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Exercise No. 9.8

Q1 Find the area of the surface of revolution generated by revolving about the x-axis the area bounded by an arc of the parabola $y^2 = 12x$ from $x = 0$ to $x = 3$

Sol.

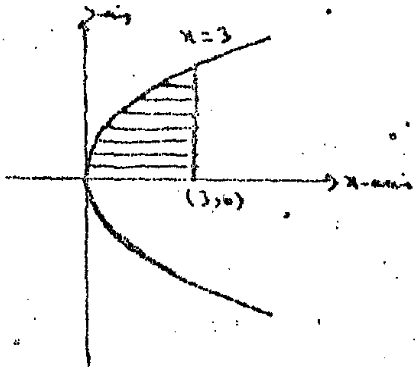
let A be the req. area

then $A = 2\pi\int_0^3 y\sqrt{1+(dy/dx)^2}\,dx$ —— ①

Here $y^2 = 12x$

$2y.\frac{dy}{dx} = 12$

$\frac{dy}{dx} = \frac{6}{y}$



$\sqrt{1+(\frac{dy}{dx})^2} = \sqrt{1+\frac{36}{y^2}}$

$= \sqrt{1+\frac{36}{12x}}$

$= \sqrt{1+\frac{3}{x}}$

$= \sqrt{\frac{x+3}{x}}$

So from ①

$A = 2\pi\int_0^3 \sqrt{12}\sqrt{x}\sqrt{\frac{x+3}{x}}\,dx$

$= 2\pi\sqrt{12}\int_0^3 \sqrt{x+3}\,dx$

$= 2\pi\sqrt{12}\left| \frac{(x+3)^{3/2}}{3/2} \right|_0^3$

$= \frac{4\pi\sqrt{12}}{3}\left[(6)^{3/2} - (3)^{3/2}\right]$

$$= \frac{4\pi\, 2\sqrt{3}}{3}\left[(2\times 3)^{3/2} - (3)^{3/2}\right]$$

$$= \frac{8\pi}{\sqrt{3}}\left[2\sqrt{2}\cdot 3\sqrt{3} - 3\sqrt{3}\right]$$

$$= \frac{8\pi}{\sqrt{3}}\cdot 3\sqrt{3}\left[2\sqrt{2} - 1\right]$$

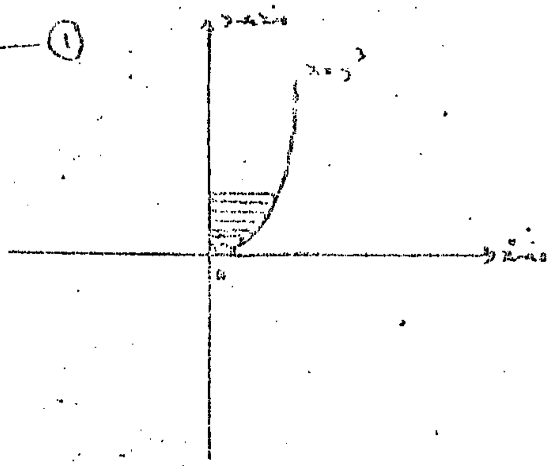
$$A = 24\pi(2\sqrt{2}-1) \text{ Sq. units}$$

Q2 Find the area of surface of revolution generated by revolving about y-axis the area enclosed by the arc of $x = y^3$ from $y = 0$ to $y = 1$

Sol:-

Let A be the req. area then

$$A = 2\pi\int_0^1 x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy \quad \text{——— (1)}$$



Here $x = y^3$

$$\frac{dx}{dy} = 3y^2$$

So $\sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1+9y^4}$

Put in (1).

$$A = 2\pi\int_0^1 y^3\sqrt{1+9y^4}\, dy$$

$$= \frac{2\pi}{36}\int_0^1 (1+9y^4)^{1/2}\cdot(36y^3)\, dy$$

$$= \frac{\pi}{18}\left|\frac{(1+9y^4)^{3/2}}{3/2}\right|_0^1$$

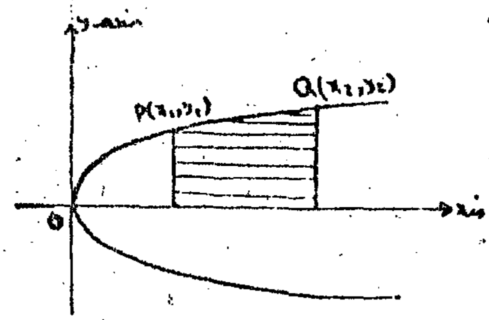
$$= \frac{\pi}{27}\left[(1+9)^{3/2} - (1)^{3/2}\right]$$

$$= \frac{\pi}{27}\left[(10)^{3/2} - 1\right] = \frac{\pi}{27}\left[10\sqrt{10} - 1\right] \text{ Sq. units} \text{ — Ans.}$$

Q3 Find the surface area of a belt of the paraboloid formed by revolving the area bounded by the curve $y^2 = 4ax$ about the x-axis. 131

Sol: Given eq. of curve is

$$y^2 = 4ax$$



Let the belt be formed when the area bounded by the curve $y^2 = 4ax$, the lines $x = x_1$, $x = x_2$ & x-axis is revolved about x-axis.

