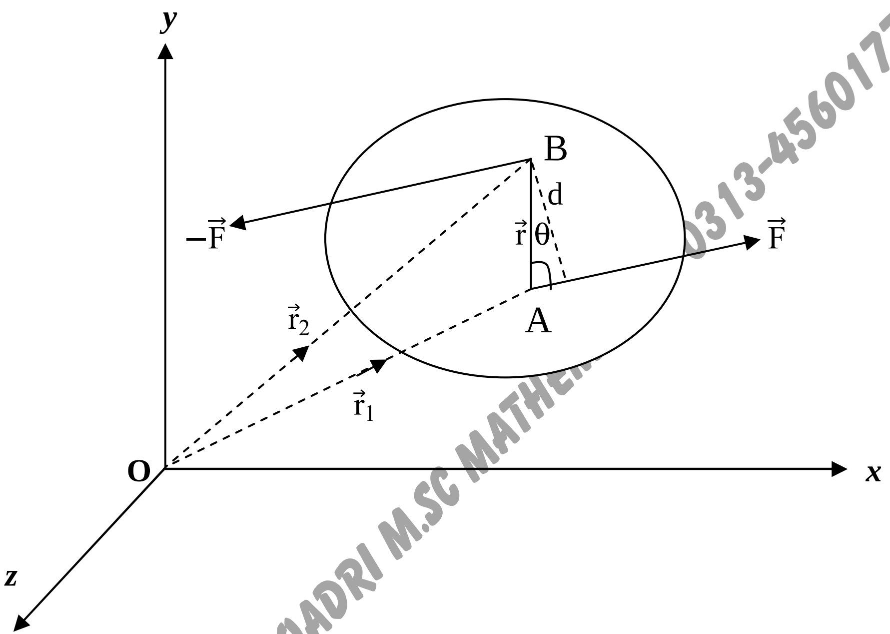
Which is required.

❖ COUPLE

A pair of forces $(\vec{F}, -\vec{F})$ of same magnitude but opposite in direction acting on a rigid body forms a couple. When couple acts on a body it rotates the body.

❖ MOMENT OF A COUPLE

Let $(\vec{F}, -\vec{F})$ be a couple. Let A and B be points on the line of action of $\vec{F}$ and $-\vec{F}$ respectively. Let $\vec{r}_1$ and $\vec{r}_2$ are the position vectors of the points A and B respectively.



Then the sum of moments of $\vec{F}$ and $-\vec{F}$ about O is

$$\vec{G} = \vec{r}_1 \times \vec{F} + \vec{r}_2 \times (-\vec{F}) = (\vec{r}_1 - \vec{r}_2) \times \vec{F}$$

From fig.

$$\vec{r}_1 - \vec{r}_2 = \vec{r}$$

So $\quad \vec{G} = \vec{r} \times \vec{F}$

The vector $\vec{G}$ is called the moment of the couple.

$$|\vec{G}| = |\vec{r} \times \vec{F}| = F\, r\sin\theta = Fd$$

$\Rightarrow \quad G = Fd$

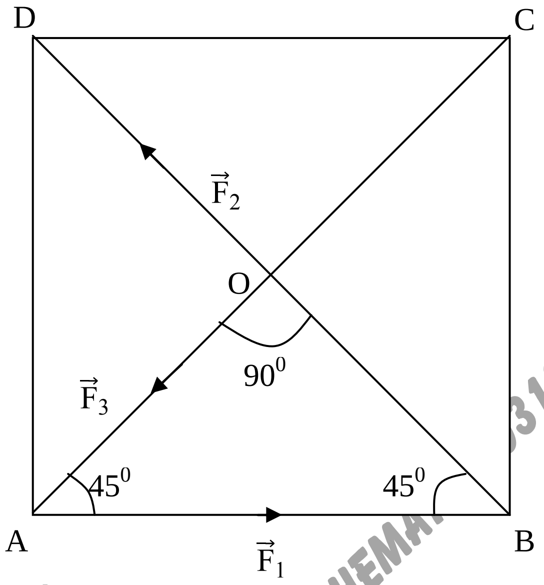
Where d is the perpendicular distance between the line of the action of the forces.

❖ QUESTION 15

A couple of moment G acts on a square board ABCD of side a. Replace the couple by the forces acting along AB BD and CA.

SOLUTION

Let ABCD be a square board of side a. Let $\vec{F}_1$, $\vec{F}_2$ and $\vec{F}_3$ be the forces acting along AB, BD and CA respectively. Where AB is side and BD and CA are the diagonals of square as shown in fig.



Take moments of all forces about A.

$$G = F_2(AO) = F_2(AB\sin 45)$$

$$= F_2 a \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad F_2 = \frac{G\sqrt{2}}{a}$$

Take moments of all forces about B.

$$G = F_3(BO) = F_3(AB\sin 45)$$

$$= F_3 a \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad F_3 = \frac{G\sqrt{2}}{a}$$

Take moment of all forces about D.

$$G = F_1(AD) - F_3(DO) = F_1(a) - F_3(AD\sin 45)$$

$$= F_1 a - F_3 \frac{a}{\sqrt{2}} = F_1 a - \frac{G\sqrt{2}}{a}\frac{a}{\sqrt{2}} = F_1 a - G$$

