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Exercise No. 9.2

Q1 Approximate $\sqrt{299^2+399^2}$ by means of differentials.

Sol.

Let $u = \sqrt{x^2+y^2}$

Let $x = 300 \Rightarrow dx = -1$

$y = 400 \Rightarrow dy = -1$

Then

$$U = \sqrt{(300)^2+(400)^2}$$

$$= \sqrt{90000+160000}$$

$$= \sqrt{250000}$$

$$u = 500$$

Now change in U is given by du where

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

Now $u = \sqrt{x^2+y^2}$

$$\frac{\partial u}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x = \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial u}{\partial y} = \frac{1}{2\sqrt{x^2+y^2}} \cdot 2y = \frac{y}{\sqrt{x^2+y^2}}$$

So $du = \frac{x}{\sqrt{x^2+y^2}}dx + \frac{y}{\sqrt{x^2+y^2}}dy$

$$= \frac{300}{500}(-1) + \frac{400}{500}(-1)$$

$$du = \frac{-300-400}{500} = -\frac{700}{500} = -1.4$$

So $\sqrt{299^2+399^2} = u + du$

$$= 500 - 1.4 = 498.6$$

Q2 If $\theta = \tan^{-1}(y/x)$, use differentials to find an approximate value of θ when $x = 0.95$ & $y = 1.05$

Sol. Given $\theta = \tan^{-1}(y/x)$.

Suppose $x = 1 \Rightarrow dx = -0.05$

$y = 1 \Rightarrow dy = 0.05$

Now $\theta = \tan^{-1}\left(\frac{1}{1}\right)$

$= \tan^{-1}(1)$

$$\boxed{\theta = \pi/4}$$

Now change in θ is given by $d\theta$ where

$$d\theta = \frac{\partial\theta}{\partial x}dx + \frac{\partial\theta}{\partial y}dy$$

$$= \frac{1}{1+\frac{y^2}{x^2}}\cdot\left(-\frac{y}{x^2}\right)dx + \frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{1}{x}dy$$

$$= \frac{-x^2}{x^2+y^2}\cdot\frac{y}{x^2}dx + \frac{x^2}{x^2+y^2}\cdot\frac{1}{x}dy$$

$$d\theta = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

$$= -\frac{1}{2}(-0.05) + \frac{1}{2}(0.05)$$

$$= \frac{0.05}{2} + \frac{0.05}{2}$$

$$= \frac{0.05+0.05}{2}$$

$$d\theta = 0.05$$

Now $\theta = \tan^{-1}\left(\frac{1.05}{0.95}\right)$

$= \theta + d\theta$

$= \frac{\pi}{4} + 0.05$

$= 0.7854 + 0.05$

$\theta = 0.8354$

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Q3 If $u = \sqrt{x+2y}$ & x changes from 3 to 2.98 while y changes from 0.5 to 0.51, find an approximate value of change in u.

Sol. Given fn. is

$$u = \sqrt{x+2y}$$

Let $x = 3$ then $dx = 2.98 - 3 = -.02$

& $y = 0.5$ then $dy = 0.51 - 0.5 = 0.01$

then change in u is given by du where

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$= \frac{1}{2\sqrt{x+2y}} dx + \frac{1}{2\sqrt{x+2y}} \cdot 2\, dy$$

$$= \frac{1}{2\sqrt{3+2(0.5)}}(-.02) + \frac{1}{\sqrt{3+2(0.5)}}(0.01)$$

$$= -\frac{.01}{\sqrt{3+1}} + \frac{0.01}{\sqrt{3+1}}$$

$$= -\frac{.01}{2} + \frac{0.01}{2}$$

$$du = 0$$

So there is no change in u.

Q4 If $u = x^2 + y^2 + z^2 + xy^2z^3$ & x changes from 2 to 2.01, y changes from 1 to 1.02 & z changes from -1 to -0.99.

Find an approximate value for change in u.

