## ※ Exercise No. 9.6 ※



Find the extrema of each of the following (Prbs 1 – 14):

**Q1** $f(x,y) = x^2 - xy + y^2 + 6x$

**Sol.** Given eq. is,

$$f(x,y) = x^2 - xy + y^2 + 6x.$$

Then

$$f_x = 2x - y + 6$$

$$f_{xx} = 2$$

$$f_y = -x + 2y$$

$$f_{yy} = 2$$

$$f_{xy} = -1$$

For critical pts., Put $f_x = 0$ & $f_y = 0$

$$\Rightarrow 2x - y + 6 = 0 \quad \text{---} (1)$$

$$-x + 2y = 0 \quad \text{---} (2)$$

Multiplying (1) by 2 & adding in (2)

$$\begin{aligned} 4x - \cancel{2y} + 12 &= 0 \\ -x + \cancel{2y} &= 0 \\ \hline 3x + 12 &= 0 \end{aligned}$$

$$\boxed{x = -4}.$$

Put in (2)

$$-(-4) + 2y = 0$$

$$4 + 2y = 0$$

$$2 + y = 0$$

$$\boxed{y = -2}$$

So $(-4, -2)$ is the critical pt.

<u>At $(-4, -2)$</u>

$A = f_{xx} = 2$

$B = f_{xy} = -1$

$C = f_{yy} = 2$

$D = B^2 - AC$

$= (-1)^2 - (2)(2)$

$= 1 - 4$

$D = -3$

Since $D < 0$ & $A$ & $C$ are both +ve

So $f$ has a local minimum at $(-4, -2)$.

Q2 $f(x,y) = \frac{1}{x} + xy - \frac{8}{y}$

Sol. Given eq. is

$f(x,y) = \frac{1}{x} + xy - \frac{8}{y}$

$\Rightarrow f_x = \frac{-1}{x^2} + y$

$f_{xx} = \frac{2}{x^3}$

$f_y = x + \frac{8}{y^2}$

$f_{yy} = -\frac{16}{y^3}$

$f_{xy} = 1$

For critical pts. $f_x = 0$, $f_y = 0$

$\Rightarrow \frac{-1}{x^2} + y = 0$ ——— (1)

$x + \frac{8}{y^2} = 0$ ——— (2)

from (1), $y = \frac{1}{x^2}$ ——— (3)

Put in (2)

$x + \frac{8}{1/x^4} = 0$

$x + 8x^4 = 0$

$x(1 + 8x^3) = 0$

$\Rightarrow x = 0$ or $x^3 = -\frac{1}{8}$

$\boxed{x = 0}$ or $\boxed{x = -\frac{1}{2}}$

But given ... is not defined at $x = 0$

Hence $\boxed{x = -\frac{1}{2}}$

Put in eq. (3)

$y = \frac{1}{(-1/2)^2}$

$y = \frac{1}{\frac{1}{4}}$

$\boxed{y = 4}$

So $(-\frac{1}{2}, 4)$ is the critical pt.

<u>At $(-\frac{1}{2}, 4)$</u>

$A = f_{xx} = \frac{2}{(-1/2)^3} = \frac{2}{-\frac{1}{8}} = -16$

$B = f_{xy} = 1$

$C = f_{yy} = -\frac{16}{(4)^3} = -\frac{16}{64} = -\frac{1}{4}$

$D = B^2 - AC$

$= (1)^2 - (-16)(-1/4)$

$= 1 - 4$

$\boxed{D = -3}$

Since $D < 0$ & both $A$ & $C$ are $-$ve

So $f$ has a local max. at $(-\frac{1}{2}, 4)$

**Q3** $f(x,y) = 2x^2 + xy^2 - 4x - 1$

**Sol:** Given eq. is

$$f(x,y) = 2x^2 + xy^2 - 4x - 1$$

Then $f_x = 4x + y^2 - 4$

$f_{xx} = 4$

$f_y = 2xy$

$f_{yy} = 2x$

$f_{xy} = 2y$



For critical pts $f_x = 0$ & $f_y = 0$

$\Rightarrow$ $4x + y^2 - 4 = 0$ ——— (1)

$2xy = 0$ ——— (2)

from (2) $xy = 0$

$\Rightarrow x = 0$ or $y = 0$

If $x = 0$ then from (1) $y^2 - 4 = 0 \Rightarrow y = \pm 2$

If $y = 0$ then from (1) $4x - 4 = 0 \Rightarrow x = 1$

Hence $(0,2), (0,-2)$ & $(1,0)$ are critical pts.

**At (0,2)**

$A = f_{xx} = 4$

$B = f_{xy} = 2(2) = 4$

$C = f_{yy} = 2(0) = 0$

$D = B^2 - AC$

$= (4)^2 - (4)(0)$

$\boxed{D = 16}$

Since $D > 0$ so $(0,2)$ is a saddle pt.

Now we take pt. $(0,-2)$

<u>At (0, -2)</u>

$A = f_{xx} = 4$

$B = f_{xy} = 2(-2) = -4$

$C = f_{yy} = 2(0) = 0$

$D = B^2 - AC$

$= (-4)^2 - 4(0)$

$\boxed{D = 16}$

Since $D > 0$. So $(0,-2)$ is a saddle pt.

<u>At (1,0)</u>

$A = f_{xx} = 4$

$B = f_{xy} = 2(0) = 0$

$C = f_{yy} = 2(1) = 2$

$D = B^2 - AC$

$= (0)^2 - (4)(2)$

$\boxed{D = -8}$

Since $D < 0$, so $f$ has a local minimum at $(1,0)$

<u>Q4</u> $f(x,y) = x^2 + 6xy + 2y^2 - 6x + 10y - 5$

<u>Sol.</u> Given:

$f(x,y) = x^2 + 6xy + 2y^2 - 6x + 10y - 5$

$f_x = 2x + 6y - 6$

$f_{xx} = 2$

$f_y = 6x + 4y + 10$

$f_{yy} = 4$

$f_{xy} = 6$

For critical pts. $f_x = 0$ & $f_y = 0$

$$\Rightarrow \quad \left.\begin{aligned} 2x+6y-6 &= 0 \\ 6x+4y+10 &= 0 \end{aligned}\right]$$

or $x+3y-3=0$ ——— ①

$3x+2y+5=0$ ——— ②

Multiplying ① by 3

$3x+9y-9=0$ ——— ①

$\underline{-3x \pm 2y \pm 5 = 0}$ ——— ②

$7y-14=0$

$7y=14$

$\boxed{y=2}$

Put in ①

$x+3(2)-3=0$

$x+3=0$

$\boxed{x=-3}$

So $(-3,2)$ is the critical pt.



At $(-3,2)$

$A = f_{xx} = 2$

$B = f_{xy} = 6$

$C = f_{yy} = 4$

Now $D = B^2 - AC$

$= (6)^2 - (2)(4)$

$= 36-8$

$\boxed{D = 28}$

Since $D>0$

So $(-3,2)$ is a saddle pt.

