Exercise No. 9.7

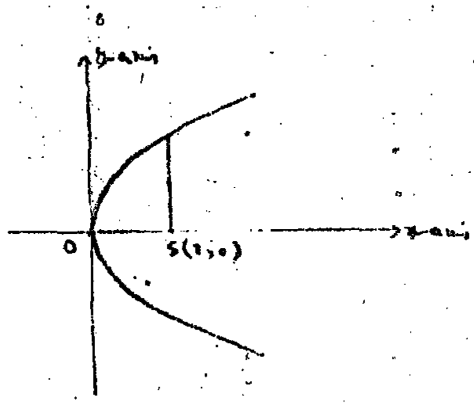
Q1 Find the volume generated by revolving the area in first quadrant bounded by the parabola $y^2 = 8x$ & its latus rectum about the x-axis.

Sol. Given eq. of parabola is

$$y^2 = 8x$$

& eq. of latus rectum is

$$x = 2$$



Let V be the req. volume

then

$$V = \pi\int_0^2 y^2\,dx$$

$$= \pi\int_0^2 8x\,dx$$

$$= 8\pi\left|\frac{x^2}{2}\right|_0^2$$

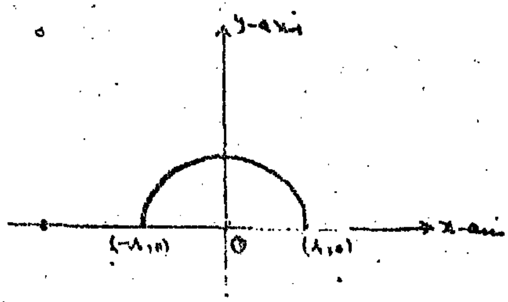
$$= \frac{8\pi}{2}\left[(2)^2-(0)^2\right]$$

$$= 4\pi(4-0)$$

$$= 16\pi \text{ cubic units.}$$

Q2 Find the volume of a sphere of radius r.

Sol. A sphere is the volume generated by a circle $x^2+y^2 = r^2$ when it is revolved about x-axis.



let V be req. volume

then

$$V = \pi\int_{-r}^{r} y^2\,dx$$

$$= \pi \int_{-r}^{r} (r^2 - x^2)\, dx$$

$$= 2\pi \int_{0}^{r} (r^2 - x^2)\, dx.$$

$$= 2\pi \left| r^2 x - \frac{x^3}{3} \right|_0^r$$

$$= 2\pi \left[r^2(r) - \frac{r^3}{3} \right]$$

$$= 2\pi \left[r^3 - \frac{r^3}{3} \right]$$

$$= 2\pi \left[\frac{3r^3 - r^3}{3} \right]$$

$$= 2\pi \left(\frac{2r^3}{3} \right)$$

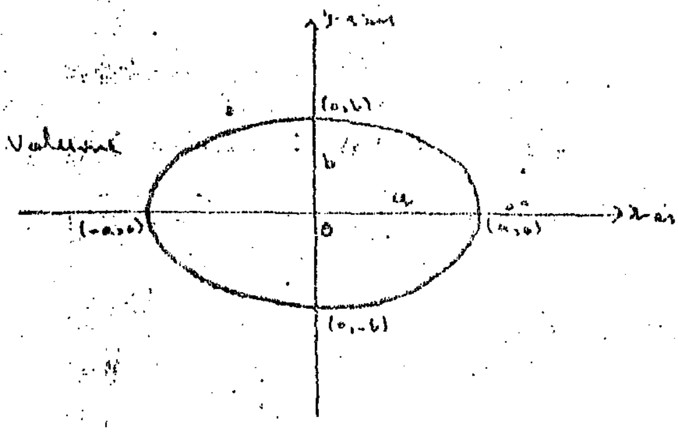
$$\therefore V = \frac{4}{3}\pi r^3 \text{ Cubic units.}$$

Q3. Find the Volume of the prolate spheroid formed by the revolution of the area bounded by ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about (i) major axis (ii) minor axis.