Sol. Given function is

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$u = x^2 + y^2 + z^2 + xy^2z^3$

Let $x = 2 \Rightarrow dx = 2.01 - 2 = 0.01$

$y = 1 \Rightarrow dy = 1.02 - 1 = 0.02$

$z = -1 \Rightarrow dz = -0.99 - (-1) = -0.99 + 1 = 0.01$

then change in u is given by du where

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

$$= (2x + y^2z^3)dx + (2y + 2xyz^3)dy + (2z + 3xy^2z^2)dz$$

$$= \{2(2) + 1.(-1)\}(0.01) + (2 + 2(2)(-1))(0.02) + (2(-1) + 3(2)(1))(0.01)$$

$$= (4-1)(0.01) + (2-4)(0.02) + (-2+6)(0.01)$$

$$= (3)(0.01) - 2(.02) + 4(.01).$$

$$= 0.03 - 0.04 + 0.04$$

$du = 0.03$

Q5 A rectangular plate expands in such a way that its length changes from 10 to 10.03 & its breadth changes from 8 to 8.02. Find an approximate value for the change in its area.

Sol: Let x & y be the length & width of plate

Here $x = 10$ then $dx = 0.03$

$y = 8$ then $dy = 0.02$

Now area of rectangular plate is

$A = xy$

then change in area is given by dA where

$$dA = \frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy$$

$dA = y\,dx + x\,dy$

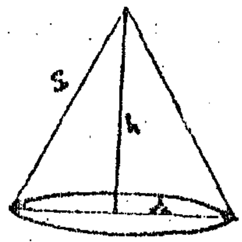
$= 8(.03) + 10(.02)$

$= 0.24 + 0.2$

$dA = 0.44$ is the approx. change in area

Q6 The lateral surface of a cone is computed from the formula $S = \pi r\sqrt{r^2+h^2}$, where r is the radius of the base & h is the height. If r is calculated as 6 with an accuracy of 1% & h as 8 with an accuracy of 0.25%. With what accuracy will be the area S.

Sol. Here



$S = \pi r\sqrt{r^2+h^2}$

Now $r = 6$ & $dr = 6 \times \frac{1}{100} = 0.06$

& $h = 8 \Rightarrow dh = 8 \times \frac{0.25}{100} = 8 \times 0.0025 = 0.02$

Now change in S is given by dS where

$$dS = \frac{\partial S}{\partial r}dr + \frac{\partial S}{\partial h}dh$$

$$= \pi\left[r \cdot \frac{1}{2\sqrt{r^2+h^2}} \cdot 2r + \sqrt{r^2+h^2} \cdot 1\right]dr + \pi r \frac{1}{2\sqrt{r^2+h^2}} \cdot 2h \cdot dh$$

$$= \pi\left[\frac{r^2}{\sqrt{r^2+h^2}} + \sqrt{r^2+h^2}\right]dr + \frac{\pi r h}{\sqrt{r^2+h^2}}dh$$

$$= \pi\left[\frac{36}{\sqrt{36+64}} + \sqrt{36+64}\right](0.06) + \frac{\pi(6)(8)}{\sqrt{36+64}}(0.02)$$

$$= \pi\left[\frac{36}{10} + 10\right](0.06) + \frac{48\pi}{10}(0.02)$$

$$= \pi(3.6+10)(0.06) + (4.8\pi)(0.02)$$

$$= (13.6\pi)(0.06) + (0.096\pi)$$

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$dS = 0.816\pi + 0.096\pi$

$dS = 0.912\pi$

Now as

$$S = \pi r\sqrt{r^2+h^2}$$

Put $r = 6$, $h = 8$

$$S = \pi(6)\sqrt{36+64}$$

$$= 6\pi(10)$$

$$S = 60\pi$$

Now 60π produces an error 0.912π

1 ———— $\dfrac{0.912\pi}{60\pi}$

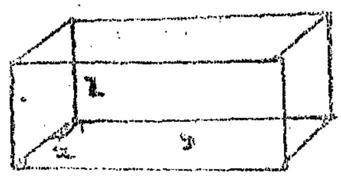
100 ———— $\dfrac{0.912\pi}{60\pi}\times 100$

$= 1.5\%$

Q7 The volume of a rectangular parallelopipped is given by the formula $V = xyz$. If this solid is compressed from above so that z is decreased by 2% while x & y increased by 0.75% approximately. What percentage change will be in V?

Sol. Given

$$V = xyz$$



where x, y & z are three edges of rectangular parallelopipped.