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**Q5** $f(x,y) = 6x^3y^2 - x^4y^2 - x^3y^3$

**Sol.** Given

$$f(x,y) = 6x^3y^2 - x^4y^2 - x^3y^3$$

$$\Rightarrow f_x = 18x^2y^2 - 4x^3y^2 - 3x^2y^3$$

$$f_{xx} = 36xy^2 - 12x^2y^2 - 6xy^3$$

$$f_y = 12x^3y - 2x^4y - 3x^3y^2$$

$$f_{yy} = 12x^3 - 2x^4 - 6x^3y$$

$$f_{xy} = 36x^2y - 8x^3y - 9x^2y^2$$

For critical pts. $f_x = 0$ & $f_y = 0$

$$\Rightarrow \left.\begin{aligned} 18x^2y^2 - 4x^3y^2 - 3x^2y^3 &= 0 \\ 12x^3y - 2x^4y - 3x^3y^2 &= 0 \end{aligned}\right]$$

or
$$\left.\begin{aligned} x^2y^2(18-4x-3y) &= 0 \\ x^3y(12-2x-3y) &= 0 \end{aligned}\right]$$

$$\Rightarrow 18 - 4x - 3y = 0 \quad \text{——— (1)}$$

$$\underline{-12 \mp 2x \mp 3y = 0} \quad \text{——— (2)}$$

$$6 - 2x = 0$$

$$2x = 6$$

$$\boxed{x = 3}$$

Put in (1)

$$18 - 4(3) - 3y = 0$$

$$18 - 12 - 3y = 0$$

$$6 - 3y = 0$$

$$3y = 6 \Rightarrow \boxed{y = 2}$$

So $(3,2)$ is the critical pt.

At $(3,2)$

$A = f_{xx} = 36(3)(2)^2 - 12(3)^2(2)^2 - 6(3)(2)^3 = 432 - 432 - 144 = -144$

$B = f_{xy} = 36(3)^2(2) - 8(3)^3(2) - 9(3)^2(2)^2 = 648 - 432 - 324 = -108$.

$C = f_{yy} = 12(3)^3 - 2(3)^4 - 6(3)^3(2) = 324 - 162 - 324 = -162$

$D = B^2 - AC$

$= (-108)^2 - (-144)(-162)$

$= 11664 - 23328$

$\boxed{D = -11664}$

Since $D < 0$ & $A, C$ are both $-ve$

So $f$ has a local max. at $(3,2)$

Q6 $\quad f(x,y) = 2x^4 + y^2 - x^2 - 2y$

Soln. Given,

$f(x,y) = 2x^4 + y^2 - x^2 - 2y$

$\Rightarrow f_x = 8x^3 - 2x$

$f_{xx} = 24x^2 - 2$

$f_y = 2y - 2$

$f_{yy} = 2$

$f_{xy} = 0$

For critical points, we have

$f_x = 0, f_y = 0$

$\Rightarrow \left.\begin{array}{l} 8x^3 - 2x = 0 \\ 2y - 2 = 0 \end{array}\right\}$

or

or $\left.\begin{aligned} 2x(4x^2-1) &= 0 \\ 2y-2 &= 0 \end{aligned}\right]$

or $x(4x^2-1) = 0$ ——— ①

$y-1 = 0$ ——— ②

from ① $x = 0$, $x = \pm\frac{1}{2}$

from ② $y = 1$

So $(0,1)$, $(\frac{1}{2},1)$ & $(-\frac{1}{2},1)$ are critical pts.

At $(0,1)$

$A = f_{xx} = 24(0)^2 - 2 = -2$

$B = f_{xy} = 0$

$C = f_{yy} = 2$

$D = B^2 - AC$

$= (0)^2 - (-2)(2)$

$\boxed{D = 4}$

Since $D > 0$ so $(0,1)$ is a saddle pt.

At $(\frac{1}{2},1)$

$A = f_{xx} = 24(\frac{1}{4}) - 2 = 6-2 = 4$

$B = f_{xy} = 0$

$C = f_{yy} = 2$

$D = B^2 - AC$

$= (0)^2 - (4)(2)$

$\boxed{D = -8}$

Since $D < 0$ & $A, C$ are both +ve

so $f$ has a local minimum at $(\frac{1}{2},1)$

At $(-\frac{1}{2},1)$

$A = f_{xx} = 24(1/4) - 2 = 6-2 = 4$

$B = f_{xy} = 0$

$C = f_{yy} = 2$

$D = B^2 - AC$

$= (0)^2 - (4)(2)$

$\boxed{D = -8}$

Since $D < 0$ & $A, C$ are both +ve

so $f$ has a local minimum at $(-\frac{1}{2}, 1)$

Q7 $f(x,y) = 18x^2 - 32y^2 - 36x - 128y$

Soln Given

$f(x,y) = 18x^2 - 32y^2 - 36x - 128y$

$\Rightarrow f_x = 36x - 36$

$f_{xx} = 36$

$f_y = -64y - 128$

$f_{yy} = -64$

$f_{xy} = 0$

For critical pts.

$f_x = 0$ & $f_y = 0$

$\Rightarrow \left.\begin{array}{l} 36x - 36 = 0 \\ -64y - 128 = 0 \end{array}\right]$

or $x - 1 = 0$ —— (1)

$y + 2 = 0$ —— (2)

from (1) $\boxed{x = 1}$

from (2) $\boxed{y = -2}$

So $(1,-2)$ is the critical pt.

At $(1,-2)$

$A = f_{xx} = 36$

$B = f_{xy} = 0$ 1586



$C = f_{yy} = -64$



$D = B^2 - AC$

$= (0)^2 - (36)(-64)$

$= 0 + 2304$

$\boxed{D = 2304}$

Since $D > 0$ so $(1,-2)$ is the saddle pt.

**Q8** $f(x,y) = e^{-(x^2+y^2+2x)}$

Sol. Given eq. is

$f(x,y) = e^{-(x^2+y^2+2x)}$

$\Rightarrow f_x = e^{-(x^2+y^2+2x)} \cdot -(2x+2) = -2(x+1)\, e^{-(x^2+y^2+2x)}$

$f_{xx} = -2\left[ (x+1)\, e^{-(x^2+y^2+2x)} \cdot -(2x+2) + e^{-(x^2+y^2+2x)} \cdot 1 \right]$

$= 4(x+1)^2 e^{-(x^2+y^2+2x)} - 2 e^{-(x^2+y^2+2x)}$

$= 2 e^{-(x^2+y^2+2x)} \left[ 2(x+1)^2 - 1 \right]$

$f_y = e^{-(x^2+y^2+2x)} \cdot -2y = -2y\, e^{-(x^2+y^2+2x)}$

$f_{yy} = -2\left[ y \cdot e^{-(x^2+y^2+2x)} \cdot (-2y) + e^{-(x^2+y^2+2x)} \cdot 1 \right]$

$f_{yy} = -2 e^{-(x^2+y^2+2x)} \left[ -2y^2 + 1 \right]$

$f_{xy} = -2(x+1) \cdot e^{-(x^2+y^2+2x)} \cdot (-2y)$

$f_{xy} = 4y(x+1) e^{-(x^2+y^2+2x)}$

For critical pts. $f_x = 0$ & $f_y = 0$

$-2(x+1) e^{-(x^2+y^2+2x)} = 0$ ——— ①

$-2y\, e^{-(x^2+y^2+2x)} = 0$ ——— ②

from ① $x+1=0$ 1587 $\because e^{-(x^2+y^2+2x)} \neq 0$

or $\boxed{x=-1}$

from ② $\boxed{y=0}$ $\because e^{-(x^2+y^2+2x)} \neq 0$

So $(-1,0)$ is a critical point.

