Differentiation of Composite functions

Chain rules:

① If z is a function of y & y is a function of x, then z is a function of x only then.

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

$$\begin{tikzcd} z \arrow[r, "\frac{dz}{dy}"'] \arrow[rr, bend left, "\frac{dz}{dx}"] & y \arrow[r, "\frac{dy}{dx}"'] & x \end{tikzcd}$$

② If $z = f(u,v)$ & $u = u(x)$, $v = v(x)$

Then z is a fn. of single variable x so

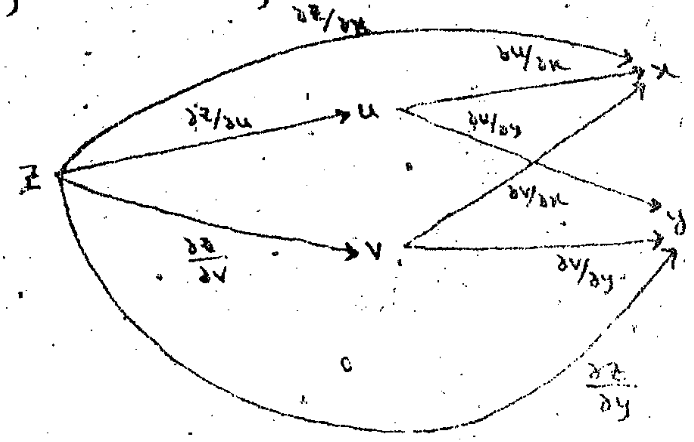
$$\frac{dz}{dx} = \frac{\partial z}{\partial u} \cdot \frac{du}{dx} + \frac{\partial z}{\partial v} \cdot \frac{dv}{dx}$$

$$\begin{tikzcd} & u \arrow[dr, "du/dx"] & \\ z \arrow[ur, "\partial z/\partial u"] \arrow[dr, "\partial z/\partial v"'] \arrow[rr, bend left=60, "dz/dx"] & & x \\ & v \arrow[ur, "dv/dx"'] & \end{tikzcd}$$

③ If $z = f(u, v)$ & $u = u(x, y)$, $v = v(x, y)$

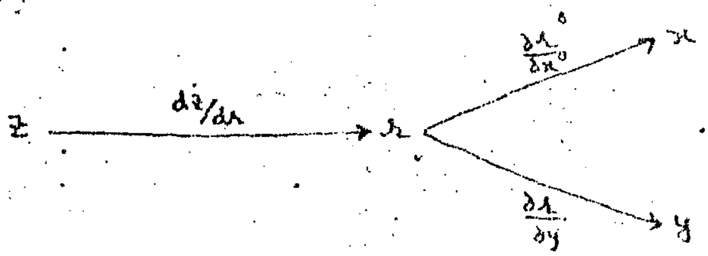
then z is a function of x & y, so

$$\left.\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} \\ \frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}\end{aligned}\right]$$



④ If $z = f(r)$ & $r = (x, y)$, then

$$\left.\begin{aligned}\frac{\partial z}{\partial x} &= \frac{dz}{dr}\cdot\frac{\partial r}{\partial x} \\ \frac{\partial z}{\partial y} &= \frac{dz}{dr}\cdot\frac{\partial r}{\partial y}\end{aligned}\right]$$



Exercise No. 9.3

Q1 If $u = x - y^2$, $x = 2r - 3s + 4$, $y = -r + 8s - 5$ then find $\frac{\partial u}{\partial r}$ & $\frac{\partial u}{\partial s}$

Sol. Given

$u = x - y^2$, $x = 2r - 3s + 4$ & $y = -r + 8s - 5$

Hence u is actually a fn. of r & s

By chain rule.

$$\left.\begin{aligned} \frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} \\ \text{\&}\ \frac{\partial u}{\partial s} &= \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s} \end{aligned}\right]$$

$$\left.\begin{aligned} \frac{\partial u}{\partial r} &= (1)(2) + (-2y)(-1) \\ \frac{\partial u}{\partial s} &= (1)(-3) + (-2y)(8) \end{aligned}\right]$$

$$\text{or}\quad \left.\begin{aligned} \frac{\partial u}{\partial r} &= 2 + 2y \\ \frac{\partial u}{\partial s} &= -3 - 16y \end{aligned}\right]$$

$$\text{or}\quad \left.\begin{aligned} \frac{\partial u}{\partial r} &= 2(1+y) \\ \frac{\partial u}{\partial s} &= -(3+16y) \end{aligned}\right]$$

Q2 If $z = \frac{\cos y}{x}$, $x = u^2 - v$, $y = e^v$ then find $\frac{\partial z}{\partial u}$, $\frac{\partial z}{\partial v}$.