Sol: Given eq. of ellipse is

(i)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



Let V be the req. Volume

then

$$V = \pi \int_{-a}^{a} y^2\, dx$$

$$= \pi \int_{-a}^{a} b^2 \left(1 - \frac{x^2}{a^2}\right) dx$$

$$= \pi \int_{-a}^{a} \frac{b^2}{a^2} (a^2 - x^2)\, dx$$

$$= \frac{\pi b^2}{a^2} \int_{-a}^{a} (a^2 - x^2)\, dx$$

$$= \frac{2\pi b^2}{a^2}\int_0^a (a^2 - x^2)\,dx$$

$$= \frac{2\pi b^2}{a^2}\left| a^2 x - \frac{x^3}{3} \right|_0^a$$

$$= \frac{2\pi b^2}{a^2}\left[a^2(a) - \frac{a^3}{3} \right]$$

$$= \frac{2\pi b^2}{a^2}\left[a^3 - \frac{a^3}{3} \right]$$

$$= \frac{2\pi b^2}{a^2}\left(\frac{2a^3}{3} \right)$$

$V = \frac{4}{3}\pi a b^2$ cubic units.

(ii) When the ellipse is rotated about y-axis then req. volume is

$$V = \pi\int_{-b}^{b} x^2\,dy$$

$$= \pi\int_{-b}^{b} a^2\left(1 - \frac{y^2}{b^2}\right)dy$$

$$= \pi\int_{-b}^{b} \frac{a^2}{b^2}(b^2 - y^2)\,dy$$

$$= \frac{\pi a^2}{b^2}\int_{-b}^{b}(b^2 - y^2)\,dy$$

$$= \frac{2\pi a^2}{b^2}\int_0^b (b^2 - y^2)\,dy$$

$$= \frac{2\pi a^2}{b^2}\left| b^2 y - \frac{y^3}{3} \right|_0^b$$

$$= \frac{2\pi a^2}{b^2}\left[b^3 - \frac{b^3}{3} \right]$$

$$= \frac{2\pi a^2}{b^2}\left(\frac{2b^3}{3} \right)$$

$V = \frac{4}{3}\pi b a^2$ cubic units.

Q4 Find the volume of the solid generated by revolving the area enclosed by $y = 2x$ & $y = x^2$ about the y-axis.

Sol. Given eqs. are,

$$y = 2x \quad \text{------} \quad (1)$$

$$y = x^2 \quad \text{------} \quad (2)$$

To find pt. of intersection solving (1) & (2)

Put $y = 2x$ in (2)

$$2x = x^2$$

$$x^2 - 2x = 0$$

$$x(x-2) = 0$$

$$\Rightarrow \boxed{x = 0, 2}$$

Put in (1)

For $x = 0$, $y = 0$

For $x = 2$, $y = 4$

So pts. of intersection are $(0,0)$ & $(2,4)$.

Let V be the req. volume then

$$V = \int_0^4 \left[\pi(\sqrt{y})^2 - \pi\left(\frac{y}{2}\right)^2\right] dy$$

$$= \pi \int_0^4 \left(y - \frac{y^2}{4}\right) dy$$

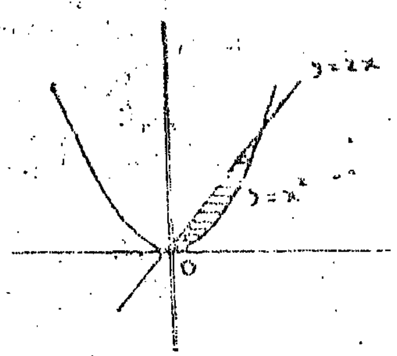
$$= \pi \left| \frac{y^2}{2} - \frac{y^3}{12} \right|_0^4$$

$$= \pi \left\{ \frac{16}{2} - \frac{64}{12} \right\}$$

$$= \pi \left\{ 8 - \frac{16}{3} \right\}$$

$$= \pi \left(\frac{24-16}{3} \right)$$

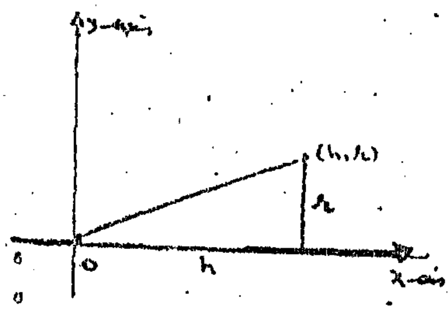
$$V = \frac{8}{3}\pi \text{ Cubic units}$$



Q5 Find the volume of a right circular cone is having base radius r & height h.