If A be the req. area then

$$A = 2\pi \int_{x_1}^{x_2} y \sqrt{1+(dy/dx)^2}\, dx \quad \text{—} \quad (1)$$

Here $y^2 = 4ax$

$$2y \frac{dy}{dx} = 4a$$

$$\frac{dy}{dx} = \frac{2a}{y}.$$

$$\text{So } \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{4a^2}{y^2}}$$

$$= \sqrt{1+\frac{4a^2}{4ax}}$$

$$= \sqrt{1+\frac{a}{x}}$$

$$= \sqrt{\frac{x+a}{x}}$$

Put in (1)

$$A = 2\pi \int_{x_1}^{x_2} \sqrt{4ax} \cdot \sqrt{\frac{x+a}{x}} \, dx$$

$$= 2\pi \cdot 2\sqrt{a} \int_{x_1}^{x_2} \sqrt{x+a} \, dx$$

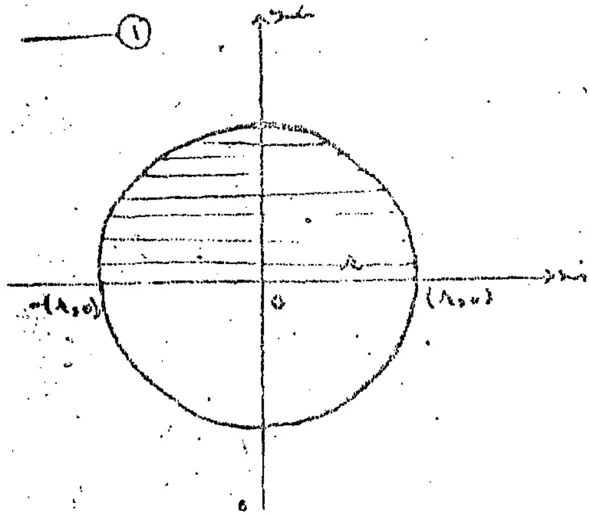
$$= 4\pi\sqrt{a} \left| \frac{(x+a)^{3/2}}{3/2} \right|_{x_1}^{x_2}$$

$$= \frac{8\pi\sqrt{a}}{3} \left[(x_2+a)^{3/2} - (x_1+a)^{3/2} \right]$$

Q4 Find the surface area of a sphere of radius r.

Sol: The req. surface area is generated when a circle $x^2+y^2=r^2$ is revolved about x-axis then the area is

$$A = 2\pi \int_{-r}^{r} y \sqrt{1+(dy/dx)^2} \, dx \quad \text{——} \textcircled{1}$$



Here $x^2+y^2 = r^2$

or $y^2 = r^2 - x^2$

$$2y \cdot \frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{x}{y}$$

So $\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{x^2}{y^2}}$

$$= \sqrt{\frac{y^2+x^2}{y^2}}$$

$$= \sqrt{\frac{r^2}{r^2-x^2}}$$

$$= \frac{r}{\sqrt{r^2-x^2}}$$

Put in (1)

$$A = 2\pi \int_{-r}^{r} \sqrt{r^2 - x^2} \cdot \frac{r}{\sqrt{r^2-x^2}}\, dx$$

$$= 2\pi \cdot 2 \int_{0}^{r} r\, dx$$

$$= 4\pi \left| r x \right|_{0}^{r}$$

$$= 4\pi (r^2 - 0)$$

$$A = 4\pi r^2 \text{ Sq. units}$$

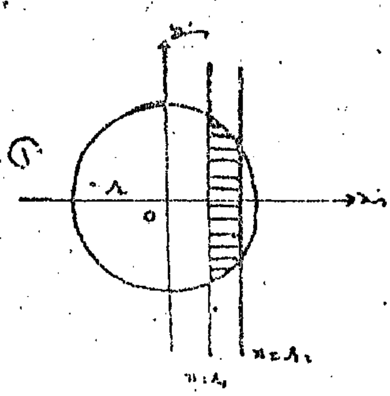
Q5. Find the area on a sphere of radius r included between two parallel planes at distances r_1 & r_2 from the Centre ($r_1 < r_2 < r$).

Sol: A sphere of radius r is formed by revolving the circle $x^2+y^2 = r^2$ about x-axis.

We want to find the surface area included b/w planes AB & CD.

Then req. area is

$$A = 2\pi \int_{r_1}^{r_2} y \sqrt{1+(dy/dx)^2}\, dx \quad \text{——— (1)}$$



Here $x^2+y^2 = r^2$

$$y^2 = r^2 - x^2$$

$$2y \frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{x}{y}$$

So $\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{x^2}{y^2}}$

$$= \sqrt{\frac{y^2+x^2}{y^2}}$$

$$= \sqrt{\frac{r^2}{r^2-x^2}}$$

$$= \frac{r}{\sqrt{r^2-x^2}}$$

Put in ①

$$A = 2\pi \int_{r_1}^{r_2} \cancel{\sqrt{r^2-x^2}} \cdot \frac{r}{\cancel{\sqrt{r^2-x^2}}}\, dx$$

$$= 2\pi \int_{r_1}^{r_2} r\, dx$$

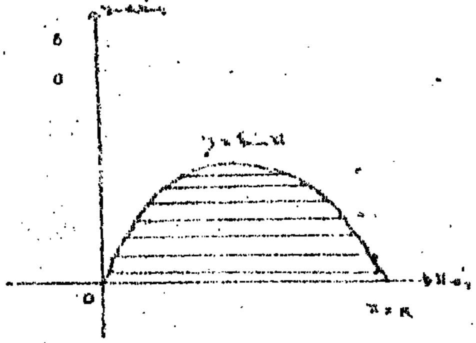
$$= 2\pi \left| r x \right|_{r_1}^{r_2}$$

$$A = 2\pi r (r_2 - r_1) \text{ Sq. units}$$

Q6 Show that the surface of the solid obtained by revolving the area bounded by the arc of the curve $y = \sin x$ from $x = 0$ to $x = \pi$ about x-axis is $2\pi\left[\sqrt{2} + \ln(\sqrt{2}+1)\right]$