Here $dx = \dfrac{75}{100\times 100}x = \dfrac{75x}{10000} = 0.0075x$

$$dy = \frac{75}{100\times100}y = \frac{75y}{10000} = 0.0075y$$

& $$dz = -\frac{2}{100}z = -0.02z$$

Now change in V is given by dV where

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

$$= yz\,dx + xz\,dy + xy\,dz$$

$$= yz(0.0075x) + xz(0.0075y) + xy(-0.02z)$$

$$= 0.0075xyz + 0.0075xyz - 0.02xyz$$

$$dV = -0.005xyz$$

Now percentage change in V is

$$= \frac{dV}{V}\times100$$

$$= \frac{-0.005xyz}{xyz}\times100$$

$$= -0.5\%$$

Which shows a decrease of 0.5% in Volume.

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Q8 A formula for the area Δ of a triangle is $\Delta = \frac{1}{2}ab\sin C$. Approximately what error is ..... in computing Δ if a is taken to be 9.1 instead of 9, b is taken to be 4.08 instead of 4 & C is taken to be $30^\circ 3'$ instead of 30°.

Sol. Given that

$$\Delta = \frac{1}{2}ab\sin C$$

Here $a = 9 \Rightarrow da = 0.1$

$b = 4 \Rightarrow db = 0.08$

$C = 30^\circ = 30 \times \frac{\pi}{180} = \frac{\pi}{6}$ rad

$\Rightarrow dC = 3' = \left(\frac{3}{60}\right)^\circ = \frac{3}{60} \times \frac{\pi}{180} = \frac{\pi}{3600}$ rad.

Now as $\Delta = \frac{1}{2} ab \sin C$

So change in Δ is given by $d\Delta$ where

$$d\Delta = \frac{\partial \Delta}{\partial a} da + \frac{\partial \Delta}{\partial b} db + \frac{\partial \Delta}{\partial c} dc$$

$$= \frac{1}{2} b \sin C \, da + \frac{1}{2} a \sin C \, db + \frac{1}{2} ab \cos C \, dC$$

$$= \frac{1}{2}(4)\sin\frac{\pi}{6}(0.1) + \frac{1}{2}(9)\sin\frac{\pi}{6}(0.08) + \frac{1}{2}(9)(4)\cos\frac{\pi}{6} \cdot \frac{\pi}{3600}$$

$$= 2 \cdot \frac{1}{2}(0.1) + \frac{9}{2} \cdot \frac{1}{2}(0.08) + (18)\left(\frac{\pi}{3600}\right)$$

$$= 0.1 + 9(0.02) + \frac{\pi}{200}$$

$$= 0.1 + 0.18 + \frac{22}{7 \times 200}$$

$$= 0.28 + 0.015$$

$$d\Delta = 0.293$$

Now $\Delta = \frac{1}{2} ab \sin C$

$$= \frac{1}{2}(9)(4)\sin \pi/6$$

$$= (18)(1/2)$$

$$\Delta = 9$$

Now percentage error in computing Δ is

$$\frac{d\Delta}{\Delta} \times 100 = \frac{0.293}{9} \times 100 = 3.25\%$$

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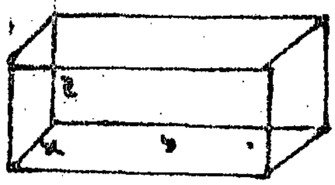
Q9 The dimensions of a box ~~are~~ measured to be 10 in, 12 in & 15 in & the measurement are correct to 0.02 in. Find the max. error if the volume of box is calculated from the given measurements. Also find the percentage error. (25)

Sol. Let x, y, z be the dimensions of the box then volume of box is



$$V = xyz$$

Here
$$x = 10'' \Rightarrow dx = 0.02''$$
$$y = 12'' \Rightarrow dy = 0.02''$$
$$z = 15'' \Rightarrow dz = 0.02''$$

Then error in V is given by dV where

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

$$= yz\,dx + xz\,dy + xy\,dz$$

$$= (12)(15)(0.02) + (10)(15)(0.02) + (10)(12)(0.02)$$

$$dV = 3.6 + 3 + 2.4 = 9$$

$$dV = 9$$

Now $V = (10)(12)(15)$

$$V = 1800$$

So percentage error in V is

$$= \frac{dV}{V} \times 100$$

$$= \frac{9}{1800} \times 100$$

$$= 0.5\%$$

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1530

Q10 Evaluate $\sin 29^\circ \cos 28^\circ \tan 44^\circ$ by using differentials. (21)

Sol.