At $(-1,0)$

$A = f_{xx} = 2e^{-(1+0-2)}\left[2(0)^2-1\right] = 2e^{1}(-1) = -2e$

$B = f_{xy} = 4(0)(-1+1)e^{-(1+0-2)} = 0$

$C = f_{yy} = -2e^{-(1+0-2)}\left[-2(0)^2+1\right] = -2e(1) = -2e$

$D = B^2 - AC$

$= (0)^2 - (-2e)(-2e)$

$\boxed{D = -4e^2}$

Since $D<0$ & $A, C$ are both $-ve$.

so $f$ has a relative max. at $(-1,0)$.

Q9 $f(x,y) = 2x^3+y^2-9x^2-4y+12x-2$

Sol. Given

$f(x,y) = 2x^3+y^2-9x^2-4y+12x-2$

$\Rightarrow f_x = 6x^2-18x+12$

$f_{xx} = 12x-18$

$f_y = 2y-4$

$f_{yy} = 2$

$f_{xy} = 0$

For critical pts. $f_x = 0$ & $f_y = 0$

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$$\left.\begin{aligned} 6x^2-18x+12 &= 0 \\ 2y-4 &= 0 \end{aligned}\right]$$

or $x^2-3x+2=0$ ——— ①

$y-2=0$ ——— ②

from ① $x^2-2x-x+2=0$

$x(x-2)-1(x-2)=0$

$(x-2)(x-1)=0$

$\Rightarrow \boxed{x=1,2}$

from ② $\boxed{y=2}$

Hence $(1,2)$ & $(2,2)$ are critical points.

<u>At $(1,2)$</u>

$A = f_{xx} = 12(1)-18 = -6$

$B = f_{xy} = 0$

$C = f_{yy} = 2$

$D = B^2-AC$

$= (0)^2-(-6)(2)$

$\boxed{D = 12}$

Since $D>0$

So $(1,2)$ is a saddle pt.

<u>At $(2,2)$</u>

$A = f_{xx} = 12(2)-18 = 6$

$B = f_{xy} = 0$

$C = f_{yy} = 2$

$D = B^2-AC$

$= (0)^2-(6)(2)$

$\boxed{D = -12}$

Since $D < 0$ & $A, C$ are both +ve

so $f$ has a local minimum at $(2,2)$

Q10 $f(x,y) = x^2 - e^{y^2}$

Soln Given

$$f(x,y) = x^2 - e^{y^2}$$

$$\Rightarrow f_x = 2x$$

$$f_{xx} = 2$$

$$f_y = -2y e^{y^2}$$

$$f_{yy} = -2\left[y e^{y^2} \cdot 2y + e^{y^2} \cdot 1\right]$$

$$= -4y^2 e^{y^2} - 2e^{y^2}$$

$$f_{yy} = -2e^{y^2}\left[2y^2 + 1\right]$$

$$f_{xy} = 0$$

For critical pts. $f_x = 0$ & $f_y = 0$

$$\Rightarrow 2x = 0 \quad \text{——— (1)}$$

$$-2y e^{y^2} = 0 \quad \text{——— (2)}$$

from (1) $\boxed{x = 0}$

from (2) $\boxed{y = 0}$ $\quad \because e^{y^2} \neq 0$

So $(0,0)$ is a critical pt.

At $(0,0)$

$A = f_{xx} = 2$

$B = f_{xy} = 0$

$C = f_{yy} = -2$

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$D = B^2 - AC$

$= (0)^2 - (2)(-2)$

$\boxed{D = 4}$

Since $D > 0$, so $(0,0)$ is a saddle pt.

Q11 $f(x,y) = \sin x + \sin y$

Sol. Given

$f(x,y) = \sin x + \sin y$

$\Rightarrow f_x = \cos x$

$f_{xx} = -\sin x$

$f_y = \cos y$

$f_{yy} = -\sin y$

$f_{xy} = 0$

For critical pts. $f_x = 0$ & $f_y = 0$

$\Rightarrow \cos x = 0$ —— (1)

& $\cos y = 0$ —— (2)

from (1) $x = m\pi + \pi/2$

from (2) $y = n\pi + \pi/2$ where $m, n \in \mathbb{Z}$

So $(m\pi + \pi/2, n\pi + \pi/2)$ is a critical pt.

At $(m\pi + \pi/2, n\pi + \pi/2)$

$A = f_{xx} = -\sin(m\pi + \pi/2) = \begin{cases} 1 & \text{if } m \text{ is odd} \\ -1 & \text{if } m \text{ is even} \end{cases}$

$B = f_{xy} = 0$

$C = f_{yy} = -\sin(n\pi + \pi/2) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ -1 & \text{if } n \text{ is even} \end{cases}$

Now $D = B^2 - AC$

$= (0)^2 - [-\sin(m\pi + \pi/2)][-\sin(n\pi + \pi/2)]$

$= 0 - \sin(m\pi + \pi/2)\cdot\sin(n\pi + \pi/2)$

$$\boxed{D = -\sin(m\pi + \pi/2)\sin(n\pi + \pi/2)}$$

Case (i)

If m, n are even then $D < 0$ & A, C are both -ve

So f has a local max. at $(m\pi + \pi/2, n\pi + \pi/2)$

Case (ii)

If m, n are odd, then $D < 0$ & A, C are both +ve

So f has a local min. at $(m\pi + \pi/2, n\pi + \pi/2)$

Case (iii)

If one of m & n is even & other odd

then $D > 0$ & so $(m\pi + \pi/2, n\pi + \pi/2)$ is a saddle pt.

Q12 $f(x, y) = y^2 - 6y\cos x + 6$

Sol. Given

$f(x, y) = y^2 - 6y\cos x + 6$

$\Rightarrow$ $f_x = 6y\sin x$

$f_{xx} = 6y\cos x$

$f_y = 2y - 6\cos x$

$f_{yy} = 2$

$f_{xy} = 6\sin x$

For critical pts. $f_x = 0$ & $f_y = 0$

$\Rightarrow \quad 6y\sin x = 0$ ——— ① 1592 88

$2y - 6\cos x = 0$ ——— ②

from ① $y\sin x = 0$

$\Rightarrow \boxed{y = 0}$ or $\boxed{x = 2n\pi}$

Put $y=0$ in ②

$0 - 6\cos x = 0$

$6\cos x = 0$

$\cos x = 0 \Rightarrow x = (2n+1)\pi/2$

Put $x = 2n\pi$ in ②

$2y - 6\cos(2n\pi) = 0$

$2y - 6 = 0$

$2y = 6$

$\boxed{y = 3}$

So $((2n+1)\pi/2, 0)$ & $(2n\pi, 3)$ are Critical pts.

