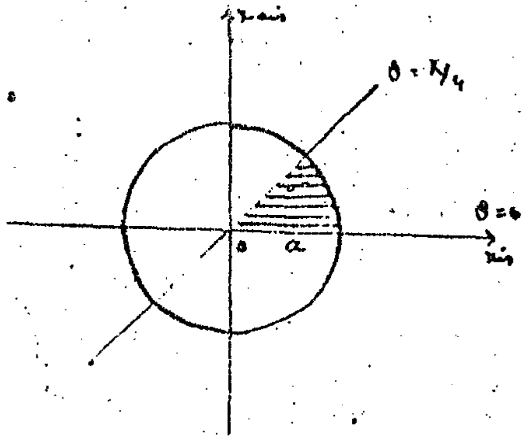
Exercise No. 9.9

Q1 Find the area of the solid generated by revolving the circle $r = a$, $0 \le \theta \le \frac{\pi}{4}$ about polar axis.

Sol. Given eq. of circle is

$$r = a$$



so parametric eqs. are

$$x = a\cos\theta, \quad y = a\sin\theta$$

let A be req. area then

$$A = 2\pi\int_0^{\pi/4} r\sin\theta\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta \quad \text{———} (1)$$

Here $\frac{dx}{d\theta} = -a\sin\theta, \quad \frac{dy}{d\theta} = a\cos\theta$

So
$$\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2} = \sqrt{a^2\sin^2\theta + a^2\cos^2\theta}$$

$$= \sqrt{a^2(\sin^2\theta+\cos^2\theta)}$$

$$= a$$

So from (1) :

$$A = 2\pi\int_0^{\pi/4} a\sin\theta \cdot a\,d\theta$$

$$= 2\pi a^2\int_0^{\pi/4}\sin\theta\,d\theta$$

$$= 2\pi a^2\Big|-\cos\theta\Big|_0^{\pi/4}$$

$$= -2\pi a^2\Big|\cos\theta\Big|_0^{\pi/4}$$

$$= -2\pi a^2\left[\cos\pi/4 - \cos 0\right]$$

$$= -2\pi a^2\left[\frac{1}{\sqrt{2}} - 1\right] = 2\pi a^2\left(1 - \frac{1}{\sqrt{2}}\right) \text{ Sq. units.}$$

Q2 Find the area of surface generated by revolving $r = 2a\sin\theta$ about polar axis.

Sol. Given eq. of curve is

$$r = 2a\sin\theta$$

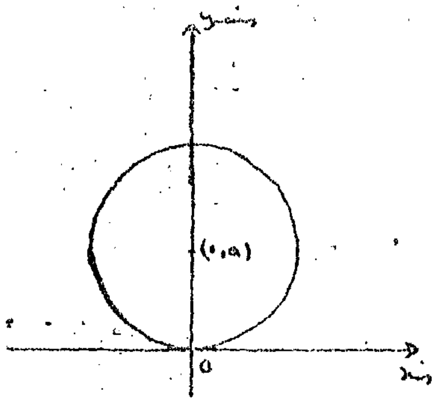
For limits of integration

Put $r = 0$

$$2a\sin\theta = 0$$

$$\sin\theta = 0$$

$$\Rightarrow \boxed{\theta = 0, \pi}$$



Let A be req. surface area then

$$A = 2\pi\int_0^\pi r\sin\theta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta \quad \text{—— (1)}$$

Here $r = 2a\sin\theta$

$$\frac{dr}{d\theta} = 2a\cos\theta$$

So

$$\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2} = \sqrt{4a^2\sin^2\theta+4a^2\cos^2\theta}$$

$$= \sqrt{4a^2}$$

$$= 2a$$

So from (1)

$$A = 2\pi\int_0^\pi 2a\sin\theta\,\sin\theta\cdot 2a\,d\theta$$

$$= 8\pi a^2\int_0^\pi \sin^2\theta\,d\theta$$

$$= 16\pi a^2\int_0^{\pi/2} \sin^2\theta\,d\theta$$

$$= 16\pi a^2\left(\frac{1}{2}\cdot\frac{\pi}{2}\right) \quad \text{(Walli's law)}$$

$$A = 4\pi^2 a^2 \text{ Sq. units}$$

Q3 The arc of the spiral $r = e^{\theta}$ from $(1,0)$ to $(e^{1},1)$ is revolved about the line $\theta = \pi/2$. Find the area of the resulting surface.

Sol.