Sol. Given $z = \frac{\cos y}{x}$, $x = u^2 - v$; $y = e^v$

So z is actually a fn. of u & v

Hence by Chain rule

1536 (30)

$$\left.\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}\\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}\end{aligned}\right]$$

as $z = \dfrac{\cos y}{x}$, $x = u^2 - v$, $y = e^v$

$$\frac{\partial z}{\partial x} = -\frac{\cos y}{x^2}, \quad \frac{\partial x}{\partial u} = 2u, \quad \frac{\partial y}{\partial u} = 0$$

$$\frac{\partial z}{\partial y} = -\frac{\sin y}{x}, \quad \frac{\partial x}{\partial v} = -1, \quad \frac{\partial y}{\partial v} = e^v$$

So

$$\frac{\partial z}{\partial u} = -\frac{\cos y}{x^2}\cdot 2u + \left(-\frac{\sin y}{x}\right)\cdot 0$$

$$\boxed{\frac{\partial z}{\partial u} = -\frac{2u\cos y}{x^2}}$$

& $$\frac{\partial z}{\partial v} = -\frac{\cos y}{x^2}\cdot(-1) + \left(-\frac{\sin y}{x}\right)\cdot e^v$$

$$= \frac{\cos y}{x^2} - \frac{\sin y\cdot e^v}{x}$$

$$= \frac{\cos y - x e^v \sin y}{x^2}$$

$$\frac{\partial z}{\partial v} = \frac{1}{x^2}\left[\cos y - xy\sin y\right]$$

<u>Note</u>: If $f(x,y) = 0$ then to find $\frac{dy}{dx}$, we use

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

or $$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

1535

Find $\frac{dy}{dx}$ (Problems 3 — 6):

Q3 $\sin(xy) - e^{xy} - x^2y = 0$

Sol.

Let $f(x,y) = \sin(xy) - e^{xy} - x^2y$

Diff. partially w.r.t. x & y

$$f_x = \cos(xy).y - e^{xy}.y - 2xy$$

$$f_x = y\cos(xy) - ye^{xy} - 2xy$$

& $$f_y = \cos(xy).x - e^{xy}.x - x^2$$

or $$f_y = x\cos(xy) - xe^{xy} - x^2$$

Now $$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

$$\frac{dy}{dx} = -\frac{y\cos(xy) - ye^{xy} - 2xy}{x\cos(xy) - xe^{xy} - x^2}$$

$$= \frac{y(\cos(xy) - e^{xy} - 2x)}{x(x + e^{xy} - \cos(xy))}$$

Q4 $3(x^2+y^2)^2 = 25(x^2-y^2)$

Sol. Given

$$3(x^2+y^2)^2 = 25(x^2-y^2)$$

or $$3(x^2+y^2)^2 - 25(x^2-y^2) = 0$$

let $$f(x,y) = 3(x^2+y^2)^2 - 25(x^2-y^2)$$

Diff. partially w.r.t. x & y

$$f_x = 6(x^2+y^2).2x - 25(2x).$$

$$= 12x(x^2+y^2) - 50x$$

$$f_y = 6(x^2+y^2)\cdot 2y - 25(-2y)$$

$$f_y = 12y(x^2+y^2) + 50y$$

Now

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

$$= -\frac{12x(x^2+y^2) - 50x}{12y(x^2+y^2) + 50y}$$

$$= -\frac{-2(25x - 6x(x^2+y^2))}{2(6y(x^2+y^2) + 25y)}$$

$$\frac{dy}{dx} = \frac{25x - 6x(x^2+y^2)}{25y + 6y(x^2+y^2)}$$

Q5 $f(x,y) = x^y - y^x = 0$

Sol. Here $f(x,y) = x^y - y^x$

Diff. partially w.r.t. x & y

$$f_x = y x^{y-1} - y^x \ln y$$

$$f_y = x^y \ln x - x y^{x-1}$$

Now $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y}$

$$= -\frac{y x^{y-1} - y^x \ln y}{x^y \ln x - x y^{x-1}}$$

$$= \frac{y^x \ln y - y x^{y-1}}{x^y \ln x - x y^{x-1}}$$

$$= \frac{y^x \ln y - y\cdot\frac{x^y}{x}}{x^y \ln x - x\cdot\frac{y^x}{y}}$$

$$= \frac{x^y \ln y - \frac{y}{x} \cdot x^y}{x^y \ln x - \frac{x}{y} \cdot x^y} \qquad (\because x^y = y^x) \qquad \underline{33}$$