Sol:-

A right circular cone is generated when the line OA is revolved about x-axis where co-ords. of pt. A is $A(h, r)$



Now eq. of line OA is

$$\frac{x-0}{h-0} = \frac{y-0}{r-0}$$

$$\frac{y}{r} = \frac{x}{h}$$

$$\Rightarrow \boxed{y = \frac{r}{h}x}$$

let V be the req. volume then

$$V = \pi\int_0^h y^2\,dx$$

$$= \pi\int_0^h \frac{r^2}{h^2}x^2\,dx$$

$$= \frac{\pi r^2}{h^2}\int_0^h x^2\,dx$$

$$= \frac{\pi r^2}{h^2}\left|\frac{x^3}{3}\right|_0^h$$

$$= \frac{\pi r^2}{h^2}\left(\frac{h^3}{3}\right)$$

$$V = \frac{\pi r^2 h}{3} \text{ cubic units}$$

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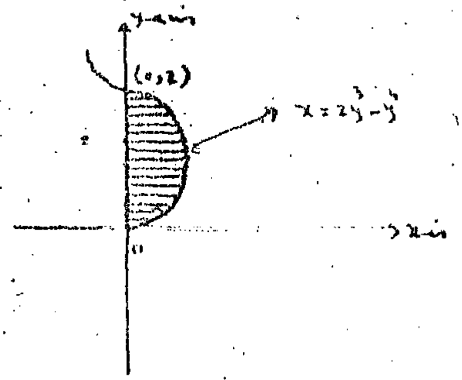
Q6 The area in the first quadrant bounded by $x = 2y^3 - y^4$ & the y-axis is revolved about x-axis. Find the volume of the resulting solid. 116

Sol: Given eq of curve is

$x = 2y^3 - y^4$



let V be the req. volume

Then

$$V = \int_0^2 2\pi y f(y)\, dy$$

$$= 2\pi \int_0^2 y(2y^3 - y^4)\, dy$$

$$= 2\pi \int_0^2 (2y^4 - y^5)\, dy$$

$$= 2\pi \left| 2\frac{y^5}{5} - \frac{y^6}{6} \right|_0^2$$

$$= 2\pi \left[\frac{2}{5}(2)^5 - \frac{(2)^6}{6} \right]$$

$$= 2\pi \left[\frac{64}{5} - \frac{64}{6} \right]$$

$$= 2\pi(64)\left[\frac{1}{5} - \frac{1}{6} \right]$$

$$= 128\pi \left(\frac{6-5}{30} \right)$$

$$= 128\pi \left(\frac{1}{30} \right)$$

$$= \frac{128\pi}{30}$$

$$V = \frac{64\pi}{15} \text{ cubic units.}$$

Q7 A basin is formed by the revolution of the area bounded by the curve $x^3 = 64y\,(y>0)$ about the axis of y. If the depth of the basin is 8 cm, how many cubic cm. of water would it hold?

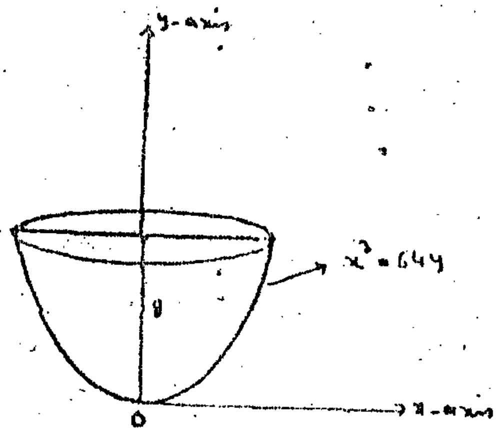
Sol. Here depth of basin is 8 cm.