$$\Rightarrow \quad 2G = F_1 a$$

$$\Rightarrow \quad F_1 = \frac{2G}{a}$$

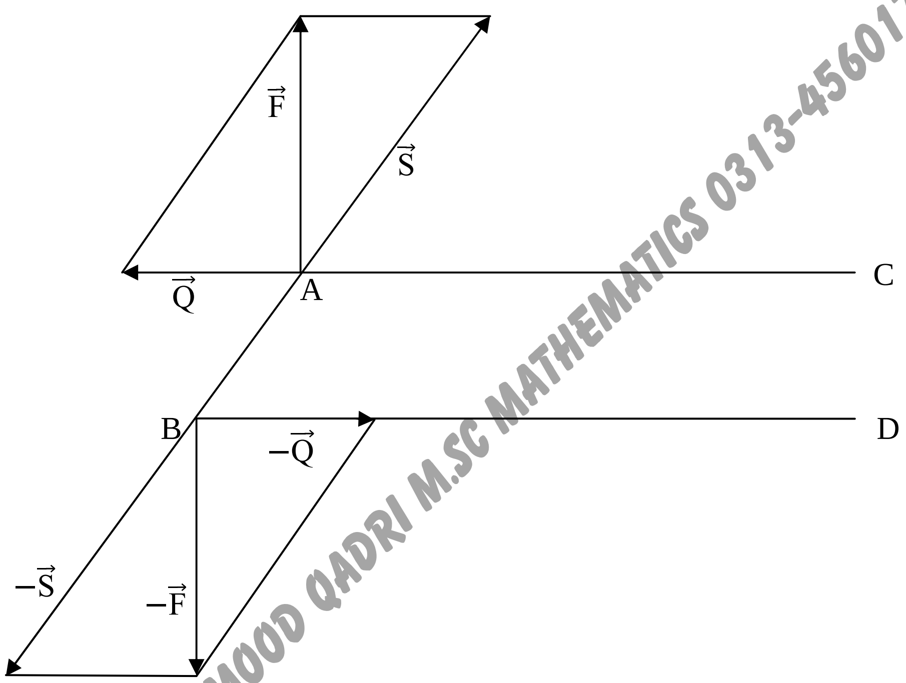
❖ EQUIVALENT COUPLE

Any two couples of equal moments lying in the same plane are called equivalent couples.

❖ THEOREM

The effect of a couple upon a rigid body is unaltered if it is replaced by any other couple of the same moment lying in the same plane.

PROOF



Let $(\vec{F}, -\vec{F})$ be a couple. Let A and B be points on the line of action of $\vec{F}$ and $-\vec{F}$ respectively. We want to replace the couple $(\vec{F}, -\vec{F})$ by any other couple. For this draw two lines AC and BD in the desire direction. Resolve the forces $\vec{F}$ and $-\vec{F}$ at the points A and B respectively into two components. Let $\vec{Q}$ and $\vec{S}$ are the resolved parts of $\vec{F}$ along CA and BA respectively. Then

$$\vec{F} = \vec{Q} + \vec{S} \qquad \text{\_\_\_\_\_\_\_\_\_\_ (i)}$$

Let $-\vec{Q}$ and $-\vec{S}$ are the resolved parts of $-\vec{F}$ along BD and AB respectively. Since the force $\vec{S}$ and $-\vec{S}$ act along the same lin. Therefore these force balance each other being equal and opposite. Thus we are left with forces $\vec{Q}$ and $-\vec{Q}$ acting at A and B along two parallel lines form a couple. So the given couple $(\vec{F}, -\vec{F})$ has been replaced by the couple $(\vec{Q}, -\vec{Q})$.

Now Moment of couple $(\vec{F}, -\vec{F}) = \overrightarrow{BA} \times \vec{F}$

$$= \overrightarrow{BA} \times \left(\vec{Q} + \vec{S}\right) \qquad \text{By (i)}$$

$$= \overrightarrow{BA} \times \vec{Q} + \overrightarrow{BA} \times \vec{S}$$

$$= \overrightarrow{BA} \times \vec{Q} + 0$$

$$= \overrightarrow{BA} \times \vec{Q}$$

$$= \text{Moment of couple } (\vec{Q}, -\vec{Q})$$

Thus we see that a couple acting on a rigid body can be replaced by another couple of the same moment lying in the same plane. This completes the proof.

❖ COMPOSITIONS OF COUPLES

Coplanar couples of moments $G_1, G_2, G_3, \ldots, G_n$ are equivalent to a single couple lying in the same plane, whose moment G is given by

$$G = G_1 + G_2 + G_3 + \ldots + G_n$$

PROOF

Replace couples of moments *$G_1, G_2, G_3, \ldots, G_n$ by couples* $(\vec{F}_1, -\vec{F}_1)$, $(\vec{F}_2, -\vec{F}_2)$, $(\vec{F}_3, -\vec{F}_3)$, …, $(\vec{F}_n, -\vec{F}_n)$ respectively with common arm d. Then

$$G_1 = F_1d,\ G_2 = F_2d,\ G_3 = F_3d,\ \ldots,\ G_n = F_nd$$

Now the forces $\vec{F}_1, \vec{F}_2, \vec{F}_3, \ldots, \vec{F}_n$ act along one straight line and $-\vec{F}_1, -\vec{F}_2, -\vec{F}_3, \ldots, -\vec{F}_n$ act along a parallel line. So we have a single couple $(\vec{F}, -\vec{F})$ with

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots + \vec{F}_n \text{ and } -\vec{F} = -(\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots + \vec{F}_n)$$

and the moment G of couple is

$$\begin{aligned} G &= Fd \\ &= (F_1 + F_2 + F_3 + \ldots + F_n)d \\ &= F_1d + F_2d + F_3d + \ldots + F_nd \\ &= G_1 + G_2 + G_3 + \ldots + G_n \end{aligned}$$

This completes the proof.

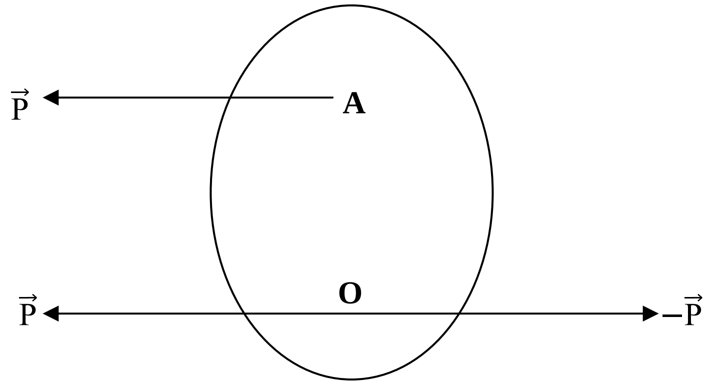
❖ A FORCE AND A COUPLE

A force $\vec{P}$ acting on a rigid body can be moved to any point O of the rigid body, provided a couple is added, whose moment is equal to the moment of $\vec{P}$ about O.