Let $f(x,y,z) = \sin x \cos y \tan z$

take $x = 30^\circ = 30 \times \frac{\pi}{180} = \frac{\pi}{6}$ rad

$\Rightarrow dx = -1^\circ = -1 \times \frac{\pi}{180} = -\frac{\pi}{180}$ rad

$y = 30^\circ = 30 \times \frac{\pi}{180} = \frac{\pi}{6}$ rad

$\Rightarrow dy = -2^\circ = -2 \times \frac{\pi}{180} = -\frac{\pi}{90}$ rad

& $z = 45^\circ = 45 \times \frac{\pi}{180} = \frac{\pi}{4}$ rad

$\Rightarrow dz = -1^\circ = -1 \times \frac{\pi}{180} = -\frac{\pi}{180}$ rad.

Now $f(x,y,z) = f(30^\circ, 30^\circ, 45^\circ)$

$= \sin 30^\circ \cos 30^\circ \tan 45^\circ$

$= \frac{1}{2} \cdot \frac{\sqrt{3}}{2} \cdot 1$

$f(x,y,z) = \frac{\sqrt{3}}{4} = 0.433$

Now change in f is given by df where

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

$$= \cos x \cdot \cos y \tan z \, dx + \sin x(-\sin y)\tan z \, dy + \sin x \cos y \sec^2 z \, dz$$

$$= \cos\frac{\pi}{6} \cdot \cos\frac{\pi}{6} \cdot \tan\frac{\pi}{4} \cdot \left(-\frac{\pi}{180}\right) - \sin\frac{\pi}{6} \cdot \sin\frac{\pi}{6} \cdot \tan\frac{\pi}{4}\left(-\frac{\pi}{90}\right) + \sin\frac{\pi}{6} \cos\frac{\pi}{6} \sec^2\frac{\pi}{4} \cdot \frac{-\pi}{180}$$

$$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} \cdot 1\left(-\frac{\pi}{180}\right) - \frac{1}{2} \cdot \frac{1}{2} \cdot 1 \cdot \left(-\frac{\pi}{90}\right) + \frac{1}{2} \cdot \frac{\sqrt{3}}{\cancel{2}} \cdot \cancel{2}\left(-\frac{\pi}{180}\right)$$

$$= -\frac{3\pi}{4 \times 180} + \frac{\pi}{4 \times 90} - \frac{\sqrt{3}\pi}{2 \times 180}$$

$$= \frac{\pi}{180}\left[-\frac{3}{4} + \frac{1}{2} - \frac{\sqrt{3}}{2}\right]$$

$$df = 0.01745\pi\left[-0.75 + 0.5 - 0.866\right]$$

$$= 0.01745\pi(-1.116)$$

$$df = -0.0195$$

Now

$$\sin 29^\circ \cos 28^\circ \tan 44^\circ = f(x,y,z) + df$$

$$= 0.4330 + (-0.0195)$$

$$= 0.4330 - 0.0195$$

$$= 0.4135$$

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Differentiation of Composite functions

Chain rules:

(1) If z is a function of y & y is a function of x, then z is a function of x only then.

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

$$\begin{tikzcd} z \arrow[r, "\frac{dz}{dy}"'] \arrow[rr, bend left, "\frac{dz}{dx}"] & y \arrow[r, "\frac{dy}{dx}"'] & x \end{tikzcd}$$

(1) If $z = f(u,v)$ & $u = u(x)$, $v = v(x)$

Then z is a fn. of single variable x so

$$\frac{dz}{dx} = \frac{\partial z}{\partial u} \cdot \frac{du}{dx} + \frac{\partial z}{\partial v} \cdot \frac{dv}{dx}$$

$$\begin{tikzcd} & u \arrow[dr, "du/dx"] & \\ z \arrow[ur, "\partial z/\partial u"] \arrow[dr, "\partial z/\partial v"'] \arrow[rr, bend left=60, "\partial z/\partial x"] & & x \\ & v \arrow[ur, "dv/dx"'] & \end{tikzcd}$$