& the curve passes through origin

Therefore limits for y are $y=0$ & $y=8$

let V be the req. Volume then

$$V = \pi\int_0^8 x^2\,dy$$



$$= \pi\int_0^8 (64y)^{2/3}\,dy$$

$$= \pi\int_0^8 16\,y^{2/3}\,dy$$

$$= 16\pi\int_0^8 y^{2/3}\,dy$$

$$= 16\pi\left|\frac{y^{2/3+1}}{\frac{2}{3}+1}\right|_0^8$$

$$= 16\pi\left|\frac{y^{5/3}}{5/3}\right|_0^8$$

$$= 16\pi\times\frac{3}{5}\left|y^{5/3}\right|_0^8$$

$$= \frac{48\pi}{5}\left[(8)^{5/3}\right]$$

$$= \frac{48\pi}{5}\left(2^5\right)$$

$$= \frac{48\pi}{5}(32)$$

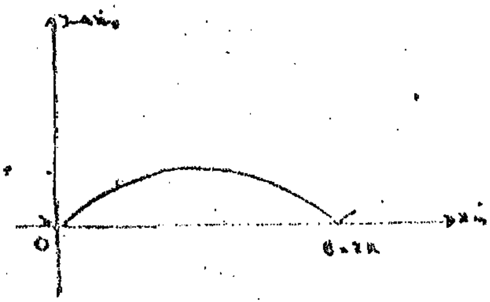
$$V = \frac{1536\pi}{5} \text{ Cubic Cm.}$$

Q8 Show that the volume generated by revolving area bounded by an arc of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$ about its base is $5\pi^2 a^3$.

Sol.
Given eq. of cycloid is

$$\left.\begin{aligned} x &= a(\theta - \sin\theta) \\ y &= a(1-\cos\theta) \end{aligned}\right]$$

Let V be the req. volume



then $V = \pi\int_0^{2\pi} y^2\, dx$

$$= \pi\int_0^{2\pi} a^2(1-\cos\theta)^2 \cdot a(1-\cos\theta)\, d\theta$$

$$= \pi a^3 \int_0^{2\pi} (1-\cos\theta)^3\, d\theta$$

$$= 2\pi a^3 \int_0^{\pi} (1-\cos\theta)^3\, d\theta$$

$$= 2\pi a^3 \int_0^{\pi} (2\sin^2\theta/2)^3\, d\theta$$

$$= 2\pi a^3 (8)\int_0^{\pi} \sin^6\theta/2\, d\theta$$

Put $\frac{\theta}{2} = t$ as $\theta = 0$, $t = 0$; $\theta = \pi$, $t = \pi/2$

$\theta = 2t$

$d\theta = 2dt$

So $V = 16\pi a^3 \int_0^{\pi/2} \sin^6 t \cdot 2dt$

$$= 32\pi a^3 \int_0^{\pi/2} \sin^6 t\, dt$$

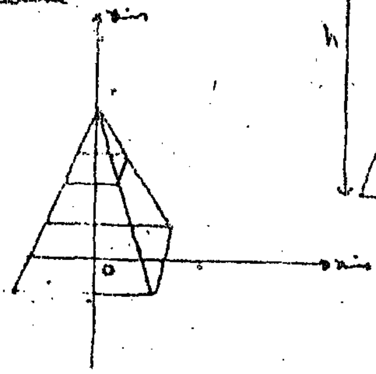
$$= 32\pi a^3 \left[\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\pi/2\right] \quad \text{(By Walli's law)}$$

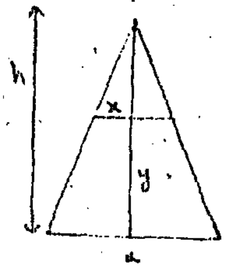
$$V = \frac{32\times 15}{96}\pi^2 a^3 = 5\pi^2 a^3 \text{ cubic units.}$$

Q9. Find the volume of a right pyramid whose height is h & a square base with each side of length a.

Sol. Suppose the axis of pyramid is taken as y-axis & centre of base square of pyramid is at origin O.

Let x be the length of one side of the representative square.