At $((2n+1)\pi/2, 0)$:

$A = f_{xx} = 6(0)\cos(2n+1)\pi/2 = 0$

$B = f_{xy} = 6\sin(2n+1)\pi/2 = 6(\pm1) = \pm 6$

$C = f_{yy} = 2$

$D = B^2 - AC$

$= (\pm 6)^2 - (0)(2)$

$\boxed{D = 36}$

Since $D > 0$, so $((2n+1)\pi/2, 0)$ is a saddle pt.

At $(2n\pi, 3)$

$A = f_{xx} = 6(3)\cos(2n\pi) = 18(1) = 18$

$B = f_{xy} = 6\sin(2n\pi) = 0$

$C = f_{yy} = 2$

$$D = B^2 - AC$$
$$= (0)^2 - (18)(2)$$
$$\boxed{D = -36}$$

Since $D < 0$ & $A, C$ are both +ve

so $f$ has a local min. at $(2n\pi, 3)$

Q13 $f(x,y) = \cos x + \cos y + \cos(x+y)$

Sol. Given

$$f(x,y) = \cos x + \cos y + \cos(x+y)$$

$$\Rightarrow f_x = -\sin x - \sin(x+y)$$

$$f_{xx} = -\cos x - \cos(x+y)$$

$$f_y = -\sin y - \sin(x+y)$$

$$f_{yy} = -\cos y - \cos(x+y)$$

$$f_{xy} = -\cos(x+y)$$

For critical pts $f_x = 0$ & $f_y = 0$

$$\Rightarrow \left.\begin{array}{l} -\sin x - \sin(x+y) = 0 \\ -\sin y - \sin(x+y) = 0 \end{array}\right] \begin{array}{l} \text{———} (1) \\ \text{———} (2) \end{array}$$

Subt. we get

$$-\sin x + \sin y = 0$$

or $\sin x = \sin y$ . or $x = y$

Put in (1)

$$-\sin x - \sin x \cos y - \cos x \sin y = 0$$

$$\sin x + \sin x \cos x + \cos x \sin x = 0$$

$$\sin x + 2 \sin x \cos x = 0$$

$$\sin x (1 + 2\cos x) = 0$$

$$\Rightarrow \sin x = 0 \quad \text{or} \quad 1 + 2\cos x = 0$$

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$\Rightarrow \sin x = 0$ or $\cos x = -\frac{1}{2}$

$\Rightarrow x = 0, \pi$ or $x = \frac{2\pi}{3}$

So $\boxed{x = 0, \pi, 2\pi/3}$

Put in $x = y$

$\Rightarrow \boxed{y = 0, \pi, 2\pi/3}$

Hence $(0,0)$, $(\pi,\pi)$ & $(\frac{2\pi}{3}, 2\pi/3)$ are critical pts

<u>At $(0,0)$</u>

$A = f_{xx} = -\cos 0 - \cos 0 = -1-1 = -2$

$B = f_{xy} = -\cos(0+0) = -1$

$C = f_{yy} = -\cos 0 - \cos(0+0) = -1-1 = -2$

$D = B^2 - AC$

$= (-1)^2 - (-2)(-2)$

$= 1-4$

$\boxed{D = -3}$

Since $D < 0$ & $A, C$ are both $-ve$

So $f$ has a local max. at $(0,0)$

<u>At $(\pi,\pi)$</u>

$A = f_{xx} = -\cos\pi - \cos 2\pi = -(-1) - (1) = 1-1 = 0$

$B = f_{xy} = -\cos 2\pi = -1$

$C = f_{yy} = -\cos\pi - \cos 2\pi = -(-1) - 1 = 1-1 = 0$

$D = B^2 - AC$

$= (-1)^2 - (0)(0)$

$\boxed{D = 1}$

Since $D > 0$ so $(\pi,\pi)$ is a saddle pt.

At $(2\pi/3, 2\pi/3)$

$$A = f_{xx} = -\cos\tfrac{2\pi}{3} - \cos(4\pi/3) = -(-\tfrac{1}{2}) - (-1/2) = \tfrac{1}{2} + \tfrac{1}{2} = 1$$

$$B = f_{xy} = -\cos(\tfrac{4\pi}{3}) = -(-1/2) = 1/2$$

$$C = f_{yy} = -\cos\tfrac{2\pi}{3} - \cos(\tfrac{4\pi}{3}) = -(-1/2) - (-1/2) = \tfrac{1}{2} + \tfrac{1}{2} = 1$$

$$D = B^2 - AC$$
$$= \left(\tfrac{1}{2}\right)^2 - (1)(1)$$
$$= \tfrac{1}{4} - 1$$
$$\boxed{D = -3/4}$$

Since $D < 0$ & $A, C$ are both +ve
so $f$ has a local min. at $(\tfrac{2\pi}{3}, \tfrac{2\pi}{3})$

**Q14** $f(x,y) = (x+1)(y+1)(x+y+1)$

**Sol.** Given $f(x,y) = (x+1)(y+1)(x+y+1)$

$$\Rightarrow f_x = (y+1)[(x+1).1 + (x+y+1).1]$$
$$= (y+1)(x+1+x+y+1)$$
$$f_x = (y+1)(2x+y+2)$$
$$f_{xx} = 2(y+1)$$
$$f_y = (x+1)[(y+1).1 + (x+y+1).1]$$
$$= (x+1)[y+1+x+y+1]$$
$$= (x+1)(x+2y+2)$$
$$f_{yy} = 2(x+1)$$
$$f_{xy} = (y+1).1 + (2x+y+2).1$$
$$= y+1+2x+y+2$$
$$f_{xy} = 2x+2y+3$$

For critical pts. $f_x = 0$ & $f_y = 0$

$\Rightarrow (y+1)(2x+y+2) = 0$ ——— ①

$(x+1)(x+2y+2) = 0$ ——— ②

from ① $y+1=0$ or $2x+y+2=0$

$\Rightarrow y=-1$ or $y=-2x-2$.

$y=-1$

Put in ②

$(x+1)(x-\cancel{2}+\cancel{2}) = 0$

$(x+1)(x) = 0$

$\Rightarrow \boxed{x = 0, -1}$

$y=-2x-2$

Put in ②

$(x+1)(x-4x-4+2) = 0$

$(x+1)(-3x-2) = 0$

$(x+1)(3x+2) = 0$

$\Rightarrow \boxed{x = -1, -2/3}$

Put in above eq.

If $x=-1$, $y = -2(-1)-2$

$= 2-2$

$= 0$

If $x = -\frac{2}{3}$, $y = -2\left(-\frac{2}{3}\right)-2$

$= \frac{4}{3} - 2$

$= \frac{4-6}{3}$

$y = -2/3$



Hence critical pts are

$(0,-1), (-1,-1), (-1,0)$ & $\left(-\frac{2}{3}, -\frac{2}{3}\right)$

At $(0,-1)$

$A = f_{xx} = 2(-\cancel{1}+\cancel{1}) = 0$

$B = f_{xy} = 2(0) + 2(-1) + 3 = 1$

1597



$C = f_{yy} = 2(0+1) = 2$

$D = B^2 - AC$

$= (1)^2 - (0)(2)$

$\boxed{D = 1}$

Since $D > 0$, so $(0, -1)$ is a saddle pt.