Sol. Let A be the req. surface area then

$$A = 2\pi \int_0^{\pi} y \sqrt{1+(dy/dx)^2}\, dx \quad \text{—— ①}$$



Here $y = \sin x$

$$\frac{dy}{dx} = \cos x$$

So $\sqrt{1+(dy/dx)^2} = \sqrt{1+\cos^2 x}$

Put in ①

$$A = 2\pi\int_0^{\pi} \sin x\sqrt{1+\cos^2 x}\,dx$$

$$= 4\pi\int_0^{\pi/2} (1+\cos^2 x)^{1/2}\cdot \sin x\,dx$$

Put $\cos x = t$ | as $x=0;\ t=1$

$-\sin x\,dx = dt$ | $x=\pi/2,\ t=0$

$\Rightarrow \sin x\,dx = -dt$

So $$A = 4\pi\int_1^0 (1+t^2)^{1/2}\cdot(-dt)$$

$$= 4\pi\int_0^1 \sqrt{1+t^2}\,dt$$

$$= 4\pi\left|\frac{t\sqrt{1+t^2}}{2} + \frac{1}{2}\ln\left(\frac{t+\sqrt{1+t^2}}{1}\right)\right|_0^1$$

$$= 4\pi\left[\frac{1\sqrt{1+1}}{2} + \frac{1}{2}\ln\left(1+\sqrt{1+1}\right)\right]$$

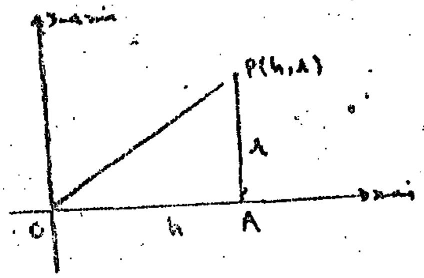
$$= 4\pi\left[\frac{\sqrt{2}}{2} + \frac{1}{2}\ln\left(1+\sqrt{2}\right)\right]$$

$$A = 2\pi\left(\sqrt{2} + \ln(\sqrt{2}+1)\right) \text{ sq. units — Ans.}$$

Q7 Find the lateral surface area of a right circular cone of height h & base radius r.

Sol.

A right circular cone is generated when the line OP is revolved about x-axis



Now eq. of line OP is

$$y-0 = \frac{r-0}{h-0}(x-0)$$

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or $y = \frac{r}{h}x$

let A be the req. area then

$$A = 2\pi\int_0^h y\sqrt{1+(dy/dx)^2}\,dx \quad \text{------ (1)}$$

Here $y = \frac{r}{h}x$

$$\frac{dy}{dx} = \frac{r}{h}$$

Now $\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{r^2}{h^2}}$

$$= \sqrt{\frac{h^2+r^2}{h^2}}$$

$$= \frac{\sqrt{r^2+h^2}}{h}$$

Put in (1)

$$A = 2\pi\int_0^h \frac{r}{h}x\cdot\frac{\sqrt{r^2+h^2}}{h}\,dx$$

$$\frac{2\pi r\sqrt{r^2+h^2}}{h^2}\int_0^h x\,dx$$

$$= \frac{2\pi r\sqrt{r^2+h^2}}{h^2}\left|\frac{x^2}{2}\right|_0^h$$

$$= \frac{\cancel{2}\pi r\sqrt{r^2+h^2}}{\cancel{h^2}}\left(\frac{\cancel{h^2}}{\cancel{2}}\right)$$

$$A = \pi r\sqrt{r^2+h^2} \quad \text{Sq. units.}$$

Q8 Find the surface area generated by revolving the line segment b/w $(x_1, 0)$ & (x_2, h) about y-axis.

Sol:-

The req. surface area is generated if the line segment AB is revolved about y-axis.

Now eq. of line AB is

$$y - 0 = \frac{h-0}{x_2 - x_1}(x - x_1)$$

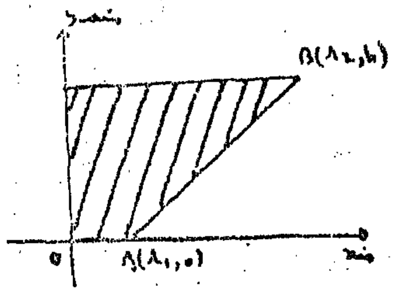
or $$y = \frac{h}{x_2 - x_1}(x - x_1)$$

$$x - x_1 = \frac{(x_2 - x_1)y}{h}$$

$$x = x_1 + \left(\frac{x_2 - x_1}{h}\right)y$$

Let A be the req. surface area then

$$A = 2\pi \int_0^h x\sqrt{1 + (dx/dy)^2}\, dy \quad \text{———— (1)}$$



Here $x = x_1 + \left(\frac{x_2 - x_1}{h}\right)y$

$$\frac{dx}{dy} = \frac{x_2 - x_1}{h}$$

So $$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + \left(\frac{x_2 - x_1}{h}\right)^2}$$