Now from fig.

$$\frac{x}{a} = \frac{y}{h}$$

or $\boxed{x = \frac{a}{h}y}$

Area of the cross-section $= x^2$

$$A(y) = \frac{a^2}{h^2}y^2$$

Let V be the req. volume then

$$V = \int_0^h A(y)\,dy$$

$$= \int_0^h \frac{a^2}{h^2}y^2\,dy$$

$$= \frac{a^2}{h^2}\left|\frac{y^3}{3}\right|_0^h$$

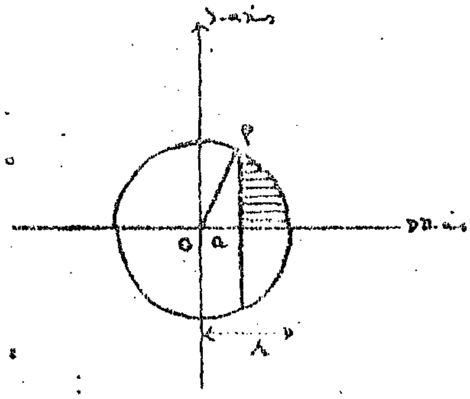
$$= \frac{a^2}{3h^2}(h^3-0)$$

$$V = \frac{1}{3}a^2h \text{ cubic units}$$

Q10 Find the volume of the solid that remains 110 after boring a hole of radius a through the centre of a solid sphere of radius $r > a$.

Sol.. Let the solid sphere is generated by the right half of the circular disc $x^2+y^2 \leq r^2$ about y-axis. Here the axis of the hole coincides with y-axis

Then the volume of solid that remains after boring after revolving shaded part about y-axis is



$$V = 2\int_a^r 2\pi x f(x)\,dx$$

$$= 4\pi \int_a^r x\sqrt{r^2-x^2}\,dx$$

$$= -2\pi \int_a^r (r^2-x^2)^{1/2}(-2x)\,dx$$

$$= -2\pi \left| \frac{(r^2-x^2)^{3/2}}{3/2} \right|_a^r$$

$$= -\frac{4\pi}{3}\left[(r^2-r^2)^{3/2} - (r^2-a^2)^{3/2}\right]$$

$$= -\frac{4\pi}{3}\left[0-(r^2-a^2)^{3/2}\right]$$

$$V = \frac{4\pi}{3}(r^2-a^2)^{3/2} \text{ cubic units.}$$

Q11 Find the volume formed by the revolution of the area enclosed by the loop of the curve $y^2 = x^2\frac{a-x}{a+x}$ about x-axis.

Sol.

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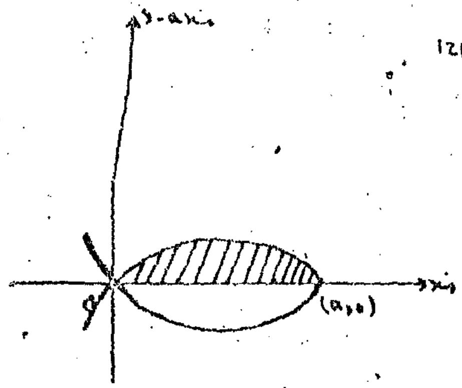
Given eq. of Curve is

$$y^2 = x^2 \frac{a-x}{a+x}$$



To find x-int. Put $y = 0$,

$$\frac{x^2(a-x)}{(a+x)} = 0.$$

$$\Rightarrow x^2(a-x) = 0$$

$$\Rightarrow \boxed{x = 0, a}$$

The Curve is symmetrical about x-axis.

Let V be the req. Volume then

$$V = \pi \int_0^a y^2\, dx$$

$$= \pi \int_0^a \frac{x^2(a-x)}{a+x}\, dx$$

$$= \pi \int_0^a \frac{-x^3 + ax^2}{x+a}\, dx$$

$$= \pi \int_0^a \left(-x^2 + 2ax - 2a^2 + \frac{2a^3}{x+a}\right) dx$$

$-x^2 + 2ax - 2a^2$
$x+a \,\big)\, -x^3 + ax^2$
$\quad -x^3 - ax^2$
$\quad 2ax^2$
$\quad 2ax^2 + 2a^2x$
$\quad -2a^2x$
$\quad -2a^2x - 2a^3$
$\quad 2a^3$

$$= \pi \left| -\frac{x^3}{3} + ax^2 - 2a^2x + 2a^3 \ln(x+a) \right|_0^a$$

$$= \pi \left[\left(-\frac{a^3}{3} + a^3 - 2a^3 + 2a^3 \ln 2a\right) - \left(0 + 0 - 0 + 2a^3 \ln a\right)\right]$$