Let A be the req. area then

$$A = 2\pi \int_0^1 r\cos\theta \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta \quad \text{———} (1)$$

Here $r = e^{\theta}$

$$\frac{dr}{d\theta} = e^{\theta}$$

$$\Rightarrow \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2} = \sqrt{e^{2\theta}+e^{2\theta}} = \sqrt{2e^{2\theta}} = \sqrt{2}\, e^{\theta}$$

So from (1)

$$A = 2\pi \int_0^1 e^{\theta}\cos\theta \sqrt{2}\, e^{\theta}\, d\theta$$

$$= 2\sqrt{2}\pi \int_0^1 e^{2\theta}\cos\theta\, d\theta$$

$$= 2\sqrt{2}\pi \left| \frac{e^{2\theta}}{(2^2+1^2)}\left(2\cos\theta + \sin\theta\right)\right|_0^1 \qquad \because \int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

$$= 2\sqrt{2}\pi \left[\frac{e^{2\theta}}{5}(2\cos\theta + \sin\theta)\right]_0^1$$

$$= \frac{2\sqrt{2}\pi}{5}\left[e^{2}(2\cos 1 + \sin 1) - (2\cos 0 + \sin 0)\right]$$

$$A = \frac{2\sqrt{2}\pi}{5}\left[e^{2}(2\cos 1 + \sin 1) - 2\right] \text{ Sq. units}$$

Q4 Find the surface area of the solid generated by revolving about the polar axis, the curve $r = e^{\theta/2}$; $0 \le \theta \le \pi$.

Sol.

Sol: let A be the req. surface area then

$$A = 2\pi\int_0^{\pi} r\sin\theta\cdot\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta \quad \text{—— (1)}$$

Here $r = e^{\theta/2}$

$$\frac{dr}{d\theta} = \frac{1}{2}e^{\theta/2}$$

Now

$$\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2} = \sqrt{e^{\theta}+\frac{1}{4}e^{\theta}} = \sqrt{\frac{4e^{\theta}+e^{\theta}}{4}} = \sqrt{\frac{5e^{\theta}}{4}} = \frac{\sqrt{5}\,e^{\theta/2}}{2}$$

So from (1)

$$A = 2\pi\int_0^{\pi} e^{\theta/2}\cdot\sin\theta\cdot\frac{\sqrt{5}}{2}e^{\theta/2}\,d\theta$$

$$= \sqrt{5}\,\pi\int_0^{\pi} e^{\theta}\sin\theta\,d\theta$$

$$= \sqrt{5}\,\pi\left|\frac{e^{\theta}}{1^2+1^2}(\sin\theta-\cos\theta)\right|_0^{\pi}$$

$$= \sqrt{5}\,\pi\left|\frac{e^{\theta}}{2}(\sin\theta-\cos\theta)\right|_0^{\pi}$$

$$= \frac{\sqrt{5}\,\pi}{2}\left|e^{\theta}(\sin\theta-\cos\theta)\right|_0^{\pi}$$

$$= \frac{\sqrt{5}\,\pi}{2}\left[e^{\pi}(\sin\pi-\cos\pi)-e^{0}(\sin 0-\cos 0)\right]$$

$$= \frac{\sqrt{5}\,\pi}{2}\left[e^{\pi}(0-(-1))-(0-1)\right]$$

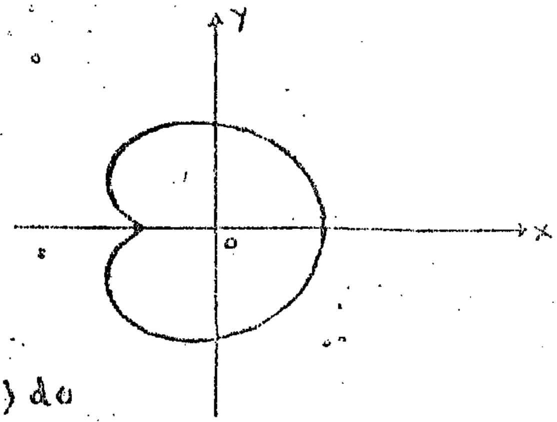
$$= \frac{\sqrt{5}\,\pi}{2}\left[e^{\pi}(1)+1\right]$$

$$A = \frac{\sqrt{5}\,\pi}{2}\left(e^{\pi}+1\right) \text{ sq. units}$$

Q5 Prove that the volume generated by revolution of the area enclosed by the limacon $r = a + b\cos\theta\ (a>b)$ is $\frac{4}{3}\pi a(a^2+b^2)$. 157