PROOF

Let the given force $\vec{P}$ act a point A. We want to shift this force $\vec{P}$ to point O of the body. At point O we introduce two equal and opposite force $\vec{P}$ and $-\vec{P}$. These forces being equal and opposite balance each other. The force $\vec{P}$ act at point A and $-\vec{P}$ act at O form a couple $(\vec{P}, -\vec{P})$

and we get a force $\vec{P}$ at point O. Therefore the force $\vec{P}$ acting at the point A is shifted to the point O and a couple has been introduced.



Also

Moment of couple $(\vec{P}, -\vec{P}) = Pd$ = Moment of the force $\vec{P}$ at A about O

This completes the proof.

❖ REDUCTION OF A SYSTEM OF COPLANAR FORCES TO ONE FORCE AND ONE COUPLE

Any system of coplanar forces acting on a rigid body can be reduced to a single force at any arbitrary point in the plane of the forces together with a couple.

PROOF

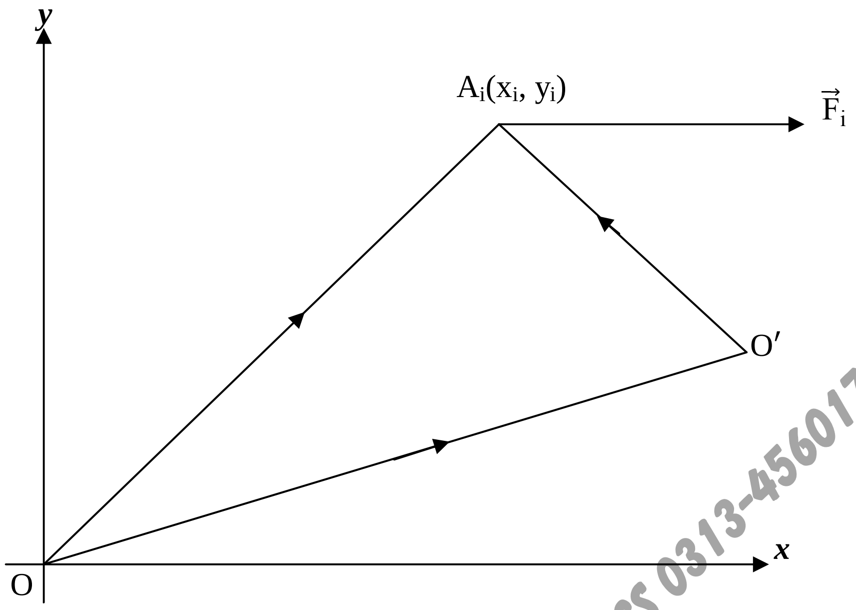
Let $\vec{F}_1, \vec{F}_2, \vec{F}_3, \dots, \vec{F}_n$ be a system of forces acting at a points $A_1, A_2, A_3, \dots, A_n$ respectively. Let O be a point in the same plane. By shifting the force $\vec{F}_i$ acting at A_i to point O, we get a force $\vec{F}_i$ at O together with a couple whose moment $\vec{G}_i$ is equal to the moment of the force $\vec{F}_i$ about O. Thus by shifting the forces to the point O, we get system of forces $\vec{F}_1, \vec{F}_2, \vec{F}_3, \dots, \vec{F}_n$ acting at O together with a system of couples of moments $G_1, G_2, G_3, \dots, G_n$.

The forces $\vec{F}_1, \vec{F}_2, \vec{F}_3, \dots, \vec{F}_n$ acting at O can be replaced by their resultant forces $\vec{F}$ acting at the same point O. Similarly, by the theorem of the composition of couples, all the coplanar couples can be replaced by a single couple of moment G. The force $\vec{F}$ and the couple of the equivalent system are given by

$$\vec{F} = \sum_{i=1}^{n} \vec{F}_i \quad \text{and} \quad G = \sum_{i=1}^{n} G_i$$

This completes the proof.

❖ EQUATION OF LINE OF ACTION OF RESULTANT



Let (X_i, Y_i) be the component of the force $\vec{F}_i$ act at the point A_i whose coordinates are (x_i, y_i) where i = 1, 2, 3, …, n. Let the reduction be made at origin O, we get a single force $\vec{F}$ acting at O together with a couple $\vec{G}$ so that

$$\vec{F} = \sum_{i=1}^{n} \vec{F}_i$$

$$= \sum_{i=1}^{n} \left(X_i\hat{i} + Y_i\hat{j}\right)$$

$$= \sum_{i=1}^{n} X_i\hat{i} + \sum_{i=1}^{n} Y_i\hat{j}$$

$$= \sum_{i=1}^{n} X_i\,\hat{i} + \sum_{i=1}^{n} Y_i\,\hat{j} = F_x\hat{i} + F_y\hat{j}$$

Where F_x and F_y are the component of the resultant $\vec{F}$.

And

$$\vec{G} = \sum_{i=1}^{n} \left(\overrightarrow{OA_i} \times \vec{F}_i\right)$$

Let the reduction be made at $O'(x, y)$. The resultant $\vec{F}$ remains same but moment $\vec{G}$ of the couple changes. Let $\vec{G}'$ be the moment of new couple then

$$\vec{G}' = \text{sum of moments about } O' \text{ of } \vec{F}_i$$

$$= \sum_{i=1}^{n} \left(\overrightarrow{O'A_i} \times \vec{F}_i\right)$$

$$= \sum_{i=1}^{n} \left(\overrightarrow{OA_i} - \overrightarrow{OO'}\right) \times \vec{F}_i$$

$$= \sum_{i=1}^{n} \overrightarrow{OA_i} \times \vec{F}_i - \sum_{i=1}^{n} \overrightarrow{OO'} \times \vec{F}_i$$