$$= \frac{\ln y - \frac{y}{x}}{\ln x - \frac{x}{y}}$$

$$= \frac{\frac{x\ln y - y}{x}}{\frac{y\ln x - x}{y}}$$

$$= \frac{y(x\ln y - y)}{x(y\ln x - x)}$$

$$\frac{dy}{dx} = \frac{y(y - x\ln y)}{x(x - y\ln x)}$$

Q6 $(\tan x)^y + y^{\cot x} = a$

Sol. Given

$$(\tan x)^y + y^{\cot x} = a$$

or $(\tan x)^y + y^{\cot x} - a = 0$

Here $f(x,y) = (\tan x)^y + y^{\cot x} - a$

Diff. partially w.r.t. x & y

$$f_x = y(\tan x)^{y-1} \cdot \sec^2 x + y^{\cot x} \cdot \ln y \cdot -\operatorname{cosec}^2 x$$

$$f_x = y\sec^2 x(\tan x)^{y-1} - \ln y \cdot \operatorname{cosec}^2 x \cdot y^{\cot x}$$

& $f_y = (\tan x)^y \cdot \ln \tan x + \cot x \cdot y^{\cot x - 1}$

Now $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y}$

or $$\frac{dy}{dx} = -\frac{y\sec^2 x(\tan x)^{y-1} - \ln y \cdot \operatorname{cosec}^2 x \cdot y^{\cot x}}{(\tan x)^y \cdot \ln \tan x + \cot x \cdot y^{\cot x - 1}} \quad \text{—— Ans.}$$

Q7 If $F(x,y,z) = 0$, find $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$

Sol. Let $w = F(x,y,z)$

then its differential is

$$dw = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz$$

But $w = 0$

$$\Rightarrow \quad dw = 0$$

Hence above eq. becomes

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz = 0 \quad \text{———} (1)$$

But we know that when z be defined explicitly as a function of two independent variables x & y

i.e., $z = f(x,y)$

then its differential is

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

Put in (1)

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}\left(\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy\right) = 0$$

$$\left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x}\right)dx + \left(\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial y}\right)dy = 0 \quad \text{———} (2)$$

To find $\frac{\partial z}{\partial x}$, we keep y constt. so $dy = 0$

Hence eq. (2) becomes

$$\left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x}\right)dx = 0$$

$\therefore \quad dx \neq 0$

$$\Rightarrow \quad \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \quad \boxed{\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z}}$$

Now to find $\frac{\partial z}{\partial y}$ we keep x constt. so $dx = 0$

Hence eq. ② becomes

$$\left(\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial y}\right) dy = 0$$

$$\because dy \neq 0$$

$$\Rightarrow \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial y} = 0$$

or $$\boxed{\frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z}}$$

Q8. If $f(x,y) = 0$ & $\phi(y,z) = 0$, show that

$$\frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z} \cdot \frac{dz}{dx} = \frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y}$$

Sol.- Given that $f(x,y) = 0$

$$\Rightarrow \frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \quad \text{——— ①}$$

Also given

$$\phi(y,z) = 0$$

$$\Rightarrow \frac{dz}{dy} = -\frac{\frac{\partial \phi}{\partial y}}{\frac{\partial \phi}{\partial z}} \quad \text{——— ②}$$

Multiplying ① & ②

$$\frac{dy}{dx} \cdot \frac{dz}{dy} = -\frac{\partial f/\partial x}{\partial f/\partial y} \cdot -\frac{\partial \phi/\partial y}{\partial \phi/\partial z}$$

$$\frac{dz}{dx} = \frac{\partial f/\partial x \cdot \frac{\partial \phi}{\partial y}}{\partial f/\partial y \cdot \partial \phi/\partial z}$$

$$\frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z} \cdot \frac{dz}{dx} = \frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y} \quad \text{——— Ans.}$$