Hence from (1)

$$A = 2\pi \int_0^h \left[x_1 + \left(\frac{x_2 - x_1}{h}\right)y\right] \cdot \sqrt{1 + \left(\frac{x_2 - x_1}{h}\right)^2}\, dy$$

$$= 2\pi \int_0^h \left[x_1 + \left(\frac{x_2 - x_1}{h}\right)y\right] \sqrt{\frac{h^2 + (x_2 - x_1)^2}{h^2}}\, dy$$

$$= \frac{2\pi\sqrt{h^2 + (x_2 - x_1)^2}}{h} \int_0^h \left\{x_1 + \left(\frac{x_2 - x_1}{h}\right)y\right\} dy$$

$$= \frac{2\pi\sqrt{h^2+(r_2-r_1)^2}}{h}\left| r_1 y + \left(\frac{r_2-r_1}{h}\right)\cdot\frac{y^2}{2}\right|_0^h$$

$$= \frac{2\pi\sqrt{h^2+(r_2-r_1)^2}}{h}\left[r_1 h + \left(\frac{r_2-r_1}{h}\right)\cdot\frac{h^2}{2}\right]$$

$$= 2\pi\sqrt{h^2+(r_2-r_1)^2}\cdot\left[r_1 + \frac{(r_2-r_1)}{2}\right]$$

$$= 2\pi\sqrt{h^2+(r_2-r_1)^2}\left[\frac{2r_1+r_2-r_1}{2}\right]$$

$$= \pi\sqrt{h^2+(r_2-r_1)^2}\cdot(r_2+r_1)$$

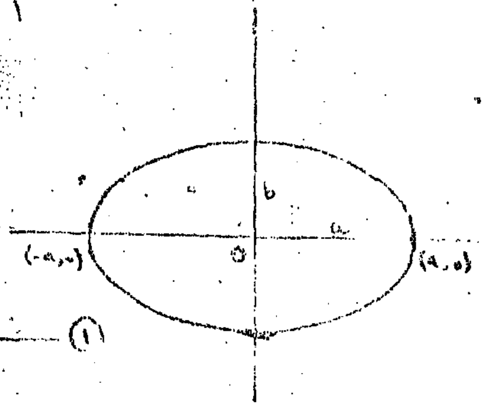
$$A = \pi(r_1+r_2)\sqrt{h^2+(r_2-r_1)^2} \text{ Sq. units}$$

Q9 Prove that the surface area of the prolate spheroid formed by the revolution of an area enclosed by an ellipse of eccentricity e about its major axis is

$$2 \times \text{area of ellipse} \times \left(\sqrt{1-e^2} + \frac{\sin^{-1} e}{e}\right)$$

Sol. Let the eq. of ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



Suppose A be the req. surface area then

$$A = 2\pi\int_{-a}^{a} y\sqrt{1+(dy/dx)^2}\,dx \quad \text{---} (1)$$

Here $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$

Diff. w.r.t. x

$$\frac{2x}{a^2} + \frac{2y}{b^2}\cdot\frac{dy}{dx} = 0$$

$$\frac{2y}{b^2}\cdot\frac{dy}{dx} = -\frac{2x}{a^2}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{b^2x}{a^2y}$$

So

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{b^4x^2}{a^4y^2}}$$

$$= \frac{\sqrt{a^4y^2+b^4x^2}}{a^2y}$$

So from ①

$$A = 2\pi\int_{-a}^{a} y\cdot\frac{\sqrt{a^4y^2+b^4x^2}}{a^2y}\,dx$$

$$= \frac{2\pi}{a^2}\int_{-a}^{a}\sqrt{a^4y^2+b^4x^2}\,dx$$

$$= \frac{4\pi}{a^2}\int_{0}^{a}\sqrt{a^4y^2+b^4x^2}\,dx$$

$$= \frac{4\pi}{a^2}\int_{0}^{a}\sqrt{a^4\cdot\frac{b^2}{a^2}(a^2-x^2)+b^4x^2}\,dx$$

$\because \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\therefore \frac{y^2}{b^2} = 1-\frac{x^2}{a^2}$

$y^2 = \frac{b^2}{a^2}(a^2-x^2)$

$$= \frac{4\pi}{a^2}\int_{0}^{a}\sqrt{a^2b^2(a^2-x^2)+b^4x^2}\,dx$$

$$= \frac{4\pi b}{a^2}\int_{0}^{a}\sqrt{a^2(a^2-x^2)+b^2x^2}\,dx$$

$$= \frac{4\pi b}{a^2}\int_{0}^{a}\sqrt{a^4-a^2x^2+b^2x^2}\,dx$$

$$A = \frac{4\pi b}{a^2}\int_{0}^{a}\sqrt{a^4-(a^2-b^2)x^2}\,dx$$

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$\because\ b^2 = a^2(1-e^2)$

$b^2 = a^2 - a^2e^2$

$\therefore\ a^2e^2 = a^2 - b^2$

So $A = \dfrac{4\pi b}{a^2}\displaystyle\int_0^a \sqrt{a^4 - a^2e^2x^2}\,dx$

$= \dfrac{4\pi ba}{a^2}\displaystyle\int_0^a \sqrt{a^2 - e^2x^2}\,dx$

$= \dfrac{4\pi be}{a}\displaystyle\int_0^a \sqrt{\dfrac{a^2}{e^2} - x^2}\,dx$

$= \dfrac{4\pi be}{a}\displaystyle\int_0^a \sqrt{\left(\dfrac{a}{e}\right)^2 - (x)^2}\,dx$

$= \dfrac{4\pi be}{a}\left| \dfrac{x\sqrt{\left(\frac{a}{e}\right)^2 - (x)^2}}{2} + \dfrac{a^2}{2e^2}\sin^{-1}\left(\dfrac{x}{\frac{a}{e}}\right)\right|_0^a$

$= \dfrac{4\pi be}{a}\left[\dfrac{a\sqrt{\frac{a^2}{e^2} - a^2}}{2} + \dfrac{a^2}{2e^2}\sin^{-1}\left(\dfrac{a}{\frac{a}{e}}\right)\right]$

$= \dfrac{4\pi be}{a}\left[\dfrac{a}{2e}\sqrt{a^2 - a^2e^2} + \dfrac{a^2}{2e^2}\sin^{-1}e\right]$