Sol. Let V be the req. volume then

$$V = \frac{2\pi}{3}\int_0^\pi r^3 \sin\theta\, d\theta \quad \text{——— (1)}$$



Here $r = a + b\cos\theta$

So from (1)

$$V = \frac{2\pi}{3}\int_0^\pi (a+b\cos\theta)^3 \cdot \sin\theta\, d\theta$$

$$= -\frac{2\pi}{3b}\int_0^\pi (a+b\cos\theta)^3 \cdot (-b\sin\theta)\, d\theta$$

$$= -\frac{2\pi}{3b}\left| \frac{(a+b\cos\theta)^4}{4} \right|_0^\pi$$

$$= -\frac{\pi}{6b}\left| (a+b\cos\theta)^4 \right|_0^\pi$$

$$= -\frac{\pi}{6b}\left[(a+b\cos\pi)^4 - (a+b\cos 0)^4\right]$$

$$= -\frac{\pi}{6b}\left[(a-b)^4 - (a+b)^4\right]$$

$$= \frac{\pi}{6b}\left[(a+b)^4 - (a-b)^4\right]$$

$$= \frac{\pi}{6b}\left[(a^4+4a^3b+6a^2b^2+4ab^3+b^4) - (a^4-4a^3b+6a^2b^2-4ab^3+b^4)\right]$$

$$= \frac{\pi}{6b}\left[\cancel{a^4}+4a^3b+\cancel{6a^2b^2}+4ab^3+\cancel{b^4}-\cancel{a^4}+4a^3b-\cancel{6a^2b^2}+4ab^3-\cancel{b^4}\right]$$

$$= \frac{\pi}{6b}\left[8a^3b+8ab^3\right]$$

$$= \frac{8ab\pi}{6b}\cdot(a^2+b^2) = \frac{4\pi a}{3}(a^2+b^2) \text{ Cubic units.}$$

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Q6 Find the Volume & area of the surface of the solid generated by the revolution of the area enclosed by the lemniscate $r^2 = a^2\cos 2\theta$ about the initial line. 158

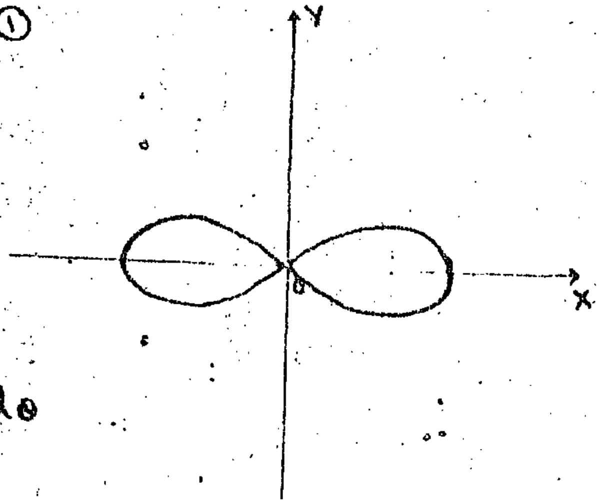
Sol. Let V be the req. Volume then

$$V = 2\cdot\frac{2\pi}{3}\int_0^{\pi/4} r^3\sin\theta\,d\theta \quad \text{---} (1)$$

Here $r^2 = a^2\cos 2\theta$

$\Rightarrow \quad r = (a^2\cos 2\theta)^{1/2}$

So from (1)



$$V = \frac{4\pi}{3}\int_0^{\pi/4} (a^2\cos 2\theta)^{3/2}\cdot\sin\theta\,d\theta$$

$$\frac{4\pi}{3}\int_0^{\pi/4} a^3(\cos 2\theta)^{3/2}\cdot\sin\theta\,d\theta$$

$$V \quad \frac{4\pi a^3}{3}\int_0^{\pi/4} (2\cos^2\theta - 1)^{3/2}\cdot\sin\theta\,d\theta$$

Put $2\cos^2\theta = t^2$

$\Rightarrow \quad \sqrt{2}\cos\theta = t$

$-\sqrt{2}\sin\theta\,d\theta = dt$

$\sin\theta\,d\theta = -\frac{1}{\sqrt{2}}\,dt$

as $\theta = 0,\ t = \sqrt{2}$

$\theta = \frac{\pi}{4},\ t = 1.$

So

$$V = \frac{4\pi a^3}{3}\int_{\sqrt{2}}^{1} (t^2-1)^{3/2}\cdot\frac{-1}{\sqrt{2}}\,dt$$

$$= \frac{4\pi a^3}{3\sqrt{2}}\int_1^{\sqrt{2}} (t^2-1)^{3/2}\,dt \quad \text{---} (2)$$