$$= \sum_{i=1}^{n} \overrightarrow{OA_i} \times \vec{F}_i - \sum_{i=1}^{n} \overrightarrow{OO'} \times \vec{F}_i$$

$$= \vec{G} - \overrightarrow{OO'} \times \sum_{i=1}^{n} \vec{F}_i$$

$$= \vec{G} - (x\hat{i} + y\hat{j}) \times (F_x\hat{i} + F_y\hat{j})$$

$$= \vec{G} - \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & 0 \\ F_x & F_y & 0 \end{vmatrix}$$

$$= \vec{G} - (xF_y - yF_x)\hat{k}$$

$\Rightarrow \quad G'\,\hat{k} = G\hat{k} - (xF_y - yF_x)\hat{k}$

$\Rightarrow \quad G' = G - xF_y + yF_x$

If the resultant passes through O′ then $G' = 0$

$\Rightarrow \quad G - xF_y + yF_x = 0$ or $G - xY + yX = 0$ (take $F_x = X$ and $F_y = Y$)

Which is the equation of the line of action of the resultant.

Note:

- A system is in equilibrium if $R = G = 0$
- A system is equivalent to a couple if $R = 0$ and $G \neq 0$

QUESTION 16

A and B are any two points in a lamina on which a system of forces coplanar with it are acting, and when the forces are reduced to a single force at each of these points and a couple, the moments of the couple are G_a and G_b respectively. Prove that when the reduction is made to be a force at the middle of AB and a couple, the moment of the couple is

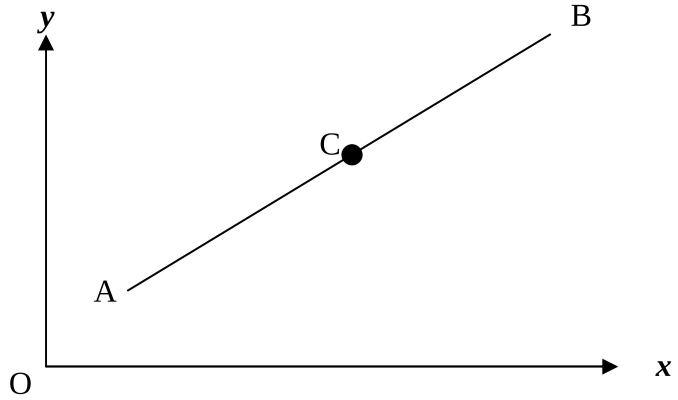
$$\frac{1}{2}(G_a + G_b)$$

SOLUTION

Let the coordinates of A and B are (x_1, y_1) and (x_2, y_2). Let C be the midpoint of AB then the coordinates of C are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Suppose the given system of forces is reduced to single force acting at O together with a couple G. Let X and Y be the component of the reduced force.



When the same system of forces is reduced to a single force acting at A, the resultant force will remain unchanged, whereas the moment will changed and is given by

$$G_a = G - x_1Y + y_1X$$

Similarly when the same system of forces is reduced to a single force acting at B, the resultant force will remain unchanged, whereas the moment will changed and is given by

$$G_b = G - x_2Y + y_2X$$

Suppose that the same system of forces is reduced to a single force acting at C, the resultant force will remain unchanged, whereas the moment will changed and is given by

$$G_c = G - \frac{x_1 + x_2}{2}Y + \frac{y_1 + y_2}{2}X = \frac{1}{2}\left(2G - x_1Y - x_2Y + y_1X + y_2X\right)$$

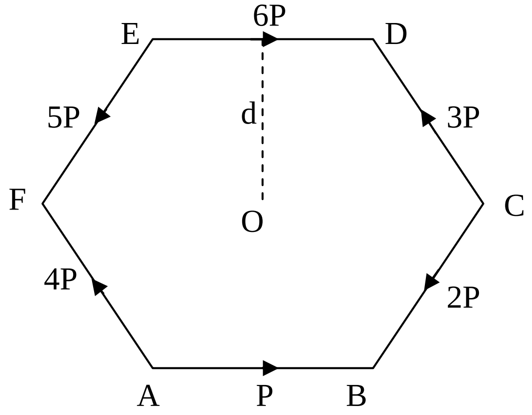
$$= \frac{1}{2}\left(G - x_1Y + y_1X + G - x_2Y + y_2X\right) = \frac{1}{2}(G_a + G_b)$$

Which is the required.

❖ QUESTION 17

Forces P, 2P, 3P, 6P, 5P and 4P act respectively along the sides AB, CB, CD, ED, EF and AF of a regular hexagon of side a, the sense of the forces being indicated by the order of the letters. Prove that the six forces are equivalent to a couple.

SOLUTION



Let ABCDEFA be a regular hexagon of side a. Forces P, 2P, 3P, 6P, 5P and 4P act along the sides AB, CB, CD, ED, EF and AF respectively. Take AB along x–axis. Take O is the centre hexagon and d is perpendicular distance of each force from O.

Let the R be the magnitude of the resultant of the forces, then

$$R = \sqrt{R_x^2 + R_y^2} \qquad \text{\_\_\_\_\_\_\_\_(i)}$$

Now by theorem of resolved parts

R_x = Sum of the resolved parts of the forces along x-axis

$= P\cos 0^0 + 2P\cos(180^0 + 60^0) + 3P\cos(180^0 - 60^0) + 4P\cos(180^0 - 60^0)$
$+ 5P\cos(180^0 + 60^0) + 6P\cos 0^0$

$= P - 2P\cos 60^0 - 3P\cos 60^0 - 4P\cos 60^0 - 5P\cos 60^0 + 6P$

$= 7P - 14P\cos 60^0$

$= 7P - 14P\left(\frac{1}{2}\right) = 7P - 7P = 0$

And R_y = Sum of the resolved parts of the forces along y-axis

$= P\sin 0^0 + 2P\sin(180^0 + 60^0) + 3P\sin(180^0 - 60^0) + 4P\sin(180^0 - 60^0)$
$+ 5P\sin(180^0 + 60^0) + 6P\sin 0^0$

$= 0 - 2P\sin 60^0 + 3P\sin 60^0 + 4P\sin 60^0 - 5P\sin 60^0 + 0$

$= 7P\sin 60^0 - 7P\sin 60^0 = 0$

Using values of R_x and R_y in (i), we get

$R = 0$

Take moment of all forces about point O.