$= \dfrac{4\pi be}{a}\left[\dfrac{a^2}{2e}\sqrt{1-e^2} + \dfrac{a^2}{2e^2}\sin^{-1}e\right]$

$= \dfrac{4\pi be}{a}\cdot\dfrac{a^2}{2e}\left[\sqrt{1-e^2} + \dfrac{1}{e}\sin^{-1}e\right]$

$= 2\pi ab\left[\sqrt{1-e^2} + \dfrac{1}{e}\sin^{-1}e\right]$

$= 2$ (area of ellipse) $\times \left[\sqrt{1-e^2} + \dfrac{1}{e}\sin^{-1}e\right]$

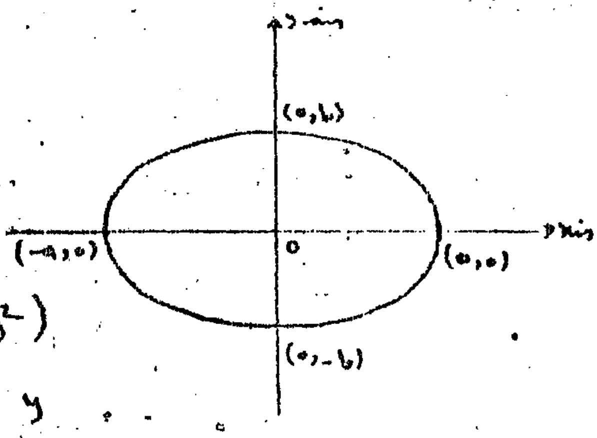
Q10 Prove that the surface area of the ellipsoid formed by the revolution of the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ round its minor axis is

$$2\pi\left[a^2+\frac{b^2}{e}\ln\sqrt{\frac{1+e}{1-e}}\right]$$

e being the eccentricity of ellipse.

Sol: Let A be the req. surface area then

$$A = 2\pi\int_{-b}^{b} x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy \quad \text{——— (1)}$$



Here $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

or $\frac{x^2}{a^2} = 1-\frac{y^2}{b^2}$

$$x^2 = \frac{a^2}{b^2}(b^2-y^2)$$

Diff. w.r.t. y

$$2x\frac{dx}{dy} = \frac{a^2}{b^2}(-2y)$$

$$\frac{dx}{dy} = \frac{-2a^2y}{2b^2x}$$

$$\frac{dx}{dy} = \frac{a^2y}{b^2x}$$

Now

$$\sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1+\frac{a^4y^2}{b^4x^2}}$$

$$= \frac{\sqrt{b^4x^2+a^4y^2}}{b^2x}$$

So from (i) 1646

$$A = 2\pi\int_{-b}^{b} x \cdot \frac{\sqrt{b^4x^2+a^4y^2}}{b^2 x}\,dy$$

$$= \frac{2\pi}{b^2}\int_{-b}^{b}\sqrt{b^4x^2+a^4y^2}\,dy$$

$$= \frac{4\pi}{b^2}\int_{0}^{b}\sqrt{b^4\cdot\frac{a^2}{b^2}(b^2-y^2)+a^4y^2}\,dy$$

$$= \frac{4\pi}{b^2}\int_{0}^{b}\sqrt{a^2b^2(b^2-y^2)+a^4y^2}\,dy$$

$$= \frac{4\pi a}{b^2}\int_{0}^{b}\sqrt{b^2(b^2-y^2)+a^2y^2}\,dy$$

$$= \frac{4\pi a}{b^2}\int_{0}^{b}\sqrt{b^4-b^2y^2+a^2y^2}\,dy$$

$$= \frac{4\pi a}{b^2}\int_{0}^{b}\sqrt{b^4+(a^2-b^2)y^2}\,dy$$

$$\because\ b^2 = a^2(1-e^2)$$

$$b^2 = a^2-a^2e^2$$

$$\Rightarrow a^2e^2 = a^2-b^2$$

So $$A = \frac{4\pi a}{b^2}\int_{0}^{b}\sqrt{b^4+a^2e^2y^2}\,dy$$

$$= \frac{4\pi a}{b^2}\cdot ae\int_{0}^{b}\sqrt{\frac{b^4}{a^2e^2}+y^2}\,dy$$

$$= \frac{4\pi a^2 e}{b^2}\int_{0}^{b}\sqrt{\left(\frac{b^2}{ae}\right)^2+y^2}\,dy$$

$$A = \frac{4\pi a^2 e}{b^2}\left|\frac{y\sqrt{\left(\frac{b^2}{ae}\right)^2+y^2}}{2} + \frac{b^4}{2a^2e^2}\ln\left(\frac{y+\sqrt{\frac{b^4}{a^2e^2}+y^2}}{\frac{b^2}{ae}}\right)\right|_0^b$$

$$= \frac{4\pi a^2 e}{b^2}\left|\frac{y\sqrt{b^4+a^2e^2y^2}}{2ae} + \frac{b^4}{2a^2e^2}\ln\left(\frac{aey+\sqrt{b^4+a^2e^2y^2}}{b^2}\right)\right|_0^b$$

$$= \frac{4\pi a^2 e}{b^2}\left[\frac{b\sqrt{b^4+a^2e^2b^2}}{2ae} + \frac{b^4}{2a^2e^2}\ln\left(\frac{aeb+\sqrt{b^4+a^2e^2b^2}}{b^2}\right)\right]$$

$$= \frac{4\pi a^2 e}{b^2(2ae)}\left[b^2\sqrt{b^2+a^2e^2} + \frac{b^4}{ae}\ln\left(\frac{ae+\sqrt{b^2+a^2e^2}}{b}\right)\right]$$