$$= \pi \left[\frac{-a^3 + 3a^3 - 6a^3}{3} + 2a^3 \ln 2a - 2a^3 \ln a\right]$$

$$= \pi \left[-\frac{4a^3}{3} + 2a^3(\ln 2a - \ln a)\right]$$

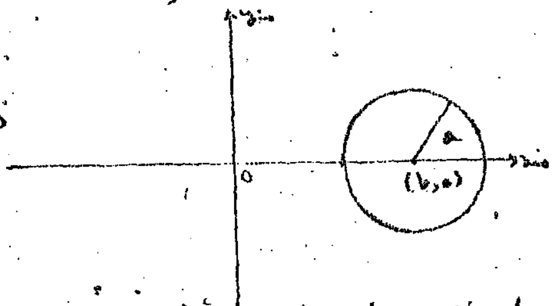
$$= \pi a^3 \left[-\frac{4}{3} + 2 \ln\left(\frac{2a}{a}\right)\right]$$

$$= \pi a^3 \left[-\frac{4}{3} + 2 \ln 2\right]$$

$$= 2\pi a^3 \left[-\frac{2}{3} + \ln 2\right]$$

$$V = 2\pi a^3 (\ln 2 - 2/3)$$ Ans.

Q12. A doughnut-shaped solid called torus (or anchor right) is generated by revolving an area enclosed by a circle about a line that does not intersect the circle. Find the volume of the torus if the circle is $(x-b)^2+y^2=a^2$ & the line is the y-axis $(0<a<b)$.



Sol: Given eq. of circle is

$$(x-b)^2+y^2=a^2.$$

Its centre is $(b,0)$

which is also the centre of gravity of given circle. When the area of the given circle is rotated about y-axis. Then the req. volume of torus formed by using first theorem of Pappus is

$$V = (\text{area})(\text{length of path described by centroid of area})$$

$$= (\pi a^2)(2\pi b)$$

$$V = 2\pi^2 a^2 b$$

Q13 A rectangular area whose length & breadth are 4 & 2 respectively & whose centroid is (4,3) is revolved about

(i) The st. line $x=9$

(ii) The st. line $y=-5$

(iii) The st. line $y=-x$

Find the volume generated in each case.

Sol:(i) Given that length & breadth of rectangular area are 4 & 2 resp.

then area of rectangle $= 4\times 2 = 8$ sq. units

As $C(4,3)$ be the centroid of rectangular area 123 then its distance from the St. line $x = 9$ or $x-9=0$

is $= \dfrac{|4-9|}{\sqrt{1^2+0^2}} = 5$

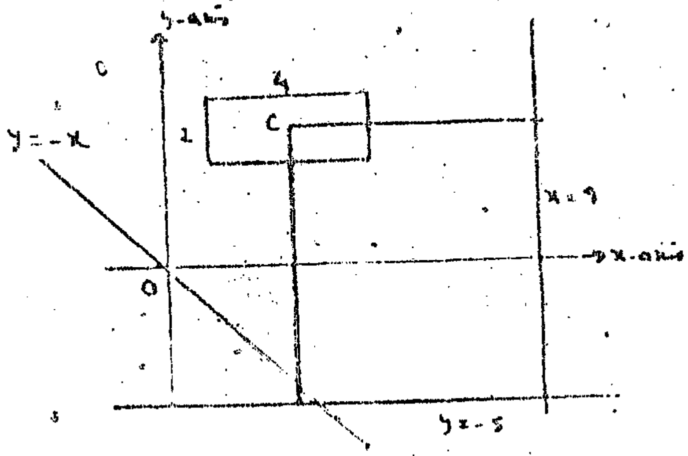
Now the distance covered by the pt. $C(4,3)$ about the line $x = 9$ in one revolution $= 2.\pi.5 = 10\pi$

Let V_1 be the req. volume then by first theorem of Pappus

V_1 = (area)(length of path described by centroid of area)

$= 8 \times 10\pi$

$V_1 = 80\pi$ cubic units



(ii) Now distance of the centroid $C(4,3)$ from St. line $y=-5$ or $y+5=0$ is

$= \dfrac{|3+5|}{\sqrt{0^2+1^2}} = 8$

Here the distance covered by the centroid $C(4,3)$ about the line $y=-5$ is $= 2\pi(8) = 16\pi$

Let V_2 be the req. volume then

by first theorem of Pappus

V_2 = (area)(length of path described by centroid of area)