Consider $I = \int (t^2-1)^{3/2}\,dt$

$$= \int (t^2-1)^{3/2}\cdot 1\,dt$$

Integ. by parts

$$= (t^2-1)^{3/2}\cdot t - \int t\cdot\tfrac{3}{2}(t^2-1)^{1/2}\cdot 2t\,dt$$

$$= t(t^2-1)^{3/2} - 3\int t^2(t^2-1)^{1/2}\,dt$$

$$= t(t^2-1)^{3/2} - 3\int (t^2-1+1)(t^2-1)^{1/2}\,dt$$

$$= t(t^2-1)^{3/2} - 3\int (t^2-1)^{3/2}\,dt - 3\int (t^2-1)^{1/2}\,dt$$

$$I = t(t^2-1)^{3/2} - 3I - 3\int (t^2-1)^{1/2}\,dt$$

$$4I = t(t^2-1)^{3/2} - 3\int \sqrt{t^2-1}\,dt$$

$$4I = t(t^2-1)^{3/2} - 3\left[\frac{t\sqrt{t^2-1}}{2} - \frac{1}{2}\ln\left(t+\sqrt{t^2-1}\right)\right]$$

$$I = \frac{t(t^2-1)^{3/2}}{4} - \frac{3}{8}\left[t\sqrt{t^2-1} - \ln\left(t+\sqrt{t^2-1}\right)\right]$$

Hence

$$V = \frac{4\pi a^3}{3\sqrt{2}}\left|\frac{t(t^2-1)^{3/2}}{4} - \frac{3}{8}\left(t\sqrt{t^2-1} - \ln\left(t+\sqrt{t^2-1}\right)\right)\right|^{\sqrt{2}}$$

$$= \frac{4\pi a^3}{3\sqrt{2}}\left[\frac{\sqrt{2}(1)}{4} - \frac{3}{8}\left(\sqrt{2}\right) + \frac{3}{8}\ln(\sqrt{2}+1)\right]$$

$$= \frac{\pi a^3}{3\sqrt{2}}\left[\sqrt{2} - \frac{3}{2}\sqrt{2} + \frac{3}{2}\ln(\sqrt{2}+1)\right]$$

$$= \frac{\pi a^3}{3\sqrt{2}}\left[-\frac{\sqrt{2}}{2} + \frac{3}{2}\ln(\sqrt{2}+1)\right]$$

$$= \frac{\pi a^3}{3\cdot 2\sqrt{2}}\left[-\sqrt{2} + 3\ln(\sqrt{2}+1)\right]$$

$$= \frac{\pi a^3}{2}\left[-\frac{1}{3} + \frac{1}{\sqrt{2}}\ln(\sqrt{2}+1)\right]$$ 1664 $\qquad$ 16.

$$V = \frac{\pi a^3}{2}\left[\frac{1}{\sqrt{2}}\ln(\sqrt{2}+1) - \frac{1}{3}\right] \text{ Cubic units.}$$

Now let A be the required area then

$$A = 2\int_0^{\pi/4} 2\pi(r\sin\theta)\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta \quad \text{------ (A)}$$

Here $r^2 = a^2\cos 2\theta$

$$2r\frac{dr}{d\theta} = -a^2\sin 2\theta\cdot 2$$

$$r\frac{dr}{d\theta} = -a^2\sin 2\theta$$

$$\Rightarrow \frac{dr}{d\theta} = -\frac{a^2\sin 2\theta}{r} = -\frac{a^2\sin 2\theta}{a\sqrt{\cos 2\theta}} = -\frac{a\sin 2\theta}{\sqrt{\cos 2\theta}}$$

Now

$$\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2} = \sqrt{a^2\cos 2\theta + \frac{a^2\sin^2 2\theta}{\cos 2\theta}}$$

$$= \sqrt{\frac{a^2\cos^2 2\theta + a^2\sin^2 2\theta}{\cos 2\theta}} = \frac{a}{\sqrt{\cos 2\theta}}$$