$$= \frac{2\pi a}{b^2}\left[b^2\sqrt{b^2+a^2e^2} + \frac{b^4}{ae}\ln\left(\frac{ae+\sqrt{b^2+a^2-b^2}}{b}\right)\right]$$

$$= 2\pi a\left[\sqrt{b^2+a^2-b^2} + \frac{b^2}{ae}\ln\left(\frac{ae+a}{b}\right)\right]$$

$$= 2\pi a\left[a + \frac{b^2}{ae}\ln\frac{a(1+e)}{a\sqrt{1-e^2}}\right]$$

$$= 2\pi\left[a^2 + \frac{b^2}{e}\ln\left(\frac{\sqrt{1+e}\sqrt{1+e}}{\sqrt{1-e}\sqrt{1+e}}\right)\right]$$

$$= 2\pi\left[a^2 + \frac{b^2}{e}\ln\sqrt{\frac{1+e}{1-e}}\right]$$

Q11 Find the area of the surface generated by revolving the curve $x = e^{\theta}\sin\theta$, $y = e^{\theta}\cos\theta$ from $\theta = 0$ to $\theta = \pi$ about the y-axis.

Sol:- Let A be the req. surface area then

$$A = 2\pi\int_0^{\pi} x\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta \quad \text{——— (1)}$$

Here. $x = e^{\theta}\sin\theta$, $y = e^{\theta}\cos\theta$

$$\frac{dx}{d\theta} = e^{\theta}\cos\theta + \sin\theta\, e^{\theta} \quad , \quad \frac{dy}{d\theta} = -e^{\theta}\sin\theta + \cos\theta\, e^{\theta}$$

$$= e^{\theta}(\cos\theta + \sin\theta) \qquad\qquad = e^{\theta}(\cos\theta - \sin\theta)$$

Now

$$\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{e^{2\theta}(\cos\theta+\sin\theta)^2 + e^{2\theta}(\cos\theta-\sin\theta)^2}$$

$$= e^{\theta}\sqrt{\cos^2\theta + \sin^2\theta + \cancel{2\sin\theta\cos\theta} + \cos^2\theta + \sin^2\theta - \cancel{2\sin\theta\cos\theta}}$$

$$= e^{\theta}\sqrt{2\cos^2\theta + 2\sin^2\theta}$$

$$= e^{\theta}\sqrt{2}$$

$$= \sqrt{2}\, e^{\theta}$$

So from (1)

$$A = 2\pi\int_0^{\pi} e^{\theta}\sin\theta \cdot \sqrt{2}\cdot e^{\theta}\, d\theta$$

$$= 2\sqrt{2}\,\pi\int_0^{\pi} e^{2\theta}\sin\theta\, d\theta$$

$$= 2\sqrt{2}\pi\left|\frac{e^{2\theta}}{2^2+1^2}(2\sin\theta - \cos\theta)\right|_0^{\pi} \qquad \because \int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

$$= 2\sqrt{2}\pi\left[\frac{e^{2\pi}}{5}(2\sin\pi - \cos\pi) - \frac{e^{0}}{5}(2\sin 0 - \cos 0)\right]$$

$$= 2\sqrt{2}\pi\left[\frac{e^{2\pi}}{5}(1) - \frac{1}{5}(-1)\right]$$

$$= \frac{2\sqrt{2}\pi}{5}\left(e^{2\pi} + 1\right) \text{ sq. units.}$$

Q12 Find the area of the surface generated by revolving the area enclosed by the cycloid. '15

$x = a(\theta - \sin\theta)$; $y = a(1-\cos\theta)$ about $y=0$

Sol.

Let A be the req. area then

$$A = 2\int_0^{\pi} 2\pi . y\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\, d\theta \quad \text{——— (1)}$$

Here $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$

$$\frac{dx}{d\theta} = a(1-\cos\theta) \quad ; \quad \frac{dy}{d\theta} = a\sin\theta$$

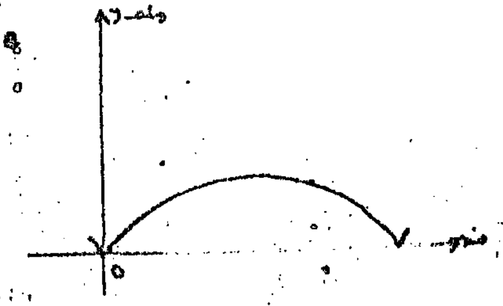
$$\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2} = \sqrt{a^2(1-\cos\theta)^2+a^2\sin^2\theta}$$

$$= a\sqrt{1-2\cos\theta+\cos^2\theta+\sin^2\theta}$$

$$= a\sqrt{2-2\cos\theta}$$

$$= a\sqrt{2(1-\cos\theta)}$$

$$= \sqrt{2}\,a\sqrt{1-\cos\theta}$$



Put in (1)

$$A = 4\pi\int_0^{\pi} a(1-\cos\theta).\sqrt{2}\,a\sqrt{1-\cos\theta}\, d\theta$$