G = sum of moments of all forces about O.

$= Pd - 2Pd + 3Pd - 4Pd + 5Pd - 6Pd = 9Pd - 12Pd = -3Pd$

Since $R = 0$ and $G \neq 0$. Therefore the system of the given coplanar forces is equivalent to a couple.

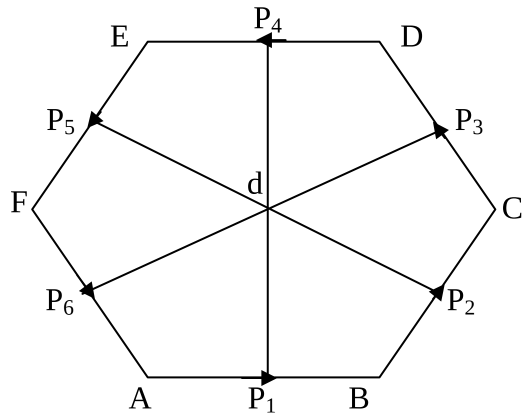
❖ QUESTION 18

Forces P_1, P_2, P_3, P_4, P_5 and P_6 act along the sides of a regular hexagon taken in order. Show that they will be in equilibrium if

$$\sum P = 0 \text{ and } P_1 - P_4 = P_3 - P_6 = P_5 - P_2$$

SOLUTION

Let ABCDEFA be a regular hexagon. Forces P_1, P_2, P_3, P_4, P_5 and P_6 act along the its sides taken one way round. O is the centre of hexagon and d is perpendicular distance of all forces from O. Take AB along x–axis.



Take moment of all forces about O.

$$G = P_1d + P_2d + P_3d + P_4d + P_5d + P_6d$$
$$= (P_1 + P_2 + P_3 + P_4 + P_5 + P_6)d$$
$$= d\sum P$$

Let X and Y be the resolved parts of the resultant of the forces then by the theorem of the resolved parts.

$$X = \text{Sum of the resolved parts of the forces along x-axis}$$
$$= P_1\cos 0^0 + P_2\cos 60 + P_3\cos 120 + P_4\cos 180 + P_5\cos 240 + P_6\cos 300$$
$$= P_1 + P_2\left(\frac{1}{2}\right) - P_3\left(\frac{1}{2}\right) - P_4 - P_5\left(\frac{1}{2}\right) + P_6\left(\frac{1}{2}\right)$$
$$= P_1 - P_4 + \frac{1}{2}(P_2 - P_3 - P_5 + P_6)$$

And $\quad Y = \text{Sum of the resolved parts of the forces along y-axis}$

$$= P_1\sin 0^0 + P_2\sin 60 + P_3\sin 120 + P_4\sin 180 + P_5\sin 240 + P_6\sin 300$$
$$= 0 + P_2\left(\frac{\sqrt{3}}{2}\right) + P_3\left(\frac{\sqrt{3}}{2}\right) - 0 - P_5\left(\frac{\sqrt{3}}{2}\right) - P_6\left(\frac{\sqrt{3}}{2}\right)$$
$$= \frac{\sqrt{3}}{2}(P_2 + P_3 - P_5 - P_6)$$

System of forces is in equilibrium if and only if X = Y = 0 and G = 0

If G = 0 then

$$d\sum P = 0$$

$\Rightarrow \quad \sum P = 0 \quad \text{or} \quad P_1 + P_2 + P_3 + P_4 + P_5 + P_6 = 0$

If Y = 0 then

$$\frac{\sqrt{3}}{2}(P_2 + P_3 - P_5 - P_6) = 0$$

$\Rightarrow \quad P_5 - P_2 = P_3 - P_6$ ____________(i)

If X = 0 then

$$P_1 - P_4 + \frac{1}{2}(P_2 - P_3 - P_5 + P_6) = 0$$

$$\Rightarrow \quad P_1 - P_4 + \frac{1}{2}\left(P_2 - P_5 - (P_3 - P_6)\right) = 0$$

$$\Rightarrow \quad P_1 - P_4 + \frac{1}{2}\left(P_2 - P_5 - (P_5 - P_2)\right) = 0 \quad \text{By (i)}$$

$$\Rightarrow \quad P_1 - P_4 + \frac{1}{2}(P_2 - P_5 - P_5 + P_2) = 0$$

$$\Rightarrow \quad P_1 - P_4 + \frac{1}{2}(2P_2 - 2P_5) = 0 \Rightarrow \quad P_1 - P_4 + P_2 - P_5 = 0$$