Q9 If. $x\sqrt{1-y^2} + y\sqrt{1-x^2} = a$, show that

$$\frac{d^2y}{dx^2} = \frac{a}{(1-x^2)^{3/2}}$$

Sol. Given

$$x\sqrt{1-y^2} + y\sqrt{1-x^2} = a$$

or $x\sqrt{1-y^2} + y\sqrt{1-x^2} - a = 0$

Here $f(x,y) = x\sqrt{1-y^2} + y\sqrt{1-x^2} - a$

Diff. partially w.r.t. x & y

$$f_x = \sqrt{1-y^2} + y \cdot \frac{1}{\cancel{2}\sqrt{1-x^2}}(-\cancel{2}x)$$

$$= \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}}$$

$$f_x = \frac{\sqrt{1-x^2}\sqrt{1-y^2} - xy}{\sqrt{1-x^2}}$$

Now $f_y = x \cdot \frac{1}{2\sqrt{1-y^2}}(-2y) + \sqrt{1-x^2}$

$$= -\frac{xy}{\sqrt{1-y^2}} + \sqrt{1-x^2}$$

$$= \frac{-xy + \sqrt{1-x^2}\sqrt{1-y^2}}{\sqrt{1-y^2}}$$

$$f_y = \frac{\sqrt{1-x^2}\sqrt{1-y^2} - xy}{\sqrt{1-y^2}}$$

Now $\frac{dy}{dx} = -\frac{f_x}{f_y}$

$$= -\frac{\dfrac{(\sqrt{1-x^2}\sqrt{1-y^2} - xy)}{\sqrt{1-x^2}}}{\dfrac{(\sqrt{1-x^2}\sqrt{1-y^2} - xy)}{\sqrt{1-y^2}}}$$

$$\frac{dy}{dx} = -\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

Diff: w. r. t. x

$$\frac{d^2y}{dx^2} = -\frac{\sqrt{1-x^2}\cdot\left(\frac{1}{2\sqrt{1-y^2}}\cdot -2y\cdot\frac{dy}{dx}\right) - \sqrt{1-y^2}\cdot\left(\frac{1}{2\sqrt{1-x^2}}(-2x)\right)}{(1-x^2)}$$

$$= -\frac{\cancel{\sqrt{1-x^2}}\cdot\frac{-y}{\sqrt{1-y^2}}\cdot -\frac{\cancel{\sqrt{1-y^2}}}{\sqrt{1-x^2}} - \sqrt{1-y^2}\cdot\frac{-x}{\sqrt{1-x^2}}}{(1-x^2)}$$

$$= -\frac{y + \frac{x\sqrt{1-y^2}}{\sqrt{1-x^2}}}{(1-x^2)}$$

$$= -\frac{y\sqrt{1-x^2} + x\sqrt{1-y^2}}{(1-x^2)\sqrt{1-x^2}}$$

$$\frac{d^2y}{dx^2} = -\frac{a}{(1-x^2)^{3/2}}$$

Q10 If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, prove that

$$\frac{d^2y}{dx^2} = \frac{abc + 2fgh - af^2 - bg^2 - ch^2}{(hx+by+f)^2}$$

Sol. Given that

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Here $f(x,y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$

Then $f_x = 2ax + 2hy + 2g$

$$f_y = 2by + 2hx + 2f$$

Now $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y}$

$$= -\frac{2ax+2hy+2g}{2by+2hx+2f}$$

$$\frac{dy}{dx} = -\frac{ax+hy+g}{hx+by+f}$$

Diff. w.r.t. x

$$\frac{d^2y}{dx^2} = -\frac{(hx+by+f).(a+h\,{}^{dy}/_{dx}) - (ax+hy+g)(h+b\frac{dy}{dx})}{(hx+by+f)^2}$$

$$= -\frac{\frac{dy}{dx}[h^2x + h\cancel{by} + hf - abx - h\cancel{by} - gb] + [a\cancel{hx} + aby + af - a\cancel{hx} - h^2y - gh]}{(hx+by+f)^2}$$

$$= -\frac{\left(-\frac{ax+hy+g}{hx+by+f}\right)[h^2x + hf - abx - gb] + [aby + af - h^2y - gh]}{(hx+by+f)^2}$$

$$= \frac{(ax+hy+g)(h^2x+hf-abx-gb) - (aby+af-h^2y-gh)(hx+by+f)}{(hx+by+f)^3}$$

$$= \frac{\begin{array}{r} ah^2x^2 + \cancel{ahfx} - a^2bx^2 - abgx + h^3xy + h^2fy - abhxy - \cancel{ghby} + h^2gx + ghf - abgx \\ - g^2b \\ -abhxy - ab^2y^2 - abfy - \cancel{afhx} - abfy - af^2 + h^3xy + h^2by^2 + h^2fy + gh^2x + \cancel{ghby} + ghf \end{array}}{(hx+by+f)^3}$$