$= 8 \times 16\pi$

$\therefore V_2 = 128\pi$ cubic units

(iii) Distance of centroid $C(4,3)$ from st. line $y=-x$ or $x+y=0$ is

$$= \frac{|4+3|}{\sqrt{1^2+1^2}} = \frac{7}{\sqrt{2}}$$

Now the distance covered by the centroid $C(4,3)$ about line $y=-x$ is $= 2\pi.\frac{7}{\sqrt{2}} = 7\sqrt{2}\pi$

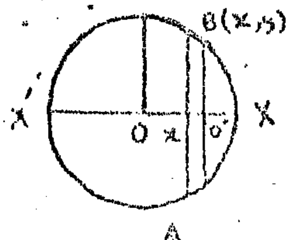
Let V_3 be the req. volume then by first theorem of Pappus

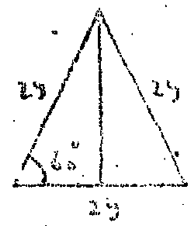
V = (area)(length of path described by centroid of area)

$= 8 \times 7\sqrt{2}\pi$

$= 56\sqrt{2}\pi$ cubic units.

Q14 A solid has a circular base of radius 4 units. Find the volume of the solid if every plane section perpendicular to a fixed distance is an equilateral triangle





Sol. Since the base of the solid is circular, so we take fixed diameter $X'OX$, i.e., x-axis with centre at origin

If AB represents one section with breadth δx then AB also forms one side of the equilateral Δ. The eq. of circle is $x^2+y^2=16$ 125

& $|AB| = 2y$

Now area of equilateral Δ whose each side $2y$ is

$= \frac{1}{2}\cdot 2y\cdot 2y\sin 60^\circ$

$= 2y^2\cdot\frac{\sqrt{3}}{2}$

$= \sqrt{3}y^2$

Then volume of one triangular strip is $= \sqrt{3}y^2\delta x$

Hence the req. volume $= \int_{-4}^{4}(\sqrt{3}y^2)dx$

$$V = 2\int_0^4 \sqrt{3}(16-x^2)\,dx$$

$$= 2\sqrt{3}\int_0^4 (16-x^2)\,dx$$

$$= 2\sqrt{3}\left| 16x - \frac{x^3}{3}\right|_0^4$$

$$= 2\sqrt{3}\left[16(4) - \frac{(4)^3}{3}\right]$$

$$= 2\sqrt{3}\left[64 - \frac{64}{3}\right]$$

$$= 2\sqrt{3}\left[\frac{192-64}{3}\right]$$

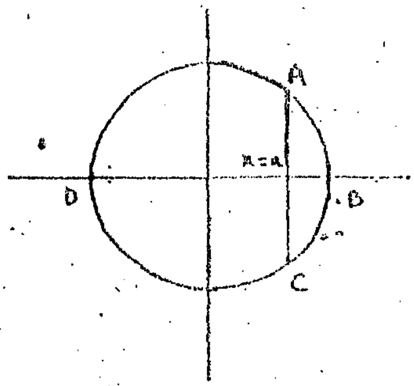
$$= 2\sqrt{3}\left[\frac{128}{3}\right]$$

$$V = \frac{256\sqrt{3}}{3} \text{ Cubic units.}$$

Q15 Find the volumes of the two portions in which a sphere of radius r is divided by the plane $x = a$ $(a < r)$

Sol. We know that a sphere is a surface which is generated by revolving the circle $x^2+y^2=r^2$ about x-axis.

When this sphere is cut by the plane $x = a$ is divided into two portions ABC & ADC.



Now volume of portion ABC is $V_1 = \pi\int_a^r y^2\,dx$

$$= \pi\int_a^r (r^2-x^2)\,dx$$

$$= \pi\left|r^2x - \frac{x^3}{3}\right|_a^r$$

$$= \pi\left[\left(r^3 - \frac{r^3}{3}\right) - \left(r^2a - \frac{a^3}{3}\right)\right]$$