So from (A)

$$A = 4\pi\int_0^{\pi/4} a\cancel{\sqrt{\cos 2\theta}}\cdot\sin\theta\cdot\frac{a}{\cancel{\sqrt{\cos 2\theta}}}\,d\theta$$

$$= 4\pi a^2\int_0^{\pi/4}\sin\theta\,d\theta$$

$$= 4\pi a^2\Big|-\cos\theta\Big|_0^{\pi/4}$$

$$= 4\pi a^2\left[-\cos\pi/4 + \cos 0\right]$$

$$= 4\pi a^2\left(-\frac{1}{\sqrt{2}} + 1\right)$$

$$A = 4\pi a^2\left(1 - \frac{1}{\sqrt{2}}\right)$$

$$A = 4\pi a^2\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)$$

$$= \pi a^2\left(\frac{4\sqrt{2}-4}{\sqrt{2}}\right)$$

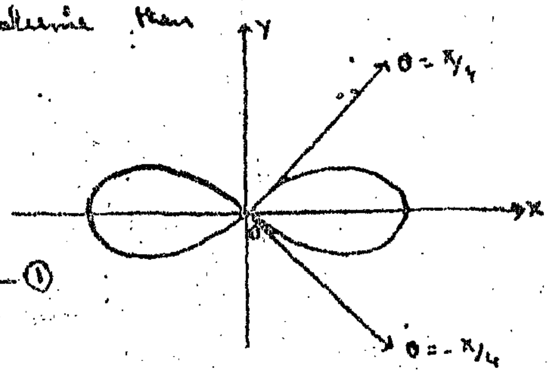
$$= \pi a^2\left(\frac{4\sqrt{2}-2\cdot\sqrt{2}\cdot\sqrt{2}}{\sqrt{2}}\right)$$

$$= \pi a^2\left(4-2\sqrt{2}\right)$$

$$A = (4-2\sqrt{2})\pi a^2 \text{ sq. units.}$$

Q7. Show that the volume of the solid formed by revolving the area bounded by one loop of the curve $r^2 = a^2\cos 2\theta$ about the line $\theta = \frac{\pi}{2}$ is $\frac{\pi a^3}{4\sqrt{2}}$.

Sol:- Let V be the req. volume then



$$V = \frac{2\pi}{3}\int_{-\frac{\pi}{4}}^{\pi/4} r^3\cos\theta\, d\theta$$

$$V = \frac{4\pi}{3}\int_{0}^{\pi/4} r^3\cos\theta\, d\theta \quad \text{——— (1)}$$

Here $r^2 = a^2\cos 2\theta$

$$\Rightarrow r = (a^2\cos 2\theta)^{1/2}$$

So from (1)

$$V = \frac{4\pi}{3}\int_{0}^{\pi/4} (a^2\cos 2\theta)^{3/2}\cdot\cos\theta\, d\theta$$

$$V = \frac{4\pi a^3}{3}\int_{0}^{\pi/4} (1-2\sin^2\theta)^{3/2}\cdot\cos\theta\, d\theta$$

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Put $2\sin^2\theta = \sin^2 t$

or $\sqrt{2}\sin\theta = \sin t$ | as $\theta = 0, t = 0$

$\sqrt{2}\cos\theta\, d\theta = \cos t\, dt$ | $\theta = \frac{\pi}{4}, t = \pi/2$

So

$$V = \frac{4\pi a^3}{3}\int_0^{\pi/2} (1-\sin^2 t)^{3/2} \cdot \frac{\cos t}{\sqrt{2}}\, dt$$

$$= \frac{4\pi a^3}{3\sqrt{2}}\int_0^{\pi/2} (\cos^2 t)^{3/2} \cdot \cos t\, dt$$

$$= \frac{4\pi a^3}{3\sqrt{2}}\int_0^{\pi/2} \cos^3 t \cdot \cos t\, dt$$

$$= \frac{4\pi a^3}{3\sqrt{2}}\int_0^{\pi/2} \cos^4 t\, dt$$

$$= \frac{4\pi a^3}{3\sqrt{2}}\left[\frac{3}{4}\cdot\frac{1}{2}\cdot\pi/2\right] \quad \text{(By Wallis's law)}$$

$$= \frac{\cancel{4}\pi a^3}{\cancel{3}\sqrt{2}} \cdot \frac{\cancel{3}\pi}{\cancel{16}_4}$$

$$V = \frac{\pi^2 a^3}{4\sqrt{2}} \text{ cubic units.}$$

Q8 The area enclosed by the lemniscate $r^2 = a^2\cos 2\theta$ revolves about a tangent at the pole. Show that the volume & area of the surface of the solid generated are respectively $\frac{1}{4}\pi^2 a^3$ & $4\pi a^2$.