$$= \frac{\begin{array}{r} ah^2x^2 - a^2bx^2 - 2abgx + 2h^3xy + 2h^2fy - 2abhxy + 2h^2gx + 2ghf - g^2b - ab^2y^2 - 2abfy \\ - af^2 + h^2by^2 \end{array}}{(hx+by+f)^3}$$

$$\frac{d^2y}{dx^2} = \frac{h^2(ax^2+2hxy+2fy+2gx+by^2) - ab(ax^2+2gx+2hxy+by^2+2fy) + 2ghf - g^2b - af^2}{(hx+by+f)^3}$$

$$= \frac{h^2(-c) - ab(-c) + 2ghf - g^2b - af^2}{(hx+by+f)^3}$$

$$\frac{d^2y}{dx^2} = \frac{abc + 2fgh - af^2 - bg^2 - ch^2}{(hx+by+f)^3}$$

Q11 Find $\frac{d^2y}{dx^2}$ if $x^3+y^3 = 3axy$

Sol: Given

$$x^3+y^3 = 3axy$$

or $$x^3+y^3 - 3axy = 0$$

let $$f(x,y) = x^3+y^3-3axy$$

then $$f_x = 3x^2-3ay$$

$$f_y = 3y^2-3ax$$

Now $$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

$$= -\frac{3x^2-3ay}{3y^2-3ax}$$

$$= -\frac{x^2-ay}{y^2-ax}$$

$$\frac{dy}{dx} = \frac{ay-x^2}{y^2-ax}$$

Diff. w.r.t. x

$$\frac{d^2y}{dx^2} = \frac{(y^2-ax)\left[a\frac{dy}{dx}-2x\right]-(ay-x^2)\left[2y\frac{dy}{dx}-a\right]}{(y^2-ax)^2}$$

$$\frac{d^2y}{dx^2} = \frac{ay^2\frac{dy}{dx} - 2xy^2 - a^2x\frac{dy}{dx} + 2ax^2 - 2ay^2\frac{dy}{dx} + a^2y + 2x^2y\frac{dy}{dx} - ax^2}{(y^2-ax)^2} \qquad \underline{40}$$

$$= \frac{\frac{dy}{dx}\left(ay^2 - a^2x - 2ay^2 + 2x^2y\right) - \left(2xy^2 - 2ax^2 - a^2y + ax^2\right)}{(y^2-ax)^2}$$

$$= \frac{\left(\frac{ay-x^2}{y^2-ax}\right)\left(2x^2y - ay^2 - a^2x\right) - \left(2xy^2 - ax^2 - a^2y\right)}{(y^2-ax)^2}$$

$$= \frac{(ay-x^2)(2x^2y - ay^2 - a^2x) - (2xy^2 - ax^2 - a^2y)(y^2-ax)}{(y^2-ax)^3}$$

$$= \frac{2ax^2y^2 - a^2y^3 - a^3xy - 2x^4y + ax^2y^2 + a^2x^3 - (2xy^4 - 2ax^2y^2 - ax^2y^2 + a^2x^3 - a^2y^3 + a^3xy)}{(y^2-ax)^3}$$

$$= \frac{3ax^2y^2 - a^2y^3 - a^3xy - 2x^4y + a^2x^3 - \left(2xy^4 - 3ax^2y^2 + a^2x^3 - a^2y^3 + a^3xy\right)}{(y^2-ax)^3}$$

$$= \frac{3ax^2y^2 - \cancel{a^2y^3} - a^3xy - 2x^4y + \cancel{a^2x^3} - 2xy^4 + 3ax^2y^2 - \cancel{a^2x^3} + \cancel{a^2y^3} - a^3xy}{(y^2-ax)^3}$$

$$= \frac{6ax^2y^2 - 2a^3xy - 2xy(x^3+y^3)}{(y^2-ax)^3}$$

$$= \frac{6ax^2y^2 - 2a^3xy - 2xy(3axy)}{(y^2-ax)^3}$$

$$= \frac{\cancel{6ax^2y^2} - 2a^3xy - \cancel{6ax^2y^2}}{(y^2-ax)^3}$$

$$= -\frac{2a^3xy}{(y^2-ax)^3}$$

$$\frac{d^2y}{dx^2} = \frac{2a^3xy}{(ax-y^2)^3}$$