At $(-1, -1)$

$A = f_{xx} = 2(-1+1) = 0$

$B = f_{xy} = 2(-1) + 2(-1) + 3 = -2 - 2 + 3 = -1$

$C = f_{yy} = 2(-1+1) = 0$

$D = B^2 - AC$

$= (-1)^2 - (0)(0)$

$\boxed{D = 1}$

Since $D > 0$, so $(-1, -1)$ is a saddle pt.

At $(-1, 0)$

$A = f_{xx} = 2(0+1) = 2$

$B = f_{xy} = 2(-1) + 2(0) + 3 = 1$

$C = f_{yy} = 2(-1+1) = 0$

$D = B^2 - AC$

$= (1)^2 - (2)(0)$

$\boxed{D = 1}$



Since $D > 0$, so $(-1, 0)$ is a saddle pt.

At $\left(-\frac{2}{3}, -\frac{2}{3}\right)$

1598

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$A = f_{xx} = 2\left(-\frac{2}{3}+1\right) = 2\left(\frac{1}{3}\right) = \frac{2}{3}$

$B = f_{xy} = 2\left(-\frac{2}{3}\right)+2\left(-\frac{2}{3}\right)+3 = -\frac{4}{3}-\frac{4}{3}+3$

$= -\frac{8}{3}+3 = \frac{1}{3}$

$C = f_{yy} = 2\left(-\frac{2}{3}+1\right) = 2\left(\frac{1}{3}\right) = \frac{2}{3}$

$D = B^2 - AC$

$= \left(\frac{1}{3}\right)^2 - \left(\frac{2}{3}\right)\left(\frac{2}{3}\right)$

$= \frac{1}{9} - \frac{4}{9}$

$= \frac{1-4}{9}$

$\boxed{D = -\frac{1}{3}}$

Since $D<0$ & $A, C$ are both +ve

so $f$ has a local minimum at $\left(-\frac{2}{3}, -\frac{2}{3}\right)$

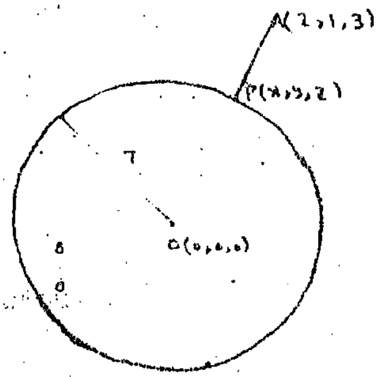
<u>Q15</u> Find the point on the sphere $x^2+y^2+z^2 = 49$ that is nearest to the point $(2,1,3)$.

<u>Sol.</u> Given eq. of sphere is

$x^2+y^2+z^2 = 49$ ——— ①

let $P(x,y,z)$ be the pt. on the sphere that is nearest to the pt $A(2,1,3)$.

then




$|AP| = \sqrt{(x-2)^2+(y-1)^2+(z-3)^2}$

$|AP|^2 = (x-2)^2+(y-1)^2+(z-3)^2$

$f(x,y) = (x-2)^2+(y-1)^2+(z-3)^2$

from ① $z^2 = 49-x^2-y^2$

$$z = \sqrt{49-x^2-y^2}$$

Put in above eq.

$$f(x,y) = (x-2)^2 + (y-1)^2 + \left(\sqrt{49-x^2-y^2} - 3\right)^2$$

$$= \cancel{x^2} - 4x + 4 + \cancel{y^2} - 2y + 1 + 49 - \cancel{x^2} - \cancel{y^2} + 9 - 6\sqrt{49-x^2-y^2}$$

$$f(x,y) = -4x - 2y + 63 - 6\sqrt{49-x^2-y^2}$$

Now we want that values of $x$ & $y$ for which $f(x,y)$ is minimum.

Now $f_x = -4 - 6 \cdot \dfrac{1}{2\sqrt{49-x^2-y^2}} \cdot (-2x) = -4 + \dfrac{6x}{\sqrt{49-x^2-y^2}}$

$$f_y = -2 - 6 \cdot \frac{1}{2\sqrt{49-x^2-y^2}} (-2y) = -2 + \frac{6y}{\sqrt{49-x^2-y^2}}$$

For critical pts. $f_x = 0$ & $f_y = 0$

$$\Rightarrow \left.\begin{array}{l} -4 + \dfrac{6x}{\sqrt{49-x^2-y^2}} = 0 \\ \& \quad -2 + \dfrac{6y}{\sqrt{49-x^2-y^2}} = 0 \end{array}\right\}$$

$$\text{or} \quad \left.\begin{array}{l} 6x - 4\sqrt{49-x^2-y^2} = 0 \\ 6y - 2\sqrt{49-x^2-y^2} = 0 \end{array}\right\}$$

$$\text{or} \quad 3x - 2\sqrt{49-x^2-y^2} = 0$$

$$3y - \sqrt{49-x^2-y^2} = 0$$

$$\text{or} \quad 3x = 2\sqrt{49-x^2-y^2} \quad \text{——— (1)}$$

$$3y = \sqrt{49-x^2-y^2} \quad \text{——— (2)}$$

Sq. eqs. (1) & (2)

1600

$$9x^2 = 4(49-x^2-y^2) \quad \text{—— (1)}$$

$$9y^2 = (49-x^2-y^2) \quad \text{—— (2)}$$

from (2) $49-x^2-y^2 = 9y^2$

Put in (1)

$$9x^2 = 4(9y^2)$$

$$9x^2 = 36y^2$$

$$y^2 = \frac{9}{36}x^2$$

$$y^2 = \frac{1}{4}x^2$$

$$\boxed{y = \pm\frac{x}{2}}$$

Put value in (2)

$$9\left(\pm\frac{x}{2}\right)^2 = 49-x^2-\left(\pm\frac{x}{2}\right)^2$$

$$9\frac{x^2}{4} = 49-x^2-\frac{x^2}{4}$$

$$9x^2 = 196-4x^2-x^2$$

$$9x^2+5x^2 = 196$$

$$14x^2 = 196$$

$$x^2 = 14$$

$$x = \pm\sqrt{14}$$

$$\boxed{x = \pm\frac{14}{\sqrt{14}}}$$

Put in eq. $y = \pm\frac{x}{2}$

$$y = \pm\left(\pm\frac{14}{\sqrt{14}\times 2}\right)$$

$$\boxed{y = \pm\frac{7}{\sqrt{14}}}$$

Put in given eq

$$z^2 = 49-x^2-y^2$$

$$z^2 = 49 - \frac{196}{14} - \frac{49}{14}$$

$$= \frac{686 - 196 - 49}{14}$$

$$z^2 = \frac{441}{14}$$

$$\boxed{z = \pm \frac{21}{\sqrt{14}}}$$

So req. pt. is $P\left(\frac{14}{\sqrt{14}}, \frac{7}{\sqrt{14}}, \frac{21}{\sqrt{14}}\right)$

Q16 Find the pt. on the plane $3x + y - 2z = 4$ nearest to the pt. $P(1, -1, 2)$.