Sol. Given eq. of curve is

$$r^2 = a^2\cos 2\theta$$

For limits of integration of one loop:

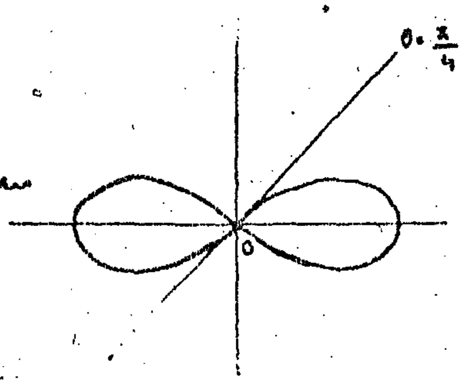
Put $r = 0$

$a^2\cos 2\theta = 0$

$\Rightarrow \cos 2\theta = 0$

$$2\theta = \pm\frac{\pi}{2}$$

$$\boxed{\theta = \pm \pi/4}$$



Let V be the req. Volume then

$$V = 2\cdot\frac{2\pi}{3}\int_{-\pi/4}^{\pi/4} r^3 \sin(\theta - \pi/4)\, d\theta$$

$$= \frac{4\pi}{3}\int_{-\pi/4}^{\pi/4} (a^2\cos 2\theta)^{\frac{3}{2}}\left[\sin\theta \cos\pi/4 - \cos\theta \sin\pi/4\right] d\theta$$

$$= \frac{4\pi a^3}{3}\int_{-\pi/4}^{\pi/4} (\cos 2\theta)^{\frac{3}{2}}\left[\sin\theta\cdot\frac{1}{\sqrt{2}} - \cos\theta\cdot\frac{1}{\sqrt{2}}\right] d\theta$$

$$= \frac{4\pi a^3}{3\sqrt{2}}\int_{-\pi/4}^{\pi/4} (\cos 2\theta)^{3/2}(\sin\theta - \cos\theta)\, d\theta$$

$$= \frac{4\pi a^3}{3\sqrt{2}}\left[\underbrace{\int_{-\frac{\pi}{4}}^{\pi/4}(\cos 2\theta)^{\frac{3}{2}}\sin\theta\, d\theta}_{\text{odd fn.}} - \underbrace{\int_{-\pi/4}^{\pi/4}(\cos 2\theta)^{\frac{3}{2}}\cos\theta\, d\theta}_{\text{even fn.}}\right]$$

$$= \frac{4\pi a^3}{3\sqrt{2}}\left[0 - 2\int_0^{\pi/4}(\cos 2\theta)^{\frac{3}{2}}\cos\theta\, d\theta\right]$$

$$= -\frac{8\pi a^3}{3\sqrt{2}}\int_0^{\pi/4}(\cos 2\theta)^{\frac{3}{2}}\cos\theta\, d\theta$$

$$V = -\frac{8\pi a^3}{3\sqrt{2}}\int_0^{\pi/4}(1 - 2\sin^2\theta)^{\frac{3}{2}}\cdot\cos\theta\, d\theta$$

Put $2\sin^2\theta = \sin^2 t$

$\sqrt{2}\sin\theta = \sin t$

$\sqrt{2}\cos\theta\, d\theta = \cos t\, dt$

$\cos\theta\, d\theta = \cos t\, dt/\sqrt{2}$

as $\theta = 0$, $t = 0$

$\theta = \frac{\pi}{4}$, $t = \frac{\pi}{2}$

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So $V = -\dfrac{8\pi a^3}{3\sqrt{2}}\int_0^{\pi/2}(1-\sin^2 t)^{3/2}\cdot\dfrac{\cos t\,dt}{\sqrt{2}}$

$= -\dfrac{8\pi a^3}{3\times 2}\int_0^{\pi/2}(\cos^2 t)^{3/2}\cdot\cos t\,dt$

$= -\dfrac{4\pi a^3}{3}\int_0^{\pi/2}\cos^3 t\cdot\cos t\,dt$

$= -\dfrac{4\pi a^3}{3}\int_0^{\pi/2}\cos^4 t\,dt$

$= -\dfrac{4\pi a^3}{3}\left(\dfrac{3}{4}\cdot\dfrac{1}{2}\cdot\pi/2\right)$ (Walli's law)

$= -\dfrac{\pi^2 a^3}{4}$

$V = \dfrac{\pi^2 a^3}{4}$ (In magnitude)

Now let A be the req. surface area generated by given curve. Then

$$A = 2\cdot 2\pi\int_{-\pi/4}^{\pi/4} r\sin(\theta-\pi/4)\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

$$= 4\pi\int_{-\pi/4}^{\pi/4} r\sin(\theta-\pi/4)\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta \quad \text{——— (2)}$$

Here $r^2 = a^2\cos 2\theta$.