$$\Rightarrow \quad P_5 - P_2 = P_1 - P_4 \qquad \text{\_\_\_\_\_\_\_\_(ii)}$$

From (i) and (ii), we get

$$P_1 - P_4 = P_5 - P_2 = P_3 - P_6$$

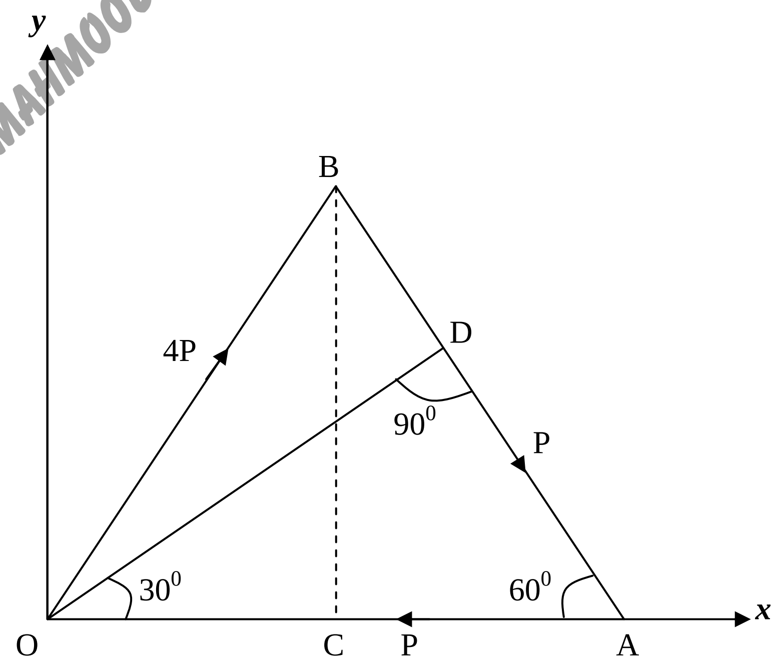
Hence the system is in equilibrium if

$$\sum P = 0 \text{ and } P_1 - P_4 = P_3 - P_6 = P_5 - P_2$$

❖ QUESTION 19

OAB is an equilateral triangle of a side a ; C is the mid-point of OA. Forces 4P, P and P act along the sides OB, BA and AO respectively. If OA and OY (parallel to BC) are taken as x- and y-axis. Prove that the resultant of the forces is 3P and the equation of its line of action is $3y = \sqrt{3}\,(3x + a)$

SOLUTION



Let the R be the magnitude of the resultant of the forces, then

$$R = \sqrt{R_x{}^2 + R_y{}^2} \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_(i)}$$

Now by theorem of resolved parts

R_x = Sum of the resolved parts of the forces along x-axis

$= P\cos 180^0 + P\cos 300 + 4P\cos 60$

$= -P + P\left(\frac{1}{2}\right) + 4P\left(\frac{1}{2}\right)$

$= -P + 5P\left(\frac{1}{2}\right)$

$= \frac{3}{2}P$

And R_y = Sum of the resolved parts of the forces along y-axis

$= P\sin 180^0 + P\sin 300 + 4P\sin 60$

$= -P\left(\frac{\sqrt{3}}{2}\right) + 4P\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{3\sqrt{3}}{2}P$

Using values of F_x and F_y in (i), we get

$$R = \sqrt{\left(\frac{3}{2}P\right)^2 + \left(\frac{3\sqrt{3}}{2}P\right)^2}$$

$$= \sqrt{\frac{9}{4}P^2 + \frac{27}{4}P^2} = \sqrt{\frac{36}{4}P^2} = \sqrt{9P^2} = 3P$$

Let G be the sum of moments of all forces about O. Then

$G = -P(OD)$

$= -P(OA\cos 30)$

$= -P\frac{a\sqrt{3}}{2}$

The equation of line of action of resultant of resultant is

$G - xR_y + yR_x = 0$

$\Rightarrow \quad -P\frac{a\sqrt{3}}{2} - x\left(\frac{3\sqrt{3}}{2}P\right) + y\left(\frac{3}{2}P\right) = 0$

$\Rightarrow \quad -a\sqrt{3} - 3\sqrt{3}x + 3y = 0$

$\Rightarrow \quad 3y = 3\sqrt{3}x + a\sqrt{3}$

$\Rightarrow \quad 3y = \sqrt{3}(3x + a)$ Which is required.

❖ DISTANCE OF A POINT FROM A LINE

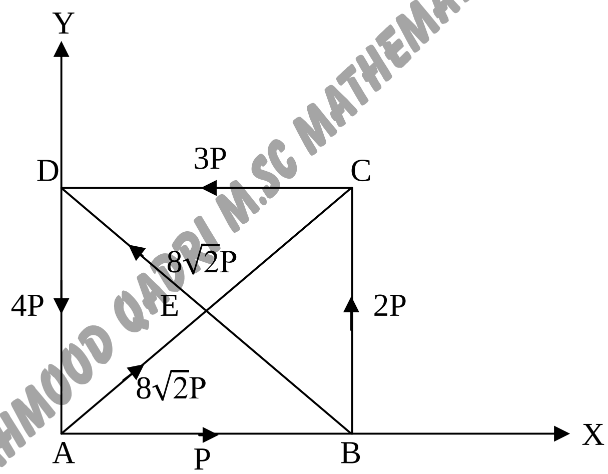
The distance d from the point $P(x_1, y_1)$ to the line $ax + by + c = 0$ is given by

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

❖ QUESTION 20

Forces of magnitude P, 2P, 3P and 4P act respectively along the sides AB, BC, CD and DA of a square ABCD of side 'a' and forces each of magnitude $8\sqrt{2}P$ act along the diagonals BD and AC. Find the magnitude of the resultant force and the distance of its line of action from A.

SOLUTION



Let ABCD be a square of side *a*. Forces act along its sides according to given conditions taking one way round. Take AB along x-axis and AD along y-axis.

Let the R be the magnitude of the resultant of the forces, then

$$R = \sqrt{R_x^2 + R_y^2} \qquad \text{\_\_\_\_\_\_\_\_(i)}$$

Now by theorem of resolved parts

R_x = Sum of the resolved parts of the forces along x-axis

$= P\cos 0^0 + 2P\cos 90^0 + 3P\cos 180^0 + 4P\cos 270^0 + 8\sqrt{2}P\cos 45 + 8\sqrt{2}P\cos 135$

$= P + 0 - 3P + 0 + 8\sqrt{2}P\left(\frac{1}{\sqrt{2}}\right) + 8\sqrt{2}P\left(-\frac{1}{\sqrt{2}}\right)$

$= -2P + 8P - 8P = -2P$

And R_y = Sum of the resolved parts of the forces along y-axis

$$= P\sin 0^0 + 2P\sin 90^0 + 3P\sin 180^0 + 4P\sin 270^0 + 8\sqrt{2}P\cos 45 + 8\sqrt{2}P\cos 135$$

$$= 0 + 2P + 0 - 4P + 8\sqrt{2}P\left(\frac{1}{\sqrt{2}}\right) + 8\sqrt{2}P\left(\frac{1}{\sqrt{2}}\right)$$

$$= -2P + 8P + 8P = 14P$$

Using values of R_x and R_y in (i), we get

$$R = \sqrt{(-2P)^2 + (14P)^2} = \sqrt{200P^2} = 10\sqrt{2}P$$

Now

G = sum of the moments of the forces about A.