$$= \pi\left[\frac{2r^3}{3} - r^2a + \frac{a^3}{3}\right]$$

$$= \frac{2\pi}{3}r^3 - \pi r^2 a + \frac{\pi a^3}{3}$$

$$V_1 = \frac{2\pi}{3}r^3 - \frac{a\pi}{3}(3r^2 - a^2)$$

But total volume of sphere of radius r is $\frac{4}{3}\pi r^3$

So volume of portion ADC

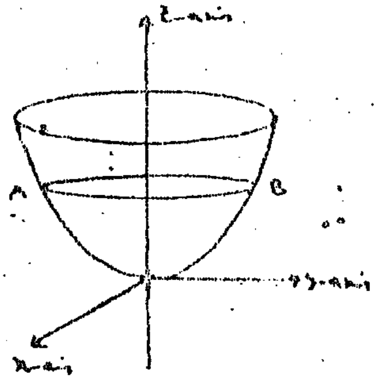
$$= V_2 = \frac{4}{3}\pi r^3 - \frac{2\pi}{3}r^3 + \frac{a\pi}{3}(3r^2-a^2) = \frac{2\pi}{3}r^3 + \frac{a\pi}{3}(3r^2-a^2)$$

Q16 Find the volume of the solid cut off from 127 the paraboloid $\frac{x^2}{16} + \frac{y^2}{25} = z$ by the plane $z = 10$

Sol.

Given eq. of surface is

$$\frac{x^2}{16} + \frac{y^2}{25} = z$$



Here the [illegible] is cut by a plane $z = 10$ & suppose AB be a representative ellipse on the paraboloid then

$$\frac{x^2}{16} + \frac{y^2}{25} = z$$

$$\Rightarrow \frac{x^2}{16z} + \frac{y^2}{25z} = 1 \quad \text{or} \quad \frac{x^2}{(4\sqrt{z})^2} + \frac{y^2}{(5\sqrt{z})^2} = 1$$

is the eq. of ellipse AB

Its area $= \pi(4\sqrt{z})(5\sqrt{z})$

$$A(z) = 20\pi z$$

Then the vol. of the elliptic disc with breadth δz is

$$V = \int_0^{10} A(z)\,dz$$

$$= \int_0^{10} 20\pi z\,dz$$

$$= 20\pi \int_0^{10} z\,dz$$

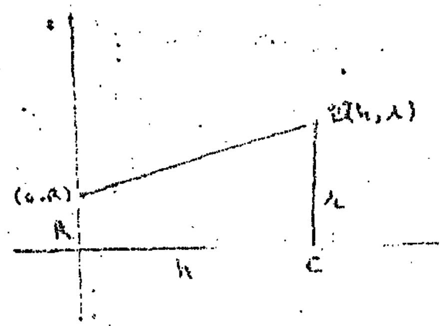
$$= 20\pi \cdot \left|\frac{z^2}{2}\right|_0^{10}$$

$$= 10\pi(100 - 0) = 1000\pi \text{ cubic units}$$

Q17 Find the volume of a frustum of a right circular cone of altitude h, lower base radius R & upper base radius r. 128

Sol. The frustum of the cone is generated when the area bounded by the line AB, $x=0$, $x=h$ & x-axis is revol.

Now eq. of line AB is



$$y - R = \frac{r-R}{h-0}(x-0)$$

$$y - R = \frac{r-R}{h}x$$

or $$y = R + \left(\frac{r-R}{h}\right)x$$

Let V be the req. volume then

$$V = \pi\int_0^h y^2\,dx$$

$$= \pi\int_0^h \left(R + \frac{r-R}{h}x\right)^2 dx$$

$$= \frac{\pi h}{r-R}\int_0^h \left(R + \frac{r-R}{h}x\right)^2 \cdot \frac{r-R}{h}\,dx$$

$$= \frac{\pi h}{r-R}\left|\frac{\left(R + \frac{r-R}{h}x\right)^3}{3}\right|_0^h$$

$$= \frac{\pi h}{3(r-R)}\left[\left(R + \frac{r-R}{h}h\right)^3 - (R+0)^3\right]$$

$$= \frac{\pi h}{3(r-R)}\left[\left(\frac{R\cancel{h} + rh - R\cancel{h}}{h}\right)^3 - R^3\right]$$

$$= \frac{\pi h}{3(r-R)}\left[r^3 - R^3\right] = \frac{1}{3}\pi h\left(r^2 + rR + R^2\right) \text{ — Ans.}$$