Sol. Given eq. is

$$3x + y - 2z = 4 \quad \text{———} \quad (A)$$

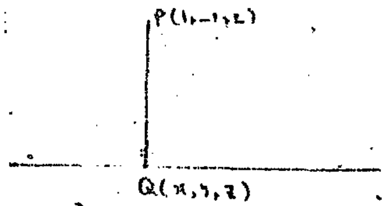
let $Q(x, y, z)$ be the pt. nearest to $P(1, -1, 2)$

Now $|PQ| = \sqrt{(x-1)^2 + (y+1)^2 + (z-2)^2}$

or $|PQ|^2 = (x-1)^2 + (y+1)^2 + (z-2)^2$

from (A) $\quad y = 4 - 3x + 2z \quad \text{———} \quad (B)$

Put in above eq.




$$|PQ|^2 = (x-1)^2 + (4 - 3x + 2z + 1)^2 + (z-2)^2$$

$$|PQ|^2 = (x-1)^2 + (5 - 3x + 2z)^2 + (z-2)^2$$

$$f(x,z) = (x-1)^2 + (5 - 3x + 2z)^2 + (z-2)^2$$

$$= x^2 - 2x + 1 + 25 + 9x^2 + 4z^2 - 30x - 12xz + 20z + z^2 - 4z + 4$$

$$f(x,z) = 10x^2 + 5z^2 - 12xz - 32x + 16z + 30$$

Now we want to find local min. for $f(x,z)$

$f_x = 20x - 12z$

$f_{xx} = 20$

1600



$f_z = 10z - 12x + 16$

$f_{zz} = 10$

$f_{xz} = -12$

For critical pts. $f_x = 0$ & $f_z = 0$

$\Rightarrow$ $\left.\begin{array}{l} 20x - 12z - 32 = 0 \\ 10z - 12x + 16 = 0 \end{array}\right]$

$5x - 3z - 8 = 0$ ——— (1)

$-6x + 5z + 8 = 0$ ——— (2)

Multiplying (1) by 6 & (2) by 5

$\cancel{30x} - 18z - 48 = 0$ ——— (1)

$\cancel{-30x} + 25z + 40 = 0$ ——— (2)

$7z - 8 = 0$

$\boxed{z = \frac{8}{7}}$

Put in (1)

$5x - 3\left(\frac{8}{7}\right) - 8 = 0$

$5x = 8 + \frac{24}{7}$

$5x = \frac{56+24}{7}$

$5x = \frac{80}{7}$

$\boxed{x = \frac{16}{7}}$ So $\left(\frac{16}{7}, \frac{8}{7}\right)$ is the critical pt.

<u>At $x = \frac{16}{7}$, $z = \frac{8}{7}$</u>

$A = f_{xx} = 20$

$B = f_{xz} = -12$

$C = f_{zz} = 10$

$D = B^2 - AC$

$D = (-12)^2 - (20)(10)$

$D = 144 - 200$

$\boxed{D = -56}$

Since $D < 0$ & $A, C$ are both +ve

So $f$ has a local min. at $x = \frac{16}{7}$, $z = \frac{8}{7}$

To find value of $y$ put values of $x$ & $z$ in (B)

$y = 4 - 3\left(\frac{16}{7}\right) + 2\left(\frac{8}{7}\right)$

$= 4 - \frac{48}{7} + \frac{16}{7}$

$= \frac{28 - 48 + 16}{7}$

$\boxed{y = -\frac{4}{7}}$

So $\left(\frac{16}{7}, -\frac{4}{7}, \frac{8}{7}\right)$ is the req. pt.

**Q17** Show that the volume of the largest parallelopipped with faces parallel to the co-ord. planes that can be inscribed in the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{is} \quad \frac{8abc}{3\sqrt{3}}$$

**Sol:** Let the co-ords. of one vertex of parallelopipped be $P(x, y, z)$ then the dimensions of the parallelopipped become $2x, 2y$ & $2z$. If $V$ is the volume of parallelopipped then

$V = (2x)(2y)(2z)$

$V = 8xyz$

Since $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$\Rightarrow \quad \frac{z^2}{c^2} = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}$

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$$z^2 = \frac{c^2\left[a^2b^2 - b^2x^2 - a^2y^2\right]}{a^2b^2}$$

$$\therefore\ z = \frac{c}{ab}\left(a^2b^2 - b^2x^2 - a^2y^2\right)^{\frac{1}{2}} \quad \text{——— (A)}$$

So $V = 8xy \cdot \frac{1}{ab}\left(a^2b^2 - b^2x^2 - a^2y^2\right)^{1/2}$

$$f(x,y) = \frac{8xy}{ab}\left(a^2b^2 - b^2x^2 - a^2y^2\right)^{1/2}$$

Now we want to find max. value of $f(x,y)$

Now

$$f_x = \frac{8xy}{ab} \cdot \frac{1}{2}\left(a^2b^2 - b^2x^2 - a^2y^2\right)^{-\frac{1}{2}}(-2b^2x) + \left(a^2b^2 - b^2x^2 - a^2y^2\right)^{\frac{1}{2}} \cdot \frac{8y}{ab}$$

$$= \frac{8y}{ab}\left(a^2b^2 - b^2x^2 - a^2y^2\right)^{-\frac{1}{2}}\left[\frac{x}{2}\cdot(-2b^2x) + a^2b^2 - b^2x^2 - a^2y^2\right]$$

$$= \frac{8y}{ab}\left(a^2b^2 - b^2x^2 - a^2y^2\right)^{-\frac{1}{2}}\left[-b^2x^2 + a^2b^2 - b^2x^2 - a^2y^2\right].$$

$$= \frac{8y}{ab}\left(a^2b^2 - b^2x^2 - a^2y^2\right)^{-1/2}\left[a^2b^2 - 2b^2x^2 - a^2y^2\right]$$

$$f_x = \frac{8y\left(a^2b^2 - 2b^2x^2 - a^2y^2\right)}{ab\left(a^2b^2 - b^2x^2 - a^2y^2\right)^{1/2}}$$

& $f_y = \frac{8xy}{ab} \cdot \frac{1}{2}\left(a^2b^2 - b^2x^2 - a^2y^2\right)^{-\frac{1}{2}}(-2a^2y) + \left(a^2b^2 - b^2x^2 - a^2y^2\right)^{\frac{1}{2}} \cdot \frac{8x}{ab}$

$$= \frac{8x}{ab}\left(a^2b^2 - b^2x^2 - a^2y^2\right)^{-\frac{1}{2}}\left[\frac{y}{2}(-2a^2y) + \left(a^2b^2 - b^2x^2 - a^2y^2\right)\right]$$

$$= \frac{8x}{ab}\left(a^2b^2 - b^2x^2 - a^2y^2\right)^{-\frac{1}{2}}\left[-a^2y^2 + a^2b^2 - b^2x^2 - a^2y^2\right]$$

$$f_y = \frac{8x\left(a^2b^2 - b^2x^2 - 2a^2y^2\right)}{ab\left(a^2b^2 - b^2x^2 - a^2y^2\right)^{1/2}}$$