$$= 2P(AB) + 3P(AD) + 8\sqrt{2}P(AE)$$

$$= 2Pa + 3Pa + 8\sqrt{2}P(AB\sin 45^0)$$

$$= 5Pa + 8\sqrt{2}Pa\left(\frac{1}{\sqrt{2}}\right) = 13Pa$$

The equation of line of action of resultant is

$$G - xR_y + yR_x = 0$$

$\Rightarrow$ $13Pa - x(14P) + y(-2P) = 0$

$\Rightarrow$ $13a - 14x - 2y = 0$

The distance of line of action of resultant from A is:

$$\frac{|0 + 0 + 13a|}{\sqrt{(-14)^2 + (-2)^2}} = \frac{13a}{\sqrt{200}} = \frac{13a}{10\sqrt{2}}$$

Which is required.

❖ CIRCUMCENTRE OF A TRIANGLE

Circumcentre of the triangle is a point at which right bisector of the triangle meet with one another.

❖ QUESTION 21

The three forces P, Q and R act along the sides BC, CA and AB respectively of a triangle ABC. Prove that if

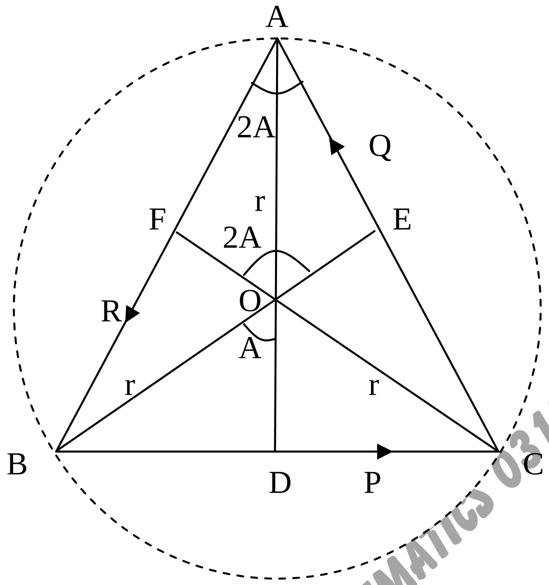
$$\mathbf{P\cos A + Q\cos B + R\cos C = 0}$$

Then the line of the action of the resultant passes through the circumcentre of the triangle.

SOLUTION

Let ABC be a triangle and AD, BE and CF are right bisector of the triangle. O be the circumcentre and r be the circumradius of the triangle. Then

$$AO = BO = CO = r$$



Let G be the moment of all forces about O. Then

$$G = P(OD) + Q(OE) + R(OF) \qquad \text{(i)}$$

From fig.

$$OD = BO\cos A = r\cos A$$

$$OE = CO\cos B = r\cos B$$

$$OF = AO\cos C = r\cos C$$

Using these values in (i), we get

$$G = P(r\cos A) + Q(r\cos B) + R(r\cos C) = r(P\cos A + Q\cos B + R\cos C)$$

The line of the action of the resultant passes through O if G = 0.

i.e. $\quad r(P\cos A + Q\cos B + R\cos C) = 0$

$\Rightarrow \quad P\cos A + Q\cos B + R\cos C = 0 \qquad \because r \neq 0$

Thus if

$$P\cos A + Q\cos B + R\cos C = 0$$

Then the line of the action of the resultant passes through O the circumcentre of the triangle.

❖ ORTHOCENTRE OF A TRIANGLE

Orthocentre of the triangle is a point at which altitudes (i.e. perpendicular from the vertices to the opposite sides) of the triangle meet with one another.

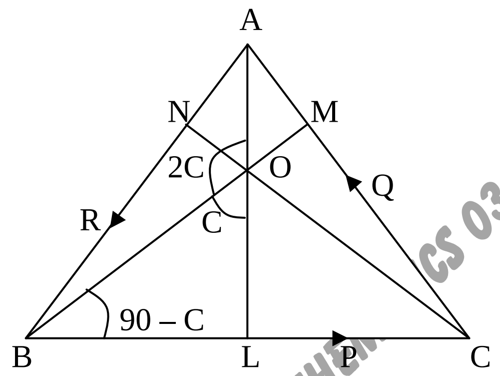
QUESTION 22

The three forces P, Q and R act along the sides BC, CA and AB respectively of a triangle ABC. Prove that if

$$\textbf{P secA + Q secB + R secC = 0}$$

Then the line of the action of the resultant passes through the orthocentre of the triangle.

SOLUTION



Let ABC be a triangle and O be the orthocentre. Draw perpendiculars AL on BC, BM on AC and CN on AB. These are also called altitudes. Let a, b and c be the lengths of the sides BC, CA and AB respectively.

Let G be the moment of all forces about O. Then

$$G = P(OL) + Q(OM) + R(ON) \qquad \text{\_\_\_\_\_\_\_\_\_\_(i)}$$

From fig.

$$\angle LBO = 90^0 - C$$

$$\frac{OL}{BL} = \tan(90^0 - C)$$

$$\Rightarrow \quad \frac{OL}{BL} = \cot C$$

$$\Rightarrow \quad OL = BL \cot C \qquad \text{\_\_\_\_\_\_\_\_\_\_(ii)}$$

In ΔABL

$$\frac{BL}{AB} = \cos B$$

$$\Rightarrow \quad \frac{BL}{c} = \cos B$$

$$\Rightarrow \quad BL = c\cos B$$

Using value of BL in (ii), we get

$$OL = c\cos B\,\frac{\cos C}{\sin C}$$

$$\Rightarrow \quad OL = \frac{c}{\sin C}\cos B\cos C \qquad \text{\_\_\_\_\_\_\_\_\_\_(iii)}$$