$2r\cdot\dfrac{dr}{d\theta} = a^2(-\sin 2\theta\cdot 2)$

$$r\frac{dr}{d\theta} = -a^2\sin 2\theta$$

$$\frac{dr}{d\theta} = -\frac{a^2\sin 2\theta}{a\sqrt{\cos 2\theta}}$$

$$\Rightarrow \frac{dr}{d\theta} = -\frac{a\sin 2\theta}{\sqrt{\cos 2\theta}}$$

Now

$$\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2} = \sqrt{a^2\cos 2\theta + \frac{a^2\sin^2 2\theta}{\cos 2\theta}}$$

$$= \sqrt{\frac{a^2\cos^2 2\theta + a^2\sin^2 2\theta}{\cos 2\theta}} = \frac{a}{\sqrt{\cos 2\theta}}$$

So from ②

$$A = 4\pi a \int_{-\pi/4}^{\pi/4} a\sqrt{\cancel{\cos 2\theta}} \cdot \sin(\theta-\pi/4) \cdot \frac{a}{\sqrt{\cancel{\cos 2\theta}}}\, d\theta$$

$$= 4\pi a^2 \int_{-\pi/4}^{\pi/4} \sin(\theta-\pi/4)\, d\theta$$

$$= 4\pi a^2 \int_{-\pi/4}^{\pi/4} (\sin\theta \cos\pi/4 - \cos\theta \cdot \sin\pi/4)\, d\theta$$

$$= 4\pi a^2 \int_{-\pi/4}^{\pi/4} \sin\theta \cdot \frac{1}{\sqrt{2}} - \cos\theta \cdot \frac{1}{\sqrt{2}}\, d\theta$$

$$= \frac{4\pi a^2}{\sqrt{2}} \int_{-\pi/4}^{\pi/4} (\sin\theta - \cos\theta)\, d\theta$$

$$= \frac{4\pi a^2}{\sqrt{2}} \Big| -\cos\theta - \sin\theta \Big|_{-\pi/4}^{\pi/4}$$

$$= \frac{4\pi a^2}{\sqrt{2}}\Big|\cos\theta + \sin\theta\Big|_{-\pi/4}^{\pi/4}$$

$$= \frac{4\pi a^2}{\sqrt{2}}\left[(\cos\pi/4 + \sin\pi/4) - (\cos(-\pi/4) + \sin(-\pi/4))\right]$$

$$= \frac{4\pi a^2}{\sqrt{2}}\left[\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - \left(\cancel{\frac{1}{\sqrt{2}}} - \cancel{\frac{1}{\sqrt{2}}}\right)\right]$$

$$= \frac{4\pi a^2}{\sqrt{2}}\left[\frac{2}{\sqrt{2}}\right]$$

$$= \frac{8\pi a^2}{2}$$

$$A = 4\pi a^2 \text{ Sq. units}$$

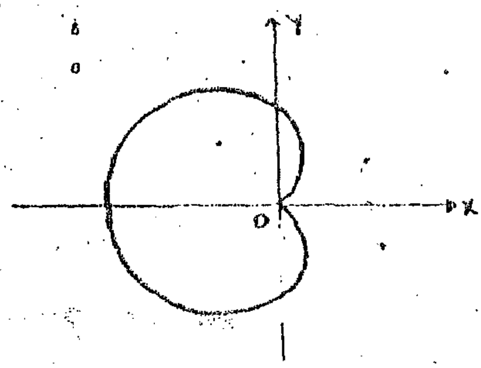
Q9. Find the area of the surface generated by revolving the area enclosed by the upper half of the Cardiod $r = a(1-\cos\theta)$ about the initial line.

Sol. Let A be the required area then

$$A = 2\pi\int_0^{\pi} r\sin\theta\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \quad \text{———} (1)$$

Here $r = a(1-\cos\theta)$

$$\frac{dr}{d\theta} = a\sin\theta$$



Now

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}$$

$$= a\sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}$$

$$= a\sqrt{2 - 2\cos\theta}$$

$$= a\sqrt{2(1-\cos\theta)}$$

$$= a\sqrt{2\cdot 2\sin^2\theta/2}$$

So

$$\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2} = 2a\sin\theta/2$$

So from ①

$$A = 2\pi\int_0^\pi a(1-\cos\theta)\cdot\sin\theta\cdot 2a\sin\theta/2\, d\theta$$

$$= 4\pi a^2\int_0^\pi 2\sin^2\theta/2\cdot 2\sin\theta/2\cos\theta/2\cdot\sin\theta/2\, d\theta$$

$$A = 16\pi a^2\int_0^\pi \sin^4\theta/2\cdot\cos\theta/2\, d\theta$$

$$= 32\pi a^2\int_0^\pi \sin^4\theta/2\cdot\left(\cos\theta/2\cdot\frac{1}{2}\right)d\theta$$