$$= 4\pi.\sqrt{2}\,a^2\int_0^{\pi}(1-\cos\theta)^{3/2}\, d\theta$$

$$= 4\sqrt{2}\,\pi a^2\int_0^{\pi}\left(2\sin^2\theta/2\right)^{3/2} d\theta$$

$$= 4\sqrt{2}\,\pi a^2 . 2\sqrt{2}\int_0^{\pi}\sin^3\theta/2\, d\theta$$

$$A = 16\pi a^2 \int_0^{\pi} \sin^3 \theta/2 \, d\theta$$

Put $\frac{\theta}{2} = t$ | as $\theta = 0, t = 0$; $\theta = \pi, t = \pi/2$

$\theta = 2t$

$d\theta = 2dt$

So $A = 16\pi a^2 \int_0^{\pi/2} \sin^3 t \cdot 2dt$

$$= 32\pi a^2 \int_0^{\pi/2} \sin^3 t \, dt$$

$$= 32\pi a^2 \left[\frac{2}{3}\right] \quad \text{(By walli's law)}$$

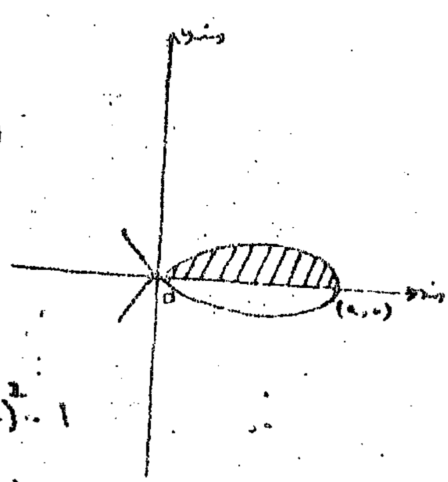
$$A = \frac{64\pi a^2}{3} \text{ sq. units.}$$

Q13 Show that the surface area formed by revolving the area enclosed by the loop of the curve $3ay^2 = x(x-a)^2$ about x-axis is $\frac{1}{3}\pi a^2$.

Sol.

let A be the req. area then

$$A = 2\pi \int_0^a y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx \quad \text{——— (1)}$$



Here $3ay^2 = x(x-a)^2$

Diff. w.r.t. x

$$6ay \cdot \frac{dy}{dx} = x \cdot 2(x-a) + (x-a)^2 \cdot 1$$

$$= (x-a)(2x + x - a)$$

$$6ay \frac{dy}{dx} = (x-a)(3x-a)$$

$$\Rightarrow \frac{dy}{dx} = \frac{(x-a)(3x-a)}{6a^2y}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{(x-a)^2(3x-a)^2}{36a^2y^2}}$$

$$= \sqrt{1+\frac{(x-a)^2(3x-a)^2}{12a\cdot(3ay^2)}}$$

$$= \sqrt{1+\frac{(3x-a)^2}{12ax}}$$

$$= \sqrt{\frac{12ax+9x^2-6ax+a^2}{12ax}}$$

$$= \sqrt{\frac{9x^2+6ax+a^2}{12ax}}$$

$$= \sqrt{\frac{(3x+a)^2}{12ax}}$$

$$= \frac{3x+a}{\sqrt{12ax}}$$

So from ①

$$A = 2\pi\int_0^a \frac{\sqrt{x}}{\sqrt{3a}}(x-a)\cdot\frac{(3x+a)}{\sqrt{12ax}}\,dx$$

$$= \frac{2\pi}{\sqrt{36a^2}}\int_0^a (x-a)(3x+a)\,dx$$

$$= \frac{2\pi}{6a}\int_0^a (3x^2+ax-3ax-a^2)\,dx$$

$$= \frac{\pi}{3a}\int_0^a (3x^2 - 2ax - a^2)\,dx$$

$$= \frac{\pi}{3a}\Big| x^3 - ax^2 - a^2x \Big|_0^a$$

$$= \frac{\pi}{3a}\left[\cancel{a^3} - \cancel{a^3} - a^3\right]$$

$$= -\frac{\pi a^3}{3a}$$

$A = \frac{1}{3}\pi a^2$ sq. units (in magnitude)

Q14. Find the surface area of torus by revolving a disc of radius a about a st. line in its plane at a distance b $(b > a)$ from the centre.

Sol. Since the radius of disc is a

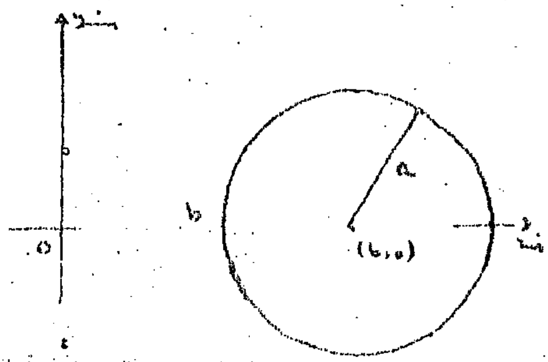
So its circumference is $= 2\pi a$

Since the centre of gravity (centre) of disc describes a circle of radius b, hence if A is the req. area then by first theorem of Pappus

A = (Centre of gravity)(length of path described by c.g.)

$= (2\pi a)(2\pi b)$

$A = 4\pi^2 ab$ sq units

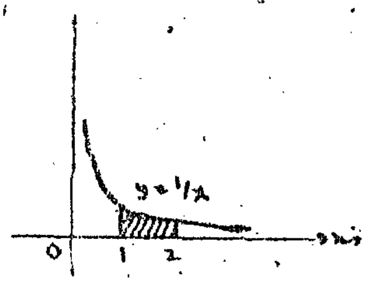


Q15 The curve $y = \frac{1}{x}$, $1 \le x \le 2$ is rotated about x-axis. Find the area of the resulting surface. If $1 \le x < \infty$, show that the volume of the solid generated is finite but its area is infinite.