By law of sine

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k \qquad \text{\_\_\_\_\_\_\_\_\_\_(iv)}$$

From (iii) and (iv), we get

$$OL = k\cos B\cos C$$

Similarly

$$OM = k\cos A\cos C$$

$$ON = k\cos A\cos B$$

Using values of OL, OM and ON in (i), we get

$$G = P(k\cos B\cos C) + Q(k\cos A\cos C) + R(k\cos A\cos B)$$

$$= k(P\cos B\cos C + Q\cos A\cos C) + R\cos A\cos B)$$

The line of the action of the resultant passes through O if G = 0.

i.e. $k(P\cos B\cos C + Q\cos A\cos C + R\cos A\cos B) = 0$

$\Rightarrow$ $P\cos B\cos C + Q\cos A\cos C + R\cos A\cos B = 0$

Dividing by $\cos A\cos B\cos C$, we get

$$\frac{P}{\cos A}+\frac{Q}{\cos B}+\frac{R}{\cos C}=0$$

$\Rightarrow$ $P\sec A + Q\sec B + R\sec C = 0$

Thus if

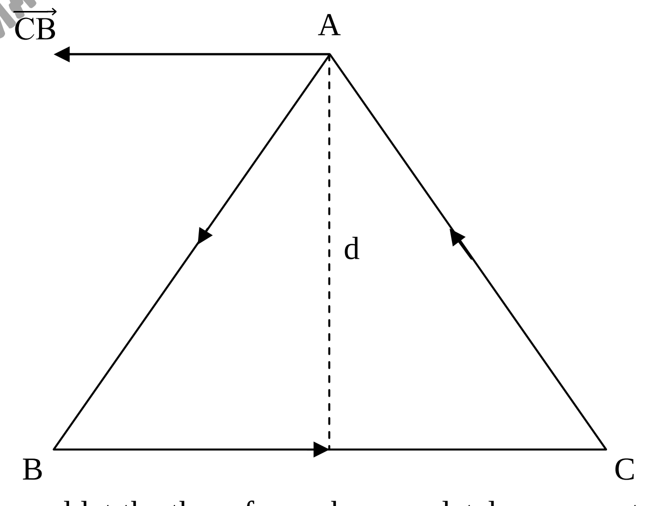
$$P\sec A + Q\sec B + R\sec C = 0$$

Then the line of the action of the resultant passes through O the orthocentre of the triangle.

❖ THEOREM

If three forces are represented in magnitude, direction and position by the sides of a triangle taken in order. They are equivalent to a couple. The magnitude of the moment of the couple is equal to the twice the area of the triangle.

SOLUTION



Let ABC be a triangle and let the three forces be completely represented by $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$ as shown in figure. Then

$$\overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB} = 0$$

$$\Rightarrow \quad \overrightarrow{CA} + \overrightarrow{AB} = -\overrightarrow{BC}$$

$$\Rightarrow \quad \overrightarrow{CA} + \overrightarrow{AB} = \overrightarrow{CB}$$

Which shows that the forces $\overrightarrow{CA}$ and $\overrightarrow{AB}$ acting at A are equivalent to a force $\overrightarrow{CB}$ which acts at A. Thus the given three forces are equivalent to two forces $\overrightarrow{CB}$ acting at A and $\overrightarrow{BC}$ along the side BC of the triangle. These two forces form a couple. If d denotes the length of the perpendicular from A to BC. Then

$$\text{Magnitude of moment of the couple} = (BC)d$$
$$= 2\left(\frac{1}{2}BC.d\right)$$
$$= 2(\text{Area of the triangle})$$

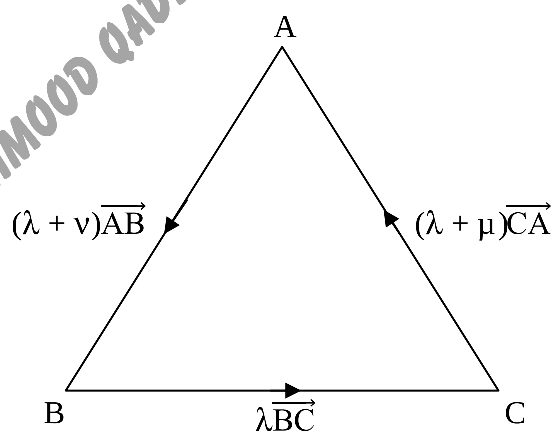
This completes the proof.

QUESTION 23

Forces act along the sides BC, CA and AB of a triangle. Show that they are equivalent to a couple only if the forces are proportional to the sides.

SOLUTION

Let forces $\lambda\overrightarrow{BC}$, $(\lambda + \mu)\overrightarrow{CA}$ and $(\lambda + \nu)\overrightarrow{AB}$ act along the sides of a triangle ABC.



Then

$$\lambda\overrightarrow{BC} + (\lambda + \mu)\overrightarrow{CA} + (\lambda + \nu)\overrightarrow{AB} = \lambda(\overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB}) + \mu\overrightarrow{CA} + \nu\overrightarrow{AB}$$

Since the forces $\overrightarrow{BC}$, $\overrightarrow{CA}$ and $\overrightarrow{AB}$ are equivalent to a couple whose moment is twice the area of the triangle ABC, we have

$$\lambda\overrightarrow{BC} + (\lambda + \mu)\overrightarrow{CA} + (\lambda + \nu)\overrightarrow{AB} = \text{a couple} + \mu\overrightarrow{CA} + \nu\overrightarrow{AB}$$

The system is equivalent to a couple only if

$\mu\overrightarrow{CA} + \nu\overrightarrow{AB} = 0$

Which holds only if $\mu = \nu = 0$

Thus the forces along the sides of the triangle are $\lambda\overrightarrow{BC}$, $\lambda\overrightarrow{CA}$ and $\lambda\overrightarrow{AB}$.

Hence forces acting along the sides of a triangle are equivalent to a couple only if they are proportional to the sides of triangle.

%%%% ***End of The Chapter # 1%%%%***