For critical pts. $f_x = 0$ & $f_y = 0$

$$\Rightarrow \left.\begin{aligned} &\frac{8y\left(a^2b^2 - 2b^2x^2 - a^2y^2\right)}{ab\left(a^2b^2 - b^2x^2 - a^2y^2\right)^{1/2}} = 0 \\ \& \quad &\frac{8x\left(a^2b^2 - b^2x^2 - 2a^2y^2\right)}{ab\left(a^2b^2 - b^2x^2 - a^2y^2\right)^{1/2}} = 0 \end{aligned}\right\}$$

or $\quad a^2b^2 - 2b^2x^2 - a^2y^2 = 0$ ——— (1)

$a^2b^2 - b^2x^2 - 2a^2y^2 = 0$ ——— (2)

Multiplying (1) by 2

$2a^2b^2 - 4b^2x^2 - \cancel{2a^2y^2} = 0$ ——— (1)

$\underline{-a^2b^2 \mp b^2x^2 \mp \cancel{2a^2y^2} = 0}$ ——— (2)

$a^2b^2 - 3b^2x^2 = 0$

$3b^2x^2 = a^2b^2$

$x^2 = \dfrac{a^2}{3}$

$$\boxed{x = \frac{a}{\sqrt{3}}}$$

Put in (1)

$a^2b^2 - \dfrac{2a^2b^2}{3} - a^2y^2 = 0$

$a^2y^2 = a^2b^2 - \dfrac{2a^2b^2}{3}$

$a^2y^2 = \dfrac{3a^2b^2 - 2a^2b^2}{3}$

$a^2y^2 = \dfrac{a^2b^2}{3}$

$y^2 = \dfrac{b^2}{3}$

$$\boxed{y = \frac{b}{\sqrt{3}}}$$



Put in (A)

$z = \dfrac{c}{ab}\left(a^2b^2 - b^2\left(\dfrac{a^2}{3}\right) - a^2\left(\dfrac{b^2}{3}\right)\right)^{1/2}$

$= \dfrac{c}{ab}\left(a^2b^2 - \dfrac{a^2b^2}{3} - \dfrac{a^2b^2}{3}\right)^{1/2}$

$= \dfrac{abc}{ab}\left(1 - \dfrac{1}{3} - \dfrac{1}{3}\right)^{1/2} = c\left(1 - \dfrac{2}{3}\right)^{1/2}$

$z = c\left(\dfrac{1}{3}\right)^{\frac{1}{2}} = \dfrac{c}{\sqrt{3}}$

Hence req. Volume $= 8\left(\dfrac{a}{\sqrt{3}}\right)\left(\dfrac{b}{\sqrt{3}}\right)\left(\dfrac{c}{\sqrt{3}}\right)$

$= \dfrac{8abc}{3\sqrt{3}}$

**Q18** Find the minimum distance b/w the lines 102

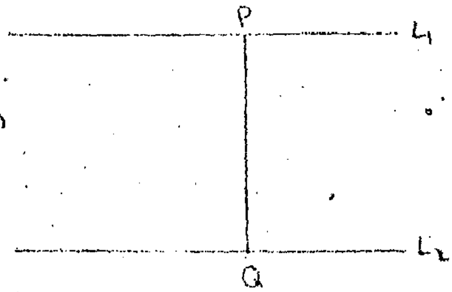
$x = t\ ;\ y = 3-2t\ ,\ z = 1+2t$

$x = -1-s\ , y = s\ ,\ z = 4-3s$

**Sol.** Given lines are

$x = t,\ y = 3-2t,\ z = 1+2t \quad \text{---}\ (1)$

$x = -1-s,\ y = s\ ,\ z = 4-3s \quad \text{---}\ (2)$




Any pt. on line (1) is

$P(t, 3-2t, 1+2t)$

Any pt. on line (2) is,

$Q(-1-s, s, 4-3s)$

Now

$$|PQ| = \sqrt{(t+1+s)^2 + (3-2t-s)^2 + (1+2t-4+3s)^2}$$

$$|PQ|^2 = (t+s+1)^2 + (3-2t-s)^2 + (2t+3s-3)^2$$

$$= t^2+s^2+1+2ts+2s+2t + 9+4t^2+s^2-12t+4ts-6s$$
$$+4t^2+9s^2+9+12ts-18s-12t$$

$$|PQ|^2 = 9t^2+11s^2+18ts-22t-22s+19$$

or $f(t,s) = 9t^2+11s^2+\ \ ts-22t-22s+19$

We want to find minm value of $f(t,s)$

Now $f_t = 18t+18s-22$

$f_{tt} = 18$

$f_s = 22s + 18t - 22$

$f_{ss} = 22$

$f_{ts} = 12$

For critical pts. $f_t = 0$ & $f_s = 0$

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$$\Rightarrow \quad 18\cancel{t} + 18s - 2\cancel{2} = 0 \quad \text{———} \quad (1)$$
$$\underline{-\cancel{18}t \pm 22s \mp 22 = 0} \quad \text{———} \quad (2)$$
$$-4s = 0$$
$$\boxed{s = 0}$$

Put in (1)

$$18t + 18(0) - 22 = 0$$
$$18t = 22$$
$$\boxed{t = \frac{11}{9}}$$



So $\left(\frac{11}{9}, 0\right)$ is the critical pt.

At $\left(\frac{11}{9}, 0\right)$

$A = f_{tt} = 18$

$B = f_{ts} = 12$

$C = f_{ss} = 22$

$D = B^2 - AC$

$= (12)^2 - (18)(22)$

$= 144 - 396$

$\boxed{D = -252}$

Since $D < 0$ & $A, C$ are both +ve

So $f(t, s)$ has minimum value at $\left(\frac{11}{9}, 0\right)$

Then Co-ords. of pts. P & Q are

$$P\left(\frac{11}{9}, 3 - 2\left(\frac{11}{9}\right), 1 + 2\left(\frac{11}{9}\right)\right) \;\&\; Q\left(-1-0, 0, 4-3(0)\right)$$

$$P\left(\frac{11}{9}, 3 - \frac{22}{9}, 1 + \frac{22}{9}\right) \;\&\; Q(-1, 0, 4)$$

$$P\left(\frac{11}{9}, \frac{5}{9}, \frac{31}{9}\right) \;\&\; Q(-1, 0, 4)$$

Now the req. minimum distance b/w lines is

$$|PQ|^2 = \left(\frac{11}{9} + 1\right)^2 + \left(\frac{5}{9} - 0\right)^2 + \left(\frac{31}{9} - 4\right)^2$$

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104

$$|PQ| = \sqrt{\left(\frac{20}{9}\right)^2 + \left(\frac{5}{9}\right)^2 + \left(-\frac{5}{9}\right)^2}$$

$$= \sqrt{\frac{400}{81} + \frac{25}{81} + \frac{25}{81}}$$

$$= \sqrt{\frac{400+25+25}{81}}$$

$$= \sqrt{\frac{450}{81}}$$

$$= \sqrt{\frac{50}{9}}$$

$$= \frac{\sqrt{50}}{3}$$

$$|PQ| = \frac{5\sqrt{2}}{3}$$

is the req. minimum distance b/w lines.