$$= 32\pi a^2\left|\frac{\sin^5\theta/2}{5}\right|_0^\pi$$

$$= \frac{32\pi a^2}{5}\left|\sin^5\theta/2\right|_0^\pi$$

$$= \frac{32\pi a^2}{5}\left[\sin^5\pi/2 - \sin 0\right]$$

$$= \frac{32\pi a^2}{5}\left[1-0\right]$$

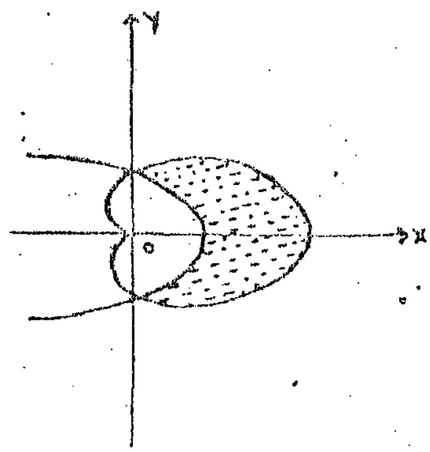
$$A = \frac{32}{5}\pi a^2 \text{ Sq. units}$$

Q10 The upper half of the area inside the Cardioid $r = 2a(1+\cos\theta)$ & outside the parabola $\frac{2a}{r} = 1+\cos\theta$ is revolved about the initial line. Show that the volume of the solid generated is $18\pi a^3$.

Sol: Given eqs. of curves are

$$r = 2a(1+\cos\theta) \quad \text{——} (1)$$

$$r = \frac{2a}{1+\cos\theta} \quad \text{——} (2)$$



For pt. of intersection solving (1) & (2)

from (1) & (2)

$$2a(1+\cos\theta) = \frac{2a}{1+\cos\theta}$$

$$1+\cos\theta = \frac{1}{1+\cos\theta}$$

$$\Rightarrow (1+\cos\theta)^2 = 1$$

$$\cancel{1} + 2\cos\theta + \cos^2\theta = \cancel{1}\ 0$$

$$\cos\theta(2+\cos\theta) = 0$$

$$\Rightarrow \cos\theta = 0 \text{ or } \cos\theta = -2$$

But $\cos\theta = -2$ is not possible

so $\cos\theta = 0$

$$\Rightarrow \boxed{\theta = \pm\frac{\pi}{2}}$$

Let V be the req. volume then

$$V = \frac{2\pi}{3}\int_0^{\pi/2}\left\{(2a(1+\cos\theta))^3 - \left(\frac{2a}{1+\cos\theta}\right)^3\right\}\sin\theta\, d\theta$$

$$\frac{\pi}{3}\cdot 8a^3\int_0^{\pi/2}\left\{(1+\cos\theta)^3\cdot\sin\theta\,d\theta-\frac{1}{(1+\cos\theta)^3}\cdot\sin\theta\right\}d\theta$$

$$=\frac{16\pi a^3}{3}\left[-\int_0^{\pi/2}(1+\cos\theta)^3\cdot(-\sin\theta)\,d\theta+\int_0^{\pi/2}(1+\cos\theta)^{-3}\cdot(-\sin\theta)\,d\theta\right]$$

$$=\frac{16\pi a^3}{3}\left[-\left|\frac{(1+\cos\theta)^4}{4}\right|_0^{\pi/2}+\left|\frac{(1+\cos\theta)^{-2}}{-2}\right|_0^{\pi/2}\right]$$

$$=-\frac{16\pi a^3}{3}\left[\left|\frac{(1+\cos\theta)^4}{4}\right|_0^{\pi/2}+\left|\frac{1}{2(1+\cos\theta)^2}\right|_0^{\pi/2}\right]$$

$$=-\frac{16\pi a^3}{3}\left[\left(\frac{1}{4}-\frac{16}{4}\right)+\left(\frac{1}{2}-\frac{1}{8}\right)\right]$$

$$=-\frac{16\pi a^3}{3}\left[-\frac{15}{4}+\frac{4-1}{8}\right]$$

$$=-\frac{16\pi a^3}{3}\left[-\frac{15}{4}+\frac{3}{8}\right]$$

$$=-\frac{16\pi a^3}{3}\left[\frac{-30+3}{8}\right]$$

$$=-\frac{2\pi a^3}{3}(-27)$$

$$=(-2\pi a^3)(-9)$$

$$V=18\pi a^3 \text{ cubic units}$$

(End of ch. 4)