Sol: Given eq. of curve is $y = \frac{1}{x}$

Let A be the area of resulting surface then

$$A = 2\pi \int_1^2 y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \quad \text{——} \ (1)$$



Here $y = \frac{1}{x}$

$$\frac{dy}{dx} = \frac{-1}{x^2}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{1}{x^4}} = \frac{\sqrt{x^4+1}}{x^2}$$

Put in (1)

$$A = 2\pi \int_1^2 \frac{1}{x} \cdot \frac{\sqrt{x^4+1}}{x^2}\, dx$$

$$A = 2\pi \int_1^2 \frac{\sqrt{1+x^4}}{x^3}\, dx$$

Consider $\int \frac{\sqrt{1+(x^2)^2}}{x^3}\, dx = \int \frac{\sqrt{1+(x^2)^2}}{x^4}\, x\,dx$

Put $x^2 = \tan\theta$

$2x\,dx = \sec^2\theta\, d\theta$

$x\,dx = \frac{\sec^2\theta\, d\theta}{2}$

So $\int \frac{\sqrt{1+x^4}}{x^3}\, dx = \int \frac{\sqrt{1+\tan^2\theta}}{\tan^2\theta} \cdot \frac{\sec^2\theta\, d\theta}{2}$

$$= \frac{1}{2}\int \cot^2\theta \cdot \sec^3\theta\, d\theta$$

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$$= \frac{1}{2}\int \frac{\cos^2\theta}{\sin^2\theta} \cdot \frac{1}{\cos^3\theta}\, d\theta$$

$$= \frac{1}{2}\int \frac{1}{\sin^2\theta\cos\theta}\, d\theta$$

$$= \frac{1}{2}\int \frac{1}{\sin^2\theta\cos^2\theta}\cos\theta\, d\theta$$

$$= \frac{1}{2}\int \frac{1}{\sin^2\theta(1-\sin^2\theta)} \cdot \cos\theta\, d\theta$$

Put $\sin\theta = t$

$\cos\theta\, d\theta = dt$

So

$$\int \frac{\sqrt{1+x^4}}{x^3}dx = \frac{1}{2}\int \frac{1}{t^2(1-t^2)}dt$$

$$= \frac{1}{2}\int \frac{1}{t^2} + \frac{1}{1-t^2}\, dt$$

$$= \frac{1}{2}\left[\frac{-1}{t} + \frac{1}{2}\ln\left(\frac{1+t}{1-t}\right)\right]$$

$$= \frac{1}{2}\left[\frac{-1}{\sin\theta} + \frac{1}{2}\ln\left(\frac{1+\sin\theta}{1-\sin\theta}\right)\right]$$

Now $\sin\theta = \frac{\sin\theta}{\cos\theta}\cdot\cos\theta$

$$= \tan\theta \cdot \cos\theta$$

$$= \frac{\tan\theta}{\sec\theta}$$

$$= \frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$$

$$\sin\theta = \frac{x^2}{\sqrt{1+x^4}}$$

So

$$\int \frac{\sqrt{1+x^4}}{x^3}dx = \frac{1}{2}\left[-\frac{\sqrt{1+x^4}}{x^2} + \frac{1}{2}\ln\left(\frac{1+\frac{x^2}{\sqrt{1+x^4}}}{1-\frac{x^2}{\sqrt{1+x^4}}}\right)\right]$$

$$\int\frac{\sqrt{1+x^4}}{x^3}dx = \frac{1}{2}\left[-\frac{\sqrt{1+x^4}}{x^2}+\frac{1}{2}\ln\left(\frac{\sqrt{1+x^4}+x^2}{\sqrt{1+x^4}-x^2}\right)\right]$$

Put value in above eq.

$$A = 2\pi\cdot\frac{1}{2}\left|-\frac{\sqrt{1+x^4}}{x^2}+\frac{1}{2}\ln\left(\frac{\sqrt{1+x^4}+x^2}{\sqrt{1+x^4}-x^2}\right)\right|_1^2$$

$$= \pi\left[\left(-\frac{\sqrt{17}}{4}+\frac{1}{2}\ln\left(\frac{\sqrt{17}+4}{\sqrt{17}-4}\right)\right)-\left(-\frac{\sqrt{2}}{1}+\frac{1}{2}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\right)\right]$$

$$A = \pi\left[-\frac{\sqrt{17}}{4}+\frac{1}{2}\ln\left(\frac{\sqrt{17}+4}{\sqrt{17}-4}\right)+\sqrt{2}-\frac{1}{2}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\right]$$ sq. units — Ans

If $1 \le x < \infty$ then Volume of generated solid is

$$V = \pi\int_1^\infty y^2\,dx$$

$$= \pi\int_1^\infty \frac{1}{x^2}\,dx$$

$$= \pi\left|\frac{-1}{x}\right|_1^\infty$$

$$= -\pi\left|\frac{1}{x}\right|_1^\infty$$

$$= -\pi\left(\frac{1}{\infty}-1\right)$$

$$= -\pi(0-1)$$

$V = \pi$ cubic units. (which is finite)

& area of generated solid is

$$A = 2\pi\int_1^\infty \frac{\sqrt{1+x^4}}{x^3}\,dx$$

we know that if $f(x) \le g(x)$

& if $\int_1^\infty f(x)dx = \infty$ then $\int_1^\infty g(x)dx = \infty$

Now

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$$\frac{\sqrt{1+x^4}}{x^3} - \frac{1}{x} = \frac{\sqrt{1+x^4} - x^2}{x^3} \geq 0$$

$$\Rightarrow \frac{\sqrt{1+x^4}}{x^3} - \frac{1}{x} \geq 0$$

$$\Rightarrow \frac{\sqrt{1+x^4}}{x^3} \geq \frac{1}{x}$$

or $\frac{1}{x} \leq \frac{\sqrt{1+x^4}}{x^3}$

But $\int_1^{\infty} \frac{1}{x}\,dx = \infty$

$$\Rightarrow \int_1^{\infty} \frac{\sqrt{1+x^4}}{x^3}\,dx = \infty$$

Hence Area is <u>infinite</u>