Q19 Find the dimensions of the largest rectangular box with three of its faces in the Co-ord. planes & the vertex opposite to origin in the first octant & on the plane $2x+y+3z = 6$ 15

Sol. Let $(x,y,z)$ be the Co-ords. of the vertex opposite to the origin then dimensions of the box are $x, y$ & $z$. then Volume of box is

$V = xyz$

& eq. of given plane is

$2x+y+3z = 6$

or $y = 6-2x-3z$ —— (A)

Put in above eq.

$V = xz(6-2x-3z) = f(x,z)$

Now we want to find the values of $x, y$ & $z$ which maximize $f(x,z)$

As $f(x,z) = 6xz - 2x^2z - 3xz^2$

$f_x = 6z - 4xz - 3z^2$

$f_{xx} = -4z$

$f_z = 6x - 2x^2 - 6xz$

$f_{zz} = -6x$

$f_{xz} = 6-4x-6z$

For critical pts.

$f_x = 0$ & $f_z = 0$

$$\Rightarrow \left.\begin{array}{l} 6z - 4xz - 3z^2 = 0 \\ 6x - 2x^2 - 6xz = 0 \end{array}\right]$$

$z(6-4x-3z)=0$

$x(6-2x-6z)=0$

or $6-4x-3z=0$ ——— ① $\quad$ ∵ $x\neq 0$ & $z\neq 0$.

$6-2x-6z=0$ ——— ②

Multiplying ① by 2

$12-8x-\cancel{6z}=0$ ——— ①

$\underline{\phantom{}^{-}6\ {}^{+}\!\!-2x\ {}^{+}\!\!\cancel{-6z}=0}$ ——— ②

$6-6x=0$

$6x=6$

$\boxed{x=1}$

Put in ①

$6-4(1)-3z=0$

$2-3z=0$

$3z=2$

$\boxed{z=\frac{2}{3}}$

So $(1,\frac{2}{3})$ is the critical pt.

<u>At $(1,\frac{2}{3})$</u>

$A=f_{xx}=-4(\frac{2}{3})=-\frac{8}{3}$

$B=f_{xz}=6-4(1)-6(\frac{2}{3})=2-4=-2$

$C=f_{zz}=-6(1)=-6$

$D=B^2-AC$

$=(-2)^2-(-8/3)(-6)$

$=4-16$

$\boxed{D=-12}$

Since $D<0$ & $A, C$ are both $-$ve

So $f(x,z)$ is max. at $(1,\frac{2}{3})$.

Put $x = 1, z = \frac{2}{3}$ in (A)

$y = 6 - 2(1) - 3\left(\frac{2}{3}\right)$

$= 4 - 2$

$\boxed{y = 2}$

So req. dimensions of the box are

$x = 1, \; y = 2, \; z = \frac{2}{3}$

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**Q20** A closed rectangular box to contain $16 ft^3$ is to be made of three different materials. The cost of the material for the top & the bottom is Rs. 9 per square ft, the cost of the material for the front & the back is Rs. 8 per sq.ft & the cost of material for the other two sides is Rs. 6 per sq.ft. Find the dimensions of the box so that the cost of the material is minimum.

**Sol.** Let $x$ & $y$ be length & width of box & let $z$ be the height of box then

$V = xyz$

$V = xyz$

But $V = 16$

$\Rightarrow xyz = 16$ ——— (A)

Let $C$ denote the total cost of material for the box then

$C$ = Cost of top & bottom + Cost of front side & backside + Cost of other two sides

$= 9(2xy) + 8(2xz) + 6(2yz)$

$C = 18xy + 16xz + 12yz$

from (A) $z = \frac{16}{xy}$

Put in above eq.

$$C = 18xy + 16x\left(\frac{16}{xy}\right) + 12y\left(\frac{16}{xy}\right)$$

$$C = 18xy + \frac{256}{y} + \frac{192}{x}$$

or $f(x,y) = 18xy + \frac{256}{y} + \frac{192}{x}$

We want that value of $x$ & $y$ which maximize $f$

Now

$$f_x = 18y - \frac{192}{x^2}$$

$$f_{xx} = \frac{384}{x^3}$$

$$f_y = 18x - \frac{256}{y^2}$$

$$f_{yy} = \frac{512}{y^3}$$

$$f_{xy} = 18$$

For critical pts.

$f_x = 0$ & $f_y = 0$

$$\Rightarrow \quad \left.\begin{aligned} 18y - \frac{192}{x^2} &= 0 \\ 18x - \frac{256}{y^2} &= 0 \end{aligned}\right]$$

$$18x^2y - 192 = 0 \quad \text{--- (1)}$$

$$18xy^2 - 256 = 0 \quad \text{--- (2)}$$

from (1) $18x^2y = 192$

$$y = \frac{192}{18x^2}$$

1613

$$y = \frac{32}{3x^2} \quad \text{------} \quad (3)$$

Put in (2)

$$18x \cdot \left(\frac{32}{3x^2}\right)^2 - 256 = 0$$

$$18x \cdot \frac{1024}{9x^4} - 256 = 0$$

$$\frac{2(1024)}{x^3} - 256 = 0$$

$$2048 - 256x^3 = 0$$

$$256x^3 = 2048$$

$$x^3 = 8$$

$$x^3 = (2)^3$$

$$\Rightarrow \boxed{x = 2}$$

Put in (3)

$$y = \frac{32}{3(4)}$$

$$\boxed{y = 8/3}$$



So $(2, 8/3)$ is the critical pt.

<u>At $(2, 8/3)$</u>

$$A = f_{xx} = \frac{384}{8} = 48$$

$$B = f_{xy} = 18$$

$$C = f_{yy} = \frac{512}{(8/3)^3} = \frac{\cancel{512} \times 27}{\cancel{512}}$$

$$C = 27$$

$$D = B^2 - AC$$

$$= (18)^2 - (48)(27)$$

$$= 324 - 1296$$

1614

$\boxed{D = -97.2}$

Since $D < 0$ & $A, C$ are both +ve

So $f(x,y)$ is min. at $(2, 8/3)$

Hence Cost is min. at $x = 2$, $y = 8/3$

Put in (A)

$(2)(8/3)(z) = 16$

$\frac{16z}{3} = 16$

$\frac{z}{3} = 1$

$\boxed{z = 3}$

Hence req. dimensions of box are

$$\left.\begin{aligned} \text{length of box} &= x = 2 \\ \text{width of box} &= y = 8/3 \\ \text{height of box} &= z = 3 \end{aligned}\right\}$$

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